Two Sequences

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abstract

In this note we presents some formulas related with the recurrences:

(i) $u_{n+5} = u_{n+3} + u_{n+2} + u_n \ , u_0 = u_1 = u_2 = u_3 = 0, u_4 = 1$

(ii) $v_{n+5} = v_{n+4} + v_{n+1} + v_n \ , v_0 = v_1 = v_2 = v_3 = 0, v_4 = 1$

1. Introducción

Para $n \in \mathbb{N} \cup \{0\}$, sean u_n, v_n las sucesiones definidas por:

$$u_{n+5} = u_{n+3} + u_{n+2} + u_n \ , u_0 = u_1 = u_2 = u_3 = 0, u_4 = 1 \tag{1}$$

$$v_{n+5} = v_{n+4} + v_{n+1} + v_n \ , v_0 = v_1 = v_2 = v_3 = 0, v_4 = 1 \tag{2}$$

Algunos valores de u_n, v_n son:

$$\{u_n\} = \{0,0,0,0,1,0,1,1,1,3,2,5,6,8,14,16,27,36,...\} \tag{3}$$

$$\{v_n\} = \{0,0,0,0,1,1,1,1,2,4,6,8,11,17,27,41,60,88,...\} \tag{4}$$

En esta nota mostramos algunas fórmulas relacionadas con las sucesiones (1) y (2).

2. Los números α y β

Definición: Se definen los números α y β como sigue:

$$\alpha = \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 0.6997370221126801934... \tag{5}$$

$$\beta = \lim_{n\to\infty} \frac{v_n}{v_{n+1}} = 0.6679607074962346057... \tag{6}$$

El número α es una raíz de la ecuación:

$$z^5 + z^3 + z^2 - 1 = 0 \quad (7)$$

El número β es una raíz de la ecuación:

$$z^5 + z^4 + z - 1 = 0 \quad (8)$$

3. Fórmulas cerradas para u_n y v_n

Para $n \in \mathbb{N} \cup \{0\}$, se tiene:

$$u_n = \sum_{z^5+z^3+z^2-1=0} \frac{z^{-n+2}}{5z^3+3z+2} \quad (9)$$

$$v_n = \sum_{z^5+z^4+z-1=0} \frac{z^{-n+3}}{5z^4+4z^3+1} \quad (10)$$

Las fórmulas (9) y (10) se pueden escribir como:

$$u_n = \sum_{z^5-z^3-z^2-1=0} \frac{z^{n+1}}{2z^3+3z^2+5} \quad (11)$$

$$v_n = \sum_{z^5-z^4-z-1=0} \frac{z^{n+1}}{z^4+4z+5} \quad (12)$$

Otra representación alternativa para u_n es:

Si $\alpha \in \mathbb{R}, z_1, \overline{z}_1, z_2, \overline{z}_2 \in \mathbb{C}$ son las raíces de la ecuación:

$$z^5 + z^3 + z^2 - 1 = 0 \quad (13)$$

donde $\overline{z}_1, \overline{z}_2$ son complejos conjugados de z_1, z_2 respectivamente , entonces:

$$u_n = \frac{\alpha^{-n+2}}{5\alpha^3+3\alpha+2} + 2\operatorname{Re}\left(\frac{z_1^{-n+2}}{5z_1^3+3z_1+2}\right) + 2\operatorname{Re}\left(\frac{z_2^{-n+2}}{5z_2^3+3z_2+2}\right) \quad ,n \in \mathbb{N} \cup \{0\} \quad (14)$$

Para v_n se tiene:

$$v_n = \frac{\beta^{-n+3}}{5\beta^4+4\beta^3+1} + 2\operatorname{Re}\left(\frac{w_1^{-n+3}}{5w_1^4+4w_1^3+1}\right) + 2\operatorname{Re}\left(\frac{w_2^{-n+3}}{5w_2^4+4w_2^3+1}\right) \quad ,n \in \mathbb{N} \cup \{0\} \quad (15)$$

donde $\beta \in \mathbb{R}, w_1, \overline{w}_1, w_2, \overline{w}_2 \in \mathbb{C}$ son las raíces de la ecuación:

$$z^5 + z^4 + z - 1 = 0 \quad (16)$$

y $\overline{w}_1, \overline{w}_2$ son complejos conjugados de w_1 y w_2 respectivamente.

4. Las sucesiones α_n y β_n

Definición: Se definen las sucesiones α_n y β_n como sigue:

$$\alpha_n = \frac{u_n}{u_{n+1}} = \left\{0,1,1,\frac{1}{3},\frac{3}{2},\frac{2}{5},\frac{5}{6},\frac{3}{4},\frac{4}{7},\frac{7}{8},\ldots\right\} \tag{17}$$

$$\beta_n = \frac{v_n}{v_{n+1}} = \left\{0,1,1,1,\frac{1}{2},\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{8}{11},\frac{11}{17},\ldots\right\} \tag{18}$$

En (17) y (18) no se consideran los casos de división por cero.

Por definición se verifica que:

$$\alpha = \lim_{n\to\infty} \alpha_n \tag{19}$$

$$\beta = \lim_{n\to\infty} \beta_n \tag{20}$$

5. Recurrencias para α_n y β_n

Para $n \in \mathbb{N} \cup \{0\}$ se tiene:

$$\alpha_{n+4} = \frac{1}{\alpha_{n+3}\left(1+\alpha_{n+2}\left(1+\alpha_{n+1}\alpha_n\right)\right)}, \alpha_0 = 0, \alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1/3 \tag{21}$$

$$\beta_{n+4} = \frac{1}{1+\left(1+\beta_n\right)\beta_{n+1}\beta_{n+2}\beta_{n+3}}, \beta_0 = 0, \beta_1 = 1, \beta_2 = 1, \beta_3 = 1 \tag{22}$$

(21) y (22) generan los conjuntos (17) y (18) respectivamente.

6. Dos sucesiones alternativas para α y β

Sean U_n y V_n definidas por:

$$U_{n+5} = 10U_{n+4} - 28U_{n+3} + 44U_{n+2} - 40U_{n+1} + 16U_n \tag{23}$$

$$U_0 = U_1 = U_2 = U_3 = 0, U_4 = 1 \tag{24}$$

$$V_{n+5} = 10V_{n+4} - 32V_{n+3} + 56V_{n+2} - 48V_{n+1} + 16V_n \tag{25}$$

$$V_0 = V_1 = V_2 = V_3 = 0, V_4 = 1 \tag{26}$$

Las sucesiones U_n y V_n satisfacen las siguientes relaciones:

$$\alpha = \lim_{n\to\infty}\left(1-\frac{2U_n}{U_{n+1}}\right) \tag{27}$$

$$\beta = \lim_{n\to\infty}\left(1-\frac{2V_n}{V_{n+1}}\right) \tag{28}$$

7. Dos limites

$$\lim_{n\to\infty}\alpha^n u_n = \frac{\alpha^2}{5\alpha^3+3\alpha+2} \tag{29}$$

$$\lim_{n\to\infty}\beta^n v_n = \frac{\beta^3}{5\beta^4+4\beta^3+1} \tag{30}$$

8. Integrales para α y β

$$\alpha = -\frac{1}{2\pi}\int_0^{2\pi}\frac{\left(12+9e^{ix}+2e^{2ix}\right)e^{2ix}}{2+10e^{ix}+14e^{2ix}+11e^{3ix}+5e^{4ix}+e^{5ix}}dx \tag{31}$$

$$\beta = -1+\frac{1}{4\pi}\int_0^{2\pi}\frac{26-152e^{ix}+28e^{2ix}-24e^{3ix}-6e^{4ix}}{13-29e^{ix}-22e^{2ix}-18e^{3ix}-7e^{4ix}-e^{5ix}}dx \tag{32}$$

9. Algunas relaciones

$$\lim_{n\to\infty}\frac{u_{2n}}{u_{2n+2}} = \alpha^2 \tag{33}$$

$$\lim_{n\to\infty}\frac{u_{3n}}{u_{3n+3}} = \alpha^3 \tag{34}$$

$$\lim_{n\to\infty}\frac{v_{4n}}{v_{4n+4}} = \beta^4 \tag{35}$$

$$\lim_{n\to\infty}\alpha_{2n}\alpha_{2n+1} = \alpha^2 \tag{36}$$

$$\lim_{n\to\infty}\alpha_{3n}\alpha_{3n+1}\alpha_{3n+2} = \alpha^3 \tag{37}$$

$$\lim_{n\to\infty}\beta_{4n}\beta_{4n+1}\beta_{4n+2}\beta_{4n+3} = \beta^4 \tag{38}$$

10. Sucesiones para $\alpha^2, \alpha^3, \beta^4$

Sea r_n la sucesión definida por:

$$r_{n+5} = 2r_{n+4} - r_{n+3} + r_{n+2} + 2r_{n+1} + r_n \tag{39}$$

$$r_0 = r_1 = r_2 = r_3 = 0, r_4 = 1 \tag{40}$$

Se tiene:

$$\lim_{n\to\infty} \frac{r_n}{r_{n+1}} = \alpha^2 \tag{41}$$

Sea s_n la sucesión definida por:

$$s_{n+5} = 3s_{n+4} - 2s_{n+3} + 4s_{n+2} + 3s_{n+1} + s_n \tag{42}$$

$$s_0 = s_1 = s_2 = s_3 = 0, s_4 = 1 \tag{43}$$

Se tiene:

$$\lim_{n\to\infty} \frac{s_n}{s_{n+1}} = \alpha^3 \tag{44}$$

Sea t_n la sucesión definida por:

$$t_{n+5} = 5t_{n+4} - 2t_{n+3} + 10t_{n+2} + 3t_{n+1} + t_n \tag{45}$$

$$t_0 = t_1 = t_2 = t_3 = 0, t_1 = 1 \tag{46}$$

Se tiene:

$$\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = \beta^4 \tag{47}$$

11. Algunas fórmulas para pi

$$\pi = 4\tan^{-1}\left(\alpha^2\right) + 4\tan^{-1}\left(\alpha^3\right) \tag{48}$$

$$\pi = 4\tan^{-1}\left(\beta\right) + 4\tan^{-1}\left(\beta^4\right) \tag{49}$$

$$\pi = 4\sum_{n=1}^{\infty} \frac{c_n}{n+1} \alpha^{n+1} \tag{50}$$

donde

$$c_{n+10} = -\left(c_{n+6} + c_{n+4} + c_n\right) \quad , n \in \mathbb{N} \cup \{0\} \tag{51}$$

$$c_0 = 0, c_1 = 2, c_2 = 3, c_3 = 0, c_4 = 0, c_5 = -2, c_6 = 0, c_7 = 0, c_8 = -3, c_9 = 2 \tag{52}$$

$$\pi = 4\sum_{n=0}^{\infty} \frac{c_n}{n+1} \beta^{n+1} \tag{53}$$

donde

$$c_{n+10} = -\left(c_{n+8} + c_{n+2} + c_n\right) \quad , n \in \mathbb{N} \cup \{0\} \tag{54}$$

$$c_0 = 1, c_1 = 0, c_2 = -1, c_3 = 4, c_4 = 1, c_5 = 0, c_6 = -1, c_7 = 0, c_8 = 1, c_9 = 0 \tag{55}$$

$$\pi = 4\sum_{n=0}^{\infty} c_n \alpha^{n+2} \left(\frac{2}{n+2} + \frac{3\alpha}{n+3} + \frac{3\alpha^5}{n+7} + \frac{2\alpha^{n+6}}{n+8} \right) \tag{56}$$

donde

$$c_{n+10} = -\left(c_{n+6} + c_{n+4} + c_n\right) \quad , n \in \mathbb{N} \cup \{0\} \tag{57}$$

$$c_0 = 1, c_1 = 0, c_2 = 0, c_3 = 0, c_4 = -1, c_5 = 0, c_6 = -1, c_7 = 0, c_8 = 1, c_9 = 0 \tag{58}$$

$$\pi = 4\sum_{n=0}^{\infty} c_n \beta^{n+1} \left(\frac{1}{n+1} + \frac{4\beta^3}{n+4} + \frac{4\beta^5}{n+6} + \frac{\beta^8}{n+9} \right) \tag{59}$$

donde

$$c_{n+10} = -\left(c_{n+8} + c_{n+2} + c_n\right) \quad , n \in \mathbb{N} \cup \{0\} \tag{60}$$

$$c_0 = 1, c_1 = 0, c_2 = -1, c_3 = 0, c_4 = 1, c_5 = 0, c_6 = -1, c_7 = 0, c_8 = 0, c_9 = 0 \tag{61}$$

$$\pi = 4\sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n \alpha^{4n+2k+2} \left(\frac{1}{2n+k+1} + \frac{3\alpha}{4n+2k+3} + \frac{3\alpha^5}{4n+2k+7} + \frac{\alpha^6}{2n+k+4} \right) \tag{62}$$

$$\pi = 4\sum_{n=0}^{\infty} (-1)^n c_n \beta^{2n+1} \left(\frac{1}{2n+1} + \frac{2\beta^3}{n+2} + \frac{2\beta^5}{n+3} + \frac{\beta^8}{2n+9} \right) \tag{63}$$

donde

$$c_n = \sum_{k=0}^{[n/4]} (-1)^k \tag{64}$$

$$\pi = 4\sin^{-1}\left(\frac{\alpha^2}{\sqrt{1+\alpha^4}} \right) + 4\sin^{-1}\left(\frac{\alpha^3}{\sqrt{1+\alpha^6}} \right) \tag{65}$$

$$\pi = 4\sin^{-1}\left(\frac{\beta}{\sqrt{1+\beta^2}}\right) + 4\sin^{-1}\left(\frac{\beta^4}{\sqrt{1+\beta^8}}\right) \tag{66}$$

$$\pi = 4\sum_{n=0}^{\infty} \alpha^{12n+2}\left(\frac{(-1)^n}{6n+1} + \frac{\alpha}{4n+1} - \frac{(-1)^n \alpha^4}{6n+3} - \frac{\alpha^7}{4n+3} + \frac{(-1)^n \alpha^8}{6n+5}\right) \tag{67}$$

$$\pi = 4\sum_{n=0}^{\infty} \beta^{8n+1}\left(\frac{1}{8n+1} - \frac{\beta^2}{8n+3} + \frac{(-1)^n \beta^3}{2n+1} + \frac{\beta^4}{8n+5} - \frac{\beta^6}{8n+7}\right) \tag{68}$$

$$\frac{\pi}{4} = \lim_{n\to\infty}\left(\tan^{-1}\left(\frac{r_n}{r_{n+1}}\right) + \tan^{-1}\left(\frac{s_n}{s_{n+1}}\right)\right) \tag{69}$$

donde r_n y s_n se definen por (39) y (42).

$$\frac{\pi}{4} = \lim_{n\to\infty}\left(\tan^{-1}(\beta_n) + \tan^{-1}\left(\frac{t_n}{t_{n+1}}\right)\right) \tag{70}$$

donde β_n y t_n se definen por (22) y (45).

$$\pi = 4\int_0^{\alpha} \frac{2x+3x^2+3x^6+2x^7}{(1+x^4)(1+x^6)}\,dx \tag{71}$$

$$\pi = 4\int_0^{\beta} \frac{1+4x^3+4x^5+x^8}{(1+x^2)(1+x^8)}\,dx \tag{72}$$

$$\pi = 4\int_{1/\alpha}^{\infty} \frac{2x+3x^2+3x^6+2x^7}{(1+x^4)(1+x^6)}\,dx \tag{73}$$

$$\pi = 4\int_{1/\beta}^{\infty} \frac{1+4x^3+4x^5+x^8}{(1+x^2)(1+x^8)}\,dx \tag{74}$$

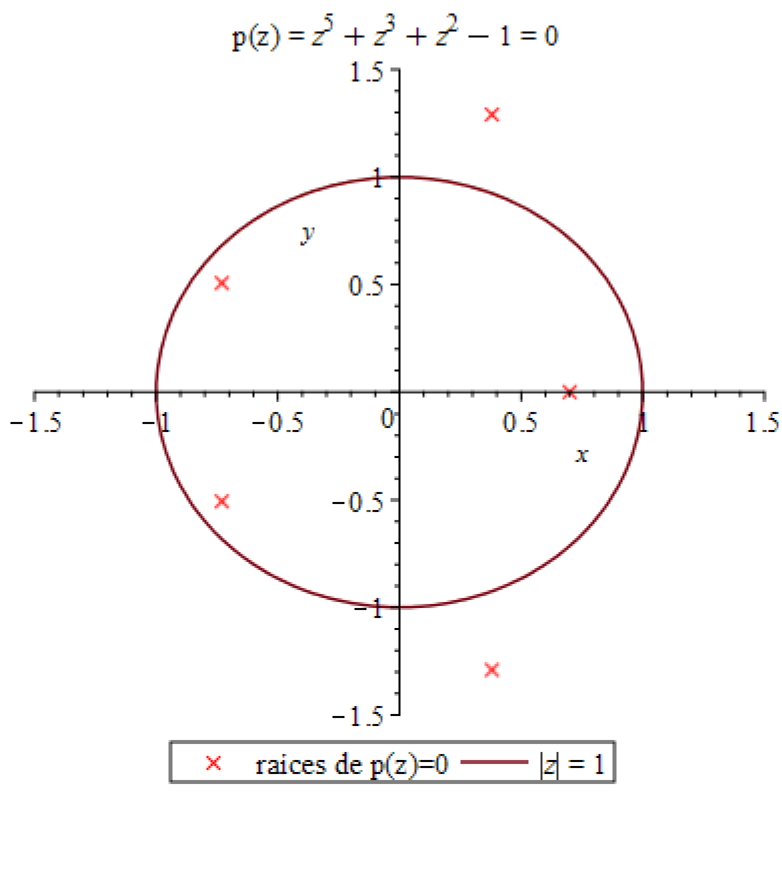
$$\pi = 2\int_{\alpha}^{1/\alpha} \frac{2x+3x^2+3x^6+2x^7}{(1+x^4)(1+x^6)}\,dx \tag{75}$$

$$\pi = 2\int_{\beta}^{1/\beta} \frac{1+4x^3+4x^5+x^8}{(1+x^2)(1+x^8)}\,dx \tag{76}$$

$$\pi = \int_0^{\infty} \frac{2x+3x^2+3x^6+2x^7}{(1+x^4)(1+x^6)}\,dx = \int_0^{\infty} \frac{1+4x^3+4x^5+x^8}{(1+x^2)(1+x^8)}\,dx \tag{77}$$

12. Raíces del polinomio $p(z)=z^5+z^3+z^2-1=0$

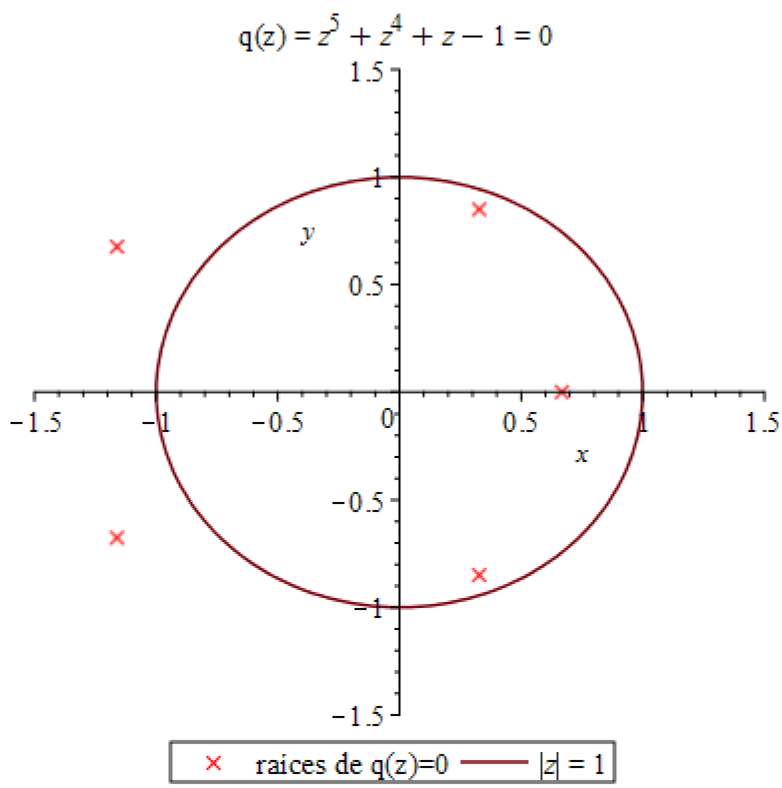
$$p(z)=z^5+z^3+z^2-1=0 \Rightarrow \begin{cases} \alpha = 0.699737... \\ z_1 = 0.380122...+i\times 1.290554... \\ \overline{z}_1 = 0.380122...-i\times 1.290554... \\ z_2 = -0.729991...+i\times 0.506621... \\ \overline{z}_2 = -0.729991...-i\times 0.506621... \end{cases} \tag{78}$$



Observación: $|z_1|=|\overline{z}_1|>1, |z_2|=|\overline{z}_2|<1$.

13. Raíces del polinomio $q(z)=z^5+z^4+z-1=0$

$$q(z)=z^5+z^4+z-1=0 \Rightarrow \begin{cases} \beta = 0.667960... \\ w_1 = 0.327319...+i\times 0.849777... \\ \bar{w}_1 = 0.327319...-i\times 0.849777... \\ w_2 = -1.161299...+i\times 0.675809... \\ \bar{w}_2 = -1.161299...-i\times 0.675809... \end{cases} \qquad (79)$$



Observación: $|w_1|=|\bar{w}_1|<1, |w_2|=|\bar{w}_2|>1$.

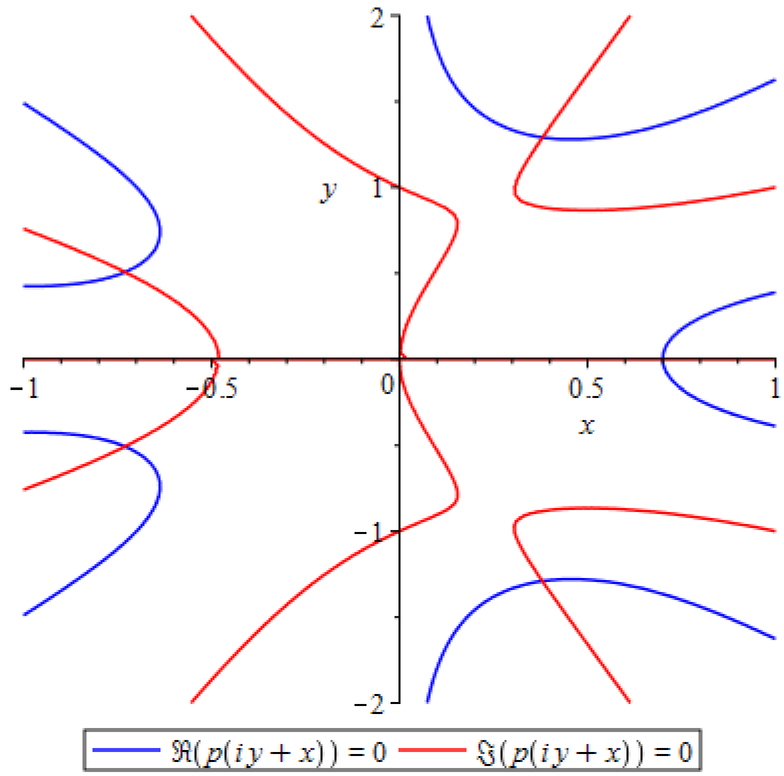
$$14.\ \mathrm{Re}\big(p(x+iy)\big) \quad , \quad \mathrm{Im}(p(x+iy))$$

Sea $p(z)=z^5+z^3+z^2-1$, se tiene:

$$\mathrm{Re}(p(x+iy)) = x^5-10x^3y^3+5xy^4+x^3-3xy^2+x^2-y^2-1 \tag{80}$$

$$\mathrm{Im}(p(x+iy)) = 5x^4y-10x^2y^3+y^5+3x^2y-y^3+2xy \tag{81}$$

$$\{\mathrm{Re}(p(x+iy))=0\}\cap\{\mathrm{Im}(p(x+iy))=0\}=\{\alpha, z_1, \overline{z}_1, z_2, \overline{z}_2\} \tag{82}$$



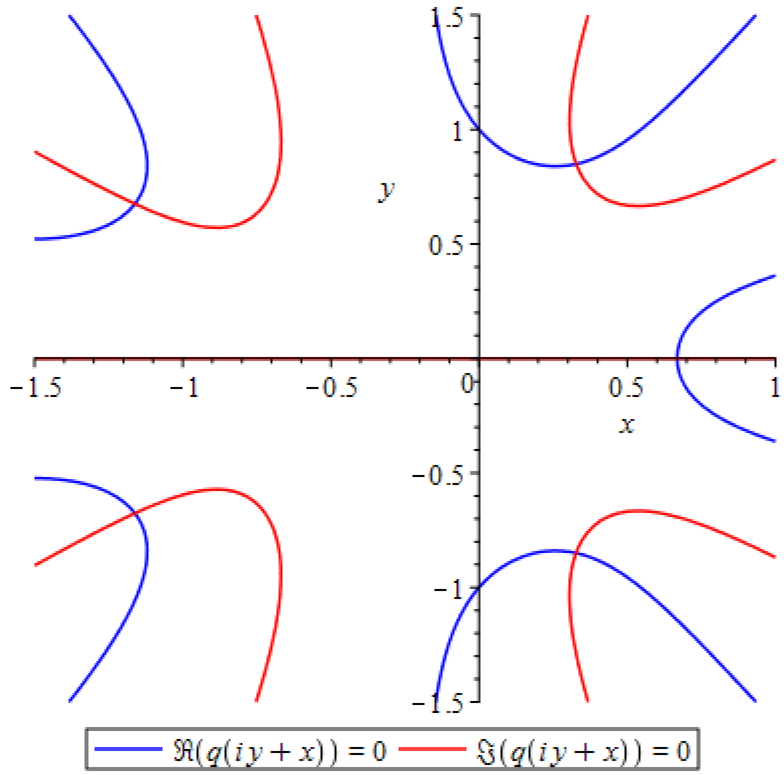
15. $\mathrm{Re}(q(x+iy)), \mathrm{Im}(q(x+iy))$

Sea $q(z) = z^5 + z^4 + z - 1$, se tiene:

$$\mathrm{Re}(q(x+iy)) = x^5 - 10x^3y^2 + 5xy^4 + x^4 - 6x^2y^2 + y^4 + x - 1 \tag{83}$$

$$\mathrm{Im}(q(x+iy)) = 5x^4y - 10x^2y^3 + y^5 + 4x^3y - 4xy^3 + y \tag{84}$$

$$\{\mathrm{Re}(q(x+iy)) = 0\} \cap \{\mathrm{Im}(q(x+iy)) = 0\} = \{\beta, w_1, \overline{w}_1, w_2, \overline{w}_2\} \tag{85}$$



16. Método de Newton para las raíces de la ecuación:

$$z^5 + z^3 + z^2 - 1 = 0$$

$$x_{n+1} = \frac{4x_n^5 + 2x_n^3 + x_n^2 + 1}{5x_n^4 + 3x_n^2 + 2x_n} \quad , x_1 = 1 \Rightarrow \lim_{n\to\infty} x_n = \alpha \tag{86}$$

$$x_{n+1} = \frac{4x_n^5 + 2x_n^3 + x_n^2 + 1}{5x_n^4 + 3x_n^2 + 2x_n} \quad , x_1 = \frac{1}{2} + i \Rightarrow \lim_{n\to\infty} x_n = z_1 \tag{87}$$

$$x_{n+1} = \frac{4x_n^5 + 2x_n^3 + x_n^2 + 1}{5x_n^4 + 3x_n^2 + 2x_n} \quad , x_1 = \frac{1}{2} - i \Rightarrow \lim_{n\to\infty} x_n = \overline{z}_1 \tag{88}$$

$$x_{n+1} = \frac{4x_n^5 + 2x_n^3 + x_n^2 + 1}{5x_n^4 + 3x_n^2 + 2x_n} \quad , x_1 = -\frac{1}{2} + \frac{i}{2} \Rightarrow \lim_{n\to\infty} x_n = z_2 \tag{89}$$

$$x_{n+1} = \frac{4x_n^5 + 2x_n^3 + x_n^2 + 1}{5x_n^4 + 3x_n^2 + 2x_n} \quad , x_1 = -\frac{1}{2} - \frac{i}{2} \Rightarrow \lim_{n\to\infty} x_n = \overline{z}_2 \tag{90}$$

17. Método de Newton para las raíces de la ecuación:

$$z^5 + z^4 + z - 1 = 0$$

$$x_{n+1} = \frac{4x_n^5 + 3x_n^4 + 1}{5x_n^4 + 4x_n^3 + 1} \quad , x_1 = 1 \Rightarrow \lim_{n\to\infty} x_n = \beta \tag{91}$$

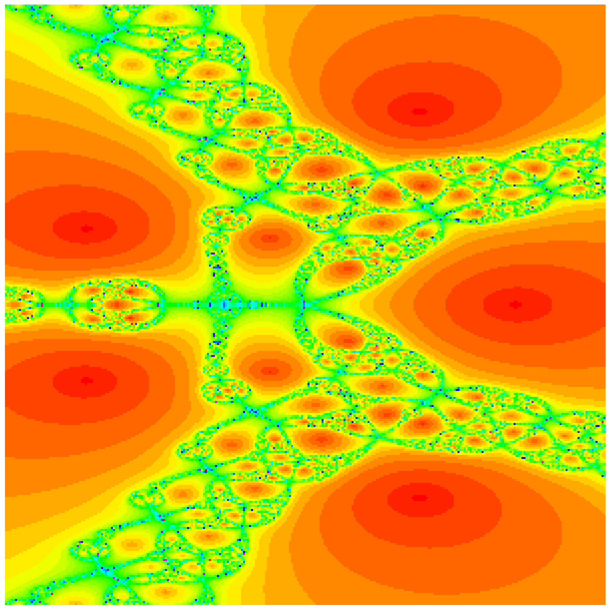
$$x_{n+1} = \frac{4x_n^5 + 3x_n^4 + 1}{5x_n^4 + 4x_n^3 + 1} \quad , x_1 = \frac{1}{2} + i \Rightarrow \lim_{n\to\infty} x_n = w_1 \tag{92}$$

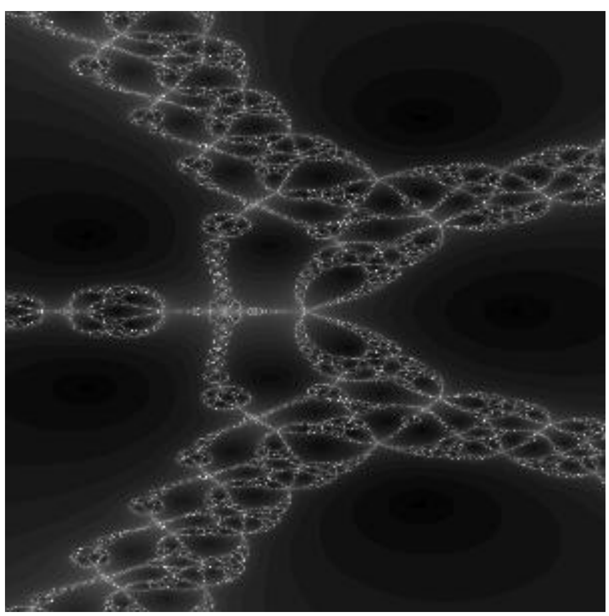
$$x_{n+1} = \frac{4x_n^5 + 3x_n^4 + 1}{5x_n^4 + 4x_n^3 + 1} \quad , x_1 = \frac{1}{2} - i \Rightarrow \lim_{n\to\infty} x_n = \overline{w}_1 \tag{93}$$

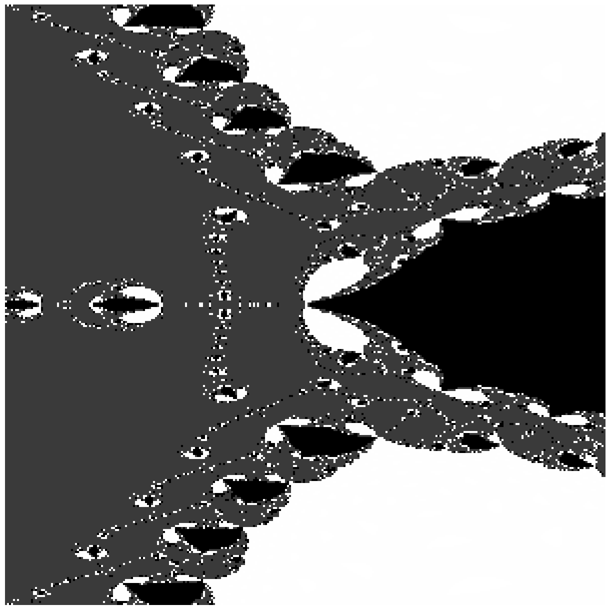
$$x_{n+1} = \frac{4x_n^5 + 3x_n^4 + 1}{5x_n^4 + 4x_n^3 + 1} \quad , x_1 = -1 + \frac{i}{2} \Rightarrow \lim_{n\to\infty} x_n = w_2 \tag{94}$$

$$x_{n+1} = \frac{4x_n^5 + 3x_n^4 + 1}{5x_n^4 + 4x_n^3 + 1} \quad , x_1 = -1 - \frac{i}{2} \Rightarrow \lim_{n\to\infty} x_n = \overline{w}_2 \tag{95}$$

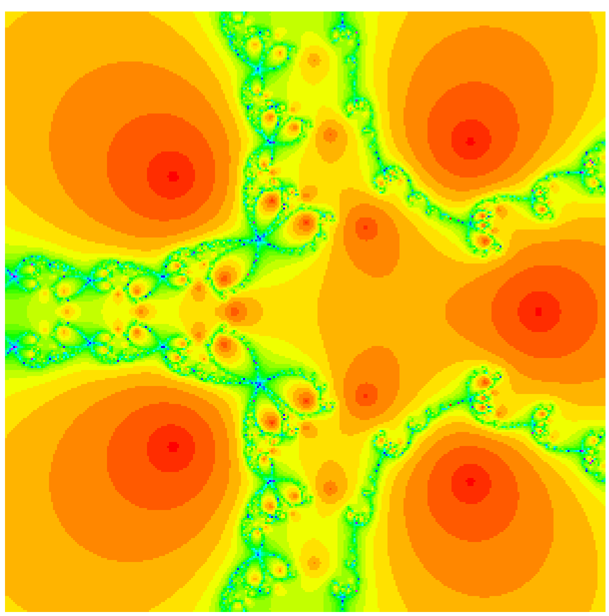
18. Fractales asociados a la función $p(z)=z^5+z^3+z^2-1$

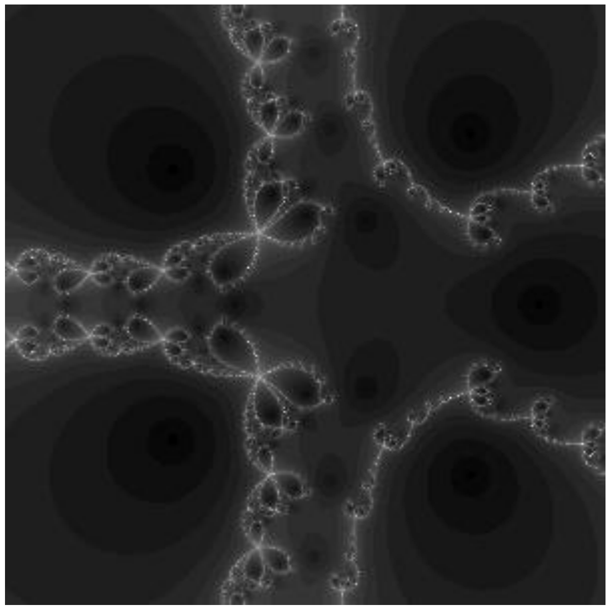


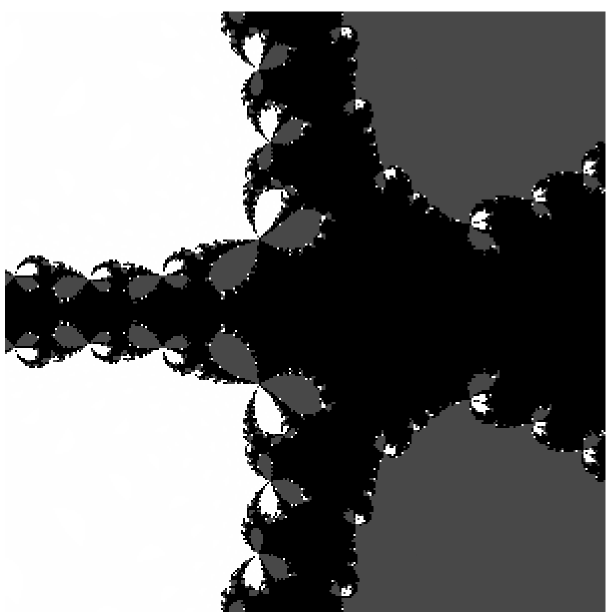




19. Fractales asociados a la función $q(z) = z^5 + z^4 + z - 1$







20. Recurrencia lineal para z_1

Sea y_n la sucesión definida por:

$$y_{n+5} = (5+5i)\,y_{n+4} - (1+20i)\,y_{n+3} - (18-23i)\,y_{n+2} + (22-4i)\,y_{n+1} - (5+4i)\,y_n \qquad (96)$$

$$y_0 = y_1 = y_2 = y_3 = 0, y_4 = 1 \qquad (97)$$

Se tiene:

$$\lim_{n\to\infty}\left(-1-i+\frac{y_{n+1}}{y_n}\right) = -1-i+\lim_{n\to\infty}\frac{y_{n+1}}{y_n} = z_1 \qquad (98)$$

21. Recurrencia lineal para z_2

Sea y_n la sucesión definida por:

$$\begin{aligned} y_{n+5} = -(10+5i)\,y_{n+4} - (31+40i)\,y_{n+3} - (27+113i)\,y_{n+2} \\ + (22-134i)\,y_{n+1} + (34-56i)\,y_n \end{aligned} \qquad (99)$$

$$y_0 = y_1 = y_2 = y_3 = 0, y_4 = 1 \qquad (100)$$

Se tiene:

$$\lim_{n\to\infty}\left(2+i+\frac{y_{n+1}}{y_n}\right) = 2+i+\lim_{n\to\infty}\frac{y_{n+1}}{y_n} = z_2 \qquad (101)$$

22. Recurrencia lineal para w_1

Sea y_n la sucesión definida por:

$$y_{n+5} = (4+5i)\,y_{n+4} + (4-16i)\,y_{n+3} - (20-8i)\,y_{n+2} + (11+8i)\,y_{n+1} + (2-3i)\,y_n \qquad (102)$$

$$y_0 = y_1 = y_2 = y_3 = 0, y_4 = 1 \qquad (103)$$

Se tiene:

$$\lim_{n\to\infty}\left(-1-i+\frac{y_{n+1}}{y_n}\right) = -1-i+\lim_{n\to\infty}\frac{y_{n+1}}{y_n} = w_1 \qquad (104)$$

23. Recurrencia lineal para w_2

Sea y_n la sucesión definida por:

$$y_{n+5} = -(6-5i)\,y_{n+4} - (4-24i)\,y_{n+3} + (20+32i)\,y_{n+2} + (27+8i)\,y_{n+1} + (8-3i)\,y_n \quad (105)$$

$$y_0 = y_1 = y_2 = y_3 = 0, y_4 = 1 \quad (106)$$

Se tiene:

$$\lim_{n\to\infty}\left(1-i+\frac{y_{n+1}}{y_n}\right) = 1-i+\lim_{n\to\infty}\frac{y_{n+1}}{y_n} = w_2 \quad (107)$$

24. Otras fórmulas

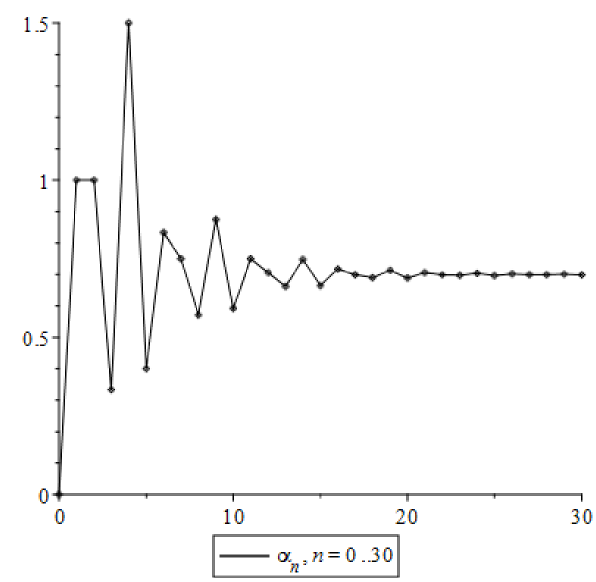
$$\pi = 4\sum_{n=0}^{\infty}\frac{c_n}{n+1} z_2^{n+1} \quad (108)$$

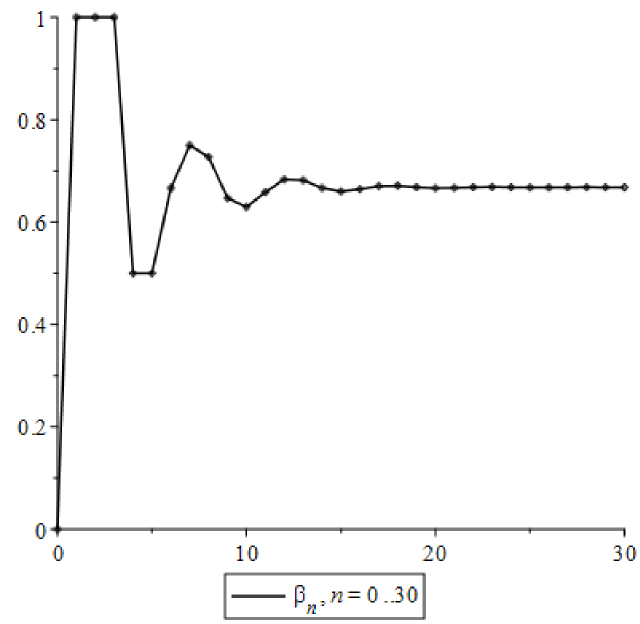
donde c_n se definen por (51) y z_2 por (78).

$$\pi = 4\sum_{n=0}^{\infty}\frac{c_n}{n+1} w_1^{n+1} \quad (109)$$

donde c_n se definen por (54) y w_1 por (79).

25. Gráficos de las sucesiones α_n y β_n





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