# Draft Introduction to Abstract Kinematics. (Version 1.0)

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**Abstract**. This work lays the foundations of the theory of *kinematic changeable sets* ("abstract kinematics"). Theory of kinematic changeable sets is based on the theory of *changeable sets*. From an intuitive point of view, changeable sets are sets of objects which, unlike elements of ordinary (static) sets, may be in the process of continuous transformations, and which may change properties depending on the point of view on them (that is depending on the reference frame). From the philosophical and imaginative point of view the changeable sets may look like as "worlds" in which evolution obeys arbitrary laws.
Kinematic changeable sets are the mathematical objects, consisting of changeable sets, equipped by different geometrical or topological structures (namely metric, topological, linear, Banach, Hilbert and other spaces). In author opinion, theories of changeable and kinematic changeable sets (in the process of their development and improvement), may become some tools of solving the sixth Hilbert problem at least for physics of macrocosm. Investigations in this direction may be interesting for astrophysics, because there exists the hypothesis, that in the large scale of Universe, physical laws (in particular, the laws of kinematics) may be different from the laws, acting in the neighborhood of our solar System. Also these investigations may be applied for the construction of mathematical foundations of tachyon kinematics.
We believe, that theories of changeable and kinematic changeable sets may be interesting not only for theoretical physics but also for other fields of science as some, new, mathematical apparatus for description of evolution of complex systems.

**Key words**: changeable sets, movement, evolution, sixth Hilbert's problem, kinematic changeable sets, kinematics, mathematical foundations of special relativity, mathematical foundations of tachyon kinematics, mathematical description of evolution of complex systems

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# Contents

# Preface

As it was noted in the abstract, the theories of changeable and kinematic changeable sets may become important not only for mathematics but also for physics, and other branches of science, which deal with the evolution of complex systems. But, at the present time, the main notions and results of these theories are scattered in many different papers. This fact makes these theories difficult for understanding. The purpose of this work is to bring together the different results of the theories of changeable and kinematic changeable sets and to expose these theories from a single point of view.

Our main aim is to depict a single picture from the separated puzzles, contained in the papers [1–15].

# Introduction

In spite of huge success of modern theoretical physics and the mightiness of mathematical tools, applied by it, the foundations of theoretical physics remain unclear. Well-known sixth Hilbert's problem of mathematically strict formulation of the foundations of theoretical physics, posed in 1900 [16], is not completely solved to this day [17, 18]. Some attempts to formalize certain physical theories were done in many papers (for example see [19–24]). The main defect of these works is the absence of a single abstract and systematic approach, and, consequently, insufficiency of flexibility of the mathematical apparatus of these works, excessive its adaptability to the specific physical theories under consideration. Moreover the attempts in [21] to immediately formalize the maximum number of known physical objects, without creating a hierarchy of elementary abstract mathematical concepts have led to the not very easy for analysis mathematical object [21, page. 177, definition 4.1]. In general, it should be noted, that the main feature of existing mathematically strict models of theoretical physics is that the investigators try to find intuitively the mathematical tools to describe physical phenomena under consideration, and only then they try to formalize the description of this phenomena, identifying physical objects with some constructs, generated by these mathematical tools, for example, with solutions of some differential equations on some space or manifold. As a result, quite complicated mathematical structures appear, whereas most elementary physical concepts and postulates, obtained by a help of experiments, life experience or common sense (which led to the appearance of these mathematical models), remain not formulated mathematically strictly. In works [25, 26] it is expressed the view that, in the general case, it is impossible to solve this problem by means of existing mathematical theories. Also in [25, 26] it is posed the problem of constructing the theory of "dynamic sets", that is the theory of new abstract mathematical structures for modeling various processes in physical, biological and other complex systems.

In the present work the foundations of the theory of changeable sets are laid and the basic properties of these sets are established. The theory of changeable sets can be considered as attempt to give a solution of the problem, posed in [25, 26].

From an intuitive point of view, changeable sets are sets of objects which, unlike the elements of ordinary (static) sets may be in the process of continuous transformations, and which may change properties depending on the point of view on them (the area of observation or reference frame). From the philosophical and imaginative point of view, changeable sets may look like as "worlds" in which changes obey arbitrary laws.

Another approach to formalization of physical theories (namely, the theory of relativity) was developed by the group of Hungarian mathematicians (Hajnal Andreka, Judit Madarasz, Istvan Nemeti, Gergely Szekely and others) in [27–30] and many other papers of these authors. This approach is based on using the apparatus of mathematical logic. The mathematical logic tools allows to avoid quite complex and burdensome mathematical structures and notions in the fundamentals of the theory. However, on the other hand, this approach results to the appearance of certain artificial axioms and concepts, which cause some unnecessary reflections and discussions of "philosophical type". For example, the Axiom AxPh in [30, page 18] demands the existence of light sphere in every point of Space-time. This axiom is required only for receiving of Lorentz-Poincare transformations between inertial reference frames. But it is unnecessary for solving the most of concrete problems, appearing in the framework of Special Relativity and leads to some excessive philosophical reflections (whether in real World every point of Space-time is penetrated by photons in all directions?). Thus, we have seen, that the concrete task of receiving of Lorentz-Poincare transformations between reference frames leads to appearance of artificial axiom of kind AxPh. And we can do a more general conclusion, that axiomatic approach leads to necessity of own system of axioms for each physical problem. The reason for this situation is that, unlike the school course in geometry (where Euclidean plane or Euclidean space is a repository of all possible geometrical figures), the building of a single repository for all variants of the evolution of physical systems is the very difficult task. And any attempt of solving the last task must lead to artificial mathematical an logical constructions. In view of the above, the proposition to apply for this task the "modal logic framework", appeared in [29, page 210] also sems to be not very helpful.

Note, that some attempts to construct the mathematical objects, bit like to the changeable sets (namely — variable sets) were made in [31, 32]. In comparison with the changeable sets, more primitive mathematical objects have been proposed in these works. For example, "variable sets" from [31] may only change their composition over time, but elements of these "variable sets" can not evolve. The same can be said about the categories $\mathbf{Bun}(X)$ (bundles over X) and $\mathbf{Shv}(X)$ (sheaves over X) from [32]. Elements of the category **Biv** from [32] may evolve but only by means of "leap" within of two discrete time points. Moreover, the authors of [31, 32] have not gone further than philosophical considerations and some definitions or axioms.

Thus, we may summarize, that (in our opinion) there are the following main causes of the lack of productivity of approaches, considered in the papers, analyzed above:

**1)** the absence of a single abstract and systematic approach;

**2)** attempts to construct the mathematical theories of physical objects "from zero" using axiomatic method;

**3)** involvement the existing mathematical structures and universal classes (such as categories or bundles) as basic objects.

In the present paper we repepresent a single abstract approach for formalization of physical theories, based on the theory of changeable sets. For the construction of this theory we don't review or complement the axiomatic foundations of classical set theory. Changeable sets are defined as a new abstract universal class of objects within the framework of classical set theory (just as are defined groups, rings, fields, lattices, linear spaces, fuzzy sets, etc.). Of course, we can not guarantee the applicability of changeable set theory for formalizatin of all branches of theoretical physics (for example for quantum mechanics). But, author hopes, that the apparatus of the theory of changeable sets will be able to generate the necessary mathematical structures at least for physics and some other natural sciences in macrocosm.

The main feature of our approach is that more complex mathematical objects are built on the basis of simpler ones. Part I sets forth the theory of changeable sets. We start our consideration with the most simple mathematical objects — oriented sets and primitive changeable sets

(Sections 1–5 of Part I). Fundamentals of the theory of primitive changeable sets also have been presented in [2]. Further we introduce and investigate the more complex objects: base changeable sets (Sections 6–8 of Part I) and (general) changeable sets (Sections 10–12 of Part I). Theory of base changeable sets also is contained in [5, 9]. General theory of changeable sets also is located in [4, 8]. The main statements of the changeable set theory have been announced in [1]. Most of main results from the abovementioned papers are collected in the preprint [3].

Part II deals with the kinematic changeable sets. Kinematic changeable sets are the mathematical objects, in which changeable sets are equipped by different geometrical or topological structures, namely metric, topological, linear, Banach, Hilbert and other spaces. Kinematic changeable sets are designed for mathematical modeling of physical evolution in a spatial environment under various kinematic laws. The main results in this direction have been announced in [11] and expounded in [10, 12].

In Part III we consider kinematic changeable sets with given universal coordinate transforms (universal kinematics). Universal coordinate transforms are coordinate transforms, under which the geometrically-time provision of an arbitrary material object in any reference frame is determined by geometrically-time position of this object in a certain, fixed frame, independently of any internal properties of the object. The main results of Part III were expounded in [13–15].

Kinematic changeable sets and universal kinematics may be interesting for astrophysics, because there exists the hypothesis, that in the large scale of Universe, physical laws (in particular, the laws of kinematics) may be different from the laws, acting in the neighborhood of our solar System. And "subuniverses" with physical laws, different from our, may also exist. Hence, we hope, that development of the theories of changeable and kinematic changeable sets may lead to elaboration of mathematical tools, necessary for "construction" of such "worlds" with physical laws different from our.

# Part I
# Abstract Theory of Changeable Sets

## 1 Oriented Sets and their Properties

When we try to see on any picture of reality (area of reality) from the most abstract point of view, we can only say that this picture consists at any time of its existence of certain things (objects). During the investigation of this area of reality, the objects participating in it can be divided into smaller elementary objects, which we call elementary states. Method of division of a given area of reality into elementary states depends on our knowledge about this area, the level of research detailing, required for practice, or the level of physical and/or mathematical idealization of the analyzed system. Depending on these factors, we can use as elementary states, for example, the position of a material point (or an elementary particle) at given time, the value of scalar, vector or tensor field at a given point of space-time, the state of an individual of a species at given time (in mathematical models of biology) and others. If the picture of reality does not change with time, then this picture of reality can be described (in the most abstract form) in the terms of classical set theory, when elementary states are interpreted as elements of a certain set. However, the reality is changeable. Elementary states may change their properties in the process of evolution (and thus lose their formal mathematical self-identity). Also elementary states may born or disappear, decompose into several elementary states, or, conversely, several elementary states may merge into a single one. But, whenever it is possible to trace "evolution lines" of the analyzed system, we can give a define answer to the question whether the elementary state "$y$" is the result of transformations (ie, is a "transformation offspring") of the elementary state "$x$". Therefore, the next definition may be considered as the simplest (starting) model of a set of changing objects.

**Definition I.1.1.** *Let, $M$ be any non-empty set ($M \neq \emptyset$).*

*An arbitrary reflexive binary relation $\leftarrowtriangle$ on $M$ (that is a relation satisfying $\forall x \in M \; x \leftarrowtriangle x$) we name an **orientation**, and the pair $\mathcal{M} = (M, \leftarrowtriangle)$ we call an **oriented set**. In this case the set $M$ is named the **basic** set or the set of all **elementary states** of oriented set $\mathcal{M}$ and it is denoted by $\mathfrak{Bs}(\mathcal{M})$. The relation $\leftarrowtriangle$ we name the **directing relation of changes (transformations)** of $\mathcal{M}$, and denote it by $\underset{\mathcal{M}}{\leftarrow}$.*

In the case where the oriented set $\mathcal{M}$ is known in advance, the char $\mathcal{M}$ in the denotation $\underset{\mathcal{M}}{\leftarrow}$ will be released, and we will use denotation $\leftarrow$ instead. For the elements $x, y \in \mathfrak{Bs}(\mathcal{M})$ the record $y \leftarrow x$ should be understood as "the elementary state $y$ is the result of transformations (or the transformation offspring) of the elementary state $x$"

*Remark* I.1.1. **1.** Some attempts to construct abstract mathematical structures for modeling physical systems were made in [23, 24]. In these works as a basic abstract model it is proposed to consider a pair of kind $(M, \prec)$, where $M$ is some set and $\prec$ is the local sequence relation (in the sense of [24, page 28]), which satisfies the additional axioms $TK_1$-$TK_3$ [24]. Recall that, according to [24, page 28], local sequence relation is the relation $\prec$ on $M$, satisfying the following conditions:

**(Pm1)** there not exist $x, y \in M$ such, that $x \prec y$ and $y \prec x$;

**(Pm2)** for each $p \in M$ the relation $\prec$ is transitive on the sets:

$$p_+ = \{x \in M \mid p \prec x\}, \quad p_- = \{x \in M \mid x \prec p\}. \tag{I.1}$$

The main deficiency of this approach is, that it is not motivated by abstract philosophical arguments, while the main motivation is provided by the specific example of order relation, generated by the "light cone" in Minkowski space-time. Due to these factors, the model, suggested in [23, 24], is not enough flexible. In particular, due to the axioms $TK_1$-$TK_3$ from [24], this model is unusable for the description of discrete processes. Also, due to the axiom (Pm2) (weak version of transitivity), this model is not enough comfortable for consideration (at the abstract level) of complex branched processes, where different "branches" of the process can "intersect" or "merge" during transformations. Moreover the construction of mathematical model of the special relativity theory, based on the order relation of "light cone" makes impossible the mathematically strict study of tachyons under this model, while building a formal theory of tachyons is one of the actual areas of modern theoretical and mathematical physics [33–38].

**2.** Note, that there is a certain "ideological" difference between our model and the model proposed in [23, 24] in the sense of way of interpretation. Namely, the directing relation of changes in Definition I.1.1 displays only real transformations, of the elementary states which have appeared in the oriented set, while the the local sequence relation [23, 24] (in particular "light cone" order relation), display all potentially possible transformations. So, considering an oriented set or mathematical objects, generated by it, we always mean a specific evolution of a particular system, but we should not imagine all potentially possible directions of evolution of a group of systems, satisfying certain conditions. But, from the other hand, models of the works [23, 24] can be interpreted as partial cases oriented sets specified in Definition I.1.1. Indeed, let us consider the following binary relation om $M$:

$$\preceq = \prec \cup \{(x,x) \mid x \in M\}.$$

Hence for arbitrary $x, y \in M$ the condition $x \preceq y$ holds if and only if $x \prec y$ or $x = y$. Taking into account the the condition (Pm1) we can see, that the binary relation $\prec$ can be uniquely restored by the relation $\preceq$. Hence, any model $(M, \prec)$ from [23,24] is equivalent to the oriented set $(M, \preceq)$. But oriented sets of kind $(M, \preceq)$, generated by the models from [23, 24] are only particular cases of general oriented sets described in Definition I.1.1.

Let $\mathcal{M}$ be an oriented set.

**Definition I.1.2.** *The subset $N \subseteq \mathfrak{Bs}(\mathcal{M})$ will be referred to as* ***transitive*** *in $\mathcal{M}$ if for any $x, y, z \in N$ such, that $z \leftarrow y$ and $y \leftarrow x$ we have $z \leftarrow x$.*

*The transitive subset $N \subseteq \mathfrak{Bs}(\mathcal{M})$ will be called* ***maximum transitive*** *if there not exist a transitive set $N_1 \subseteq \mathfrak{Bs}(\mathcal{M})$, such, that $N \subset N_1$ (where the symbol $\subset$ denotes the strict inclusion, that is $N \neq N_1$).*

*The transitive subset $L \subseteq \mathfrak{Bs}(\mathcal{M})$ will be referred to as* ***chain*** *in $\mathcal{M}$ if for any $x, y \in L$ at least one of the relations $y \leftarrow x$ or $x \leftarrow y$ is true. The chain $L \subseteq \mathfrak{Bs}(\mathcal{M})$ we will name by the* ***maximum chain*** *if there not exist a chain $L_1 \subseteq \mathfrak{Bs}(\mathcal{M})$, such, that $L \subset L_1$.*

**Assertion I.1.1.** *Let $\mathcal{M}$ be an oriented set.*

1. *Any non-empty subset $N \subseteq \mathfrak{Bs}(\mathcal{M})$, containing not more than, two elements is transitive.*
2. *Any non-empty subset $L = \{x, y\} \subseteq \mathfrak{Bs}(\mathcal{M})$, containing not more than, two elements is a chain if and only if $y \leftarrow x$ or $x \leftarrow y$. In particular, any singleton $L = \{x\} \subseteq \mathfrak{Bs}(\mathcal{M})$ is a chain.*

The proof of Assertion I.1.1 is reduced to trivial verification.

**Denotation I.1.1.** *Further we denote by $2^{\mathbf{M}}$ the set of all subsets of any set* **M**.

**Lemma I.1.1.** *Let $\mathcal{M}$ be an oriented set.*

1. *Union of an arbitrary family of transitive sets of $\mathcal{M}$, linearly ordered by the inclusion relation, is a transitive set in $\mathcal{M}$.*
2. *Union of an arbitrary family of chains of $\mathcal{M}$, linearly ordered by the inclusion relation, is a chain in $\mathcal{M}$.*

*Proof.* 1. Let $\mathfrak{N} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ be a family of transitive sets of $\mathcal{M}$, linearly ordered by the inclusion relation. Denote:

$$\widetilde{N} := \bigcup_{N\in\mathfrak{N}} N.$$

Consider any elementary states $x, y, z \in \widetilde{N}$ such, that $z \leftarrow y$ and $y \leftarrow x$. Since $x, y, z \in \widetilde{N} = \bigcup_{N\in\mathfrak{N}} N$, then there exist $N_x, N_y, N_z \in \mathfrak{N}$ such, that $x \in N_x$, $y \in N_y$, $z \in N_z$. Since the family of sets $\mathfrak{N}$ is linearly ordered by the inclusion relation, then there exists the set $N_0 \in \{N_x, N_y, N_z\}$ such, that $N_x, N_y, N_z \subseteq N_0$. So, we have $x, y, z \in N_0$. Since $N_0 \in \{N_x, N_y, N_z\} \subseteq \mathfrak{N}$, then $N_0$ is the transitive set. Therefore from conditions $z \leftarrow y$ and $y \leftarrow x$ it follows, that $z \leftarrow x$. Thus $\widetilde{N}$ is the transitive set.

2. Let $\mathfrak{L} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ be a family of chains of $\mathcal{M}$, linearly ordered by the inclusion relation. Denote:

$$\widetilde{L} := \bigcup_{L\in\mathfrak{L}} L.$$

By the post 1, $\widetilde{L}$ is the transitive set. Consider any elementary states $x, y \in \widetilde{L}$. Since the family of sets $\mathfrak{L}$ is linearly ordered by the inclusion relation, then, similarly as in the post 1, there exists a chain $L_0 \in \mathfrak{L}$ such, that $x, y \in L_0$. And, because $L_0$ is chain, at least one of the relations $y \leftarrow x$ or $x \leftarrow y$ is true. Thus $\widetilde{L}$ is the chain of $\mathcal{M}$. □

Using Lemma I.1.1 and the Zorn's lemma, we obtain the following assertion.

**Assertion I.1.2.**

1. *For any transitive set $N$ of oriented set $\mathcal{M}$ there exists a maximum transitive set $N_{\max}$ such, that $N \subseteq N_{\max}$.*
2. *For any chain $L$ of oriented set $\mathcal{M}$ there exists a maximum chain $L_{\max}$ such, that $L \subseteq L_{\max}$.*

It should be noted that the second post of Assertion I.1.2 can be interpreted to as the generalization of the Hausdorff maximal principle in the framework of oriented set theory.

The following corollaries result from assertions I.1.2 and I.1.1.

**Corollary I.1.1.** *For any two elements $x, y \in \mathfrak{Bs}(\mathcal{M})$ in the oriented set $\mathcal{M}$ there exists a maximum transitive set $N \subseteq \mathfrak{Bs}(\mathcal{M})$ such that $x, y \in N$.*

**Corollary I.1.2.** *For any two elements $x, y \in \mathfrak{Bs}(\mathcal{M})$, such that $y \leftarrow x$, there exists a maximum chain $L$ of the oriented set $\mathcal{M}$ such that $x, y \in L$.*

If we put $x = y \in \mathfrak{Bs}(\mathcal{M})$ (by Definition I.1.1 $\mathfrak{Bs}(\mathcal{M}) \neq \emptyset$), we obtain, that maximum transitive sets and maximum chains must exist in any oriented set $\mathcal{M}$.

Main results of this Section were anonced in [1] and published in [2, Section 2].

## 2 Definition of the Time. Primitive Changeable Sets

In theoretical physics, scientists tend to think, that the moments of time are real numbers. But the abstract mathematics deal with objects of an arbitrarily large cardinality. With this in mind, in the papers of Hajnal Andreka, Judit Madarasz, Istvan Nemeti, Gergely Szekely it is proposed to consider any ordered field as the scale of time points (see [27–30] and many other papers of these authors). In our abstract theory we will not be restricted to the moments of time belonging to the set of real numbers $\mathbb{R}$ or some other ordered field. And, as it will be seen further, we need not any assumptions about algebraic structure on time scale for obtaining many interesting abstract results. In the next definition, moments of time are elements of any linearly (totally) ordered set (in the sense of [40, p. 12]). Such definition of time is close to the philosophical conception of time as some “chronological order”, somehow agreed with the processes of transformations.

**Definition I.2.1.** *Let $\mathcal{M}$ be an oriented set and $\mathbb{T} = (\mathbf{T}, \leq)$ be a linearly ordered set. A maping $\psi : \mathbf{T} \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ is referred to as* ***time*** *on $\mathcal{M}$ if the following conditions are satisfied:*

*1) For any elementary state $x \in \mathfrak{Bs}(\mathcal{M})$ there exists an element $t \in \mathbf{T}$ such that $x \in \psi(t)$.*

*2) If $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$, $x_2 \leftarrow x_1$ and $x_1 \neq x_2$, then there exist elements $t_1, t_2 \in \mathbf{T}$ such that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 < t_2$ (this means that there is a temporal separateness of successive unequal elementary states).*

*In this case the elements $t \in \mathbf{T}$ we call the moments of time, the pair*

$$\mathcal{H} = (\mathbb{T}, \psi) = ((\mathbf{T}, \leq), \psi)$$

*we name by* ***chronologization*** *of $\mathcal{M}$ and the triple*

$$\mathcal{P} = (\mathcal{M}, \mathbb{T}, \psi) = (\mathcal{M}, (\mathbf{T}, \leq), \psi)$$

*we call* ***primitive changeable set****.*

*Remark* I.2.1. In [23, 24] linearly ordered sets has been used as time-scales also. But the conception of time in Definition I.2.1 is significantly different from [23, 24]. Note, that the definition of time in [23, 24] is less general, then Definition I.2.1 due to less generality of the model, suggested in [23, 24] (recall that according to Remark I.1.1, the models of the works [23, 24] can be interpreted as partial cases oriented sets specified in Definition I.1.1).

We say that an oriented set $\mathcal{M}$ ***can be chronologized*** if there exists at least one chronologization of $\mathcal{M}$. It turns out that any oriented set can be chronologized. To make sure this we may consider any linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$, which contains at least two elements and put:

$$\psi(t) := \mathfrak{Bs}(\mathcal{M}), \quad t \in \mathbf{T}.$$

The conditions of Definition I.2.1 for the function $\psi(\cdot)$ apparently are satisfied. More non-trivial methods to chronologize an oriented set we will consider in Section 3.

The following two assertions (I.2.1 and I.2.2) are trivial consequences of Definition I.2.1.

**Assertion I.2.1.** *Let $\mathcal{M}$ and $\mathcal{M}_1$ be oriented sets, and while $\mathfrak{Bs}(\mathcal{M}) \subseteq \mathfrak{Bs}(\mathcal{M}_1)$ and $\underset{\mathcal{M}}{\leftarrow} \subseteq \underset{\mathcal{M}_1}{\leftarrow}$ (last inclusion means that for $x, y \in \mathfrak{Bs}(\mathcal{M})$ the condition $y \underset{\mathcal{M}}{\leftarrow} x$ implies $y \underset{\mathcal{M}_1}{\leftarrow} x$).*

*If a mapping $\psi_1 : \mathbf{T} \mapsto 2^{\mathfrak{Bs}(\mathcal{M}_1)}$ (where $\mathbb{T} = (\mathbf{T}, \leq)$ is a linearly ordered set) is a time on $\mathcal{M}_1$ then the mapping:*

$$\psi(t) = \psi_1(t) \cap \mathfrak{Bs}(\mathcal{M})$$

*is the time om $\mathcal{M}$.*

**Assertion I.2.2.** *Let $\mathcal{M}$ be an oriented set and and $\psi : \mathbf{T} :\mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ be a time om $\mathcal{M}$.*

*(1) If $\mathbf{T}_1 \subseteq \mathbf{T}$, $\mathbf{T}_1 \neq \emptyset$ and $\psi(t) = \emptyset$ for $t \in \mathbf{T} \setminus \mathbf{T}_1$, then the mapping $\psi_1 = \psi \restriction \mathbf{T}_1$, which is the restriction of $\psi$ on the set $\mathbf{T}_1$ also is time on $\mathcal{M}$.*

*(2) If the linearly ordered set $(\mathbf{T}, \leq)$ is embedded in a linearly ordered set $\left(\tilde{\mathbf{T}}, \leq_1\right)$ (preserving order)*[1]*, then the mapping $\tilde{\psi} : \tilde{\mathbf{T}} \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$:*

$$\tilde{\psi}(t) = \begin{cases} \psi(t), & t \in \mathbf{T} \\ \emptyset, & t \in \tilde{\mathbf{T}} \setminus \mathbf{T} \end{cases}$$

*also is time on $\mathcal{M}$.*

Assertion I.2.2 affirms, that "moments of full death" may be erased from or added to "chronological history" of primitive changeable set.

Main results of this Section were anonced in [1] and published in [2, Section 3].

## 3 One-point and Monotone Time. Chronologization Theorems

**Definition I.3.1.** *Let $(\mathcal{M}, \mathbb{T}, \psi) = (\mathcal{M}, (\mathbf{T}, \leq), \psi)$ be a primitive changeable set.*

*1) The time $\psi$ will be called **quasi one-point** if for any $t \in \mathbf{T}$ the set $\psi(t)$ is a singleton.*

*2) The time $\psi$ will be called **one-point** if the following conditions are satisfied:*

*(a) The time $\psi$ is quasi one-point;*

*(b) If $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 \leq t_2$ then $x_2 \leftarrow x_1$.*

*3) The time $\psi$ will be called **monotone** if for any elementary states $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ the conditions $x_2 \leftarrow x_1$ and $x_1 \not\leftarrow x_2$ imply $t_1 < t_2$.*

*In the case, when the time $\psi$ is quasi one-point (one-point/monotone) the chronologization $(\mathbb{T}, \psi)$ of the oriented set $\mathcal{M}$ will be called quasi one-point (one-point/monotone) correspondingly.*

***Example*** I.3.1. Let us consider an arbitrary mapping $f : \mathbb{R} \mapsto \mathbb{R}^d$ $(d \in \mathbb{N})$, where $\mathbb{N}$ is the set of all natural numbers. This mapping can be interpreted as equation of motion of single material point in the space $\mathbb{R}^d$. This mapping generates the oriented set $\mathcal{M} = \left(\mathfrak{Bs}(\mathcal{M}), \underset{\mathcal{M}}{\leftarrow}\right)$, where $\mathfrak{Bs}(\mathcal{M}) = \mathfrak{R}(f) = \{f(t) \mid t \in \mathbb{R}\} \subseteq \mathbb{R}^d$ and for $x, y \in \mathfrak{Bs}(\mathcal{M})$ the correlation $y \underset{\mathcal{M}}{\leftarrow} x$ is true if and only if there exist $t_1, t_2 \in \mathbb{R}$ such, that $x = f(t_1)$, $y = f(t_2)$ and $t_1 \leq t_2$. It is easy to verify, that the mapping:

$$\psi(t) = \{f(t)\} \subseteq \mathfrak{Bs}(\mathcal{M}), \quad t \in \mathbb{R}.$$

is a one-point time on $\mathcal{M}$.

Example I.3.1 makes clear the definition of one-point time. It is evident, that *any one-point time is quasi one-point and monotone.* It turns out that a quasi one-point time need not be monotone (and thus one-point), and monotone time need not be quasi one-point (and thus one-point). The next examples prove facts, written above.

***Example*** I.3.2. Let us consider any two element set $M = \{x_1, x_2\}$ $(x_1 \neq x_2)$. We construct the oriented set $\mathcal{M} = \left(\mathfrak{Bs}(\mathcal{M}), \underset{\mathcal{M}}{\leftarrow}\right)$ by the following way:

$$\mathfrak{Bs}(\mathcal{M}) = M = \{x_1, x_2\};$$
$$\underset{\mathcal{M}}{\leftarrow} = \{(x_2, x_1), (x_1, x_1), (x_2, x_2)\}$$

[1] This means that $\mathbf{T} \subseteq \tilde{\mathbf{T}}$ and for $x, y \in \mathbf{T}$ the correlation $x \leq y$ holds if and only if $x \leq_1 y$.

(or, in other words, $x_2 \underset{\mathcal{M}}{\leftarrow} x_1$, $x_1 \underset{\mathcal{M}}{\leftarrow} x_1$, $x_2 \underset{\mathcal{M}}{\leftarrow} x_2$). Note that the directing relation of changes $\underset{\mathcal{M}}{\leftarrow}$ can be represented in more laconic form: $\underset{\mathcal{M}}{\leftarrow} = \{(x_2, x_1)\} \cup \mathsf{diag}(M)$, where $\mathsf{diag}(M) = \{(x,x) \mid x \in M\}$. As a linearly ordered set we take $\mathbb{T} = (\mathbb{R}, \leq)$ (with the usual order on the real field). On the oriented set $\mathcal{M}$ we define time by the following way:

$$\psi(t) := \begin{cases} \{x_1\}, & t \notin \mathbb{Q}; \\ \{x_2\}, & t \in \mathbb{Q}, \end{cases}$$

where $\mathbb{Q}$ is the field of rational numbers. It is easy to verify, that the mapping $\psi$ really is time on $\mathcal{M}$ in the sense of Definition I.2.1. Since $\psi(t)$ is a singleton for any $t \in \mathbb{R}$, the time $\psi$ is quasi one-point. If we put $t_1 = \sqrt{2}$, $t_2 = 1$, we obtain $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$, $x_2 \leftarrow x_1$, $x_1 \not\leftarrow x_2$, but $t_1 > t_2$. Thus the time $\psi$ is not monotone.

***Example*** I.3.3. Let us consider an arbitrary four-element set $M = \{x_1, x_2, x_3, x_4\}$ $(x_i \neq x_j$ for $i \neq j)$ and construct the oriented set $\mathcal{M} = \left(\mathfrak{Bs}(\mathcal{M}), \underset{\mathcal{M}}{\leftarrow}\right)$ by the following way:

$$\mathfrak{Bs}(\mathcal{M}) = M = \{x_1, x_2, x_3, x_4\};$$
$$\underset{\mathcal{M}}{\leftarrow} = \{(x_2, x_1), (x_4, x_3)\} \cup \mathsf{diag}(M).$$

As a linearly ordered set we take $\mathbb{T} = (\{1,2\}, \leq)$ (with the usual ordering on the real axis). Time on $\mathcal{M}$ is defined by the following way:

$$\psi(t) := \begin{cases} \{x_1, x_3\}, & t = 1 \\ \{x_2, x_4\}, & t = 2. \end{cases}$$

It is not hard to prove, that the mapping $\psi$ is a monotone time on $\mathcal{M}$. But this time, obviously, is not quasi one-point.

**Denotation I.3.1.** *Futher we denote by $\overline{m,n}$ (where $m, n \in \mathbb{N}$, $m \leq n$) the set: $\overline{m,n} = \{m, \dots, n\} \subseteq \mathbb{N}$.*

It appears that quasi one-point and monotone time need not be one-point. This fact is illustrated by the following example.

***Example*** I.3.4. Let the oriented set $\mathcal{M}$ be same as in Example I.3.3. We consider the ordered set $\mathbb{T} = (\{1,2,3,4\}, \leq) = (\overline{1,4}, \leq)$ (with the usual natural or real number ordering). Time on $\mathcal{M}$ we define by the following way:

$$\psi(t) := \{x_t\}, \quad t \in \overline{1,4}.$$

It is not hard to verify, that $\psi(\cdot)$ is quasi one-point and monotone time on $\mathcal{M}$. Although, if we put $t_1 := 2$, $t_2 := 3$, we receive, $x_2 \in \psi(t_1)$, $x_3 \in \psi(t_2)$, $t_1 \leq t_2$, but $x_3 \not\leftarrow x_2$. Thus, the time $\psi$ is not one-point.

**Definition I.3.2.** *Oriented set $\mathcal{M}$ will be called a **chain oriented set** if the set $\mathfrak{Bs}(\mathcal{M})$ is the chain of $\mathcal{M}$, that is if the relation $\leftarrow$ if transitive on $\mathfrak{Bs}(\mathcal{M})$ and for any $x, y \in \mathfrak{Bs}(\mathcal{M})$ at least one of the conditions $x \leftarrow y$ or $y \leftarrow x$ is satisfied.*

*Oriented set $\mathcal{M}$ will be called a **cyclic** if for any $x, y \in \mathfrak{Bs}(\mathcal{M})$ both of the relations $x \leftarrow y$ and $y \leftarrow x$ are true.*

It is evident, that any cyclic oriented set is a chain.

**Lemma I.3.1.** *Any cyclic oriented set can be one-point chronologized.*

*Proof.* Let $\mathcal{M}$ be a cyclic oriented set. By definition of oriented set, $\mathfrak{Bs}(\mathcal{M}) \neq \emptyset$. Choose any two disjoint sets $\mathbf{T}_1, \mathbf{T}_2$ equipotent to the set $\mathfrak{Bs}(\mathcal{M})$ ($\mathbf{T}_1 \cap \mathbf{T}_2 = \emptyset$). (Such sets must exist, because we can put $\mathbf{T}_1 := \mathfrak{Bs}(\mathcal{M})$ and construct the set $\mathbf{T}_2$ from the elements of set $\tilde{\mathbf{T}} = 2^{\mathbf{T}_1} \setminus \mathbf{T}_1$, cardinality of which is not smaller than the cardinality of $\mathbf{T}_1$.) Let $\trianglelefteq_i$ $(i = 1, 2)$ be any linear order relation on $\mathbf{T}_i$ (by Zermelo's theorem, such linear order relations necessarily exist). Denote:

$$\mathbf{T} := \mathbf{T}_1 \cup \mathbf{T}_2.$$

On the set $\mathbf{T}$ we construct the relation:

$$\leq = \trianglelefteq_1 \cup \trianglelefteq_2 \cup \{(t, \tau) \mid t \in \mathbf{T}_1,\ \tau \in \mathbf{T}_2)\},$$

or, in the other words, for $t, \tau \in \mathbf{T}$ relation $t \leq \tau$ holds if and only if one of the following conditions is satisfied:

(O1) $t, \tau \in \mathbf{T}_i$ and $t \trianglelefteq_i \tau$ for some $i \in \{1, 2\}$;
(O2) $t \in \mathbf{T}_1$, $\tau \in \mathbf{T}_2$.

The pair $(\mathbf{T}, \leq)$ is the ordered union of the linearly ordered sets $(\mathbf{T}_1, \trianglelefteq_1)$ and $(\mathbf{T}_2, \trianglelefteq_2)$. Thus, by [39, p. 208], $(\mathbf{T}, \leq)$ is a linear ordered set. Let $f : \mathbf{T}_2 \mapsto \mathbf{T}_1$ be any bijection (one-to-one correspondence) between the (equipotent) sets $\mathbf{T}_1$ and $\mathbf{T}_2$. And let $g : \mathbf{T}_1 \mapsto \mathfrak{Bs}(\mathcal{M})$ be any bijection between the (equipotent) sets $\mathbf{T}_1$ and $\mathfrak{Bs}(\mathcal{M})$.

Let us consider the following mapping $\psi : \mathbf{T} \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$:

$$\psi(t) := \begin{cases} \{g(t)\}, & t \in \mathbf{T}_1; \\ \{g(f(t))\}, & t \in \mathbf{T}_2. \end{cases} \tag{I.2}$$

We are going to prove, that $\psi$ is a time on the oriented set $\mathcal{M}$.

1) Let $x \in \mathfrak{Bs}(\mathcal{M})$. Since the mapping $g : \mathbf{T}_1 \mapsto \mathfrak{Bs}(\mathcal{M})$ is bijection between the sets $\mathbf{T}_1$ and $\mathfrak{Bs}(\mathcal{M})$, there exists the inverse mapping $g^{[-1]} : \mathfrak{Bs}(\mathcal{M}) \mapsto \mathbf{T}_1$. Let us consider the element $t_x = g^{[-1]}(x) \in \mathbf{T}_1 \subseteq \mathbf{T}$. According to (I.2):

$$\psi(t_x) = \{g(t_x)\} = \left\{g(g^{[-1]}(x))\right\} = \{x\}.$$

Therefore, $x \in \psi(t_x)$. Thus the first condition of the time Definition I.2.1 is satisfied.

2) Let $x, y \in \mathfrak{Bs}(\mathcal{M})$ be elements of $\mathfrak{Bs}(\mathcal{M})$ such, that $y \leftarrow x$ and $x \neq y$. Denote:

$$t_x := g^{[-1]}(x) \in \mathbf{T}_1;$$
$$t_y := f^{[-1]}\left(g^{[-1]}(y)\right) \in \mathbf{T}_2.$$

By (O2), $t_x \leq t_y$. Since $\mathbf{T}_1 \cap \mathbf{T}_2 = \emptyset$, we have $t_x \neq t_y$. Thus $t_x < t_y$. In accordance with (I.2), we obtain:

$$\psi(t_x) = \{g(t_x)\} = \left\{g\left(g^{[-1]}(x)\right)\right\} = \{x\};$$
$$\psi(t_y) = \{g(f(t_y))\} = \left\{g\left(f\left(f^{[-1]}\left(g^{[-1]}(y)\right)\right)\right)\right\} = \{y\}.$$

Consequently, $x \in \psi(t_x)$, $y \in \psi(t_y)$ and $t_x < t_y$. That is the second condition of the time Definition I.2.1 also is satisfied.

Thus $\psi$ is a time on $\mathcal{M}$. It remains to prove that the time $\psi$ is one-point.

According to (I.2), for any $t \in \mathbf{T}$ the set $\psi(t)$ consists of one element (is a singleton). Thus the condition (a) of the one-point time Definition I.3.1 is satisfied. Since the oriented set $\mathcal{M}$ is a cyclic, the condition (b) of Definition I.3.1 is also satisfied. Thus time $\psi$ is one-point. □

**Theorem I.3.1.** *Any chain oriented set can be one-point chronologized.*

*Proof.* Let $\mathcal{M}$ be a chain oriented set. Then the set $\mathfrak{Bs}(\mathcal{M})$ is a chain of oriented set $\mathcal{M}$, ie the relation $\leftarrow = \underset{\mathcal{M}}{\leftarrow}$ is quasi order [2] on $\mathfrak{Bs}(\mathcal{M})$.

We will say that elements $x, y \in \mathfrak{Bs}(\mathcal{M})$ are *cyclic equivalent* (denotation $x \overset{\mathbf{o}}{\equiv} y$) if $x \leftarrow y$ and $y \leftarrow x$. In accordance with [40, page 21], relation $\overset{\mathbf{o}}{\equiv}$ is equivalence relation on $\mathfrak{Bs}(\mathcal{M})$. Let $F_1$ and $F_2$ be any two classes of equivalence, generated by the relation $\overset{\mathbf{o}}{\equiv}$. We will denote $F_2 \twoheadleftarrow F_1$ if for any $x_1 \in F_1$, $x_2 \in F_2$ it is true $x_2 \leftarrow x_1$. According to [40, page 21], the relation $\twoheadleftarrow$ is ordering on the quotient set $\mathfrak{Bs}(\mathcal{M})/\overset{\mathbf{o}}{\equiv}$ of all equivalence classes, generated by $\overset{\mathbf{o}}{\equiv}$. We aim to prove, that this ordering is linear. Chose any equivalence classes $F_1, F_2 \in \mathfrak{Bs}(\mathcal{M})/\overset{\mathbf{o}}{\equiv}$. Because $F_1$,$F_2$ are equivalence classes, they are nonempty, therefore there exists the elements $x_1 \in F_1$, $x_2 \in F_2$. Since the oriented set $\mathcal{M}$ is a chain, at least one from the relations $x_2 \leftarrow x_1$ or $x_1 \leftarrow x_2$ is true. But, because any two elements, belonging to the same class of equivalence, are cyclic equivalent, in the case $x_2 \leftarrow x_1$ we will have $F_2 \twoheadleftarrow F_1$, and in the case $x_1 \leftarrow x_2$ we obtain $F_1 \twoheadleftarrow F_2$. Thus $(\mathfrak{Bs}(\mathcal{M})/\overset{\mathbf{o}}{\equiv}, \twoheadleftarrow)$ is a linearly ordered set.

Any equivalence class $F \in \mathfrak{Bs}(\mathcal{M})/\overset{\mathbf{o}}{\equiv}$ is a cyclic oriented set relatively the relation $\leftarrow$ (restricted to this class). Consequently, by Lemma I.3.1, any such equivalence class can be one-point chronologized. Let $(\mathbb{T}_F, \psi_F) = ((\mathbf{T}_F, \leq_F), \psi_F)$ be a one-point chronologization of the class of equivalence $F \in \mathfrak{Bs}(\mathcal{M})/\overset{\mathbf{o}}{\equiv}$. Without loss of generality we can assume that $\mathbf{T}_F \cap \mathbf{T}_G = \emptyset$ for $F \neq G$. Indeed, otherwise we may use the sets:

$$\tilde{\mathbf{T}}_F = \{(t, F) : t \in \mathbf{T}_F\}, \quad F \in \mathfrak{Bs}(\mathcal{M})/\overset{\mathbf{o}}{\equiv},$$

with ordering:

$$(t_1, F) \lesssim_F (t_2, F) \Longleftrightarrow t_1 \leq_F t_2, \quad t_1, t_2 \in \mathbf{T}_F \qquad (F \in \mathfrak{Bs}(\mathcal{M})/\overset{\mathbf{o}}{\equiv})$$

and times:

$$\tilde{\psi}_F((t, F)) = \psi_F(t), \quad t \in \mathbf{T}_F \qquad (F \in \mathfrak{Bs}(\mathcal{M})/\overset{\mathbf{o}}{\equiv}),$$

it is evident, that these times are one-point.

Thus, we will assume that $\mathbf{T}_F \cap \mathbf{T}_G = \emptyset$, $F \neq G$. Denote:

$$\mathbf{T} := \bigcup_{F \in \mathfrak{Bs}(\mathcal{M})/\overset{\mathbf{o}}{\equiv}} \mathbf{T}_F.$$

According to this denotation, for any element $t \in \mathbf{T}$ there exists an equivalence class $F(t) \in \mathfrak{Bs}(\mathcal{M})/\overset{\mathbf{o}}{\equiv}$ such, that $t \in \mathbf{T}_{F(t)}$. Since $\mathbf{T}_F \cap \mathbf{T}_G = \emptyset$, $F \neq G$, such equivalence class $F(t)$ is for an element $t \in \mathbf{T}$ unique, ie the following assertion is true:

**(F)** *For any element $t \in \mathbf{T}$ the condition $t \in \mathbf{T}_F$ $(F \in \mathfrak{Bs}(\mathcal{M})/\overset{\mathbf{o}}{\equiv})$ results in $F = F(t)$.*

For arbitrary elements $t, \tau \in \mathbf{T}$ we will denote $t \leq \tau$ if and only if at least one of the following conditions is true:

(O1) $F(t) \neq F(\tau)$ and $F(\tau) \twoheadleftarrow F(t)$.

(O2) $F(t) = F(\tau)$ and $t \leq_{F(t)} \tau$.

The pair $(\mathbf{T}, \leq)$ is the ordered union of the (linearly ordered) family of linearly ordered sets $(\mathbf{T}_F)_{F \in \mathfrak{Bs}(\mathcal{M})/\overset{\mathbf{o}}{\equiv}}$. Thus, by [39, p. 208], $\leq$ is a linear ordering on $\mathbf{T}$.

Denote:

$$\psi(t) := \psi_{F(t)}(t), \qquad t \in \mathbf{T}. \tag{I.3}$$

[2] In accordance with [40] any reflexive and transitive binary relation $\preceq$, defined on the some set $\mathbf{X}$ is named by quasi order on $\mathbf{X}$. That is the $\preceq$ relation on $\mathbf{X}$ is quasi order if and only if it satisfies the following conditions:

1) $\forall x \in \mathbf{X}$ $(x \preceq x)$; 2) For any $x, y, z \in \mathbf{X}$ the correlations $x \preceq y$ and $y \preceq z$ lead to correlation $x \preceq z$.

Since $\psi_{F(t)}(t) \subseteq F(t) \subseteq \mathfrak{Bs}(\mathcal{M})$, $t \in \mathbf{T}$, the mapping $\psi$ reflects $\mathbf{T}$ into $2^{\mathfrak{Bs}(\mathcal{M})}$. Now we are going to prove, that $\psi$ is one-point time.

(a) Let $x \in \mathfrak{Bs}(\mathcal{M})$. Then there exists an equivalence class $\Phi \in \mathfrak{Bs}(\mathcal{M})/\overset{\mathbf{o}}{\equiv}$, such, that $x \in \Phi$. Since the mapping $\psi_\Phi : \mathbf{T}_\Phi \mapsto 2^\Phi$ is a time on the oriented set $(\Phi, \leftarrow)$, there exists a time moment $t \in \mathbf{T}_\Phi$, such, that $x \in \psi_\Phi(t)$. Since $t \in \mathbf{T}_\Phi$, then by virtue of Assertion (F) we have $\Phi = F(t)$. Therefore:

$$\psi(t) = \psi_{F(t)}(t) = \psi_\Phi(t) \ni x.$$

Thus, the first condition of the time Definition I.2.1 is satisfied.

(b) Let $x, y \in \mathfrak{Bs}(\mathcal{M})$, $y \leftarrow x$ and $y \neq x$. According to the item (a), there exist $t, \tau \in \mathbf{T}$ such, that $x \in \psi(t)$, $y \in \psi(\tau)$. And, using (I.3), we obtain, $x \in \psi(t) = \psi_{F(t)}(t) \subseteq F(t)$, $y \in F(\tau)$. Hence, since $y \leftarrow x$, for any $x' \in F(t)$, $y' \in F(\tau)$ (taking into account, that $x' \overset{\mathbf{o}}{\equiv} x$, $y' \overset{\mathbf{o}}{\equiv} y$), we obtain $y' \leftarrow x'$. This means, that $F(\tau) \Leftarrow F(t)$. And, in the case $F(t) \neq F(\tau)$, using (O1), we obtain, $t \leq \tau$, so, taking into account, that $F(t) \neq F(\tau)$ causes $t \neq \tau$, we have $t < \tau$. Thus it remains to consider the case $F(t) = F(\tau)$. In this case we have $x, y \in F(t)$. And since $y \leftarrow x$, $y \neq x$ and $\psi_{F(t)}$ is a time on $(F(t), \leftarrow)$, there exist the elements $t', \tau' \in \mathbf{T}_{F(t)}$ such, that $x \in \psi_{F(t)}(t')$, $y \in \psi_{F(t)}(\tau')$ and $t' <_{F(t)} \tau'$. Therefore, since $t', \tau' \in \mathbf{T}_{F(t)}$, using Assertion (F), we obtain $x \in \psi_{F(t)}(t') = \psi_{F(t')}(t') = \psi(t')$ and $y \in \psi(\tau')$. Hence $x \in \psi(t')$, $y \in \psi(\tau')$, where, $t' <_{F(t)} \tau'$ (that is $t' \leq_{F(t)} \tau'$ and $t' \neq \tau'$). So, by (F) and (O2), we obtain $t' < \tau'$.

Thus $\psi$ is a time on $\mathcal{M}$.

(c) It remains to prove, that the time $\psi$ is one-point. Since for any equivalence class $G \in \mathfrak{Bs}(\mathcal{M})/\overset{\mathbf{o}}{\equiv}$ the mapping $\psi_G$ is a one-point time, by (I.3), set $\psi(t)$ is a singleton for any $t \in \mathbf{T}$. Thus, the first condition of the one-point time definition is satisfied.

Let $x \in \psi(t)$, $y \in \psi(\tau)$, where $t \leq \tau$. Then by (I.3) $x \in \psi(t) = \psi_{F(t)}(t) \subseteq F(t)$, $y \in F(\tau)$. And in the case $F(t) = F(\tau)$ the relation $y \leftarrow x$ follows from the relation $x \overset{\mathbf{o}}{\equiv} y$. Concerning the case $F(t) \neq F(\tau)$, in this case, by (O1),(O2), we obtain $F(\tau) \Leftarrow F(t)$, which involves $y \leftarrow x$. Thus, the second condition of the one-point time definition also is satisfied.

Therefore, the time $\psi$ is one-point. □

**Theorem I.3.2.** *Any oriented set can be quasi one-point chronologized.*

*Proof.* **1.** Let $\mathcal{M}$ be an oriented set. Denote by $\mathfrak{L}$ the set of all maximum chains of the $\mathcal{M}$. In accordance with Theorem I.3.1, for any chain $L \in \mathfrak{L}$ there exists an one-point chronologization $((\mathbf{T}_L, \leq_L), \psi_L)$ of the oriented set $(L, \leftarrow)$. Similarly to the proof of the Theorem I.3.1, without loss of generality we can assume, that $\mathbf{T}_L \cap \mathbf{T}_M = \emptyset$, $L \neq M$. Denote:

$$\mathbf{T} := \bigcup_{L \in \mathfrak{L}} \mathbf{T}_L. \tag{I.4}$$

Let $\trianglelefteq$ be any linear order relation on $\mathfrak{L}$ (by Zermelo's theorem, such linear order relation necessarily exists). By virtue of (I.4), for any element $t \in \mathbf{T}$ chain $L(t) \in \mathfrak{L}$ exists such, that $t \in \mathbf{T}_{L(t)}$. Since $\mathbf{T}_F \cap \mathbf{T}_G = \emptyset$ $(F \neq G)$, this chain $L(t)$ is unique. This means, that the following assertion is true:

**(L)** *For any element $t \in \mathbf{T}$ the condition $t \in \mathbf{T}_L$ $(L \in \mathfrak{L})$ causes $L = L(t)$.*

Let $t, \tau \in \mathbf{T}$. We shall put $t \leq \tau$ if and only if one of the following conditions is satisfied:

(O1) $L(t) \neq L(\tau)$ and $L(t) \trianglelefteq L(\tau)$.

(O2) $L(t) = L(\tau)$ and $t \leq_{L(t)} \tau$.

The pair $(\mathbf{T}, \leq)$ is the ordered union of the (linearly ordrered) family of linearly ordered sets $(\mathbf{T}_L)_{L \in \mathfrak{L}}$. Thus, by [39, p. 208], $(\mathbf{T}, \leq)$ is a linearly ordered set. Denote:

$$\psi(t) := \psi_{L(t)}(t) \qquad t \in \mathbf{T}. \tag{I.5}$$

Note, that $\psi(t) = \psi_{L(t)}(t) \subseteq L(t) \subseteq \mathfrak{Bs}(\mathcal{M})$, $t \in \mathbf{T}$.

**2.** We intend to prove, that the mapping $\psi : \mathbf{T} \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ is a time.

2.a) Let $x \in \mathfrak{Bs}(\mathcal{M})$. In accordance with Corollary I.1.2, there exists the maximum chain $N_x \in \mathfrak{L}$ such, that $x \in N_x$. And, since the mapping $\psi_{N_x} : \mathbf{T}_{N_x} \mapsto 2^{N_x}$ is a time, there exists an element $t_x \in \mathbf{T}_{N_x} \subseteq \mathbf{T}$, such, that $x \in \psi_{N_x}(t_x)$. Since $t_x \in \mathbf{T}_{N_x}$, by Assertion (L) (see above) we have $N_x = L(t_x)$. Therefore:

$$\psi(t_x) = \psi_{L(t_x)}(t_x) = \psi_{N_x}(t_x) \ni x.$$

Thus, for any element $x \in \mathfrak{Bs}(\mathcal{M})$ always an element $t_x \in \mathbf{T}$ exists, such, that $x \in \psi(t_x)$.

2.b) Let $x, y \in \mathfrak{Bs}(\mathcal{M})$, $y \leftarrow x$, $x \neq y$. According to Corollary I.1.2, a maximum chain $N_{xy} \in \mathfrak{L}$ exists, such, that $x, y \in N_{xy}$. Since $y \leftarrow x$, $x \neq y$ and mapping $\psi_{N_{xy}} : \mathbf{T}_{N_{xy}} \mapsto 2^{N_{xy}}$ is a time, there exist elements $t_x, t_y \in \mathbf{T}_{N_{xy}}$ such, that $t_x <_{N_{xy}} t_y$ (ie $t_x \leq_{N_{xy}} t_y$, $t_x \neq t_y$) and $x \in \psi_{N_{xy}}(t_x)$, $y \in \psi_{N_{xy}}(t_y)$. Since $t_x, t_y \in \mathbf{T}_{N_{xy}}$, in accordance with Assertion (L), we obtain $L(t_x) = L(t_y) = N_{xy}$. Therefore:

$$\psi(t_x) = \psi_{L(t_x)}(t_x) = \psi_{N_{xy}}(t_x) \ni x;$$
$$\psi(t_y) = \psi_{L(t_y)}(t_y) = \psi_{N_{xy}}(t_y) \ni y.$$

Since $L(t_x) = L(t_y) = N_{xy}$, $t_x \leq_{N_{xy}} t_y$ and $t_x \neq t_y$, by (O2) we obtain $t_x \leq t_y$ and $t_x \neq t_y$, that is $t_x < t_y$.

Consequently for any elements $x, y \in \mathfrak{Bs}(\mathcal{M})$ such, that $y \leftarrow x$, $x \neq y$ there exists elements $t_x, t_y \in \mathbf{T}$, such, that $t_x < t_y$, $x \in \psi(t_x)$, $y \in \psi(t_y)$.

Thus, the mapping $\psi : \mathbf{T} \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ really is a time on $\mathcal{M}$.

**3.** Since the times $\{\psi_L | L \in \mathfrak{L}\}$ are one point, from (I.5) it follows, that for any $t \in \mathbf{T}$ the set $\psi(t)$ is a singleton. Thus, the time $\psi$ is quasi one-point. □

It is clear, that any oriented set $\mathcal{M}$, containing elementary states $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ such, that $x_2 \not\leftarrow x_1$ and $x_1 \not\leftarrow x_2$, can not be one-point chronologized. Thus, not any oriented set can be one-point chronologized. The next assertion shows, that not any oriented set can be monotone chronologized.

**Assertion I.3.1.** *If oriented set $\mathcal{M}$ contains elementary states $x_1, x_2, x_3 \in \mathfrak{Bs}(\mathcal{M})$ such, that $x_2 \leftarrow x_1$, $x_1 \not\leftarrow x_2$, $x_3 \leftarrow x_2$, $x_2 \not\leftarrow x_3$, $x_1 \leftarrow x_3$, $x_1 \neq x_3$, then this oriented set can not be monotone chronologized.*

*Proof.* Let oriented set $\mathcal{M}$ contains elementary states $x_1, x_2, x_3 \in \mathfrak{Bs}(\mathcal{M})$, satisfying the conditions of Assertion. Suppose, that the monotone chronologization $((\mathbf{T}, \leq), \psi)$ of the oriented set $\mathcal{M}$ exists. This means, that the mapping $\psi : \mathbf{T} :\mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ is a monotone time on $\mathcal{M}$. Since $x_1 \leftarrow x_3$ and $x_1 \neq x_3$, by time Definition I.2.1, there exist time points $t_1, t_3 \in \mathbf{T}$ such, that $x_1 \in \psi(t_1)$, $x_3 \in \psi(t_3)$ and $t_3 < t_1$. Also, by Time Definition I.2.1, there exists time point $t_2 \in \mathbf{T}$, such, that $x_2 \in \psi(t_2)$. Then, by Definition of monotone time I.3.1, from conditions $x_2 \leftarrow x_1$, $x_1 \not\leftarrow x_2$, $x_3 \leftarrow x_2$, $x_2 \not\leftarrow x_3$ it follows, that $t_1 < t_2$, $t_2 < t_3$. Hence $t_1 < t_3$, which contradicts inequality above ($t_3 < t_1$). Thus, the assumption about the existence of monotone chronologization of $\mathcal{M}$ is wrong. □

**Problem I.3.1.** *Find necessary and sufficient conditions of existence of one-point chronologization for oriented set.*

**Problem I.3.2.** *Find necessary and sufficient conditions of existence of monotone chronologization for oriented set.*

Main results of this Section were anonced in [1] and published in [2, Section 4].

## 4 Time and Simultaneity. Internal Time

**Definition I.4.1.** *Let $(\mathcal{M}, \mathbb{T}, \psi) = (\mathcal{M}, (\mathbf{T}, \leq), \psi)$ be a primitive changeable set. The set*

$$Y_\psi = \{\psi(t) \mid t \in \mathbf{T}\}$$

*will be referred to as the* ***set of simultaneous states****, generated by the time $\psi$.*

Directly from the time definition (Definition I.2.1) it follows the next assertion.

**Assertion I.4.1.** *Let $(\mathcal{M}, \mathbb{T}, \psi) = (\mathcal{M}, (\mathbf{T}, \leq), \psi)$ be a primitive changeable set, and $Y_\psi$ be a set of simultaneous states, generated by the time $\psi$. Then:*

$$\bigcup_{A \in Y_\psi} A = \mathfrak{Bs}(\mathcal{M}).$$

**Definition I.4.2.** *Let $\mathcal{M}$ be an oriented set. Any family of sets $\mathbf{Y} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$, which possesses the property $\bigcup_{A \in \mathbf{Y}} A = \mathfrak{Bs}(\mathcal{M})$ we will call the* ***simultaneity*** *on $\mathcal{M}$.*

According to Assertion I.4.1, any set of simultaneous states, generated by the time $\psi$ of a primitive changeable set $(\mathcal{M}, \mathbb{T}, \psi)$ is a simultaneity.

Let $\mathbf{Y}$ be a simultaneity on an oriented set $\mathcal{M}$ and $A, B \in \mathbf{Y}$. We will denote $B \leftarrow A$ (or $B \underset{\mathcal{M}}{\leftarrow} A$) if and only if:

$$A = B = \emptyset, \text{ or } \quad \exists\, x \in A\, \exists\, y \in B \;\; (y \leftarrow x)\,.$$

The next lemma is trivial.

**Lemma I.4.1.** *Let $\mathbf{Y}$ be a simultaneity on an oriented set $\mathcal{M}$. Then the pair $(\mathbf{Y}, \leftarrow)$ itself is an oriented set.*

**Theorem I.4.1.** *Let $\mathcal{M}$ be an oriented set and $\mathbf{Y} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ be a simultaneity on $\mathcal{M}$. Then there exists time $\psi$ on the oriented set $\mathcal{M}$, such, that:*

$$\mathbf{Y} = Y_\psi,$$

*where $Y_\psi$ is the set of simultaneous states, generated by the time $\psi$.*

*Proof.* Let $\mathcal{M}$ be an oriented set and $\mathbf{Y} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ be a simultaneity on $\mathcal{M}$.

**a)** First we prove the Theorem in the case, where the simultaneity $\mathbf{Y}$ "separates" sequential unequal elementary states, that is where the following condition holds:

**(Rp)** For any $x, y \in \mathfrak{Bs}(\mathcal{M})$ such, that $y \leftarrow x$ and $x \neq y$ there exists sets $A, B \in \mathbf{Y}$ such, that $x \in A$, $y \in B$ and $A \neq B$.

By Lemma I.4.1, $(\mathbf{Y}, \leftarrow)$ is an oriented set. According to Theorem I.3.2, oriented set $(\mathbf{Y}, \leftarrow)$ can be quasi one-point chronologized. Let $\Psi : \mathbf{T} \mapsto 2^{\mathbf{Y}}$ be quasi one-point time on $(\mathbf{Y}, \leftarrow)$. By Definition I.3.1 of quasi one-point time, for any $t \in \mathbf{T}$ the set $\Psi(t)$ is a singleton. This means, that:

$$\forall\, t \in \mathbf{T}\; \exists A_t \in \mathbf{Y} \quad \Psi(t) = \{A_t\}\,.$$

Denote:

$$\psi(t) := A_t, \quad t \in \mathbf{T}.$$

The next aim is to prove, that $\psi$ is time on $\mathcal{M}$. Since $\Psi$ is time on $(\mathbf{Y}, \leftarrow)$, then $\bigcup_{t \in \mathbf{T}} \Psi(t) = \mathbf{Y}$. And, taking into account, that $\Psi(t) = \{A_t\}$, $t \in \mathbf{T}$, we obtain $\{A_t \mid t \in \mathbf{T}\} = \mathbf{Y}$. Therefore, since the family of sets $\mathbf{Y}$ is simultaneity on $\mathcal{M}$, we have, $\bigcup_{t \in \mathbf{T}} \psi(t) = \bigcup_{t \in \mathbf{T}} A_t = \bigcup_{A \in \mathbf{Y}} A =$

$\mathfrak{Bs}(\mathcal{M})$. Hence, for any $x \in \mathfrak{Bs}(\mathcal{M})$ there exists a time moment $t \in \mathbf{T}$ such, that $x \in \psi(t)$. Thus, the first condition of time Definition I.2.1 is satisfied. Now, we are going to prove, that the second condition of Definition I.2.1 also is satisfied. Let $x, y \in M$, $y \leftarrow x$ and $x \neq y$. By condition (Rp), there exist sets $A, B \in \mathbf{Y}$, such, that $x \in A$, $y \in B$ and $A \neq B$. Taking into account, that $x \in A$, $y \in B$ and $y \leftarrow x$, we obtain $B \leftarrow A$. Since $B \leftarrow A$, $A \neq B$ and $\Psi$ — time on $(\mathbf{Y}, \leftarrow)$, there exist time moments $t, \tau \in \mathbf{T}$ such, that $A \in \Psi(t)$, $B \in \Psi(\tau)$ and $t < \tau$. And, taking into account $\Psi(t) = \{A_t\}$, $\Psi(\tau) = \{A_\tau\}$, we obtain $A = A_t$, $B = A_\tau$, that is $A = \psi(t)$, $B = \psi(\tau)$. Since $x \in A$, $y \in B$, then $x \in \psi(t)$, $y \in \psi(\tau)$, where $t < \tau$.

Thus, $\psi$ is a time on $\mathcal{M}$. Moreover, taking into account that it has been proven before, we get:

$$Y_\psi = \{\psi(t) \mid t \in \mathbf{T}\} = \{A_t \mid t \in \mathbf{T}\} = \mathbf{Y}.$$

Hence, in the case, when (Rp) is true, Theorem is proved.

**b)** Now we consider the case, when the condition (Rp) is not satisfied. Chose any element $\tilde{x}$, such, that $x \notin \mathfrak{Bs}(\mathcal{M})$. Denote:

$$\tilde{M} := \mathfrak{Bs}(\mathcal{M}) \cup \{\tilde{x}\}.$$

For elements $x, y \in \tilde{M}$ we put $y \widetilde{\leftarrow} x$ if and only if one of the following conditions is satisfied:

(a) $x, y \in \mathfrak{Bs}(\mathcal{M})$ and $y \leftarrow x$; (b) $x = y = \tilde{x}$.

That is the relation $\widetilde{\leftarrow}$ can be represented by formula $\widetilde{\leftarrow} = \leftarrow \cup \{(\tilde{x}, \tilde{x})\}$. Taking into account, that for $x, y \in \mathfrak{Bs}(\mathcal{M})$ the condition $y \widetilde{\leftarrow} x$ is is equivalent to the condition $y \leftarrow x$, further for relations $\widetilde{\leftarrow}$ and $\leftarrow$ we will use the same denotation $\leftarrow$, assuming, that the relation $\leftarrow$ is expanded to the set $\tilde{M}$. It is obvious, that $\left(\tilde{M}, \leftarrow\right)$ is an oriented set. Denote:

$$\mathbf{Y}_0 := \{B \in \mathbf{Y} \mid \exists x, y \in B : x \neq y,\, y \leftarrow x\}.$$

Since Condition (Rp) is not satisfied, we have $\mathbf{Y}_0 \neq \emptyset$. For $B \in \mathbf{Y}_0$ we put:

$$\tilde{B} := B \cup \{\tilde{x}\}.$$

Also we put:

$$\tilde{\mathbf{Y}}_0 := \left\{\tilde{B} \mid B \in \mathbf{Y}_0\right\}$$
$$\tilde{\mathbf{Y}} := \mathbf{Y} \cup \tilde{\mathbf{Y}}_0.$$

Since $\mathbf{Y}$ is a simultaneity on $\mathcal{M}$, and $\tilde{x} \in \tilde{B}$ for any set $\tilde{B} \in \tilde{\mathbf{Y}}_0$, then $\tilde{\mathbf{Y}}$ is a simultaneity on $\left(\tilde{M}, \leftarrow\right)$. The simultaneity $\tilde{\mathbf{Y}}$ readily satisfies the condition (Rp). Therefore, according to result, proven in paragraph a), there exists the time $\psi_1 : \mathbf{T} \mapsto 2^{\tilde{M}}$ on $\left(\tilde{M}, \leftarrow\right)$, such, that $Y_{\psi_1} = \{\psi_1(t) \mid t \in \mathbf{T}\} = \tilde{\mathbf{Y}}$. Now, we denote:

$$\psi(t) := \psi_1(t) \cap \mathfrak{Bs}(\mathcal{M}), \quad t \in \mathbf{T}.$$

In accordance with Assertion I.2.1, $\psi$ is a time on $\mathcal{M}$. Moreover we obtain:

$$\begin{aligned} Y_\psi = \{\psi(t) \mid t \in \mathbf{T}\} = \{\psi_1(t) \cap \mathfrak{Bs}(\mathcal{M}) \mid t \in \mathbf{T}\} = \left\{A \cap \mathfrak{Bs}(\mathcal{M}) \mid A \in \tilde{\mathbf{Y}}\right\} = \\ = \{A \cap \mathfrak{Bs}(\mathcal{M}) \mid A \in \mathbf{Y}\} \cup \left\{A \cap \mathfrak{Bs}(\mathcal{M}) \mid A \in \tilde{\mathbf{Y}}_0\right\} = \\ = \{A \mid A \in \mathbf{Y}\} \cup \left\{\tilde{B} \cap \mathfrak{Bs}(\mathcal{M}) \mid B \in \mathbf{Y}_0\right\} = \mathbf{Y} \cup \{B \mid B \in \mathbf{Y}_0\} = \end{aligned}$$

$$= \mathbf{Y} \cup \mathbf{Y}_0 = \mathbf{Y}.$$

□

**Definition I.4.3.** *Let $\mathbf{Y} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ be any simultaneity on the oriented set $\mathcal{M}$. Time $\psi$ on $\mathcal{M}$ will be named the **generating time** of the simultaneity $\mathbf{Y}$ if and only if $\mathbf{Y} = Y_\psi$, where $Y_\psi$ is the set of simultaneous states, generated by the time $\psi$.*

Thus, Theorem I.4.1 asserts, that any simultaneity always has it's own generating time. Below we consider the question about uniqueness of a generating time for a simultaneity (under the certain conditions). To ensure the correctness of staging this question, first of all, we need to introduce the concept of equivalence of two chronologizations.

**Definition I.4.4.** *Let $\mathcal{M}$ be an oriented set and $\psi_1 : \mathbf{T}_1 \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$, $\psi_2 : \mathbf{T}_2 \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ be some times for $\mathcal{M}$, defined on the linear ordered sets $(\mathbf{T}_1, \leq_1)$, $(\mathbf{T}_2, \leq_2)$. We say, that the chronologizations $\mathcal{H}_1 = ((\mathbf{T}_1, \leq_1), \psi_1)$ and $\mathcal{H}_2 = ((\mathbf{T}_2, \leq_2), \psi_2)$ are **equivalent** (using the denotation $\mathcal{H}_1 \upuparrows \mathcal{H}_2$) if and only if there exist an one-to-one correspondence $\xi : \mathbf{T}_1 \mapsto \mathbf{T}_2$ such, that:*

*1) $\xi$ is order isomorphism between the linearly ordered sets $(\mathbf{T}_1, \leq_1)$, $(\mathbf{T}_2, \leq_2)$, that is for any $t, \tau \in \mathbf{T}_1$ the inequality $t \leq_1 \tau$ is equivalent to the inequality $\xi(t) \leq_2 \xi(\tau)$.*

*2) For any $t \in \mathbf{T}_1$ it is valid the equality $\psi_1(t) = \psi_2(\xi(t))$.*

**Assertion I.4.2.** *Let $\mathcal{M}$ be any oriented set and $\mathcal{W}$ is any set, which consists of chronologizations of $\mathcal{M}$. Then the relation $\upuparrows$ is an equivalence relation on $\mathcal{W}$.*

*Proof.* Throughout in this proof $\mathcal{H}_i = ((\mathbf{T}_i, \leq_i), \psi_i) \in \mathcal{W}$ $(i = 1, 2, 3)$ mean any three chronologizations of the oriented set $\mathcal{M}$.

**1)** Reflexivity. Denote $\xi_{11}(t) := t$, $t \in \mathbf{T}_1$. It is obvious that $\xi_{11}$ is a order isomorphism between $(\mathbf{T}_1, \leq_1)$ and $(\mathbf{T}_1, \leq_1)$. Besides we have $\psi(t) = \psi(\xi(t))$, $t \in \mathbf{T}$. Thus $\mathcal{H}_1 \upuparrows \mathcal{H}_1$.

**2)** Symmetry. Let $\mathcal{H}_1 \upuparrows \mathcal{H}_2$. Then, by Definition I.4.4, there exist an one-to-one correspondence $\xi_{12} : \mathbf{T}_1 \mapsto \mathbf{T}_2$ such, that:

1) $\xi_{12}$ is order isomorphism between the linearly ordered sets $(\mathbf{T}_1, \leq_1)$, $(\mathbf{T}_2, \leq_2)$.

2) $\psi_1(t) = \psi_2(\xi_{12}(t))$, for any $t \in \mathbf{T}_1$.

Since the mapping $\xi_{12}$ is bijection, there exists the inverse mapping $\xi_{21}(t) := \xi_{12}^{[-1]}(t) \in \mathbf{T}_1$, $t \in \mathbf{T}_2$. Since $\xi_{12}$ is order isomorphism between the linearly ordered sets $(\mathbf{T}_1, \leq_1)$, $(\mathbf{T}_2, \leq_2)$, then $\xi_{21}$ is order isomorphism between $(\mathbf{T}_2, \leq_2)$ and $(\mathbf{T}_1, \leq_1)$. Moreover, for any $t \in \mathbf{T}_2$ we obtain:

$$\psi_2(t) = \psi_2\left(\xi_{12}\left(\xi_{12}^{[-1]}(t)\right)\right) = \psi_1(\xi_{21}(t)).$$

Thus, $\mathcal{H}_2 \upuparrows \mathcal{H}_1$.

**3)** Transitivity. Let $\mathcal{H}_1 \upuparrows \mathcal{H}_2$, $\mathcal{H}_2 \upuparrows \mathcal{H}_3$. Then there exist order isomorphisms $\xi_{12} : \mathbf{T}_1 \mapsto \mathbf{T}_2$ and $\xi_{23} : \mathbf{T}_2 \mapsto \mathbf{T}_3$ such, that $\psi_1(t) = \psi_2(\xi_{12}(t))$, $t \in \mathbf{T}_1$ and $\psi_2(t) = \psi_3(\xi_{23}(t))$, $t \in \mathbf{T}_2$. Denote, $\xi_{13}(t) := \xi_{23}(\xi_{12}(t))$, $t \in \mathbf{T}_1$. It is easy to verify, that $\xi_{13}$ is an order isomorphism between $(\mathbf{T}_1, \leq_1)$ and $(\mathbf{T}_3, \leq_3)$. Moreover, for any $t \in \mathbf{T}_1$ we obtain:

$$\psi_1(t) = \psi_2(\xi_{12}(t)) = \psi_3(\xi_{23}(\xi_{12}(t))) = \psi_3(\xi_{13}(t)).$$

Therefore, $\mathcal{H}_1 \upuparrows \mathcal{H}_3$. □

Now, if we consider the question about uniqueness of a generating time for a simultaneity up to equivalence of corresponding chronologizations, the answer still is negative. For example we can consider a linearly ordered sets $(\mathbf{T}, \leq)$ and $(\mathbf{T}_1, \leq)$ such, that $\emptyset \neq \mathbf{T}_1 \subset \mathbf{T}$ (more accurately linear order relation on $\mathbf{T}_1$ is a restriction of order relation on $\mathbf{T}$, and both relations are denoted by the same symbol "$\leq$"). If $\psi_1 : \mathbf{T}_1 \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ is a time on the oriented set $\mathcal{M}$, then for any fixed element $t_1 \in \mathbf{T}_1$ we can define the time:

$$\psi(t) := \begin{cases} \psi_1(t), & t \in \mathbf{T}_1 \,; \\ \psi_1(t_1)\,, & t \in \mathbf{T} \setminus \mathbf{T}_1 \end{cases} \qquad (t \in \mathbf{T})\,.$$

This time is such, that $Y_\psi = Y_{\psi_1}$, although, in the case, when the ordered sets $(\mathbf{T}, \leq)$ and $(\mathbf{T}_1, \leq)$ are not isomorphic, the chronologizations $((\mathbf{T}, \leq), \psi)$ and $((\mathbf{T}_1, \leq), \psi_1)$ are not equivalent. That is why, to obtain the positive answer for the above question, further we will impose additional conditions on simultaneity and generating time.

**Definition I.4.5.** *Let $\mathcal{M}$ be an oriented set.*

*1) We will say, that a set $B \subseteq \mathfrak{Bs}(\mathcal{M})$ is* ***monotonously sequential*** *to a set $A \in \mathfrak{Bs}(\mathcal{M})$ in the oriented set $\mathcal{M}$ if and only if there exist elements $x \in A$ and $y \in B$ such, that $y \underset{\mathcal{M}}{\leftarrow} x$ and $x \underset{\mathcal{M}}{\not\leftarrow} y$. In this case we will use the denotation $B \underset{\mathcal{M}}{\leftarrow(\mathrm{m})} A$.*

*2) Let $\mathcal{Q} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ be any system of subsets of $\mathfrak{Bs}(\mathcal{M})$. We will say, that a set $A \in \mathcal{Q}$ is* ***transitively monotonously sequential*** *to a set $B \in \mathcal{Q}$ relative to the system $\mathcal{Q}$ if and only if there exist a finite sequence of sets $C_0, C_1, \cdots, C_n \in \mathcal{Q}$ ($n \in \mathbb{N}$) such, that $C_0 = A$, $C_n = B$ and $C_k \underset{\mathcal{M}}{\leftarrow(\mathrm{m})} C_{k-1}$, for any $k \in \overline{1,n}$. In this case we will use the denotation $B \overset{\mathcal{Q}}{\underset{\mathcal{M}}{\twoheadleftarrow(\mathrm{m})}} A$.*

In the case where the oriented set $\mathcal{M}$ is known in advance, the char $\mathcal{M}$ in the denotations $\underset{\mathcal{M}}{\leftarrow(\mathrm{m})}$ and $\overset{\mathcal{Q}}{\underset{\mathcal{M}}{\twoheadleftarrow(\mathrm{m})}}$ will be released, and we will use the abbreviated denotations $\leftarrow(\mathrm{m})$ and $\overset{\mathcal{Q}}{\twoheadleftarrow(\mathrm{m})}$ (respectively).

*Remark* I.4.1. It is easy to prove, that for any system of sets $\mathcal{Q} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ (in any oriented set $\mathcal{M}$) the binary relation $\overset{\mathcal{Q}}{\twoheadleftarrow(\mathrm{m})}$ is transitive on the set $\mathcal{Q}$.

**Assertion I.4.3.** *Let $\mathcal{M}$ be an oriented set, and $\mathfrak{S}, \mathfrak{S}' \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ be systems of subsets of $\mathfrak{Bs}(\mathcal{M})$, moreover $\mathfrak{S} \sqsubseteq \mathfrak{S}'$ (this means, that for any set $A \in \mathfrak{S}$ there exist a set $A' \in \mathfrak{S}'$ such, that $A \subseteq A'$).*

*Then for any $A, B \in \mathfrak{S}$ and $A', B' \in \mathfrak{S}'$ such, that $A \subseteq A'$, $B \subseteq B'$ correlation $B \overset{\mathfrak{S}}{\twoheadleftarrow(\mathrm{m})} A$ leads to the correlation $B' \overset{\mathfrak{S}'}{\twoheadleftarrow(\mathrm{m})} A'$.*

*Proof.* Suppose that the conditions of Assertion are performed. Let $A, B \in \mathfrak{S}$, $A', B' \in \mathfrak{S}'$, $A \subseteq A'$, $B \subseteq B'$ and $B \overset{\mathfrak{S}}{\twoheadleftarrow(\mathrm{m})} A$. Then, there exists a finite sequence of sets $C_0, \cdots, C_n \in \mathfrak{S}$ ($n \in \mathbb{N}$) such, that $C_0 = A$, $C_n = B$ and $C_k \leftarrow(\mathrm{m})\, C_{k-1}$ (for any $k \in \overline{1,n}$). Since $\mathfrak{S} \sqsubseteq \mathfrak{S}'$, there exist sets $C_0', \cdots, C_n' \in \mathfrak{S}'$ such, that $C_k \subseteq C_k'$ ($k \in \overline{0,n}$). Moreover, since $C_0 = A \subseteq A' \in \mathfrak{S}'$, $C_n = B \subseteq B' \in \mathfrak{S}'$, we can consider that $C_0' = A'$, $C_n' = B'$. Taking into account that $C_k \subseteq C_k'$ ($k \in \overline{0,n}$), and $C_k \leftarrow(\mathrm{m})\, C_{k-1}$ ($k \in \overline{1,n}$), by Definition I.4.5, we obtain $C_k' \leftarrow(\mathrm{m})\, C_{k-1}'$, $k \in \overline{1,n}$ (where $C_0' = A'$, $C_n' = B'$). Thus $B' \overset{\mathfrak{S}'}{\twoheadleftarrow(\mathrm{m})} A'$. □

**Definition I.4.6.** *Let $\mathcal{M}$ be an oriented set.*

*1) System of sets $\mathfrak{S} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ will be referred to as* ***unrepeatable*** *if and only if there not exist sets $A, B \in \mathfrak{S}$ such, that $A \overset{\mathfrak{S}}{\twoheadleftarrow(\mathrm{m})} B$ and $B \overset{\mathfrak{S}}{\twoheadleftarrow(\mathrm{m})} A$. In particular, in the case, where a simultaneity $\mathbf{Y} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ is unrepeatable system of sets, this simultaneity we will call an* ***unrepeatable simultaneity****.*

*2) Simultaneity $\mathbf{Y} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ will be referred to as* ***precise*** *if and only if for any $x, y \in \mathfrak{Bs}(\mathcal{M})$ such, that $y \leftarrow x$ and $x \neq y$ there exist sets $A, B \in \mathbf{Y}$ such, that $x \in A$, $y \in B$, $A \neq B$ and $B \overset{\mathbf{Y}}{\twoheadleftarrow(\mathrm{m})} A$ (this means, that this simultaneity "fixes" all changes on the oriented set $\mathcal{M}$).*

*3) Simultaneity* $\mathbf{Y}$ *will be called* ***precisely-unrepeatable*** *if and only if it is precise and, at the same time, unrepeatable.*

**Assertion I.4.4.** *Let* $\mathcal{M}$ *be an oriented set and* $\mathfrak{S} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ *is unrepeatable system of sets. Then:*

*1) For any* $A, B \in \mathfrak{S}$*, such, that* $B \overset{\mathfrak{S}}{\twoheadleftarrow}(\mathrm{m})\, A$ *is true* $A \neq B$*.*

*2) If* $\mathfrak{S}_1 \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ *and* $\mathfrak{S}_1 \sqsubseteq \mathfrak{S}$*, then* $\mathfrak{S}_1$ *also is unrepeatable system of sets.*

*Proof.* 1) Let $\mathfrak{S} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ be unrepeatable system of sets. If we suppose, that $B \overset{\mathfrak{S}}{\twoheadleftarrow}(\mathrm{m})\, A$ and $A = B$ (for some $A, B \in \mathfrak{S}$), then we obtain $A \overset{\mathfrak{S}}{\twoheadleftarrow}(\mathrm{m})\, B$ and $B \overset{\mathfrak{S}}{\twoheadleftarrow}(\mathrm{m})\, A$, which is impossible, since the system of sets $\mathfrak{S}$ is unrepeatable.

2) Let $\mathfrak{S}_1 \sqsubseteq \mathfrak{S}$. Suppose, that the system of sets $\mathfrak{S}_1$ is not unrepeatable. Then, there exists sets $A_1, B_1 \in \mathfrak{S}_1$ such, that $A_1 \overset{\mathfrak{S}_1}{\twoheadleftarrow}(\mathrm{m})\, B_1$ and $B_1 \overset{\mathfrak{S}_1}{\twoheadleftarrow}(\mathrm{m})\, A_1$. Since $\mathfrak{S}_1 \sqsubseteq \mathfrak{S}$, there exist sets $A, B \in \mathfrak{S}$ such, that $A_1 \subseteq A$, $B_1 \subseteq B$. Hence, by Assertion I.4.3, we obtain $A \overset{\mathfrak{S}}{\twoheadleftarrow}(\mathrm{m})\, B$ and $B \overset{\mathfrak{S}}{\twoheadleftarrow}(\mathrm{m})\, A$, which is impossible, since the system of sets $\mathfrak{S}$ is unrepeatable. Thus, the system of sets $\mathfrak{S}_1$ is unrepeatable, because the opposite assumption is wrong. □

*Remark* I.4.2. From Remark I.4.1 and Assertion I.4.4 (item 1) it readily follows, that in the case, where a simultaneity $\mathbf{Y} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ is unrepeatable, the relation $\overset{\mathbf{Y}}{\twoheadleftarrow}(\mathrm{m})$ is a strict order on $\mathbf{Y}$ (ie $\overset{\mathbf{Y}}{\twoheadleftarrow}(\mathrm{m})$ is anti-reflexive and transitive relation).

**Lemma I.4.2.** *Let* $\psi : \mathbf{T} \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ *be a monotone time on an oriented set* $\mathcal{M}$*, and* $Y_\psi$ *be a simultaneity, generated by the time* $\psi$*. Then for any* $t_1, t_2 \in \mathbf{T}$ *the condition* $\psi(t_2) \overset{Y_\psi}{\twoheadleftarrow}(\mathrm{m})\, \psi(t_1)$ *leads to* $t_1 < t_2$*.*

*Proof.* 1) First we consider the case, where $\psi(t_2) \leftarrow(\mathrm{m})\, \psi(t_1)$. In this case, by Definition I.4.5, there exist elements $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ such, that $x_2 \leftarrow x_1$ and $x_1 \not\leftarrow x_2$. Hence, since the time $\psi$ is monotone, we obtain $t_1 < t_2$ (by Definition I.3.1).

Now, we consider the general case, $\psi(t_2) \overset{Y_\psi}{\twoheadleftarrow}(\mathrm{m})\, \psi(t_1)$. In this case, by Definition I.4.5, there exist time points $\tau_0, \tau_1, \cdots, \tau_n \in \mathbf{T}$ such, that $\tau_0 = t_1$, $\tau_n = t_2$ and $\psi(\tau_k) \leftarrow(\mathrm{m})\, \psi(\tau_{k-1})$ for any $k \in \overline{1,n}$. By statement 1), $\tau_{k-1} < \tau_k$, $k \in \overline{1,n}$. Thus, $t_1 = \tau_0 < \tau_1 < \cdots < \tau_n = t_2$. □

**Definition I.4.7.** *We will say, that a simultaneity* $\mathbf{Y}$ *on an oriented set is* ***monotone-connected*** *if and only if for any sets* $A, B \in \mathbf{Y}$ *such, that* $A \neq B$ *it holds one of the conditions* $A \overset{\mathbf{Y}}{\twoheadleftarrow}(\mathrm{m})\, B$ *or* $B \overset{\mathbf{Y}}{\twoheadleftarrow}(\mathrm{m})\, A$*.*

*Remark* I.4.3. Directly from Definition I.4.7 and Remark I.4.2 it follows, that if a simultaneity $\mathbf{Y} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ is unrepeatable and monotone-connected, then the relation $\overset{\mathbf{Y}}{\twoheadleftarrow}(\mathrm{m})$ is a strict linear order on $\mathbf{Y}$.

**Definition I.4.8.** *Let* $\mathcal{M}$ *be an oriented set and* $(\mathbf{T}, \leq)$ *be a linearly ordered set. Time* $\psi : \mathbf{T} \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ *will be called* ***incessant*** *if and only if there not exist time points* $t_1, t_2 \in \mathbf{T}$ *such, that* $t_1 < t_2$ *and for any* $t \in \mathbf{T}$*, satisfying* $t_1 \leq t \leq t_2$ *it is true the equality* $\psi(t) = \psi(t_1)$*. In the case, where the time* $\psi$ *is both monotone and incessant it will be called* ***strictly monotone****.*

**Lemma I.4.3.** *Let* $\mathbf{Y}$ *be precisely-unrepeatable and monotone-connected simultaneity on the oriented set* $\mathcal{M}$ *and* $\psi : \mathbf{T} \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ *is the time, generating this simultaneity.*

*1) If the time* $\psi$ *is strictly monotone, then it is* ***unrepeatable*** *(this means, that for any* $t_1, t_2 \in \mathbf{T}$ *such, that* $t_1 \neq t_2$ *the correlation* $\psi(t_1) \neq \psi(t_2)$ *is valid).*

*2) The time* $\psi$ *is strictly monotone if and only if for any* $t_1, t_2 \in \mathbf{T}$ *inequality* $t_1 < t_2$ *implies the correlation* $\psi(t_2) \overset{\mathbf{Y}}{\twoheadleftarrow}(\mathrm{m})\, \psi(t_1)$*.*

*3) If the time $\psi$ is strictly monotone, then the strictly linearly ordered sets* $(\mathbf{T}, >)$ *and* $\left(\mathbf{Y}, \overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})}\right)$ *are isomorphic relative the order, and the mapping* $\psi : \mathbf{T} \mapsto \mathbf{Y}$ *is the order isomorphism between them.*

*Proof.* 1) Let, under conditions of the Lemma, time $\psi : \mathbf{T} \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ be strictly monotone. Suppose, there exist time points $t_1, t_2 \in \mathbf{T}$ such, that $t_1 < t_2$ and $\psi(t_1) = \psi(t_2)$. Since the time $\psi$ (being strictly monotone) is incessant, there exists a time point $t_3 \in \mathbf{T}$ such, that $t_1 < t_3 < t_2$ and $\psi(t_3) \neq \psi(t_1) = \psi(t_2)$. So far as $\psi(t_3) \neq \psi(t_1)$ and the simultaneity $\mathbf{Y}$ is monotone-connected, one of the conditions $\psi(t_3) \overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})} \psi(t_1)$ or $\psi(t_1) \overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})} \psi(t_3)$ is performed. But since $t_1 < t_3$ the correlation $\psi(t_1) \overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})} \psi(t_3)$ is impossible by Lemma I.4.2. Therefore, $\psi(t_3) \overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})} \psi(t_1)$. Similarly, since $t_3 < t_2$ and $\psi(t_3) \neq \psi(t_2)$, we obtain $\psi(t_2) \overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})} \psi(t_3)$. Hence, taking into account, that $\psi(t_1) = \psi(t_2)$, we have $\psi(t_3) \overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})} \psi(t_1)$ and $\psi(t_1) \overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})} \psi(t_3)$, which is impossible, because the simultaneity $\mathbf{Y} = Y_\psi$ is unrepeatable.

2.a) Suppose, that the time $\psi : \mathbf{T} \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ is strictly monotone. Chose any time points $t_1, t_2 \in \mathbf{T}$ such, that $t_1 < t_2$. By the first statement of this Lemma, $\psi(t_1) \neq \psi(t_2)$. Since the simultaneity $\mathbf{Y}$ is monotone-connected, one of the conditions $\psi(t_2) \overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})} \psi(t_1)$ or $\psi(t_1) \overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})} \psi(t_2)$ is performed. But, so far as $t_1 < t_2$, the condition $\psi(t_1) \overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})} \psi(t_2)$ is impossible by Lemma I.4.2. Thus:

$$\forall t_1, t_2 \in \mathbf{T} \; t_1 < t_2 \Rightarrow \psi(t_2) \overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})} \psi(t_1). \tag{I.6}$$

2.b) Now we suppose, that Condition (I.6) holds. The first aim is to prove, that the time $\psi$ is monotone. Consider any elementary states $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ such, that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$, $x_2 \leftarrow x_1$ and $x_1 \not\leftarrow x_2$ (where $t_1, t_2 \in \mathbf{T}$). By Definition I.4.5, $\psi(t_2) \leftarrow_{(\mathrm{m})} \psi(t_1)$. Consequently,

$$\psi(t_2) \overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})} \psi(t_1). \tag{I.7}$$

If we suppose $t_1 \geq t_2$, we must obtain:

$$\psi(t_1) \overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})} \psi(t_2) \tag{I.8}$$

(indeed, in the case $t_1 = t_2$ the correlation (I.8) follows from (I.7), and in the case $t_1 > t_2$ the correlation (I.8) is caused by Condition (I.6)). Thus, in the case $t_1 \geq t_2$, both of the conditions (I.7) and (I.8) must be performed, which is impossible (since the simultaneity $\mathbf{Y}$ is unrepeatable). Consequently, only the inequality $t_1 < t_2$ is possible. This proves that the time $\psi$ is monotone.

Thus, it remains to prove, that the time $\psi$ is incessant. Suppose, there exist time points $t_1, t_2 \in \mathbf{T}$ such, that $t_1 < t_2$, and $\psi(t) = \psi(t_1)$ for any $t \in \mathbf{T}$, satisfying $t_1 \leq t \leq t_2$. Then, in particular, $\psi(t_1) = \psi(t_2)$ (where $t_1 < t_2$). Since $t_1 < t_2$, by Condition (I.6), correlation (I.7) must be performed. But since $\psi(t_1) = \psi(t_2)$, the correlation (I.8) also is performed, which is impossible (since the simultaneity $\mathbf{Y}$ is unrepeatable). Therefore, the time $\psi$ is incessant.

Thus, the time $\psi$ is strictly monotone.

3) Let the time $\psi : \mathbf{T} \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ be strictly monotone. According to the first statement of the Lemma, the mapping $\psi : \mathbf{T} \mapsto \mathbf{Y} = Y_\psi$ is one-to-one correspondence between $\mathbf{T}$ and $\mathbf{Y} = Y_\psi$. According to the second statement of the Lemma, for any $t_1, t_2 \in \mathbf{T}$ the inequality $t_2 > t_1$ implies the correlation $\psi(t_2) \overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})} \psi(t_1)$. Hence, taking into account, that by remark I.4.3, $\left(\mathbf{Y}, \overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})}\right)$ is a linearly ordered set (with strict order), we conclude, that the mapping

$\psi$ is isomorphism between the strictly linearly ordered sets $(\mathbf{T}, >)$ and $\left(\mathbf{Y}, \overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})}\right)$. □

*Remark* I.4.4. It turns out, that for any precisely-unrepeatable and monotone-connected simultaneity $\mathbf{Y} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ the assertion, inverse to the first statement, of Lemma I.4.3, in general, is not true. To demonstrate this we present the following example.

***Example*** I.4.1. Let us consider the following oriented set:

$$\mathfrak{Bs}(\mathcal{M}) := \{1,2,3,4\}\,;$$
$$\underset{\mathcal{M}}{\leftarrow} := \{(3,1),(4,2)\} \cup \mathbf{diag}\,(\mathfrak{Bs}(\mathcal{M}))\,,$$

that is, in the other words, $3 \leftarrow 1$, $4 \leftarrow 2$, $1 \leftarrow 1$, $2 \leftarrow 2$, $3 \leftarrow 3$, $4 \leftarrow 4$. In this oriented set we consider the following simultaneity:

$$\mathbf{Y} := \{\{1,2\}\,,\ \{3,4\}\,,\ \{2,3\}\}\,.$$

Then, we have $\{2,3\} \leftarrow_{(\mathrm{m})} \{1,2\}$, $\{3,4\} \leftarrow_{(\mathrm{m})} \{2,3\}$, $\{3,4\} \leftarrow_{(\mathrm{m})} \{1,2\}$, and $\{2,3\} \not\leftarrow (m) \{3,4\}$, $\{1,2\} \not\leftarrow (m) \{2,3\}$, $\{1,2\} \not\leftarrow (m) \{3,4\}$, moreover, any set of simultaneity $\mathbf{Y}$ is not monotonously sequential by the itself. That is, schematically:

$$\begin{array}{ccccc} \{3,4\} & \leftarrow_{(\mathrm{m})} & \{2,3\} & \leftarrow_{(\mathrm{m})} & \{1,2\} \\ \nwarrow & <-- & \leftarrow_{(\mathrm{m})} & <-- & \swarrow \end{array}\,,$$

and, moreover, the relation "$\leftarrow_{(\mathrm{m})}$" on the simultaneity $\mathbf{Y}$ is fully generated by the last scheme. And from this scheme it is evident, that the simultaneity $\mathbf{Y}$ is unrepeatable and monotone-connected. Moreover, it is easy to verify, that this simultaneity is precise.

Also we consider the following linearly ordered set:

$$\mathbf{T} := \{1,2,3\}\,,$$

with the standard linear order relation on the natural numbers $(\leq)$. The simultaneity $\mathbf{Y}$ can be generated by the following times:

$$\psi_1: \quad \psi_1(1) := \{1,2\}\,,\ \psi_1(2) := \{2,3\}\,,\ \psi_1(3) := \{3,4\}\,;$$
$$\psi_2: \quad \psi_2(1) := \{1,2\}\,,\ \psi_2(2) := \{3,4\}\,,\ \psi_2(3) := \{2,3\}\,.$$

Both of times $\psi_1$ and $\psi_2$ are, evidently, unrepeatable, but the time $\psi_2$ is not monotone, because of:

$$2 \in \psi_2(3), \quad 4 \in \psi_2(2),$$
$$4 \leftarrow 2,\ 2 \not\leftarrow 4, \quad \text{but } 3 \not< 2.$$

**Theorem I.4.2.** *For any precisely-unrepeatable and monotone-connected simultaneity* $\mathbf{Y}$ *an unique up to equivalence of chronologizations strictly monotone time* $\psi$ *exists, such, that* $\mathbf{Y} = Y_\psi$.

It should be noted, that the uniqueness up to equivalence of chronologizations in Theorem I.4.2 is understood as follows:

"if strictly monotone times $\psi_1$ and $\psi_2$, defined on linear ordered sets $\mathbb{T}_1$ and $\mathbb{T}_2$ are such, that $\mathbf{Y} = Y_{\psi_1} = Y_{\psi_2}$, then $\mathcal{H}_1 \upuparrows \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are corresponding chronologizations $(\mathcal{H}_i = (\mathbb{T}_i, \psi_i),\ i \in \{1,2\})$".

*Proof.* **1**. Let $\mathbf{Y}$ be precisely-unrepeatable and monotone-connected simultaneity on an oriented set $\mathcal{M}$. Then, by Remark I.4.3, $\overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})}$ is a strict linear order on $\mathbf{Y}$. Hence, the relation $_{(\mathrm{m})}\overset{\mathbf{Y}}{\twoheadrightarrow}$, inverse to the relation $\overset{\mathbf{Y}}{\twoheadleftarrow}_{(\mathrm{m})}$, also is a strict linear order on $\mathbf{Y}$. Denote:

$$\mathbf{T} := \mathbf{Y}.$$

For $t, \tau \in \mathbf{T} = \mathbf{Y}$ we will assume, that $t \leq \tau$ if and only if:

$$t = \tau \text{ or } t \overset{\mathbf{Y}}{\text{(m)}\twoheadrightarrow} \tau.$$

That is, $\leq$ is (non-strict) linear order, generated by the strict order $\overset{\mathbf{Y}}{\text{(m)}\twoheadrightarrow}$. Therefore, for $t, \tau \in \mathbf{T}$ the following logical equivalence is true:

$$t < \tau \iff t \overset{\mathbf{Y}}{\text{(m)}\twoheadrightarrow} \tau, \tag{I.9}$$

where "$\iff$" is the symbol of logical equivalence ("if and only if") and record $t < \tau$ means, that $t \leq \tau$ and $t \neq \tau$. Thus, $(\mathbf{T}, \leq)$ is a linearly ordered set. Denote:

$$\psi(t) := t, \quad t \in \mathbf{T} = \mathbf{Y}.$$

Since $\mathbf{T} = \mathbf{Y}$, then $\psi(t) = t \in \mathbf{Y} \subseteq 2^{\mathfrak{Bs}(\mathcal{M})}$ for $t \in \mathbf{T}$ .

**2**. The next aim is to prove, that $\psi$ is a time on $\mathcal{M}$.

(a) Since $\mathbf{Y}$ is a simultaneity, then for any $x \in \mathfrak{Bs}(\mathcal{M})$ there exists set $t_x \in \mathbf{Y} = \mathbf{T}$, such, that $x \in t_x$. Therefore, we obtain $\psi(t_x) = t_x \ni x$. Thus, the first condition of the time Definition I.2.1 is performed.

(b) Suppose, that $x, y \in \mathfrak{Bs}(\mathcal{M})$, $y \leftarrow x$ and $x \neq y$. Since the simultaneity $\mathbf{Y}$ is precise, there exist $t_x, t_y \in \mathbf{Y} = \mathbf{T}$ such, that $x \in t_x$, $y \in t_y$ and $t_y \overset{\mathbf{Y}}{\twoheadleftarrow\text{(m)}} t_x$. Then, by (I.9), $t_x < t_y$. Moreover, since $\psi(t) = t$, $t \in \mathbf{T}$, we have:

$$x \in t_x = \psi(t_x); \quad y \in t_y = \psi(t_y).$$

Consequently, the second condition of Definition I.2.1 also is satisfied.

Thus, the mapping $\psi$ is a time.

**3**. Now we aim to prove, that the time $\psi$ is strictly monotone.

(a) Let $x \in \psi(t_x)$, $y \in \psi(t_y)$, $y \leftarrow x$ and $x \not\leftarrow y$. Then $t_y = \psi(t_y) \leftarrow\text{(m)}\, \psi(t_x) = t_x$. Consequently, $t_y \overset{\mathbf{Y}}{\twoheadleftarrow\text{(m)}} t_x$, ie, by (I.9), $t_x < t_y$. Thus, the time $\psi$ is monotone.

(b) Suppose, that this time is not incessant. Then there exist $t_1, t_2 \in \mathbf{T}$ such, that $t_1 < t_2$ and $\psi(t) = \psi(t_1)$ for any $t \in \mathbf{T}$, satisfying the inequality $t_1 \leq t \leq t_2$. In particular this means, that $\psi(t_2) = \psi(t_1)$. And, since $\psi(\tau) = \tau$, $\tau \in \mathbf{T}$, we obtain $t_2 = t_1$, which contradicts the inequality $t_1 < t_2$. Therefore, the assumption is wrong, and the time $\psi$ is incessant.

Thus, the time $\psi$ is strictly monotone.

**4**. It remains to prove, that the time $\psi$ is unique up to equivalence of chronologizations. Let $\psi_1 : \mathbf{T}_1 \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ be an other strictly monotone time such, that $Y_{\psi_1} = \mathbf{Y}$ (where $(\mathbf{T}_1, \leq_1)$ is a linearly ordered set. Then, by Lemma I.4.3, the linearly ordered (by strict order) sets $(\mathbf{T}_1, >_1)$ and $\left(\mathbf{Y}, \overset{\mathbf{Y}}{\twoheadleftarrow\text{(m)}}\right)$ are isomorphic relative the order, with the mapping $\psi_1 : \mathbf{T}_1 \mapsto \mathbf{Y}$ being isomorphism, where $>_1$ is relation, inverse to the relation $<_1$, and $<_1$ is strict order, generated by non-strict order $\leq_1$. Thus, the ordered sets $(\mathbf{T}_1, \leq_1)$ and $(\mathbf{Y}, \leq) = (\mathbf{T}, \leq)$ also are isomorphic with the isomorphism $\psi_1$. Moreover, for any $t \in \mathbf{T}_1$, we have:

$$\psi_1(t) = \psi(\psi_1(t)),$$

ie, by Definition I.4.4, $((\mathbf{T}_1, \leq_1), \psi_1) \upuparrows ((\mathbf{T}, \leq), \psi)$. □

**Definition I.4.9.** *Let $\mathcal{M}$ be an oriented set, and $\psi : \mathbf{T} \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ be a time on $\mathcal{M}$.*

*A mapping $\mathbf{h} : \mathbf{T} \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ will be referred to as **chronometric process** (for the time $\psi$), if and only if:*

*1) $\mathbf{h}(t) \subseteq \psi(t)$ for any $t \in \mathbf{T}$.*

*2) For arbitrary $t, \tau \in \mathbf{T}$ inequality $t < \tau$ is valid if and only if $\mathbf{h}(\tau) \overset{\mathbf{h}(\mathbf{T})}{\twoheadleftarrow(\mathrm{m})} \mathbf{h}(t)$ and $\mathbf{h}(t) \neq \mathbf{h}(\tau)$, where $\mathbf{h}(\mathbf{T}) = \{\mathbf{h}(t) \mid t \in \mathbf{T}\}$;*

*The time $\psi$ on $\mathcal{M}$ will be referred to as **internal** if and only if there exists at least one chronometric process for this time.*

Sense of the term "internal time" lies in the fact that in the case, where a time on a primitive changeable set is internal, this time can be measured within this primitive changeable set, using the chronometric process as a "clock" and states of this process as "indicators of time points". The next aim is to establish the sufficient condition of existence and uniquness of internal time for given simultaneity.

**Lemma I.4.4.** *The generating time for precisely-unrepeatable and monotone-connected simultaneity is internal if and only if it is strictly monotone.*

*Proof.* Let $\mathcal{M}$ be an oriented set, $\mathbf{Y}$ is precisely-unrepeatable and monotone-connected simultaneity and $\psi : \mathbf{T} \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ is a time on $\mathcal{M}$, which generates $\mathbf{Y}$ (ie $\mathbf{Y} = Y_\psi$).

**1)** Suppose, that the time $\psi$ is internal. Then there exists a chronometric process $\mathbf{h} : \mathbf{T} \mapsto 2^{\mathfrak{Bs}(\mathcal{M})}$ for the time $\psi$.

1.a) First we are going to prove, that the time $\psi$ is monotone. Let $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$, $x_2 \leftarrow x_1$ and $x_1 \not\leftarrow x_2$. Then $\psi(t_2) \leftarrow(\mathrm{m}) \psi(t_1)$, ie $\psi(t_2) \overset{\mathbf{Y}}{\twoheadleftarrow(\mathrm{m})} \psi(t_1)$. Hence, since the simultaneity $\mathbf{Y}$ is unrepeatable, using Assertion I.4.4, we obtain $\psi(t_1) \neq \psi(t_2)$, ie $t_1 \neq t_2$. Thus, the possible cases are $t_1 < t_2$ or $t_2 < t_1$. Let us suppose, that $t_2 < t_1$. Then, since $\mathbf{h}$ is chronometric process, we have, $\mathbf{h}(t_1) \overset{\mathbf{h}(\mathbf{T})}{\twoheadleftarrow(\mathrm{m})} \mathbf{h}(t_2)$. From Definition I.4.9 it follows, that $\mathbf{h}(\mathbf{T}) \sqsubseteq \mathbf{Y}$ (where $\mathbf{h}(\mathbf{T}) = \{\mathbf{h}(t) \mid t \in \mathbf{T}\}$), consequently, using Assertion I.4.3, we obtain $\psi(t_1) \overset{\mathbf{Y}}{\twoheadleftarrow(\mathrm{m})} \psi(t_2)$, which is impossible, because the simultaneity $\mathbf{Y}$ is unrepeatable, and, according to the above proved, $\psi(t_2) \overset{\mathbf{Y}}{\twoheadleftarrow(\mathrm{m})} \psi(t_1)$. So only possible it remains the inequality $t_1 < t_2$, which proves, that the time $\psi$ is monotone.

1.b) Now, we are going to prove, that the time $\psi$ is incessant. Assume the contrary. Then there exist the time points $t_1, t_2 \in \mathbf{T}$ such, that $t_1 < t_2$, and for any $t \in \mathbf{T}$, satisfying $t_1 \leq t \leq t_2$, the equality $\psi(t) = \psi(t_1)$ is true. Then, in particular, $\psi(t_2) = \psi(t_1)$. But, since $\mathbf{h}$ is chronometric process, then $\mathbf{h}(t_2) \overset{\mathbf{h}(\mathbf{T})}{\twoheadleftarrow(\mathrm{m})} \mathbf{h}(t_1)$, and, by Assertion I.4.3, $\psi(t_2) \overset{\mathbf{Y}}{\twoheadleftarrow(\mathrm{m})} \psi(t_1)$. Therefore, by Assertion I.4.4, $\psi(t_2) \neq \psi(t_1)$, which contradicts the above written. Thus, the time $\psi$ is incessant. And, taking into account that has been proved in Paragraph 1.a), we have, that time $\psi$ is strictly monotone.

**2)** Now we suppose, that the time $\psi$ is strictly monotone. Then, by Lemma I.4.3, the strictly linearly ordered sets $(\mathbf{T}, >)$ and $\left(\mathbf{Y}, \overset{\mathbf{Y}}{\twoheadleftarrow(\mathrm{m})}\right) = \left(\mathbf{Y}, \overset{Y_\psi}{\twoheadleftarrow(\mathrm{m})}\right)$ are isomorphic relative the order, and the mapping $\psi : \mathbf{T} \mapsto \mathbf{Y}$ is the order isomorphism between them. That is why, for any $t_1, t_2 \in \mathbf{T}$ the conditions $t_1 < t_2$ and $\psi(t_2) \overset{Y_\psi}{\twoheadleftarrow(\mathrm{m})} \psi(t_1)$ are logically equivalent (where $Y_\psi = \mathbf{Y} = \psi(\mathbf{T})$). Thus, taking into account statement 1 of Assertion I.4.4, we conclude, that the mapping $\mathbf{h}(t) = \psi(t)$, $t \in \mathbf{T}$ is a chronometric process for the time $\psi$. Consequently, the time $\psi$ is internal. □

The next theorem follows from Lemma I.4.4 and Theorem I.4.2.

**Theorem I.4.3.** *For any precisely-unrepeatable and monotone-connected simultaneity $\mathbf{Y}$ an unique up to equivalence of chronologizations internal time $\psi$ exists, such, that $\mathbf{Y} = Y_\psi$.*

Philosophical content of Theorem I.4.3 is that the originality of pictures of reality, possibility to see any changes in the sequential simultaneous states, and connectivity of different pictures of reality by chains of changes are uniquely generating the course of "internal" time in "our" world.

*Remark* I.4.5. Further we will denote primitive changeable sets by large calligraphic letters.

Let $\mathcal{P} = (\mathcal{M}, \mathbb{T}, \phi)$ be a primitive changeable set, where $\mathbb{T} = (\mathbf{T}, \trianglelefteq)$ is a linearly ordered set. We introduce the following denotations:

$$\begin{array}{lll} \mathfrak{Bs}(\mathcal{P}) := \mathfrak{Bs}(\mathcal{M}); & \underset{\mathcal{P}}{\leftarrow} := \underset{\mathcal{M}}{\leftarrow}; & \mathbf{Tm}(\mathcal{P}) := \mathbf{T}; \\ \leq_{\mathcal{P}} := \trianglelefteq; & \psi_{\mathcal{P}} := \phi; & \mathbb{T}\mathbf{m}(\mathcal{P}) := (\mathbf{Tm}(\mathcal{P}), \leq_{\mathcal{P}}) = (\mathbf{T}, \trianglelefteq). \end{array}$$

Also we will use the records $\geq_{\mathcal{P}}, <_{\mathcal{P}}, >_{\mathcal{P}}$ to denote the inverse, strict and strict inverse orders, generated by the order $\leq_{\mathcal{P}}$. The set $\mathfrak{Bs}(\mathcal{P})$ we will name the *basic* set or the set of all *elementary states* of the primitive changeable set $\mathcal{P}$. Elements of the set $\mathfrak{Bs}(\mathcal{P})$ will be named *elementary states* of $\mathcal{P}$. The relation $\underset{\mathcal{P}}{\leftarrow}$ we will name the *directing relation of changes* of $\mathcal{P}$. The set $\mathbf{Tm}(\mathcal{P})$ will be named the *set of time points* of $\mathcal{P}$. The relations $\leq_{\mathcal{P}}, <_{\mathcal{P}}, \geq_{\mathcal{P}}, >_{\mathcal{P}}$ will be referred to as the relations of non-strict, strict, non-strict inverse and strict inverse time order on $\mathcal{P}$.

In the case, where the primitive changeable set $\mathcal{P}$ is clear, in the notations $\underset{\mathcal{P}}{\leftarrow}$, $\leq_{\mathcal{P}}$, $<_{\mathcal{P}}$, $\geq_{\mathcal{P}}$, $>_{\mathcal{P}}$, $\psi_{\mathcal{P}}$ the symbol $\mathcal{P}$ will be omitted, and the notations $\leftarrow$, $\leq$, $<$, $\geq$, $>$, $\psi$ will be used instead.

*Remark* I.4.6. From definitions of oriented and primitive changeable sets taking into account the denotations, accepted above, we conclude, that

$$\mathfrak{Bs}(\mathcal{P}) \neq \emptyset$$

for any primitive changeable set $\mathcal{P}$.

Main results of this Section were anonced in [1] and published in [2, Section 5].

## 5 Systems of Abstract Trajectories and Primitive Changeable Sets, Generated by them

**Definition I.5.1.** *Let $M$ be an arbitrary set and $\mathbb{T} = (\mathbf{T}, \leq)$ be any linearly ordered set.*

1. *Any mapping $r : \mathfrak{D}(r) \mapsto M$, where $\mathfrak{D}(r) \subseteq \mathbf{T}$, $\mathfrak{D}(r) \neq \emptyset$ will be referred to as an* ***abstract trajectory*** *from $\mathbb{T}$ to $M$ (here $\mathfrak{D}(r)$ is the domain of the abstract trajectory $r$).*
2. *Any set $\mathcal{R}$, which consists of abstract trajectories from $\mathbb{T}$ to $M$ and satisfies:*

$$\bigcup_{r \in \mathcal{R}} \mathfrak{R}(r) = M$$

*will be named by a* ***system of abstract trajectories*** *from $\mathbb{T}$ to $M$ (here $\mathfrak{R}(r)$ is the range of the abstract trajectory $r$).*

**Theorem I.5.1.** *Let $\mathcal{R}$ be a system of abstract trajectories from $\mathbb{T} = (\mathbf{T}, \leq)$ to $M$. Then there exists a unique primitive changeable set $\mathcal{P}$, which satisfies the following conditions:*

**1)** $\mathfrak{Bs}(\mathcal{P}) = M$; $\quad \mathbb{T}\mathbf{m}(\mathcal{P}) = \mathbb{T}$ *(that is $\mathbf{Tm}(\mathcal{P}) = \mathbf{T}$, $\leq_{\mathcal{P}} = \leq$).*

**2)** *For any $x, y \in \mathfrak{Bs}(\mathcal{P})$ the condition $y \leftarrow x$ is satisfied if and only if there exist an abstract trajectory $r = r_{x,y} \in \mathcal{R}$ and elements $t, \tau \in \mathfrak{D}(r) \subseteq \mathbf{T}$ such, that $x = r(t)$, $y = r(\tau)$ and $t \leq \tau$.*

**3)** *For arbitrary $x \in \mathfrak{Bs}(\mathcal{P})$ and $t \in \mathbf{Tm}(\mathcal{P})$ the condition $x \in \psi(t)$ is satisfied if and only if there exist an abstract trajectory $r = r_x \in \mathcal{R}$ such, that $t \in \mathfrak{D}(r)$ and $x = r(t)$.*

*Proof.* Let $\mathcal{R}$ be any system of abstract trajectories from $\mathbb{T} = (\mathbf{T}, \leq)$ to $M$. Define the following relation:

$$\underset{\mathcal{R}}{\twoheadleftarrow} = \{(y,x) \in M \times M \mid \exists r \in \mathcal{R}\, \exists t,\tau \in \mathfrak{D}(r):\ x = r(t),\ y = r(\tau),\ t \leq \tau\}$$

on the set $M$ (where the symbol $\times$ denotes the Cartesian product of sets). Or, in other words, for $x, y \in M$ the correlation $y \underset{\mathcal{R}}{\twoheadleftarrow} x$ is true if and only if there exist an abstract trajectory $r = r_{x,y} \in \mathcal{R}$ and elements $t, \tau \in \mathfrak{D}(r)$ such, that $x = r(t)$, $y = r(\tau)$ and $t \leq \tau$. Also we define the following mapping $\varphi_{\mathcal{R}} : \mathbf{T} \mapsto 2^M$:

$$\varphi_{\mathcal{R}}(t) = \bigcup_{r \in \mathcal{R},\, t \in \mathfrak{D}(r)} \{r(t)\} = \{r(t) \mid r \in \mathcal{R},\ t \in \mathfrak{D}(r)\}.$$

In particular, $\varphi_{\mathcal{R}}(t) = \emptyset$ in the case, where there not exist a trajectory $r \in \mathcal{R}$ such, that $t \in \mathfrak{D}(r)$.

It is not hard to verify, that the pair $\mathcal{M} = \left(M, \underset{\mathcal{R}}{\twoheadleftarrow}\right)$ is an oriented set and the mapping $\varphi_{\mathcal{R}}$ is a time on $\mathcal{M}$. Therefore, the triple:

$$\mathcal{P} = (\mathcal{M}, \mathbb{T}, \varphi_{\mathcal{R}}) = \left(\left(M, \underset{\mathcal{R}}{\twoheadleftarrow}\right), (\mathbf{T}, \leq), \varphi_{\mathcal{R}}\right)$$

is a primitive changeable set. And it is not hard to see, that this primitive changeable set satisfies the conditions 1),2),3) of this Theorem.

Inversely, if a primitive changeable set $\mathcal{P}_1$ satisfies the conditions 1),2),3) of this Theorem, then from the first condition it follows, that $\mathfrak{Bs}(\mathcal{P}_1) = M$, $\mathbf{Tm}(\mathcal{P}_1) = \mathbf{T}$, $\leq_{\mathcal{P}_1} = \leq$. And the second and third conditions imply the equalities $\underset{\mathcal{P}_1}{\leftarrow} = \underset{\mathcal{R}}{\twoheadleftarrow}$, $\psi_{\mathcal{P}_1} = \varphi_{\mathcal{R}}$. Thus,

$$\mathcal{P}_1 = \left(\left(\mathfrak{Bs}(\mathcal{P}_1), \underset{\mathcal{P}_1}{\leftarrow}\right), \left(\mathbf{Tm}(\mathcal{P}_1), \leq_{\mathcal{P}_1}\right), \psi_{\mathcal{P}_1}\right) = \left(\left(M, \underset{\mathcal{R}}{\twoheadleftarrow}\right), (\mathbf{T}, \leq), \varphi_{\mathcal{R}}\right) = \mathcal{P}.$$

□

**Definition I.5.2.** *Let $\mathcal{R}$ be any system of abstract trajectories from $\mathbb{T} = (\mathbf{T}, \leq)$ to $M$. The primitive changeable set $\mathcal{P}$, which satisfies the conditions 1),2),3) of Theorem I.5.1, will be named by primitive changeable set,* ***generated by the system of abstract trajectories*** *$\mathcal{R}$, and it will be denoted by $\mathcal{A}tp(\mathbb{T}, \mathcal{R})$:*

$$\mathcal{A}tp(\mathbb{T}, \mathcal{R}) := \mathcal{P}.$$

Thus, systems of abstract trajectories provide the simple tool for creation of primitive changeable sets.

Main results of this Section were anonced in [1] and published in [2, Section 6].

# 6 Elementary-time States and Base Changeable Sets

## 6.1 Elementary-time States of Primitive Changeable Sets and their Properties

**Definition I.6.1.** *Let $\mathcal{P}$ be a primitive changeable set. Any pair of kind $(t, x)$, where $t \in \mathbf{Tm}(\mathcal{P})$ and $x \in \psi(t)$, will be named by* ***elementary-time state*** *of $\mathcal{P}$.*

*The set of all elementary-time states of $\mathcal{P}$ will be denoted by $\mathbb{Bs}(\mathcal{P})$:*

$$\mathbb{Bs}(\mathcal{P}) := \{\omega \mid \omega = (t, x),\ where\ t \in \mathbf{Tm}(\mathcal{P}),\ x \in \psi(t)\}.$$

*Remark* I.6.1. From definitions I.1.1 and I.2.1 it follows that $\mathbb{B}\mathfrak{s}(\mathcal{P}) \neq \emptyset$ for any primitive changeable set $\mathcal{P}$.

By Definition I.2.1 $\psi(t) \subseteq \mathfrak{B}\mathfrak{s}(\mathcal{P})$ for all $t \in \mathbf{Tm}(\mathcal{P})$. That is why, we have:

$$\mathbb{B}\mathfrak{s}(\mathcal{P}) \subseteq \mathbf{Tm}(\mathcal{P}) \times \mathfrak{B}\mathfrak{s}(\mathcal{P})$$

for any primitive changeable set $\mathcal{P}$.

Let $\mathbb{T} = (\mathbf{T} \leq)$ be any linearly ordered set and $\mathcal{X}$ be any set. For any ordered pair $\omega = (\tau, \xi) \in \mathbf{T} \times \mathcal{X}$ we introduce the folliwing denotations:

$$\mathsf{bs}\,(\omega) := \xi, \quad \mathsf{tm}\,(\omega) := \tau. \tag{I.10}$$

Hence, for any $\omega \in \mathbf{T} \times \mathcal{X}$ we obtain, $\omega = (\mathsf{tm}\,(\omega)\,,\ \mathsf{bs}\,(\omega))$.

In particular, for any elementary-time state $\omega = (t, x) \in \mathbb{B}\mathfrak{s}(\mathcal{P})$ we have:

$$\mathsf{bs}\,(\omega) = x, \quad \mathsf{tm}\,(\omega) = t.$$

**Definition I.6.2.** *We say, that an elementary-time state $\omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{P})$ is **formally sequential** to an elementary-time state $\omega_1 \in \mathbb{B}\mathfrak{s}(\mathcal{P})$ if and only if $\omega_1 = \omega_2$ or $\mathsf{bs}\,(\omega_2) \underset{\mathcal{P}}{\leftarrow} \mathsf{bs}\,(\omega_1)$ and $\mathsf{tm}\,(\omega_1) <_{\mathcal{P}} \mathsf{tm}\,(\omega_2)$. For this case we use the denotation:*

$$\omega_2 \underset{\mathcal{P}}{\leftarrow}(\mathrm{f})\, \omega_1.$$

In the case, where the primitive changeable set $\mathcal{P}$, in question is known, in the denotation $\omega_2 \underset{\mathcal{P}}{\leftarrow}(\mathrm{f})\, \omega_1$ the symbol $\mathcal{P}$ will be omitted. In this case we use the abbreviated denotation $\omega_2 \leftarrow(\mathrm{f})\, \omega_1$.

**Assertion I.6.1.** *1) If $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{P})$ and $\omega_2 \leftarrow(\mathrm{f})\, \omega_1$, then $\mathsf{tm}\,(\omega_1) \leq \mathsf{tm}\,(\omega_2)$. If, in addition, $\omega_1 \neq \omega_2$, then $\mathsf{tm}\,(\omega_1) < \mathsf{tm}\,(\omega_2)$.*

*2) The relation $\leftarrow(\mathrm{f}) = \underset{\mathcal{P}}{\leftarrow}(\mathrm{f})$ is asymmetric on the set $\mathbb{B}\mathfrak{s}(\mathcal{P})$, that is if $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{P})$, $\omega_2 \leftarrow(\mathrm{f})\, \omega_1$ and $\omega_1 \leftarrow(\mathrm{f})\, \omega_2$, then $\omega_1 = \omega_2$.*

*Proof.* The first statement follows by a trivial way from Definition I.6.2, and the second statement derives from the first. □

**Definition I.6.3.** *The oriented set $\mathcal{M}$ is named **anti-cyclical** if for any $x, y \in \mathfrak{B}\mathfrak{s}(\mathcal{M})$ the conditions $x \leftarrow y$ and $y \leftarrow x$ cause the equality $x = y$.*

**Assertion I.6.2.** *Let $\mathcal{P}$ be a primitive changeable set. Then:*

*1) The pair $\mathcal{Q} = \left(\mathbb{B}\mathfrak{s}(\mathcal{P}), \underset{\mathcal{P}}{\leftarrow}(\mathrm{f})\right) = (\mathbb{B}\mathfrak{s}(\mathcal{P}), \leftarrow(\mathrm{f}))$ is an anti-cyclical oriented set.*

*2) The mapping:*

$$\tilde{\psi}(t) = \tilde{\psi}_{\mathcal{P}}(t) := \{\omega \in \mathbb{B}\mathfrak{s}(\mathcal{P}) \mid \mathsf{tm}\,(\omega) = t\} \in 2^{\mathbb{B}\mathfrak{s}(\mathcal{P})}, \quad t \in \mathbf{Tm}(\mathcal{P}) \tag{I.11}$$

*is a monotone time on $\mathcal{Q}$.*

*3) For $t_1 \neq t_2$ we have $\tilde{\psi}\,(t_1) \cap \tilde{\psi}\,(t_2) = \emptyset$.*

*4) If, in addition, $\psi(t) \neq \emptyset$, $t \in \mathbf{Tm}(\mathcal{P})$, then the time $\psi$ is strictly monotone.*

*Proof.* 1) The first statement of Assertion I.6.2 follows from Definition I.6.2 and second statement of Assertion I.6.1.

2) 2.1) Let $\omega \in \mathbb{B}\mathfrak{s}(\mathcal{P})$. Then, by (I.11), $\omega \in \tilde{\psi}(t)$, where $t = \mathsf{tm}\,(\omega)$.

2.2) Let $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{P})$, $\omega_2 \leftarrow(\mathrm{f})\, \omega_1$ and $\omega_1 \neq \omega_2$. According to (I.11), for $t_1 = \mathsf{tm}\,(\omega_1)$, $t_2 = \mathsf{tm}\,(\omega_2)$ we obtain:

$$\omega_1 \in \tilde{\psi}\,(t_1)\,, \quad \omega_2 \in \tilde{\psi}\,(t_2)\,.$$

Since $\omega_2 \leftarrow (\mathrm{f})\, \omega_1$ and $\omega_2 \neq \omega_1$, then, by Assertion I.6.1 (statement 1), $t_1 < t_2$.

From 2.1),2.2) it follows, that $\tilde{\psi}$ is a time on $\mathcal{Q}$.

2.3) Let $\omega_1 \in \tilde{\psi}(t_1)$, $\omega_2 \in \tilde{\psi}(t_2)$, $\omega_2 \leftarrow (\mathrm{f})\, \omega_1$ and $\omega_1 \not\leftarrow (\mathrm{f})\, \omega_2$. Then, by definition of time $\tilde{\psi}$ (I.11), $\mathsf{tm}(\omega_1) = t_1$, $\mathsf{tm}(\omega_2) = t_2$. Therefore, by Assertion I.6.1, statement 1, $t_1 < t_2$. Thus, the time $\tilde{\psi}$ is monotone.

3) Let $t_1, t_2 \in \mathbf{Tm}(\mathcal{P})$. Suppose, that $\tilde{\psi}(t_1) \cap \tilde{\psi}(t_2) \neq \emptyset$. Then there exists an elementary-time state $\omega \in \tilde{\psi}(t_1) \cap \tilde{\psi}(t_2)$. Hence, by (I.11), we obtain $t_1 = \mathsf{tm}(\omega) = t_2$.

4) Assume, that, in addition, $\psi(t) \neq \emptyset, t \in \mathbf{Tm}(\mathcal{P})$. Then for an arbitrary $t \in \mathbf{Tm}(\mathcal{P})$ there exists an elementary state $x_t \in \mathfrak{Bs}(\mathcal{P})$ such, that $x_t \in \psi(t)$. Consequently, the elementary-time state $\omega_t = (t, x_t) \in \mathbb{B}\mathfrak{s}(\mathcal{P})$ satisfies the condition $\mathsf{tm}(\omega_t) = t$, that is $\omega_t \in \tilde{\psi}(t)$. Thus, $\tilde{\psi}(t) \neq \emptyset$, $t \in \mathbf{Tm}(\mathcal{P})$. Hence, taking into account the statement 3) of this Assertion, we obtain, $\tilde{\psi}(t_1) \neq \tilde{\psi}(t_2)$ for $t_1, t_2 \in \mathbf{Tm}(\mathcal{P})$, $t_1 \neq t_2$. Consequently, the time $\tilde{\psi}$ is incessant, and, taking into account the statement 2) of this Assertion, we conclude, that the time $\tilde{\psi}$ is strictly monotone. □

### 6.2 Base of Elementary Processes and Base Changeable Sets

As it had been proved in Assertion I.6.2, for any primitive changeable set $\mathcal{P}$ the pair $(\mathbb{B}\mathfrak{s}(\mathcal{P}), \leftarrow (\mathrm{f}))$ is an oriented set, in which $\leftarrow (\mathrm{f})$ is the directing relation of changes. But, it turns out, that sometimes the relation $\leftarrow (\mathrm{f})$ is not quite fit for description of evolution of elementary-time states in real systems. And in the reality, this relation may generate such "transformations" of elementary-time states, which never took place in the real physical system. To illustrate this fact, we consider the following example.

***Example*** I.6.1. Let us consider the system of abstract trajectories, which describes the uniform linear motion of the system of identical material points, evenly distributed on the straight trajectory of their own motion. The identity of the material points assumes, that all characteristics of these points in a some time moment can be reduced only to their coordinates. This means, that a material point, which has a certain coordinates at a some time moment is completely mathematically identical to the one point that have the same coordinates in another time. This system of material points can be described by the following system of abstract trajectories from $\mathbb{R}$ to $\mathbb{R}$:

$$\mathcal{R} = \{r_\alpha \mid \alpha \in \mathbb{R}\}, \quad \text{where} \tag{I.12}$$
$$r_\alpha(t) := t + \alpha, \quad t \in \mathbb{R}, \ \alpha \in \mathbb{R} \qquad (\mathfrak{D}(r_\alpha) = \mathbb{R}, \ \alpha \in \mathbb{R}).$$

Denote:

$$\mathcal{P} := \mathcal{A}tp((\mathbb{R}, \leq), \mathcal{R}),$$

where "$\leq$" is the standard linear order relation on the real numbers. By Definition I.5.2 and condition 1) of Theorem I.5.1, $\mathfrak{Bs}(\mathcal{P}) = \mathbf{Tm}(\mathcal{P}) = \mathbb{R}$. We aim to prove, that for the elements $x_1, x_2 \in \mathfrak{Bs}(\mathcal{P}) = \mathbb{R}$ the condition $x_2 \leftarrow x_1$ is equivalent to the inequality $x_1 \leq x_2$. Indeed, in the case $x_1 \leq x_2$ for $t_1 = x_1$, $t_2 = x_2$ we obtain $x_1 = r_0(t_1)$, $x_2 = r_0(t_2)$, where $t_1 \leq t_2$. Therefore, by the condition 2) of Theorem I.5.1, we obtain $x_2 \leftarrow x_1$. Inversely, if $x_2 \leftarrow x_1$, then, by condition 2) of Theorem I.5.1, there exist numbers $\alpha, t_1, t_2 \in \mathbb{R}$ such, that $t_1 \leq t_2$, $x_1 = r_\alpha(t_1)$, $x_2 = r_\alpha(t_2)$, that is $x_1 = t_1 + \alpha$, $x_2 = t_2 + \alpha$, where $t_1 \leq t_2$. Hence, $x_1 \leq x_2$.

The next aim is to prove, that $\mathbb{B}\mathfrak{s}(\mathcal{P}) = \mathbb{R} \times \mathbb{R}$. Since $\mathfrak{Bs}(\mathcal{P}) = \mathbf{Tm}(\mathcal{P}) = \mathbb{R}$, we have $\mathbb{B}\mathfrak{s}(\mathcal{P}) \subseteq \mathbb{R} \times \mathbb{R}$. Thus, it remains to prove, the inverse inclusion. Let $\omega = (\tau, x) \in \mathbb{R} \times \mathbb{R}$. Denote $\alpha_\omega := x - \tau$. Then $r_{\alpha_\omega}(\tau) = \tau + (x - \tau) = x$. Therefore, by condition 3) of Theorem I.5.1, $x \in \psi_{\mathcal{P}}(\tau)$. This means, that $\omega = (\tau, x) \in \mathbb{B}\mathfrak{s}(\mathcal{P})$. The equality $\mathbb{B}\mathfrak{s}(\mathcal{P}) = \mathbb{R} \times \mathbb{R}$ has been proved.

By Definition I.6.2 of formally sequential elementary-time states, for $\omega_1 = (t_1, x_1)$, $\omega_2 = (t_2, x_2) \in \mathbb{B}\mathfrak{s}(\mathcal{P})$ the condition $\omega_2 \leftarrow (\mathrm{f})\, \omega_1$ is performed if and only if $\omega_1 = \omega_2$ or $t_1 < t_2$ and

$x_1 \leq x_2$. Hence, if we choose any elementary-time states $\omega_1 = (t_1, x_1)$, $\omega_2 = (t_2, x_2) \in \mathbb{Bs}(\mathcal{P}) = \mathbb{R} \times \mathbb{R}$, satisfying $t_1 < t_2$ and $x_1 \leq x_2$, we obtain $\omega_2 \leftarrow (\mathrm{f})\, \omega_1$. But in the case $x_1 - t_1 \neq x_2 - t_2$ there not exist an abstract trajectory $r_\alpha \in \mathcal{R}$ such, that $\omega_1, \omega_2 \in r_\alpha$. This means, that in this model of real physical process, the elementary-time state $\omega_2 = (t_2, x_2)$ may not be the result of transformations of the elementary-time state $\omega_1 = (t_1, x_1)$ under condition $x_1 - t_1 \neq x_2 - t_2$. Thus, in this example, the relation $\leftarrow (\mathrm{f})$ generates infinitely many "parasitic transformation relations", which never took place in the reality.

There is a way of overcoming the above uncomfortable situation by introducing a formal "signs of non-identity" for material points that move along the specified trajectories. For example, instead of (I.12) we may consider the system of trajectories from $\mathbb{R}$ to $\mathbb{R}^2$ of kind:

$$\mathcal{R} = \{r_\alpha \mid \alpha \in \mathbb{R}\}, \quad \text{where}$$
$$r_\alpha(t) := (t + \alpha,\, \alpha) \in \mathbb{R}^2, \quad t \in \mathbb{R} \ \ (\alpha \in \mathbb{R}).$$

Note, that the value $\alpha$ in the second coordinate of $r_\alpha(t)$ should not be understood as a space coordinate, but only as a "number" of the trajectory $r_\alpha$. However, in the abstract situation, this approach is not convenient because it could complicate description of different "branched processes" when elementary states during the evolution can be "divided" into a few, or, conversely, several elementary states may be "merged" into one.

Another (more flexible) way of overcoming the above situation is to define the directing relation of changes not only on the set of elementary states $\mathfrak{Bs}(\mathcal{P})$, but, also, on the set of elementary-time states $\mathbb{Bs}(\mathcal{P})$ of a primitive changeable set $\mathcal{P}$. Indeed, let us consider the primitive changeable set $\mathcal{P} := \mathcal{A}tp(\mathcal{R})$ from Example I.6.1. For $\omega_1, \omega_2 \in \mathbb{Bs}(\mathcal{P})$ we can put $\omega_2 \twoheadleftarrow \omega_1$ if and only if $\mathsf{tm}(\omega_1) \leq \mathsf{tm}(\omega_2)$ and there exist an abstract trajectory $r_\alpha \in \mathcal{R}$ such, that $\omega_1, \omega_2 \in r_\alpha$ (that is such, that $\mathsf{bs}(\omega_1) = r_\alpha(\mathsf{tm}(\omega_1))$, $\mathsf{bs}(\omega_2) = r_\alpha(\mathsf{tm}(\omega_2))$). Thus, we obtain the relation "$\twoheadleftarrow$", which reflects only such transformations of the elementary-time states, which actually took place in the reality.

**Definition I.6.4.** *Let $\mathcal{P}$ be a primitive changeable set.*

*1. Relation $\twoheadleftarrow$ on $\mathbb{Bs}(\mathcal{P})$ is named by* ***base of elementary processes*** *if and only if:*

**(1)** $\forall\, \omega \in \mathbb{Bs}(\mathcal{P})\ \omega \twoheadleftarrow \omega$.

**(2)** *If $\omega_1, \omega_2 \in \mathbb{Bs}(\mathcal{P})$ and $\omega_2 \twoheadleftarrow \omega_1$, then $\omega_2 \leftarrow (\mathrm{f})\, \omega_1$ (ie $\twoheadleftarrow\ \subseteq\ \leftarrow (\mathrm{f})$).*

**(3)** *For arbitrary $x_1, x_2 \in \mathfrak{Bs}(\mathcal{P})$ such, that $x_2 \leftarrow x_1$ there exist $\omega_1, \omega_2 \in \mathbb{Bs}(\mathcal{P})$ such, that $\mathsf{bs}(\omega_1) = x_1$, $\mathsf{bs}(\omega_2) = x_2$ and $\omega_2 \twoheadleftarrow \omega_1$.*

*2. In the case, where $\twoheadleftarrow$ is the base of elementary processes on the primitive changeable set $\mathcal{P}$, the pair:*

$$\mathcal{B} = (\mathcal{P}, \twoheadleftarrow)$$

*will be referred to as* ***base changeable set***[3].

### 6.3 Remarks on Denotations

For further, base changeable sets will be denoted by large calligraphy symbols.

Let $\mathcal{B} = (\mathcal{P}, \twoheadleftarrow)$ be a base changeable set. We introduce the following denotations:

$$\mathfrak{Bs}(\mathcal{B}) := \mathfrak{Bs}(\mathcal{P}); \quad \mathbb{Bs}(\mathcal{B}) := \mathbb{Bs}(\mathcal{P}); \qquad \underset{\mathcal{B}}{\leftarrow} := \underset{\mathcal{P}}{\leftarrow};$$
$$\underset{\mathcal{B}}{\leftarrow}(\mathrm{f}) := \underset{\mathcal{P}}{\leftarrow}(\mathrm{f}); \quad \overset{\mathbb{Bs}}{\underset{\mathcal{B}}{\leftarrow}} := \twoheadleftarrow \qquad \mathbf{Tm}(\mathcal{B}) := \mathbf{Tm}(\mathcal{P});$$
$$\leq_{\mathcal{B}} := \leq_{\mathcal{P}}; \quad \mathbb{Tm}(\mathcal{B}) := \mathbb{Tm}(\mathcal{P}) = (\mathbf{Tm}(\mathcal{B}), \leq_{\mathcal{B}}); \quad <_{\mathcal{B}} := <_{\mathcal{P}};$$
$$\geq_{\mathcal{B}} := \geq_{\mathcal{P}}; \quad >_{\mathcal{B}} := >_{\mathcal{P}}; \quad \psi_{\mathcal{B}} := \psi_{\mathcal{P}}.$$

[3] Note that in some early works (for example in [3]) the term "basic changeable set" is used instead of the term "base changeable set". This situation appeared due due to existence of two variants of translation of this term from Ukrainian language.

In the case, where the base changeable set $\mathcal{B}$, is clear in the denotations $\underset{\mathcal{B}}{\leftarrow}$, $\underset{\mathcal{B}}{\leftarrow}(\mathrm{f})$, $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}}$ $\leq_{\mathcal{B}}$, $<_{\mathcal{B}}$, $\geq_{\mathcal{B}}$, $>_{\mathcal{B}}$, $\psi_{\mathcal{B}}$ the symbol $\mathcal{B}$ will be omitted, and the denotations $\leftarrow$, $\leftarrow(\mathrm{f})$, $\overset{\mathbb{B}\mathfrak{s}}{\leftarrow}$, $\leq$, $<$, $\geq$, $>$, $\psi$ will be used instead.

Also for elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ we may use the denotations $\omega_2 \underset{\mathcal{B}}{\leftarrow} \omega_1$ or $\omega_2 \leftarrow \omega_1$ instead of the denotations $\omega_2 \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}} \omega_1$ or $\omega_2 \overset{\mathbb{B}\mathfrak{s}}{\leftarrow} \omega_1$ (in cases, where this does not lead to misunderstanding).

The next properties of base changeable sets follow from definitions I.6.4, I.6.2 and remarks I.4.6, I.6.1.

**Properties I.6.1.** *Let $\mathcal{B}$ be any base changeable set. Then:*

1. *The pair $\mathcal{B}_0 = (\mathfrak{B}\mathfrak{s}(\mathcal{B}), \leftarrow) = \left(\mathfrak{B}\mathfrak{s}(\mathcal{B}), \underset{\mathcal{B}}{\leftarrow}\right)$ is an oriented set (that is $\leftarrow = \underset{\mathcal{B}}{\leftarrow}$ is a reflexive binary relation on $\mathfrak{B}\mathfrak{s}(\mathcal{B})$, so, for any elementary state $x \in \mathfrak{B}\mathfrak{s}(\mathcal{B})$ the correlation $x \underset{\mathcal{B}}{\leftarrow} x$ is performed).*
2. *$\mathfrak{B}\mathfrak{s}(\mathcal{B}) \neq \emptyset$ and $\mathbb{B}\mathfrak{s}(\mathcal{B}) \neq \emptyset$.*
3. *$\leq_{\mathcal{B}}$ is relation of (not-strict) linear order defined on $\mathbf{Tm}(\mathcal{B})$ (i.e. $\mathbb{T}\mathbf{m}(\mathcal{B}) = (\mathbf{Tm}(\mathcal{B}), \leq_{\mathcal{B}})$ is linearly ordered set).*
4. *$\mathbb{B}\mathfrak{s}(\mathcal{B}) \subseteq \mathbf{Tm}(\mathcal{B}) \times \mathfrak{B}\mathfrak{s}(\mathcal{B})$.*
5. *The mapping $\psi = \psi_{\mathcal{B}}$ is a time on $\mathcal{B}_0 = (\mathfrak{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$.*
6. *$\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}}$ is reflexive binary relation, defined on $\mathbb{B}\mathfrak{s}(\mathcal{B})$. Hence, $\omega \leftarrow \omega$ for any elementary-time state $\omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})$.*
7. *If $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ and $\omega_2 \leftarrow \omega_1$, then $\omega_2 \leftarrow(\mathrm{f})\, \omega_1$, and therefore, $\mathsf{bs}(\omega_2) \leftarrow \mathsf{bs}(\omega_1)$ and $\mathsf{tm}(\omega_1) \leq \mathsf{tm}(\omega_2)$. If, in addition, $\omega_1 \neq \omega_2$, then $\mathsf{tm}(\omega_1) < \mathsf{tm}(\omega_2)$.*
8. *For arbitrary $x_1, x_2 \in \mathfrak{B}\mathfrak{s}(\mathcal{B})$ the condition $x_2 \leftarrow x_1$ holds if and only if there exist elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ such, that $\mathsf{bs}(\omega_1) = x_1$, $\mathsf{bs}(\omega_2) = x_2$ and $\omega_2 \leftarrow \omega_1$.*
9. *$\mathfrak{B}\mathfrak{s}(\mathcal{B}) = \{\mathsf{bs}(\omega) \mid \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})\}$.*

## 6.4 Examples of Base Changeable Sets

***Example*** I.6.2. Let $\mathcal{P}$ be any primitive changeable set. Then the relation $\leftarrow(\mathrm{f}) = \underset{\mathcal{P}}{\leftarrow}(\mathrm{f})$ is base of elementary processes on $\mathcal{P}$. Indeed, the conditions (1) and (2) of Definition I.6.4 for the relation $\leftarrow(\mathrm{f})$ are fulfilled by a trivial way. To verify the condition (3) we consider arbitrary $x_1, x_2 \in \mathfrak{B}\mathfrak{s}(\mathcal{P})$ such, that $x_2 \leftarrow x_1$. In the case $x_1 = x_2$ by Time Definition I.2.1, there exist a time point $t_1 \in \mathbf{Tm}(\mathcal{P})$ such, that $x_1 \in \psi(t_1)$. Hence, for $\omega_1 = \omega_2 = (t_1, x_1) \in \mathbb{B}\mathfrak{s}(\mathcal{P})$ we obtain $\mathsf{bs}(\omega_1) = \mathsf{bs}(\omega_2) = x_1 = x_2$ and $\omega_2 \leftarrow(\mathrm{f})\, \omega_1$. Thus, in the case $x_1 = x_2$ the condition (3) of Definition I.6.4 is satisfied. In the case $x_1 \neq x_2$, by Definition I.2.1, there exist time points $t_1, t_2 \in \mathbf{Tm}(\mathcal{P})$ such, that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 < t_2$. Hence, for $\omega_1 = (t_1, x_1)$, $\omega_2 = (t_1, x_2) \in \mathbb{B}\mathfrak{s}(\mathcal{P})$, we obtain $\mathsf{bs}(\omega_1) = x_1$, $\mathsf{bs}(\omega_2) = x_2$ and $\omega_2 \leftarrow(\mathrm{f})\, \omega_1$. Thus, in the case $x_1 \neq x_2$ the condition (3) of Definition I.6.4 also is satisfied.

Therefore any primitive changeable set can be interpreted as base changeable set $\mathcal{P}_{(f)} = (\mathcal{P}, \leftarrow(\mathrm{f}))$ in which the relation $\leftarrow(\mathrm{f})$ is the base of elementary processes.

***Example*** I.6.3. Let $\mathcal{R}$ be any system of abstract trajectories from $\mathbb{T} = (\mathbf{T}, \leq)$ to $M$. Denote:

$$\mathcal{P} := \mathcal{A}tp(\mathbb{T}, \mathcal{R}).$$

By Theorem I.5.1, $\mathfrak{Bs}(\mathcal{P}) = M$, $\mathbf{Tm}(\mathcal{P}) = \mathbf{T}$. Moreover, by third statement of this Theorem for $(t,x) \in \mathbf{T} \times M$ the condition $(t,x) \in \mathbb{B}\mathfrak{s}(\mathcal{P})$ holds if and only if there exist an abstract trajectory $r = r_{t,x} \in \mathcal{R}$ such, that $t \in \mathfrak{D}(r)$ and $x = r(t)$, ie such, that $\omega = (t,x) \in r$. Thus,

$$\mathbb{B}\mathfrak{s}(\mathcal{P}) = \bigcup_{r \in \mathcal{R}} r. \tag{I.13}$$

Then, for $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{P})$ we put $\omega_2 \twoheadleftarrow [\mathcal{R}]\, \omega_1$ if and only if $\mathsf{tm}(\omega_1) \leq \mathsf{tm}(\omega_2)$ and there exists an abstract trajectory $r \in \mathcal{R}$ such, that $\omega_1, \omega_2 \in \mathcal{R}$ (ie such, that $\mathsf{bs}(\omega_1) = r(\mathsf{tm}(\omega_1))$, $\mathsf{bs}(\omega_2) = r(\mathsf{tm}(\omega_2))$). We are going to prove, that the relation $\twoheadleftarrow [\mathcal{R}]$ is a base of elementary processes on $\mathcal{P}$.

(a) Let $\omega \in \mathbb{B}\mathfrak{s}(\mathcal{P})$. Then, by (I.13), there exist an abstract trajectory $r \in \mathcal{R}$ such, that $\omega \in r$. Hence, by definition of the relation "$\twoheadleftarrow [\mathcal{R}]$", we have $\omega \twoheadleftarrow [\mathcal{R}] \omega$.

(b) Let $\omega_1 = (t_1, x_1)$, $\omega_2 = (t_2, x_2) \in \mathbb{B}\mathfrak{s}(\mathcal{P})$ and $\omega_2 \twoheadleftarrow [\mathcal{R}] \omega_1$. Then, from definition of the relation "$\twoheadleftarrow [\mathcal{R}]$", it follows, that $t_1 \leq t_2$ and there exists an an abstract trajectory $r \in \mathcal{R}$ such, that $\omega_1, \omega_2 \in \mathcal{R}$ (ie such, that $x_1 = r(t_1)$, $x_2 = r(t_2)$). Consequently, by statement 2) of Theorem I.5.1, $x_2 \underset{\mathcal{A}tp(\mathcal{R})}{\leftarrow} x_1$. Therefore, in the case $t_1 \neq t_2$ we have $t_1 < t_2$ and $x_2 \leftarrow x_1$, besides in the case $t_1 = t_2$ we obtain $x_1 = r(t_1) = r(t_2) = x_2$, that is $\omega_1 = \omega_2$. But, in the both cases the correlation $\omega_2 \leftarrow (\mathrm{f})\, \omega_1$ is true.

(c) Let $x_1, x_2 \in \mathfrak{Bs}(\mathcal{P})$, $x_2 \leftarrow x_1$ (ie $x_2 \underset{\mathcal{A}tp(\mathcal{R})}{\leftarrow} x_1$). Then, by statement 2) of Theorem I.5.1, there exists an abstract trajectory $r \in \mathcal{R}$ such, that $x_1 = r(t_1)$, $x_2 = r(t_2)$ for some $t_1, t_2 \in \mathbf{Tm}(\mathcal{P})$ such, that $t_1 \leq t_2$. Denote:

$$\omega_i := (t_i, x_i), \qquad i \in \{1, 2\}.$$

Then, $\omega_1, \omega_2 \in r \subseteq \bigcup_{\rho \in \mathcal{R}} \rho = \mathbb{B}\mathfrak{s}(\mathcal{P})$, $\mathsf{bs}(\omega_i) = x_i$ $(i \in \{1,2\})$ and, by definition of the relation "$\twoheadleftarrow [\mathcal{R}]$", $\omega_2 \twoheadleftarrow [\mathcal{R}] \omega_1$.

From the items (a)-(c) it follows, that the relation $\twoheadleftarrow [\mathcal{R}]$ is base of elementary processes on $\mathcal{P} = \mathcal{A}tp(\mathbb{T}, \mathcal{R})$. Thus, the pair:

$$\mathcal{A}t(\mathbb{T}, \mathcal{R}) = (\mathcal{P}, \twoheadleftarrow [\mathcal{R}]\,) = (\mathcal{A}tp(\mathbb{T}, \mathcal{R}), \twoheadleftarrow [\mathcal{R}]\,)$$

is a base changeable set.

From Properties I.6.1(8,9) [4] it follows, that if for a some base changeable set $\mathcal{B}$ we know $\mathbf{Tm}(\mathcal{B})$, $\leq_{\mathcal{B}}$, $\mathbb{B}\mathfrak{s}(\mathcal{B})$ and base of elementary processes $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}}$, then we can we can recover the set $\mathfrak{Bs}(\mathcal{B})$, the directing relation of changes $\underset{\mathcal{B}}{\leftarrow}$ and the time $\psi_{\mathcal{B}}(t)$ (using the formula $\psi_{\mathcal{B}}(t) = \{x \in \mathfrak{Bs}(\mathcal{B}) \mid (t,x) \in \mathbb{B}\mathfrak{s}(\mathcal{B})\}$, $t \in \mathbf{Tm}(\mathcal{B})$), and thus, we can recover the whole base changable set $\mathcal{B}$. Hence from the last example it follows the next theorem.

**Theorem I.6.1.** *Let $\mathcal{R}$ be a system of abstract trajectories from $\mathbb{T} = (\mathbf{T}, \leq)$ to $M$. Then there exists a unique base changeable set $\mathcal{B} = \mathcal{A}t(\mathbb{T}, \mathcal{R})$, such, that:*

**1)** $\mathbb{T}\mathbf{m}(\mathcal{A}t(\mathbb{T}, \mathcal{R})) = \mathbb{T}$ *(that is $\mathbf{Tm}(\mathcal{A}t(\mathbb{T}, \mathcal{R})) = \mathbf{T}$, $\leq_{\mathcal{A}t(\mathbb{T},\mathcal{R})} = \leq$);*

**2)** $\mathbb{B}\mathfrak{s}(\mathcal{A}t(\mathbb{T}, \mathcal{R})) = \bigcup_{r \in \mathcal{R}} r$*;*

**3)** *For arbitrary $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{A}t(\mathbb{T}, \mathcal{R}))$ the condition $\omega_2 \underset{\mathcal{A}t(\mathbb{T},\mathcal{R})}{\leftarrow} \omega_1$ is satisfied if and only if $\mathsf{tm}(\omega_1) \leq \mathsf{tm}(\omega_2)$ and there exist an abstract trajectory $r \in \mathcal{R}$ such, that $\omega_1, \omega_2 \in r$.*

[4] Reference to Properties I.6.1(8,9) means reference to the items 8 and 9 from the group of properties "Properties I.6.1".

*Remark* I.6.2. 1. Since the construction of the base changeable set $\mathcal{A}t(\mathbb{T},\mathcal{R})$ is based on the primitive changeable set $\mathcal{A}tp(\mathbb{T},\mathcal{R})$, for any base changeable set of kind $\mathcal{B}=\mathcal{A}t(\mathbb{T},\mathcal{R})$ the statements, formulated in the items 1),2),3) of Theorem I.5.1 remain true (with replacement the character $\mathcal{P}$ by $\mathcal{B}$ or by $\mathcal{A}t(\mathbb{T},\mathcal{R})$).

2. In the case, when the linearly ordered set $\mathbb{T}$ is given in advance, we will use the denotation $\mathcal{A}t(\mathcal{R})$ instead of $\mathcal{A}t(\mathbb{T},\mathcal{R})$.

## 6.5 Another Way to Definition of Base Changeable Sets

The following theorem demonstrates another, more laconic, although more artificial, way for the definition of the base changeable set concept.

**Theorem I.6.2.** ***Let*** $\mathbb{T}=(\mathbf{T},\leq)$ *be any linearly ordered set,* $\mathcal{X}$ *be any set and* $\Leftarrow$ *be a binary relation, defined on some set* $\mathbf{B}\subseteq\mathbf{T}\times\mathcal{X}$*. Suppose, that the relation* $\Leftarrow$ *satisfies the following conditions:*

1. *Relation* $\Leftarrow$ *is reflexive on* $\mathbf{B}$*;*
2. $\forall\,\omega_1,\omega_2\in\mathbf{B}$ *the conditions* $\omega_2\Leftarrow\omega_1$ *and* $\omega_1\neq\omega_2$ *lead to* $\mathsf{tm}\,(\omega_1)<\mathsf{tm}\,(\omega_2)$*.*

***Then*** *there exists a unique base changeable set* $\mathcal{B}$*, which satisfies the following conditions:*

**a)** $\mathbb{T}\mathbf{m}(\mathcal{B})=\mathbb{T}$*;*

**b)** $\mathbb{B}\mathfrak{s}(\mathcal{B})=\mathbf{B}$*;*

**c)** $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}}=\Leftarrow$.

*Proof.* **1.** Denote:

$$\begin{aligned} r_{\omega_1,\omega_2} &:= \{\omega_1,\omega_2\}\,, \quad \omega_1,\omega_2\in\mathbf{B} \\ \mathcal{R} &:= \{r_{\omega_1,\omega_2}\,|\,\omega_1,\omega_2\in\mathbf{B},\ \omega_2\Leftarrow\omega_1\}\,. \end{aligned} \tag{I.14}$$

We are going to prove, that all elements of the set $\mathcal{R}$ are abstract trajectories from $\mathbb{T}$ to $\mathcal{X}$. Consider any fixed $\omega_1,\omega_2\in\mathbf{B}$ such, that $\omega_2\Leftarrow\omega_1$. Since $r_{\omega_1,\omega_2}=\{\omega_1,\omega_2\}\subseteq\mathbf{B}\subseteq\mathbf{T}\times\mathcal{X}$, we conclude, that $r_{\omega_1,\omega_2}$ is a binary relation from $\mathbf{T}$ to $\mathcal{X}$. We shall prove, that this relation is function. Assume the contrary. Then there exist $(t,x_1),(t,x_2)\in r_{\omega_1,\omega_2}$ such, that $x_1\neq x_2$ (and, consequently, $(t,x_1)\neq(t,x_2)$). Thus only two cases $(t,x_1)=\omega_1$, $(t,x_2)=\omega_2$ or $(t,x_1)=\omega_2$, $(t,x_2)=\omega_1$ are possible. But, since $\omega_2\Leftarrow\omega_1$, by the condition 2 of this Theorem, in the both cases we obtain wrong inequality $t<t$. Hence, the relation $r_{\omega_1,\omega_2}$ is function. This means, that $r_{\omega_1,\omega_2}$ is an abstract trajectory from $\mathbb{T}$ to $\mathcal{X}$. Thus, $\mathcal{R}$ is a system of abstract trajectories from $\mathbb{T}$ to $\bigcup_{r\in\mathcal{R}}\mathfrak{R}(r)\subseteq\mathcal{X}$. Denote:

$$\mathcal{B}:=\mathcal{A}t\,(\mathbb{T},\mathcal{R})\,.$$

**a)** By Theorem I.6.1 (item 1), $\mathbb{T}\mathbf{m}(\mathcal{B})=\mathbb{T}$.

**b)** By Theorem I.6.1 (item 2):

$$\mathbb{B}\mathfrak{s}(\mathcal{B})=\bigcup_{r\in\mathcal{R}}r=\bigcup_{\substack{\omega_1,\omega_2\in\mathbf{B}\\ \omega_2\Leftarrow\omega_1}}\{\omega_1,\omega_2\}\subseteq\mathbf{B}. \tag{I.15}$$

From the other hand, taking into account, that the relation $\Leftarrow$ is reflexive, we obtain the inverse inclusion:

$$\mathbb{B}\mathfrak{s}(\mathcal{B})=\bigcup_{\substack{\omega_1,\omega_2\in\mathbf{B}\\ \omega_2\Leftarrow\omega_1}}\{\omega_1,\omega_2\}\supseteq\bigcup_{\omega\in\mathbf{B}}\{\omega\}=\mathbf{B}. \tag{I.16}$$

Thus, $\mathbb{B}\mathfrak{s}(\mathcal{B}) = \mathbf{B}$. Hence, the base changeable set $\mathcal{B}$ satisfies the conditions a),b).

**c)** We aim to prove, that the condition c) for the base changeable set $\mathcal{B}$ also is satisfied. It is necessary to prove, that for any $\omega_1, \omega_2 \in \mathbf{B} = \mathbb{B}\mathfrak{s}(\mathcal{B})$ the condition $\omega_2 \twoheadleftarrow \omega_1$ is equivalent to the condition $\omega_2 \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}} \omega_1$ (that is to the condition $\omega_2 \leftarrow \omega_1$). Since both the relations $\twoheadleftarrow$ and $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}}$ are reflexive on $\mathbb{B}\mathfrak{s}(\mathcal{B}) = \mathbf{B}$, it is sufficient to prove the last assertion only for the case $\omega_1 \neq \omega_2$. Thus, we consider any fixed $\omega_1, \omega_2 \in \mathbf{B} = \mathbb{B}\mathfrak{s}(\mathcal{B})$ such, that $\omega_1 \neq \omega_2$.

**c.1)** Suppose, that $\omega_2 \leftarrow \omega_1$. Then, by Theorem I.6.1 (item 3)

$$\mathsf{tm}\,(\omega_1) \leq \mathsf{tm}\,(\omega_2) \tag{I.17}$$

and there exist a trajectory $r_{w_1,w_2} \in \mathcal{R}$ $(w_2 \twoheadleftarrow w_1)$ such, that $\omega_1, \omega_2 \in r_{w_1,w_2} = \{w_1, w_2\}$. Consequently, since $w_2 \twoheadleftarrow w_1$ and $\omega_1 \neq \omega_2$, one of the following conditions:

$$\omega_2 \twoheadleftarrow \omega_1 \quad \text{or} \quad \omega_1 \twoheadleftarrow \omega_2$$

must be fulfilled. But the case $\omega_1 \twoheadleftarrow \omega_2$ is impossible, because in this case, by the condition 2 of the present theorem we obtain the inequality $\mathsf{tm}\,(\omega_2) < \mathsf{tm}\,(\omega_1)$, which is in a contradiction to the inequality (I.17). Therefore, $\omega_2 \twoheadleftarrow \omega_1$.

**c.2)** Conversely, suppose, that $\omega_2 \twoheadleftarrow \omega_1$. Then, by (I.14), $r_{\omega_1,\omega_2} \in \mathcal{R}$, and, by the condition 2 of this Theorem, $\mathsf{tm}\,(\omega_1) < \mathsf{tm}\,(\omega_2)$. Hence, by Theorem I.6.1 (item 3) $\omega_2 \leftarrow \omega_1$.

The equality $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}} = \twoheadleftarrow$ have been proven. Thus, the base changeable set $\mathcal{B}$ satisfies the conditions a),b),c).

We need to prove, that the base changeable set $\mathcal{B}$, which satisfies the conditions conditions a),b),c) is unique. Assume, that a base changeable set $\mathcal{B}_1$ also satisfies the conditions a),b),c). We shall prove, that this base changeable set $\mathcal{B}_1$ must satisfy the conditions 1),2),3) of Theorem I.6.1 for the system of abstract trajectories $\mathcal{R}$, defined in (I.14).

2.1) By the condition a), $\mathbb{T}\mathbf{m}\,(\mathcal{B}_1) = \mathbb{T}$.

2.2) Using the condition b), and equalities (I.15),(I.16) we obtain:

$$\mathbb{B}\mathfrak{s}\,(\mathcal{B}_1) = \mathbf{B} = \mathbb{B}\mathfrak{s}(\mathcal{B}) = \bigcup_{r \in \mathcal{R}} r.$$

2.3) Since both base changeable sets $\mathcal{B}$ and $\mathcal{B}_1$ satisfy the condition c), we have:

$$\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_1}{\leftarrow}} = \twoheadleftarrow = \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}} = \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{A}t(\mathbb{T},\mathcal{R})}{\longleftarrow}}.$$

This means, that $\mathcal{B}_1$ satisfies the condition 3) of Theorem I.6.1.

Therefore, the base changeable set $\mathcal{B}_1$ satisfies all conditions of Theorem I.6.1 for the system of abstract trajectories $\mathcal{R}$. Thus, by Theorem I.6.1, $\mathcal{B}_1 = \mathcal{A}t(\mathbb{T}, \mathcal{R}) = \mathcal{B}$. □

*Remark* I.6.3. From Properties I.6.1 and Definition I.6.1 it follows that for base changeable set $\mathcal{B}$, which satisfies the conditions a),b),c) of Theorem I.6.2 the following propositions are true:

1. $\mathfrak{B}\mathfrak{s}(\mathcal{B}) = \mathsf{bs}\,(\mathbf{B}) = \{\mathsf{bs}\,(\omega) \mid \omega \in \mathbf{B}\}$;

2. for arbitrary $x_1, x_2 \in \mathfrak{B}\mathfrak{s}(\mathcal{B})$ the correlation $x_2 \leftarrow x_1$ holds if and only if there exist elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ such, that $\mathsf{bs}\,(\omega_1) = x_1$, $\mathsf{bs}\,(\omega_2) = x_2$ and $\omega_2 \twoheadleftarrow \omega_1$.

3. $\psi_{\mathcal{B}}(t) = \{\mathsf{bs}\,(\omega) \mid \omega \in \mathbf{B},\ \mathsf{tm}\,(\omega) = t\}$, $t \in \mathbf{Tm}(\mathcal{B})$. In particular, $\psi_{\mathcal{B}}(t) = \emptyset$ in the case, where there not exist $\omega \in \mathbf{B}$ such, that $\mathsf{tm}\,(\omega) = t$.

*Remark* I.6.4. Let $\mathcal{B}$ be any base changeable set. Denote:

$$\mathbb{T} := \mathbb{T}\mathbf{m}(\mathcal{B}); \quad \mathcal{X} := \mathfrak{B}\mathfrak{s}(\mathcal{B}); \quad \mathbf{B} := \mathbb{B}\mathfrak{s}(\mathcal{B}), \quad \twoheadleftarrow := \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}}.$$

It is obvious, that conditions 1,2 of Theorem I.6.2 for $\mathbb{T}, \mathcal{X}, \mathbf{B}, \twoheadleftarrow$ are satisfied. Moreover, $\mathcal{B}$ is a (unique) base changeable set, which satisfies conditions a),b),c) of the conclusion part of this theorem. Thus, using Theorem I.6.2 we may give the new definition of the base changeable set notion as a mathematics structure, which consists of linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$, set $\mathcal{X}$, subset $\mathbf{B} \subseteq \mathbf{T} \times \mathcal{X}$, and binary relation $\Leftarrow$, defined on $\mathbf{B}$, satisfying conditions 1,2 of Theorem I.6.2. This approach to definition of base changeable sets is implemented in [9].

From Theorem I.6.2 (taking into account Remark I.6.4) we obtain the following corollary.

**Corollary I.6.1.** *If for base changeable sets $\mathcal{B}_1, \mathcal{B}_2$ we have $\mathbb{T}\mathbf{m}(\mathcal{B}_1) = \mathbb{T}\mathbf{m}(\mathcal{B}_2)$, $\mathbb{B}\mathfrak{s}(\mathcal{B}_1) = \mathbb{B}\mathfrak{s}(\mathcal{B}_2)$, $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_1}{\leftarrow}} = \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_2}{\leftarrow}}$, then $\mathcal{B}_1 = \mathcal{B}_2$.*

Main results of this Section were anonced in [1] and published in [5, Section 2], while Theorem I.6.2 is published in [8, Theorem 2.2].

## 7 Chains in the Set of Elementary-time States. Fate Lines and their Properties

Using definition of base changeable sets as well as assertions I.6.2 and I.6.1 (item 2) we obtain the following assertion.

**Assertion I.7.1.** *Let $\mathcal{B}$ be a base changeable set. Then:*

*1) The pair $\mathcal{Q}_{\mathcal{B}} = \left(\mathbb{B}\mathfrak{s}(\mathcal{B}), \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}}\right) = (\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$ is an anti-cyclical oriented set.*

*2) The mapping*

$$\tilde{\psi}(t) = \tilde{\psi}_{\mathcal{B}}(t) := \{\omega \in \mathbb{B}\mathfrak{s}(\mathcal{B}) \mid \mathsf{tm}\,(\omega) = t\} \in 2^{\mathbb{B}\mathfrak{s}(\mathcal{B})}, \quad t \in \mathbf{Tm}(\mathcal{B}) \tag{I.18}$$

*is a monotone time on $\mathcal{Q}_{\mathcal{B}}$.*

*3) If, in addition, $\psi(t) \neq \emptyset$, $t \in \mathbf{Tm}(\mathcal{P})$, then the time $\psi$ is strictly monotone.*

According to Assertion I.7.1, for any base changeable set $\mathcal{B}$ the pair $\mathcal{Q}_{\mathcal{B}} = (\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$ is (anti-cyclical) oriented set. Therefore we may introduce transitive sets and chains in the oriented set $(\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$. From anti-cyclicity of the oriented set $(\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$ it follows the following assertion.

**Assertion I.7.2.** *Let $\mathcal{B}$ be a base changeable set.*

*1) Any transitive subset $\mathcal{N} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ of the oriented set $(\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$ is a (partially) ordered set (relatively the relation $\leftarrow$).*

*2) Any chain $\mathcal{L} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ of the oriented set $(\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$ is a linearly ordered set (relatively the relation $\leftarrow$).*

**Denotation I.7.1.** *Futher we denote by $\mathbb{L}l(\mathcal{B})$ the set of all chains of the oriented set $\left(\mathbb{B}\mathfrak{s}(\mathcal{B}), \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}}\right) = (\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$.*

**Definition I.7.1.** *Let $\mathcal{B}$ be a base changeable set.*

*1) Any maximum chain $\mathcal{L} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ of the oriented set $\left(\mathbb{B}\mathfrak{s}(\mathcal{B}), \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}}\right) = (\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$ will be named by **fate line** of $\mathcal{B}$. The set of all fate lines of $\mathcal{B}$ will be denoted by $\mathbb{L}d(\mathcal{B})$:* {[5]}

$$\mathbb{L}d(\mathcal{B}) = \{\mathcal{L} \in \mathbb{L}l(\mathcal{B}) \mid \mathcal{L} \textit{ is a fate line of } \mathcal{B}\}.$$

*2) Any fate line, which contains an elementary-time state $\omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ will be named the **(eigen)** fate line of elementary-time state $\omega$ (in $\mathcal{B}$).*

*3) A fate line $\mathcal{L} \in \mathbb{L}d(\mathcal{B})$ will be named the **(eigen)** fate line of the elementary state $x \in \mathfrak{B}\mathfrak{s}(\mathcal{B})$ if and only if there exists the elementary-time state $\omega_x \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ such, that $\mathsf{bs}\,(\omega_x) = x$ and $\mathcal{L}$ is eigen fate line of $\omega_x$.*

[5] Ukrainian name of the term "fate line" looks like as "**l**iniya **d**oli" (in English transliteration). This explains the genesis of the denotation "$\mathbb{L}d(\mathcal{B})$". Note, that some denotations in this paper are generated by Ukrainian names of corresponding terms in English transliteration.

From Definition I.7.1 it dollows, that

$$\mathbb{L}d(\mathcal{B}) \subseteq \mathbb{L}l(\mathcal{B})$$

for any base changeable set $\mathcal{B}$.

It is clear that, in the general case, an elementary (elementary-time) state may have many eigen fate lines.

**Definition I.7.2.** *We will say, that elementary (elementary-time) states $x_1, x_2 \in \mathfrak{Bs}(\mathcal{B})$, ($\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$) are* **united by fate** *if and only if there exist at least one fate line $\mathcal{L} \in \mathbb{L}d(\mathcal{B})$, which is eigen fate line of both states $x_1, x_2$ ($\omega_1, \omega_2$).*

**Assertion I.7.3.** *1) Any elementary-time state $\omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ must have at least one eigen fate line.*

*2) For elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ to be united by fate it is necessary and sufficient satisfaction one of the following conditions:*

$$\omega_2 \leftarrow \omega_1 \quad or \quad \omega_1 \leftarrow \omega_2. \tag{I.19}$$

*Proof.* 1) The first statement of this Assertion follows from Corollary I.1.2.

2) 2.a) Suppose, that for the elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ there exist a common fate line $\mathcal{L} \in \mathbb{L}d(\mathcal{B})$ such that $\omega_1, \omega_2 \in \mathcal{L}$. Then, by Assertion I.7.2, item 2, the pair $(\mathcal{L}, \leftarrow)$ is a linearly ordered set. Thus at least one of the conditions (I.19) must be fulfilled.

2.b) Let, $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ and $\omega_2 \leftarrow \omega_1$. Then, by Corollary I.1.2, there exist a maximum chain (fate line) $\mathcal{L} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ ($\mathcal{L} \in \mathbb{L}d(\mathcal{B})$) such, that $\omega_1, \omega_2 \in \mathcal{L}$. □

**Assertion I.7.4.** *1) Any elementary state $x \in \mathfrak{Bs}(\mathcal{B})$ must have at least one eigen fate line.*

*2) For elementary states $x, y \in \mathfrak{Bs}(\mathcal{B})$ to be united by fate it is necessary and sufficient satisfaction one of the following conditions:*

$$y \leftarrow x \quad or \quad x \leftarrow y. \tag{I.20}$$

*Proof.* 1) Let $x \in \mathfrak{Bs}(\mathcal{B})$. Then, by the definition of time, there exist a time point $t \in \mathbf{Tm}(\mathcal{B})$ such, that $x \in \psi(t)$. By Assertion I.7.3, the elementary-time state $\omega_x = (t, x) \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ must have an eigen fate line $\mathcal{L} \in \mathbb{L}d(\mathcal{B})$. This fate line $\mathcal{L}$ must be eigen fate line of elementary state $x$.

2) 2.a) Let $x, y \in \mathfrak{Bs}(\mathcal{B})$, $y \leftarrow x$. Then, by Property I.6.1(8) (see Properties I.6.1), there exist elementary-time states $\omega_1$,$\omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ such, that $\mathsf{bs}(\omega_1) = x$, $\mathsf{bs}(\omega_2) = y$ and $\omega_2 \leftarrow \omega_1$. By Assertion I.7.3, there exist a common fate line $\mathcal{L} \in \mathbb{L}d(\mathcal{B})$ for the elementary-time states $\omega_1$,$\omega_2$ (such, that $\omega_1, \omega_2 \in \mathcal{L}$). By Definition I.7.1, this fate line $\mathcal{L}$ must be eigen fate line of both elementary states $x$ and $y$.

2.b) Suppose, that for the elementary states $x, y \in \mathfrak{Bs}(\mathcal{B})$ there exist a common eigen fate line $\mathcal{L} \in \mathbb{L}d(\mathcal{B})$. Then, there exist elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$, such, that $\mathsf{bs}(\omega_1) = x$, $\mathsf{bs}(\omega_2) = y$ and $\omega_1, \omega_2 \in \mathcal{L}$. Hence, by Assertion I.7.3, statement 2), one of the conditions $\omega_2 \leftarrow \omega_1$ or $\omega_1 \leftarrow \omega_2$ must be satisfied. Then, by Property I.6.1(7), at least one of the conditions (I.20) must be fulfilled. □

As it was shown in Theorem I.6.1, any system of abstract trajectories, defined on some linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$, generates the base changeable set $\mathcal{A}t(\mathbb{T}, \mathcal{R})$. The next aim is to show, that any base changeable set $\mathcal{B}$ can be represented in the form $\mathcal{B} = \mathcal{A}t(\mathbb{T}, \mathcal{R})$, where $\mathcal{R}$ is some system of abstract trajectories, defined on some linearly ordered set $\mathbb{T}$.

**Definition I.7.3.** *Let $\mathcal{R}$ be a system of abstract trajectories from $\mathbb{T} = (\mathbf{T}, \leq)$ to $M$.*

1. *Trajectory $r \in \mathcal{R}$ will be named a* ***maximum trajectory*** *(relatively the $\mathcal{R}$) if and only if there not exist any trajectory $\rho \in \mathcal{R}$ ($\rho \neq r$) such, that $\mathfrak{D}(r) \subset \mathfrak{D}(\rho)$ and $r(t) = \rho(t)$ $t \in \mathfrak{D}(r)$ (that is such, that $r \subset \rho$).*

*2. The system of abstract trajectories $\mathcal{R}$ will be referred to as the* ***system of maximum trajectories*** *if and only if any trajectory $r \in \mathcal{R}$ is maximum trajectory (relatively the $\mathcal{R}$).*

Recall, that in Subsection 6.3 we introduced the following denotation (for any base changeable set $\mathcal{B}$):

$$\mathbb{T}\mathbf{m}(\mathcal{B}) := (\mathbf{Tm}(\mathcal{B}), \leq_{\mathcal{B}}) .$$

**Assertion I.7.5.** *Let $\mathcal{B}$ be a base changeable set. Then:*

*1) Any chain $\mathcal{L} \in \mathbb{L}l(\mathcal{B})$ is an abstract trajectory from $\mathbb{T}\mathbf{m}(\mathcal{B})$ to $\mathfrak{Bs}(\mathcal{B})$.*

*2) The set $\mathbb{L}l(\mathcal{B})$ is a system of abstract trajectories from $\mathbb{T}\mathbf{m}(\mathcal{B})$ to $\mathfrak{Bs}(\mathcal{B})$.*

*3) Any fate line $\mathcal{L} \in \mathbb{L}d(\mathcal{B}) \subseteq \mathbb{L}l(\mathcal{B})$ is a maximum trajectory (relatively the system of abstract trajectories $\mathbb{L}l(\mathcal{B})$).*

*4) The set $\mathbb{L}d(\mathcal{B})$ is a system of maximum trajectories (from $\mathbb{T}\mathbf{m}(\mathcal{B})$ to $\mathfrak{Bs}(\mathcal{B})$).*

*Proof.* 1) Let $\mathcal{L} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ be a chain of the oriented set $(\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$. Since $\mathbb{B}\mathfrak{s}(\mathcal{B}) \subseteq \mathbf{Tm}(\mathcal{B}) \times \mathfrak{Bs}(\mathcal{B})$ and $\mathcal{L} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$, then $\mathcal{L}$ is a binary relation from the set $\mathbf{Tm}(\mathcal{B})$ to the set $\mathfrak{Bs}(\mathcal{B})$. Thus, to make sure that $\mathcal{L}$ is an abstract trajectory from $\mathbb{T}\mathbf{m}(\mathcal{B})$ to $\mathfrak{Bs}(\mathcal{B})$, it is sufficient to prove, that this relation $\mathcal{L}$ is a function from $\mathbf{Tm}(\mathcal{B})$ to $\mathfrak{Bs}(\mathcal{B})$. Suppose contrary. Then there exist elementary-time states $\omega_1, \omega_2 \in \mathcal{L}$ of kind $\omega_1 = (t, x_1)$, $\omega_2 = (t, x_2)$, where $x_1 \neq x_2$. Since $\mathcal{L}$ is a chain, one of the conditions $\omega_2 \leftarrow \omega_1$ or $\omega_1 \leftarrow \omega_2$ must be satisfied. Assume, that $\omega_2 \leftarrow \omega_1$. Then, since $\omega_1 \neq \omega_2$, by Property I.6.1(7), we obtain $t < t$, which is impossible. Similarly the assumption $\omega_1 \leftarrow \omega_2$ also leads to contradiction. The obtained contradiction proves that the chain $\mathcal{L}$ is a function. Thus, we have proved item 1).

Taking into account, that, according to item 1), any chain $\mathcal{L} \in \mathbb{L}l(\mathcal{B})$ is an abstract trajectory, we may use the notations $\mathfrak{D}(\mathcal{L})$ for the domain of $\mathcal{L}$ and $x = \mathcal{L}(t)$ (where $t \in \mathfrak{D}(\mathcal{L})$) to indicate the fact that $(t, x) \in \mathcal{L}$.

2) Chose any elementary state $x \in \mathfrak{Bs}(\mathcal{B})$. By the time definition, there exist a time point $t \in \mathbf{Tm}(\mathcal{B})$ such, that $x \in \psi(t)$. By Assertion I.1.1, item 2, the singleton set $\mathcal{L}_x = \{(t, x)\} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ is a chain of the oriented set $(\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$. Besides, $\mathfrak{R}(\mathcal{L}_x) = \{x\} \ni x$. Thus, any elementary state $x \in \mathfrak{Bs}(\mathcal{B})$ is contained in the range of some abstract trajectory $\mathcal{L}_x \in \mathbb{L}l(\mathcal{B})$. Therefore, $\bigcup_{\mathcal{L} \in \mathbb{L}l(\mathcal{B})} \mathfrak{R}(\mathcal{L}) = \mathfrak{Bs}(\mathcal{B})$. Thus, taking into account the statement 1) of this Assertion we conclude, that $\mathbb{L}l(\mathcal{B})$ is the system of abstract trajectories from $\mathbb{T}\mathbf{m}(\mathcal{B})$ to $\mathfrak{Bs}(\mathcal{B})$.

3) Let $\mathcal{L} \in \mathbb{L}d(\mathcal{B})$ be a fate line of $\mathcal{B}$ (ie $\mathcal{L}$ is a maximum chain of the oriented set $(\mathbb{B}\mathfrak{s}(\mathcal{B}), \leftarrow)$). Then, there not exist any chain (abstract trajectory) $\mathcal{L}_1 \in \mathbb{L}l(\mathcal{B})$ such, that $\mathcal{L} \subset \mathcal{L}_1$. Hence, $\mathcal{L}$ is a maximum trajectory (relatively the system of abstract trajectories $\mathbb{L}l(\mathcal{B})$).

4) Now, we are going to prove, that $\bigcup_{\mathcal{L} \in \mathbb{L}d(\mathcal{B})} \mathfrak{R}(\mathcal{L}) = \mathfrak{Bs}(\mathcal{B})$. Since $\bigcup_{\mathcal{L} \in \mathbb{L}d(\mathcal{B})} \mathcal{L} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}) \subseteq \mathbf{Tm}(\mathcal{B}) \times \mathfrak{Bs}(\mathcal{B})$, we have $\bigcup_{\mathcal{L} \in \mathbb{L}d(\mathcal{B})} \mathfrak{R}(\mathcal{L}) \subseteq \mathfrak{Bs}(\mathcal{B})$. Thus, it remains to prove the inverse inclusion. Chose any elementary state $x \in \mathfrak{Bs}(\mathcal{B})$. By Assertion I.7.4 (item 1), the elementary state $x$ must have an eigen fate line $\mathcal{L}_x \in \mathbb{L}d(\mathcal{B})$. This (by Definition I.7.1) means, that there exist an elementary-time state $\omega_x = (t, x) \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ such, that $\omega_x \in \mathcal{L}_x$. Since $(t, x) \in \mathcal{L}_x$, then $\mathcal{L}_x(t) = x$. Therefore, $x \in \mathfrak{R}(\mathcal{L}_x) \subseteq \bigcup_{\mathcal{L} \in \mathbb{L}d(\mathcal{B})} \mathfrak{R}(\mathcal{L})$. Thus, $\bigcup_{\mathcal{L} \in \mathbb{L}d(\mathcal{B})} \mathfrak{R}(\mathcal{L}) = \mathfrak{Bs}(\mathcal{B})$. Hence, $\mathbb{L}d(\mathcal{B})$ is a system of abstract trajectories from $\mathbb{T}\mathbf{m}(\mathcal{B})$ to $\mathfrak{Bs}(\mathcal{B})$. Since (by item 3 of this Assertion) any fate line $\mathcal{L} \in \mathbb{L}d(\mathcal{B}) \subseteq \mathbb{L}l(\mathcal{B})$ is a maximum trajectory relatively the system of abstract trajectories $\mathbb{L}l(\mathcal{B})$, it is the maximum trajectory relatively the narrower system of abstract trajectories $\mathbb{L}d(\mathcal{B})$. □

**Assertion I.7.6.** *Let $\mathcal{R}$ be a system of abstract trajectories from $\mathbb{T}$ to $M$. Then*

$$\mathcal{R} \subseteq \mathbb{L}l\left(\mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)\right).$$

*Proof.* Let $\mathcal{R}$ be a system of abstract trajectories from $\mathbb{T} = (\mathbf{T}, \leq)$ to $M$. Let us consider any $r \in \mathcal{R}$. According to Theorem I.6.1, we get $r \subseteq \mathbb{B}\mathfrak{s}\left(\mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)\right)$, moreover, for any $\omega_1, \omega_2 \in r$ the condition $\omega_2 \underset{\mathcal{A}t(\mathbb{T},\mathcal{R})}{\leftarrow} \omega_1$ holds if and only if $\mathsf{tm}(\omega_1) \leq \mathsf{tm}(\omega_2)$. Hence, since $(\mathbf{T}, \leq)$ is linearly ordered set, we have, that $r$ is a chain of $\mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)$. □

The next theorem shows, that any base changeable set can be generated by some system of maximum trajectories.

**Theorem I.7.1.** *For any base changeable set $\mathcal{B}$ the following equality is true:*

$$\mathcal{A}t\left(\mathbb{T}\mathbf{m}(\mathcal{B}),\, \mathbb{L}d(\mathcal{B})\right) = \mathcal{B}.$$

*Proof.* Denote: $\mathcal{R} := \mathbb{L}d(\mathcal{B})$. We need to prove, that $\mathcal{A}t(\mathcal{R}) = \mathcal{B}$. {[6]}

1) By Assertion I.7.5, $\mathcal{R} = \mathbb{L}d(\mathcal{B})$ is the system of abstract trajectories from $\mathbb{T}\mathbf{m}(\mathcal{B}) = (\mathbf{Tm}(\mathcal{B}), \leq_{\mathcal{B}})$ to $\mathfrak{Bs}(\mathcal{B})$. Hence, by the first item of Theorem I.6.1,

$$\mathbb{T}\mathbf{m}\left(\mathcal{A}t(\mathcal{R})\right) = \mathbb{T}\mathbf{m}(\mathcal{B}).$$

2) By the second item of Theorem I.6.1:

$$\mathbb{B}\mathfrak{s}\left(\mathcal{A}t(\mathcal{R})\right) = \bigcup_{r\in\mathcal{R}} r = \bigcup_{\mathcal{L}\in\mathbb{L}d(\mathcal{B})} \mathcal{L} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}). \tag{I.21}$$

On the other hand, by Assertion I.7.3, for any $\omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ the fate line $\mathcal{L}_\omega \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ exists such, that $\omega \in \mathcal{L}_\omega$. Threfore, $\mathbb{B}\mathfrak{s}(\mathcal{B}) \subseteq \bigcup_{\mathcal{L}\in\mathbb{L}d(\mathcal{B})} \mathcal{L} = \mathbb{B}\mathfrak{s}\left(\mathcal{A}t(\mathcal{R})\right)$. And, taking into account (I.21) we obtain:

$$\mathbb{B}\mathfrak{s}\left(\mathcal{A}t(\mathcal{R})\right) = \mathbb{B}\mathfrak{s}(\mathcal{B}).$$

3) Let us consider any elementary-time states $\omega_1 = (t_1, x_1)$, $\omega_2 = (t_2, x_2) \in \mathbb{B}\mathfrak{s}(\mathcal{B}) = \mathbb{B}\mathfrak{s}\left(\mathcal{A}t(\mathcal{R})\right)$.

3.a) Suppose, that $\omega_2 \underset{\mathcal{B}}{\leftarrow} \omega_1$. By Property I.6.1(7), $\mathsf{tm}\,(\omega_1) \leq \mathsf{tm}\,(\omega_2)$. Moreover, by Assertion I.7.3 (item 2) the fate line $\mathfrak{L} \in \mathbb{L}d(\mathcal{B})$ exists such, that $\omega_1, \omega_2 \in \mathfrak{L}$. Thus, by Theorem I.6.1 (item 3), $\omega_2 \underset{\mathcal{A}t(\mathbb{L}d(\mathcal{B}))}{\leftarrow} \omega_1$, that is $\omega_2 \underset{\mathcal{A}t(\mathcal{R})}{\leftarrow} \omega_1$.

3.b) Conversely, suppose, that $\omega_2 \underset{\mathcal{A}t(\mathcal{R})}{\leftarrow} \omega_1$, scilicet $\omega_2 \underset{\mathcal{A}t(\mathbb{L}d(\mathcal{B}))}{\leftarrow} \omega_1$. Then, by Theorem I.6.1 (item 3), $\mathsf{tm}\,(\omega_1) \leq \mathsf{tm}\,(\omega_2)$ and there exists the fate line $\mathfrak{L} \in \mathbb{L}d(\mathcal{B})$ exists such, that $\omega_1, \omega_2 \in \mathfrak{L}$. Since the fate line $\mathfrak{L}$ is a chain, at least one from the correlations $\omega_2 \underset{\mathcal{B}}{\leftarrow} \omega_1$ or $\omega_1 \underset{\mathcal{B}}{\leftarrow} \omega_2$ must be true. We shall prove, that $\omega_2 \underset{\mathcal{B}}{\leftarrow} \omega_1$. Assume the contrary ($\omega_2 \underset{\mathcal{B}}{\not\leftarrow} \omega_1$). Then, we have $\omega_1 \underset{\mathcal{B}}{\leftarrow} \omega_2$ and $\omega_2 \neq \omega_1$ (because in the case $\omega_1 = \omega_2$ we have $\omega_2 \underset{\mathcal{B}}{\leftarrow} \omega_1$). Hence, by Property I.6.1(7), $\mathsf{tm}\,(\omega_2) < \mathsf{tm}\,(\omega_1)$. The last inequality is impossible, because we have proved, that $\mathsf{tm}\,(\omega_1) \leq \mathsf{tm}\,(\omega_2)$. Therefore, $\omega_2 \underset{\mathcal{B}}{\leftarrow} \omega_1$.

From the items 3.a) and 3.b) it follows, that $\underset{\mathcal{B}}{\leftarrow} = \underset{\mathcal{A}t(\mathcal{R})}{\leftarrow}$ (for the bases of elementary processes on $\mathbb{B}\mathfrak{s}\left(\mathcal{A}t(\mathcal{R})\right) = \mathbb{B}\mathfrak{s}(\mathcal{B})$).

According to the items 1),2),3) below, we have, that $\mathbb{T}\mathbf{m}\left(\mathcal{A}t(\mathcal{R})\right) = \mathbb{T}\mathbf{m}(\mathcal{B})$, $\mathbb{B}\mathfrak{s}\left(\mathcal{A}t(\mathcal{R})\right) = \mathbb{B}\mathfrak{s}(\mathcal{B})$, $\underset{\mathcal{A}t(\mathcal{R})}{\leftarrow} = \underset{\mathcal{B}}{\leftarrow}$. Hence, by Corollary I.6.1, we obtain $\mathcal{A}t(\mathcal{R}) = \mathcal{B}$. □

The following example shows that the equality $\mathbb{L}d(\mathcal{A}t(\mathcal{R})) = \mathcal{R}$ for any system of maximal trajectories $\mathcal{R}$, in the general case is not true. Moreover, in general, we can not even assert about the inclusion of one of these sets to another.

***Example*** I.7.1. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be the function of kind:

$$f(t) := \frac{|t| - t}{2}, \quad t \in \mathbb{R}.$$

We consider the system of abstract trajectories $\mathcal{R} = \{r_\alpha \,|\, \alpha \in [0, \infty)\}$, where

$$r_\alpha(t) : = f(t+\alpha),\ t \in (-\infty, \alpha] \quad (\mathfrak{D}\,(r_\alpha) = (-\infty, \alpha]), \quad \alpha \in (0, \infty);$$

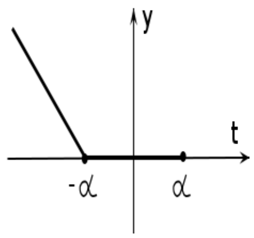
[6] We use the abbreviated denotation $\mathcal{A}t(\mathcal{R})$ instead of $\mathcal{A}t(\mathbb{T}\mathbf{m}(\mathcal{B}), \mathcal{R})$.

$$r_0(t) := 0, \qquad t \in [0,\infty) \qquad (\mathfrak{D}(r_0) = [0,\infty)), \quad \alpha = 0.$$

It is easy to verify, that $\mathcal{R}$ is the system of maximum trajectories from $\mathbb{R}$ to $[0,\infty)$. However, the trajectory $r_0 \in \mathcal{R}$ is not fate line of base changeable set $\mathcal{A}t(\mathcal{R})$. Thus, $r_0 \notin \mathbb{L}d(\mathcal{A}t(\mathcal{R}))$, and, therefore $\mathcal{R} \not\subseteq \mathbb{L}d(\mathcal{A}t(\mathcal{R}))$. From the other hand, we may consider the trajectory of kind:

$$r_0^{\sim}(t) = 0, \quad t \in \mathbb{R} \qquad (\mathfrak{D}(r_0^{\sim}) = \mathbb{R}).$$

($r_0^{\sim} = \{(t, r_0^{\sim}(t)) \mid t \in \mathbb{R}\} = \{(t,0) \mid t \in \mathbb{R}\} \subseteq \bigcup_{r\in\mathcal{R}} r = \mathbb{B}\mathfrak{s}(\mathcal{A}t(\mathcal{R}))$). It is easy to verify, that $r_0^{\sim}$ is a fate line of base changeable set $\mathcal{A}t(\mathcal{R})$, although $r_0^{\sim} \notin \mathcal{R}$. Hence, $\mathbb{L}d(\mathcal{A}t(\mathcal{R})) \not\subseteq \mathcal{R}$.




Graph of the trajectory $y = r_\alpha(t)$ for $\alpha \in (0,\infty)$.

Below it will be described the simplest class of cases, where the equality $\mathbb{L}d(\mathcal{A}t(\mathcal{R})) = \mathcal{R}$ still takes place.

**Definition I.7.4.** *System of abstract trajectories $\mathcal{R}$ from $\mathbb{T} = (\mathbf{T}, \leq)$ to $M$ will be named a* ***system of individual trajectories*** *if and only if any two different trajectories $r_1, r_2 \in \mathcal{R}$ are disjoint ($\forall r_1, r_2 \in \mathcal{R}$ $(r_1 \neq r_2 \Longrightarrow r_1 \cap r_2 = \emptyset)$).*

It is easy to see, that a system of abstract trajectories $\mathcal{R}$ from $\mathbb{T} = (\mathbf{T}, \leq)$ to $M$ is a system of individual trajectories if and only if for any $r_1, r_2 \in \mathcal{R}$ such, that $r_1 \neq r_2$ it is true one of the following propositions:

$$\mathfrak{D}(r_1) \cap \mathfrak{D}(r_2) = \emptyset \quad \text{or} \quad r_1(t) \neq r_2(t) \ (\forall t \in \mathfrak{D}(r_1) \cap \mathfrak{D}(r_2)).$$

From here, in particular, it follows, that the trajectory system $\mathcal{R}$ in Example I.6.1 is a system of individual trajectories.

**Theorem I.7.2.** *Let $\mathcal{R}$ be a system of individual trajectories from $\mathbb{T} = (\mathbf{T}, \leq)$ to $M$. Then:*

$$\mathbb{L}d(\mathcal{A}t(\mathbb{T}, \mathcal{R})) = \mathcal{R}.$$

*Proof.* Throughout this proof symbol "$\leftarrow$" will mean the directing relation of changes or the base of elementary processes in the base changeable set $\mathcal{A}t(\mathcal{R}) = \mathcal{A}t(\mathbb{T}, \mathcal{R})$.

**1**. Let $r \in \mathcal{R}$. According to Assertion I.7.6, $r \in \mathbb{L}l(\mathcal{A}t(\mathcal{R}))$, that is the trajectory $r$ is a chain of the oriented set $(\mathbb{B}\mathfrak{s}(\mathcal{A}t(\mathcal{R})), \leftarrow)$. We aim to prove, that $r$ is a fate line of $\mathcal{A}t(\mathcal{R})$) (that is $r$ is a maximum chain in $\mathbb{B}\mathfrak{s}(\mathcal{A}t(\mathcal{R}))$). Suppose opposite. Then there exists a fate line $\mathcal{L} \in \mathbb{L}d(\mathcal{A}t(\mathcal{R}))$ such, that $r \subset \mathcal{L}$. Since the inclusion $r \subset \mathcal{L}$ is strict, there exists an elementary-time state $\omega \in \mathcal{L}$ such,that $\omega \notin r$. From the other hand, by definition of abstract trajectory (Definition I.5.1, item 1), any trajectory of the system $\mathcal{R}$ is nonempty. Hence, there exists an elementary-time state $\omega_0 \in r$. Since $r \subset \mathcal{L}$, we have $\omega_0 \in \mathcal{L}$. Therefore, the elementary-time states $\omega$ and $\omega_0$ are united by fate. Thus, by Assertion I.7.3, one of the conditions $\omega \leftarrow \omega_0$ or $\omega_0 \leftarrow \omega$ must be satisfied. But, in the both cases, by Theorem I.6.1 (item 3), a trajectory $r_1 \in \mathcal{R}$ must exist such, that $\omega, \omega_0 \in r_1$. Since $\omega \notin r$ and $\omega \in r_1$, we have $r \neq r_1$. However, from the other hand, $\omega_0 \in r \cap r_1$, which is contradicts to the fact, that $\mathcal{R}$ is the system of individual trajectories. This contradiction proves, that $r$ is a fate line of $\mathcal{A}t(\mathcal{R})$). Thus:

$$\mathcal{R} \subseteq \mathbb{L}d(\mathcal{A}t(\mathcal{R})). \tag{I.22}$$

**2**. Let, $\mathcal{L} \in \mathbb{L}d\left(\mathcal{A}t(\mathcal{R})\right)$. From Remark I.6.1 and Corollary I.1.2 it follows, that any fate line of any base changeable set is nonempty set. Hence, there exists an elementary-time state $\omega \in \mathcal{L}$. Since $\omega \in \mathcal{L} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{A}t(\mathcal{R}))$, then, by Theorem I.6.1 (item 2), there exist a trajectory $r \in \mathcal{R}$ such, that $\omega \in r$. Let us consider any elementary-time state $\omega_1 \in \mathcal{L}$. And because $\omega, \omega_1 \in \mathcal{L}$, then the elementary-time states $\omega$ and $\omega_1$ — are united by fate. Hence, by Assertion I.7.3, one of the conditions $\omega_1 \leftarrow \omega$ or $\omega \leftarrow \omega_1$ must be satisfied. Therefore, by Theorem I.6.1 (item 3), the trajectory $r_1 \in \mathcal{R}$ such, that $\omega, \omega_1 \in r_1$ must exist. Thus, we have, that, $\omega \in r \cap r_1$. But, since $\mathcal{R}$ is the system of individual trajectories, the last relation is only possible when $r = r_1$. Hence, any elementary-time state $\omega_1 \in \mathcal{L}$ belongs to $r$. This means, that $\mathcal{L} \subseteq r$. But, according to the item 1 of this proof, the trajectory $r$ also is the fate line of $\mathcal{A}t(\mathcal{R})$. Since $r$ and $\mathcal{L}$ are the fate lines of $\mathcal{A}t(\mathcal{R})$, the inclusion $\mathcal{L} \subseteq r$ is possible only by condition $\mathcal{L} = r$. Thus, $\mathcal{L} = r \in \mathcal{R}$. Taking into account, that the fate line $\mathcal{L} \in \mathbb{L}d\left(\mathcal{A}t(\mathcal{R})\right)$ had been chosen by an arbitrary way, we obtain the inclusion, inverse to the (I.22). □

***Example*** I.7.2. Let $\mathfrak{X}$ be a complete metric space. Recall [41, page. 4], that a dynamic system on $\mathfrak{X}$ is any pair of kind:

$$\mathbb{S} = (\Theta, W), \quad \text{where:} \tag{I.23}$$

- $\Theta \subseteq \mathbb{R}$ is an arbitrary subset of the real axis $\mathbb{R}$;
- $W$ is an operator-valued function, defined on the set $\widetilde{\Theta} = \{(\tau, t_0) \in \mathbb{R}^2 |\, t_0, t_0 + \tau \in \Theta\}$, which maps any pair of kind $(\tau, t_0) \in \widetilde{\Theta}$ to the opetrator $W(\tau, t_0) : \mathfrak{X} \mapsto \mathfrak{X}$, and satisfies the following conditions:

$$W(0, t_0)\, x = x, \quad x \in \mathfrak{X},\ t_0 \in \Theta; \tag{I.24}$$

$$W(t + s, t_0) = W(t, t_0 + s)\, W(s, t_0), \quad t_0,\ t_0 + s,\ t_0 + t + s \in \Theta, \tag{I.25}$$

  where the product of operators is defined by the standard way $(W(t, t_0 + s)\, W(s, t_0)\, x = W(t, t_0 + s)\,(W(s, t_0)\, x)$, $x \in \mathfrak{X})$.

(Note, that the operators $W(\tau, t_0)$ $((\tau, t_0) \in \widetilde{\Theta})$ may me nonlinear.)

Any dynamic system $\mathbb{S}$ of kind (I.23) generates the system of abstract trajectories:

$$\mathcal{R}_{\mathbb{S}} = \{r_{x,t_0} \mid x \in \mathfrak{X},\ t_0 \in \Theta\},$$
$$r_{x,t_0}(t) = W(t - t_0, t_0)\, x, \quad x \in \mathfrak{X},\ t \in \Theta \tag{I.26}$$

from $\mathbb{T}_\Theta = (\Theta, \leq)$ to $\mathfrak{X}$, where $\leq$ is the standard linear order relation on the real numbers. From (I.24),(I.25) it follows, that, any trajectories from the $\mathcal{R}_{\mathbb{S}}$ possess the following properties:

$$r_{x,t_0}(t_0) = x, \quad x \in \mathfrak{X}, \quad t_0 \in \Theta,$$
$$r_{x,t_0'} = r_{\left[r_{x,t_0'}(t_0)\right],t_0}, \quad x \in \mathfrak{X},\ t_0, t_0' \in \Theta.$$

Thus, for any fixed $t_0 \in \Theta$ the system of trajectories $\mathcal{R}_{\mathbb{S}}$ can be represented in the form:

$$\mathcal{R}_{\mathbb{S}} = \{r_{x,t_0} \mid x \in \mathfrak{X}\}. \tag{I.27}$$

We are going to prove, that $\mathcal{R}_{\mathbb{S}}$ is the system of individual trajectories. Indeed, consider any fixed number $t_0 \in \Theta$. Using the equality (I.27), we may consider arbitrary trajectories $r_{x_1,t_0}, r_{x_2,t_0} \in \mathcal{R}_{\mathbb{S}}$. Suppose, that for some $t \in \Theta$ we have $r_{x_1,t_0}(t) = r_{x_2,t_0}(t)$. Then, taking into account (I.24), (I.25), (I.26), we obtain:

$$x_2 = W(0, t_0)\, x_2 = W(t_0 - t, t)\, W(t - t_0, t_0)\, x_2 = W(t_0 - t, t)\, r_{x_2,t_0}(t) =$$
$$= W(t_0 - t, t)\, r_{x_1,t_0}(t) = W(t_0 - t, t)\, W(t - t_0, t_0)\, x_1 = x_1.$$

Consequently, $r_{x_1,t_0} = r_{x_2,t_0}$. This means, that for any trajectories $r_{x_1,t_0}, r_{x_2,t_0} \in \mathcal{R}_{\mathbb{S}}$ such, that $r_{x_1,t_0} \neq r_{x_2,t_0}$ we have $r_{x_1,t_0}(t) \neq r_{x_2,t_0}(t)$ $(\forall t \in \Theta)$. Hence, $\mathcal{R}_{\mathbb{S}}$ is the system of individual trajectories. In particular, for any $x \in \mathfrak{X}$ and $t_0 \in \Theta$ there exists a unique trajectory $\rho_{x,t_0} \in \mathcal{R}_{\mathbb{S}}$ such, that $\rho_{x,t_0}(t_0) = x$ (where $\rho_{x,t_0} = r_{x,t_0}$).

The system of abstract trajectories $\mathcal{R}_{\mathbb{S}}$ generates the base changeable set $\mathcal{A}t(\mathcal{R}_{\mathbb{S}})$, moreover, by Theorem I.6.1, $\mathbf{Tm}(\mathcal{A}t(\mathcal{R}_{\mathbb{S}})) = \Theta$. From Theorem I.7.2 it follows, that if we know the base changeable set $\mathcal{A}t(\mathcal{R}_{\mathbb{S}})$, then we can recover the system of trajectories $\mathcal{R}_{\mathbb{S}}$. From here, we can uniquely restore the evolution operators $\left\{W(\tau, t_0) \mid (\tau, t_0) \in \widetilde{\Theta}\right\}$ by the help of formula:

$$W(\tau, t_0)\, x = \rho_{x,t_0}(\tau + t_0), \quad x \in \mathfrak{X},\ t_0, t_0 + \tau \in \Theta,$$

where $\rho_{x,t_0} \in \mathcal{R}_{\mathbb{S}}$ is the trajectory, satisfying the condition $\rho_{x,t_0}(t_0) = x$. Thus, the dynamic system $\mathbb{S}$ can be uniquely restored by the base changeable set $\mathcal{A}t(\mathcal{R}_{\mathbb{S}})$. Consequently, dynamic systems of kind (I.23) can be interpreted as particular cases of base changeable sets.

Main results of this Section were anonced in [1] and published in [5, Section 3].

## 8 Changeable Systems and Processes

**Definition I.8.1.** *Let $\mathcal{B}$ be a base changeable set. Any subset $S \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ we will name a **changeable system** of the base changeable set $\mathcal{B}$.*

In the mechanics the elementary states can be interpreted as the states or positions of material point in various moments of time. That is why, the concept of changeable system may be considered as the abstract generalization of the notion of physical body, which, in the general case, has not constant composition.

**Definition I.8.2.** *Let $\mathcal{B}$ be a base changeable set. Any mapping $s : \mathbf{Tm}(\mathcal{B}) :\mapsto 2^{\mathfrak{B}\mathfrak{s}(\mathcal{B})}$ such, that $s(t) \subseteq \psi(t)$, $t \in \mathbf{Tm}(\mathcal{B})$ will be referred to as a **process** of the base changeable set $\mathcal{B}$.*

Since primitive changeable sets can be interpreted as base changeable set with the base of elementary processes $\leftarrow(\mathrm{f})$, the chronometric processes, introduced in Definition I.4.9 can be considered as the particular cases of processes, introduced in Definition I.8.2.

Let $S \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ be an arbitrary changeable system of any base changeable set $\mathcal{B}$. Denote:

$$S^{\sim}(t) := \{x \in \mathfrak{B}\mathfrak{s}(\mathcal{B}) \mid (t, x) \in S\}, \quad t \in \mathbf{Tm}(\mathcal{B}) \tag{I.28}$$

(in particular $S^{\sim}(t) = \emptyset$ in the case, where there do not exist $x \in \mathfrak{B}\mathfrak{s}(\mathcal{B})$ such, that $(t, x) \in S$). It is easy to see, that $S^{\sim}(t) \subseteq \psi(t)$, $t \in \mathbf{Tm}(\mathcal{B})$. Thus, by Definition I.8.2, $S^{\sim}$ is a process of the base changeable set $\mathcal{B}$.

**Definition I.8.3.** *The process $S^{\sim}$ will be named the **process of transformations** of the changeable system $S$.*

**Assertion I.8.1.** *Let $\mathcal{B}$ be a base changeable set.*

*1. For any changeable systems $S_1, S_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ the equality $S_1^{\sim} = S_2^{\sim}$ holds if and only if $S_1 = S_2$.*

*2. For an arbitrary process $s$ of the base changeable set $\mathcal{B}$ a unique changeable system $S \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ exists such, that $s = S^{\sim}$.*

*Proof.* 1. To prove the first statement, it is enough to verify that for any $S_1, S_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ the equality $S_1^{\sim} = S_2^{\sim}$ implies the equality $S_1 = S_2$. Hence, suppose, that $S_1^{\sim} = S_2^{\sim}$. Then for any $t \in \mathbf{Tm}(\mathcal{B})$ we have $S_1^{\sim}(t) = S_2^{\sim}(t)$. Therefore, by (I.28), for an arbitrary $t \in \mathbf{Tm}(\mathcal{B})$ the condition $(t, x) \in S_1$ is equivalent to the condition $(t, x) \in S_2$. But, this means, that $S_1 = S_2$.

2. Let $s$ be a process of a base changeable set $\mathcal{B}$. Denote:

$$S := \{(t,x) \mid t \in \mathbf{Tm}(\mathcal{B}),\ x \in s(t)\} = \bigcup_{t\in\mathbf{Tm}(\mathcal{B})} (\{t\} \times s(t)),$$

where the symbol $\times$ denotes Cartesian product of sets. Since for any pair $(t,x) \in S$ it is true $x \in s(t) \subseteq \psi(t)$, we have $S \subseteq \mathbb{Bs}(\mathcal{B})$. Therefore, $S$ is a changeable system of $\mathcal{B}$. Moreover, for any $t \in \mathbf{Tm}(\mathcal{B})$ we obtain:

$$S^{\sim}(t) = \{x \in \mathfrak{Bs}(\mathcal{B}) \mid (t,x) \in S\} = \{x \in \mathfrak{Bs}(\mathcal{B}) \mid x \in s(t)\} = s(t).$$

Consequently, $S^{\sim} = s$. Suppose, an other changeable system $S_1$ exists such, that $S_1^{\sim} = s$. Then, $S^{\sim} = S_1^{\sim}$, and, by the statement 1, $S = S_1$. Thus, changeable system $S$, satisfying $S^{\sim} = s$ is unique. □

Therefore, the mapping $(\cdot)^{\sim}$ provides one-to-one correspondence between changeable systems and processes of any base changeable set. Taking into account this fact, further we will "identify" changeable systems and processes of any base changeable set, and for denotation of processes of a base changeable set we will use letters with tilde, keeping in mind, that any process is the process of transformations of some changeable system.

*We say, that a changeable system $U \subseteq \mathbb{Bs}(\mathcal{B})$ in a base changeable set $\mathcal{B}$ is a* ***subsystem*** *of a changeable system $S \subseteq \mathbb{Bs}(\mathcal{B})$ if and only if $U \subseteq S$.* The following assertion is true:

**Assertion I.8.2.** *Changeable system $U \subseteq \mathbb{Bs}(\mathcal{B})$ is a subsystem of a changeable system $S \subseteq \mathbb{Bs}(\mathcal{B})$ if and only if:*

$$\forall\, t \in \mathbf{Tm}(\mathcal{B}) \quad U^{\sim}(t) \subseteq S^{\sim}(t).$$

*Proof.* 1. Let $S, U \subseteq \mathbb{Bs}(\mathcal{B})$ and $U \subseteq S$. Then, by (I.28), for any $t \in \mathbf{Tm}(\mathcal{B})$ we obtain:

$$U^{\sim}(t) = \{x \in \mathfrak{Bs}(\mathcal{B}) \mid (t,x) \in U\} \subseteq \{x \in \mathfrak{Bs}(\mathcal{B}) \mid (t,x) \in S\} = S^{\sim}(t).$$

2. Conversely, suppose, that $U^{\sim}(t) \subseteq S^{\sim}(t)$ for any $t \in \mathbf{Tm}(\mathcal{B})$. Denote:

$$S_1 := \bigcup_{t\in\mathbf{Tm}(\mathcal{B})} \{t\} \times S^{\sim}(t); \quad U_1(t) := \bigcup_{t\in\mathbf{Tm}(\mathcal{B})} \{t\} \times U^{\sim}(t).$$

As it had been shown in the proof of statement 2 of Assertion I.8.1, $S_1^{\sim} = S^{\sim}$, $U_1^{\sim} = U^{\sim}$. Therefore, by the first item of Assertion I.8.1, $S_1 = S$, $U_1 = U$. Thus:

$$U = U_1 = \bigcup_{t\in\mathbf{Tm}(\mathcal{B})} \{t\} \times U^{\sim}(t) \subseteq \bigcup_{t\in\mathbf{Tm}(\mathcal{B})} \{t\} \times S^{\sim}(t) = S_1 = S.$$

□

**Definition I.8.4.** *We say, that the elementary state $x \in \mathfrak{Bs}(\mathcal{B})$ of a base changeable set $\mathcal{B}$* ***belongs*** *to a changeable system $S \subseteq \mathbb{Bs}(\mathcal{B})$ in a time point $t \in \mathbf{Tm}(\mathcal{B})$ if and only if $x \in S^{\sim}(t)$.*

The fact, that elementary state $x \in \mathfrak{Bs}(\mathcal{B})$ of a base changeable set $\mathcal{B}$ belongs to a changeable system $S$ in a time point $t$, will be denoted by:

$$x \in [t, \mathcal{B}]\ S,$$

and in the case, when the base changeable set is clear, we will use the denotation:

$$x \in [t]\ S.$$

By Assertion I.8.2, for any changeable systems $U, S \subseteq \mathbb{Bs}(\mathcal{B})$ the correlation $U \subseteq S$ holds if and only if for any $t \in \mathbf{Tm}(\mathcal{B})$ and $x \in \mathfrak{Bs}(\mathcal{B})$ the condition $x \in [t]\ U$ assures $x \in [t]\ S$.

The last remark indicates that a changeable system of any base changeable set can be interpreted as analog of the subset notion in the classic set theory, and the relation $\in [\cdot]$ can be interpreted as analog of the belonging relation of the classic set theory. However, the elementary-time state is not the complete analogue of the notion of element in the classic set theory, because knowing all the elementary-time states of a base changeable set, we can not fully recover this base changeable set.

It is evident, that any fate line $\mathcal{L} \in \mathbb{L}d(\mathcal{B})$ of a base changeable set $\mathcal{B}$ is the changeable system of $\mathcal{B}$.

**Definition I.8.5.** *The process $\mathcal{L}^{\sim}$, generated by a fate line $\mathcal{L} \in \mathbb{L}d(\mathcal{B})$ of the base changeable set $\mathcal{B}$ we name by the* ***elementary process*** *of $\mathcal{B}$.*

The concept of elementary process can be considered as the complete analogue of the notion of element in the classic set theory, because knowing all the elementary process of a base changeable set, we can fully recover this base changeable set, using Theorem I.7.1.

Main results of this Section were anonced in [1] and published in [5, Section 4].

# 9 Evolutional extensions and analogues of the operation of union for base changeable sets

## 9.1 Motivation

In physics we often encounter speculations, when the physical system is imaginary incorporated by additional components, not really existing in it. For example, during deduction of the formulas of Lorentz Transformations for reference frames with parallel axes it is often used the method of "light sphere". Namely, it is supposed, that on the zero time point a light had flashed in the origin of the frame, and light rays are traveling in all directions from the origin (for example see [42, page. 25]). This assumption does not imply, that in any evolution model, connected with the special relativity (SR) the light sphere must exist. But, simply, it is assumed that the coordinate transform will not be changed under the condition, that we "attach" the light sphere to any evolution model in the framework of SR, that is if we will consider the "extended" model, containing the light sphere, instead of the original model.

In the present paper we try to give mathematically strict foundation of the procedure of incorporation of new, "virtual" evolving components to the original model on the level of the theory of base changeable sets under the assumption, that incorporation of this components **do not effect** on the evolution of the original components of system. For this purpose we introduce the analogs of the set-theoretic inclusion relation and set-theoretic operation of union for base changeable sets.

Note that base changeable sets may be treated as the simplest particular cases of general changeable sets to be introduced further (in Section 10), namely as changeable sets, which have only one reference frame. Therefore, our consideration in this Section concerns only the case of single reference frame.

## 9.2 Definition and Properties of the Evolutional Extension and Evolutional Union

**Definition I.9.1.** *Base changeable sets $\mathcal{B}_0$ and $\mathcal{B}_1$ will be named* ***chronologically affined*** *if and only if $\mathbb{T}\mathbf{m}(\mathcal{B}_0) = \mathbb{T}\mathbf{m}(\mathcal{B}_1)$.*

**Definition I.9.2.** *Base changeable set $\mathcal{B}_1$ will be named by* ***evolutional extension*** *of an base changeable set $\mathcal{B}_0$ if and only if:*

1. *$\mathcal{B}_0$ and $\mathcal{B}_1$ are chronologically affined;*

*2.* $\mathbb{B}\mathfrak{s}(\mathcal{B}_0) \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$*;*

*3. For any* $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B}_0)$ *the condition* $\omega_2 \underset{\mathcal{B}_0}{\leftarrow} \omega_1$ *leads to the correlation* $\omega_2 \underset{\mathcal{B}_1}{\leftarrow} \omega_1$*. Or, in other words, we can write,* $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_0}{\leftarrow}} \subseteq \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_1}{\leftarrow}}$ *(where the binary relations* $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_0}{\leftarrow}}$ *and* $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_1}{\leftarrow}}$ *are usually understood as the sets, in particular* $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_0}{\leftarrow}} = \{(\omega_2, \omega_1) \mid \omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B}_0),\ \omega_2 \leftarrow \omega_1\}$*).*

*In the case, where the base changeable set* $\mathcal{B}_1$ *is an evolutional extension of the base changeable set* $\mathcal{B}_0$*, we also will say, that* $\mathcal{B}_0$ *is evolutionarily included in* $\mathcal{B}_1$*, using the denotation* $\mathcal{B}_0 \underset{\rightarrow}{\subset} \mathcal{B}_1$ *or* $\mathcal{B}_1 \underset{\leftarrow}{\supset} \mathcal{B}_0$*.*

In Section 8 it is explained, that the notion of elementary process (generated by some fate line) may serve as analog of the notion of element of ordinary (static) set. The last fact motivates the next definition.

**Definition I.9.3.** *Base changeable set* $\mathcal{B}_1$ *will be named as* ***super-evolutional extension*** *of an base changeable set* $\mathcal{B}_0$ *if and only if:*

*1.* $\mathcal{B}_0$ *and* $\mathcal{B}_1$ *are chronologically affined;*

*2.* $\mathbb{L}d(\mathcal{B}_0) \subseteq \mathbb{L}d(\mathcal{B}_1)$*, that is any elementary process of* $\mathcal{B}_0$ *is the elementary process of* $\mathcal{B}_1$*.*

*In the case, where the base changeable set* $\mathcal{B}_1$ *is an super-evolutional extension of the base changeable set* $\mathcal{B}_0$*, we also say, that* $\mathcal{B}_0$ *is* ***super-evolutionarily included*** *in* $\mathcal{B}_1$*, using the denotation* $\mathcal{B}_0 \underset{\rightarrow}{\sqsubset} \mathcal{B}_1$ *or* $\mathcal{B}_1 \underset{\leftarrow}{\sqsupset} \mathcal{B}_0$*.*

**Assertion I.9.1.** *If* $\mathcal{B}_0 \underset{\rightarrow}{\subset} \mathcal{B}_1$*, then:*

*1.* $\mathfrak{Bs}(\mathcal{B}_0) \subseteq \mathfrak{Bs}(\mathcal{B}_1)$*;*

*2.* $\overset{\mathfrak{Bs}}{\underset{\mathcal{B}_0}{\leftarrow}} \subseteq \overset{\mathfrak{Bs}}{\underset{\mathcal{B}_1}{\leftarrow}}$ *(that is for arbitrary* $x_1, x_2 \in \mathfrak{Bs}(\mathcal{B}_0)$ *the condition* $x_2 \underset{\mathcal{B}_0}{\leftarrow} x_1$ *leads to the correlation* $x_2 \underset{\mathcal{B}_1}{\leftarrow} x_1$*).*

*Proof.* 1. Since $\mathcal{B}_0 \underset{\rightarrow}{\subset} \mathcal{B}_1$, then, by Definition I.9.2 (item 2), we have $\mathbb{B}\mathfrak{s}(\mathcal{B}_0) \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$. Hence, using Property I.6.1(9), we obtain:

$$\mathfrak{Bs}(\mathcal{B}_0) = \{\mathsf{bs}(\omega) \mid \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B}_0)\} \subseteq \{\mathsf{bs}(\omega) \mid \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B}_1)\} = \mathfrak{Bs}(\mathcal{B}_1).$$

2. Suppose, that $x_1, x_2 \in \mathfrak{Bs}(\mathcal{B}_0)$ and $x_2 \underset{\mathcal{B}_0}{\leftarrow} x_1$. Then, according to Property I.6.1(8), there exist the elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B}_0)$ such, that $\mathsf{bs}(\omega_i) = x_i$ $(i = 1, 2)$ and $\omega_2 \underset{\mathcal{B}_0}{\leftarrow} \omega_1$. Since $\mathcal{B}_0 \underset{\rightarrow}{\subset} \mathcal{B}_1$, then, by Definition I.9.2 (items 2,3), $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$ and $\omega_2 \underset{\mathcal{B}_1}{\leftarrow} \omega_1$. Therefore, we have $\mathsf{bs}(\omega_i) = x_i$ $(i = 1, 2)$, where $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$ and $\omega_2 \underset{\mathcal{B}_1}{\leftarrow} \omega_1$, that is, by Property I.6.1(8), $x_2 \underset{\mathcal{B}_1}{\leftarrow} x_1$. □

**Assertion I.9.2.** *Any super-evolutional extension of arbitrary base changeable set* $\mathcal{B}_0$ *is its evolutional extension, that is the correlation* $\mathcal{B}_0 \underset{\rightarrow}{\sqsubset} \mathcal{B}_1$*, always leads to the correlation* $\mathcal{B}_0 \underset{\rightarrow}{\subset} \mathcal{B}_1$*.*

*Proof.* **1**. According to Theorem I.7.1, for any base changeable set $\mathcal{B}$ we have:

$$\mathcal{A}t(\mathbb{T}\mathbf{m}(\mathcal{B}), \mathbb{L}d(\mathcal{B})) = \mathcal{B}. \tag{I.29}$$

Hence, by Theorem I.6.1 (item 2), we obtain the equality:

$$\mathbb{B}\mathfrak{s}(\mathcal{B}) = \mathbb{B}\mathfrak{s}(\mathcal{A}t(\mathbb{T}\mathbf{m}(\mathcal{B}), \mathbb{L}d(\mathcal{B}))) = \bigcup_{L \in \mathbb{L}d(\mathcal{B})} L. \tag{I.30}$$

**2.1**. Suppose, that $\mathcal{B}_0 \underset{\to}{\sqsubset} \mathcal{B}_1$. Then, by definition, we have $\mathbb{L}d(\mathcal{B}_0) \subseteq \mathbb{L}d(\mathcal{B}_1)$. Hence, using the equality (I.30), we obtain:

$$\mathbb{B}\mathfrak{s}(\mathcal{B}_0) = \bigcup_{L \in \mathbb{L}d(\mathcal{B}_0)} L \subseteq \bigcup_{L \in \mathbb{L}d(\mathcal{B}_1)} L = \mathbb{B}\mathfrak{s}(\mathcal{B}_1).$$

**2.2**. Let $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B}_0)$ and $\omega_2 \underset{\mathcal{B}_0}{\leftarrow} \omega_1$. Then, from (I.29) and Theorem I.6.1 (item 3), it follows, that $\mathsf{tm}(\omega_1) \leq \mathsf{tm}(\omega_2)$ and there exist the fate line $L \in \mathbb{L}d(\mathcal{B}_0)$ such, that $\omega_1, \omega_2 \in L$. Since $\mathcal{B}_0 \underset{\to}{\sqsubset} \mathcal{B}_1$, then, by Definition I.9.3, we have, $\mathbb{L}d(\mathcal{B}_0) \subseteq \mathbb{L}d(\mathcal{B}_1)$. Therefore, we obtain $L \in \mathbb{L}d(\mathcal{B}_1)$. Thus, $\mathsf{tm}(\omega_1) \leq \mathsf{tm}(\omega_2)$ and $\omega_1, \omega_2 \in L$, where $L \in \mathbb{L}d(\mathcal{B}_1)$ ($L \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$). Hence, in accordance with formula (I.29) and Theorem I.6.1 (item 3), we get $\omega_2 \underset{\mathcal{B}_1}{\leftarrow} \omega_1$.

From the items 2.1 and 2.2 it follows, that $\mathcal{B}_0 \underset{\to}{\subseteq} \mathcal{B}_1$. □

Henceforth we use the denotation $M^{\times 2}$ for Cartesian square of the set $M$, ie $M^{\times 2} = M \times M$.

The next example shows, that the statement, inverse to Assertion I.9.2 is not true.

***Example*** I.9.1. Let, $\mathcal{R}_0 = \{r_0\}$, $\mathcal{R}_1 = \{r_1\}$ be the systems of abstract trajectories from $\mathbb{R}$ to $\mathbb{R}$, with:

$$\mathfrak{D}(r_0) = [0, \infty), \quad r_0(t) = t,\ t \in \mathfrak{D}(r_0);$$
$$\mathfrak{D}(r_1) = \mathbb{R}, \quad r_1(t) = t,\ t \in \mathfrak{D}(r_1).$$

Since $\mathcal{R}_0$ and $\mathcal{R}_1$ are composed of the single trajectory, then $\mathcal{R}_0$ and $\mathcal{R}_1$ are the systems of individual trajectories in the sense of Definition I.7.4. Denote:

$$\mathcal{B}_0 := \mathcal{A}t(\mathbb{R}_{ord}, \mathcal{R}_0); \qquad \mathcal{B}_1 := \mathcal{A}t(\mathbb{R}_{ord}, \mathcal{R}_1),$$

where $\mathbb{R}_{ord} = (\mathbb{R}, \leq)$ and $\leq$ is the standard linear order relation on the real numbers.

Since $\mathcal{R}_0$ and $\mathcal{R}_1$ are systems of individual trajectories, then, by Theorem I.7.2, we have:

$$\mathbb{L}d(\mathcal{B}_0) = \mathcal{R}_0; \quad \mathbb{L}d(\mathcal{B}_1) = \mathcal{R}_1.$$

And we get $\mathbb{L}d(\mathcal{B}_0) \not\subseteq \mathbb{L}d(\mathcal{B}_1)$, because $\mathcal{R}_0 \not\subseteq \mathcal{R}_1$. Hence, according to Definition I.9.3, $\mathcal{B}_1$ can not be super-evolutional extension of $\mathcal{B}_0$, therefore $\mathcal{B}_0 \underset{\to}{\not\sqsubset} \mathcal{B}_1$.

From the other hand, taking into account the inclusion $r_0 \subseteq r_1$ and applying Theorem I.6.1, we receive:

$$\begin{aligned}
\mathbb{T}\mathbf{m}(\mathcal{B}_0) &= \mathbb{R}_{ord} = \mathbf{Tm}(\mathcal{B}_1); \\
\mathbb{B}\mathfrak{s}(\mathcal{B}_0) &= \bigcup_{r \in \mathcal{R}_0} r = r_0 \subseteq r_1 = \bigcup_{r \in \mathcal{R}_1} r = \mathbb{B}\mathfrak{s}(\mathcal{B}_1); \\
\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_0}{\leftarrow}} &= \left\{(\omega_2, \omega_1) \in \mathbb{B}\mathfrak{s}(\mathcal{B}_0)^{\times 2} \mid (\mathsf{tm}(\omega_1) \leq \mathsf{tm}(\omega_2)) \wedge \right. \\
&\qquad\qquad \left. \wedge (\exists r \in \mathcal{R}_0\, (\omega_1, \omega_2 \in r))\right\} = \\
&= \left\{(\omega_2, \omega_1) \in \mathbb{B}\mathfrak{s}(\mathcal{B}_0)^{\times 2} \mid (\mathsf{tm}(\omega_1) \leq \mathsf{tm}(\omega_2)) \wedge (\omega_1, \omega_2 \in r_0)\right\} \subseteq \\
&\subseteq \left\{(\omega_2, \omega_1) \in \mathbb{B}\mathfrak{s}(\mathcal{B}_1)^{\times 2} \mid (\mathsf{tm}(\omega_1) \leq \mathsf{tm}(\omega_2)) \wedge (\omega_1, \omega_2 \in r_1)\right\} = \\
&= \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_1}{\leftarrow}},
\end{aligned}$$

where the symbol $\wedge$ denotes the logical operation of conjunction.

Hence, we have proved, that $\mathbb{T}\mathbf{m}(\mathcal{B}_0) = \mathbb{T}\mathbf{m}(\mathcal{B}_1)$, $\mathbb{B}\mathfrak{s}(\mathcal{B}_0) \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$ and $\underset{\mathcal{B}_0}{\leftarrow} \subseteq \underset{\mathcal{B}_1}{\leftarrow}$. Therefore, by Definition I.9.2, $\mathcal{B}_0 \underset{\to}{\subseteq} \mathcal{B}_1$. Thus, $\mathcal{B}_0 \underset{\to}{\not\sqsubset} \mathcal{B}_1$, although $\mathcal{B}_0 \underset{\to}{\subseteq} \mathcal{B}_1$.

**Assertion I.9.3.** *The evolutional inclusion possesses the following properties:*

*1. $\mathcal{B}\underset{\rightarrow}{\subsetneq}\mathcal{B}$ for an arbitrary base changeable set $\mathcal{B}$;*

*2. If $\mathcal{B}_1\underset{\rightarrow}{\subsetneq}\mathcal{B}_2$ and $\mathcal{B}_2\underset{\rightarrow}{\subsetneq}\mathcal{B}_1$ then $\mathcal{B}_1=\mathcal{B}_2$;*

*3. If $\mathcal{B}_1\underset{\rightarrow}{\subsetneq}\mathcal{B}_2$ and $\mathcal{B}_2\underset{\rightarrow}{\subsetneq}\mathcal{B}_3$ then $\mathcal{B}_1\underset{\rightarrow}{\subsetneq}\mathcal{B}_3$.*

*Proof.* **1.** The correlation $\mathcal{B}\underset{\rightarrow}{\subsetneq}\mathcal{B}$ follows by a trivial way from Definition I.9.2.

**2.** Suppose, that $\mathcal{B}_1\underset{\rightarrow}{\subsetneq}\mathcal{B}_2$ and $\mathcal{B}_2\underset{\rightarrow}{\subsetneq}\mathcal{B}_1$. Then, by Definition I.9.2:

$$\mathbb{T}\mathbf{m}\left(\mathcal{B}_1\right)=\mathbb{T}\mathbf{m}\left(\mathcal{B}_2\right); \tag{I.31}$$

$$\mathbb{B}\mathfrak{s}\left(\mathcal{B}_1\right)\subseteq\mathbb{B}\mathfrak{s}\left(\mathcal{B}_2\right);\ \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_1}{\leftarrow}}\subseteq\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_2}{\leftarrow}};\quad \mathbb{B}\mathfrak{s}\left(\mathcal{B}_2\right)\subseteq\mathbb{B}\mathfrak{s}\left(\mathcal{B}_1\right);\ \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_2}{\leftarrow}}\subseteq\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_1}{\leftarrow}}.$$

Therefore, we receive:

$$\mathbb{B}\mathfrak{s}\left(\mathcal{B}_1\right)=\mathbb{B}\mathfrak{s}\left(\mathcal{B}_2\right);\ \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_1}{\leftarrow}}=\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_2}{\leftarrow}}. \tag{I.32}$$

Thus, the equality $\mathcal{B}_1=\mathcal{B}_2$ follows from the equalities $(I.31)$,(I.32), by means of Corollary I.6.1.

**3.** Let $\mathcal{B}_1\underset{\rightarrow}{\subsetneq}\mathcal{B}_2$ and $\mathcal{B}_2\underset{\rightarrow}{\subsetneq}\mathcal{B}_3$. Then, by Definition I.9.2, the base changeable sets $\mathcal{B}_1$ and $\mathcal{B}_2$ as well as $\mathcal{B}_2$ and $\mathcal{B}_3$ are chronologically affined. Hence, $\mathcal{B}_1$ and $\mathcal{B}_3$ also are chronologically affined. According to Definition I.9.2, the evolutional inclusions $\mathcal{B}_1\underset{\rightarrow}{\subsetneq}\mathcal{B}_2$ and $\mathcal{B}_2\underset{\rightarrow}{\subsetneq}\mathcal{B}_3$ lead to the inclusions:

$$\mathbb{B}\mathfrak{s}\left(\mathcal{B}_1\right)\subseteq\mathbb{B}\mathfrak{s}\left(\mathcal{B}_2\right);\ \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_1}{\leftarrow}}\subseteq\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_2}{\leftarrow}};\quad \mathbb{B}\mathfrak{s}\left(\mathcal{B}_2\right)\subseteq\mathbb{B}\mathfrak{s}\left(\mathcal{B}_3\right);\ \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_2}{\leftarrow}}\subseteq\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_3}{\leftarrow}}.$$

Thus, $\mathbb{B}\mathfrak{s}\left(\mathcal{B}_1\right)\subseteq\mathbb{B}\mathfrak{s}\left(\mathcal{B}_3\right)$ and $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_1}{\leftarrow}}\subseteq\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_3}{\leftarrow}}$. Therefore, by Definition I.9.2, we receive $\mathcal{B}_1\underset{\rightarrow}{\subsetneq}\mathcal{B}_3$. □

In Assertion I.7.5 (item 2)) it had been proved, that for any base changeable set $\mathcal{B}$, the set $\mathbb{L}l(\mathcal{B})$ is the systems of abstract trajectories from $\mathbb{T}\mathbf{m}(\mathcal{B})$ to $\mathfrak{B}\mathfrak{s}(\mathcal{B})$.

**Assertion I.9.4.** *If for some base changeable set $\mathcal{B}$ the correlation $\mathcal{R}\subseteq\mathbb{L}l(\mathcal{B})$ holds while $\mathcal{R}\neq\emptyset$, then*

$$\mathcal{A}t\left(\mathbb{T}\mathbf{m}(\mathcal{B}),\mathcal{R}\right)\underset{\rightarrow}{\subsetneq}\mathcal{B}.$$

*Proof.* Suppose, that the condition of this Assertion is satisfied. Denote:

$$\mathcal{B}_1:=\mathcal{A}t\left(\mathbb{T}\mathbf{m}(\mathcal{B}),\mathcal{R}\right).$$

Since $\mathcal{R}\subseteq\mathbb{L}l(\mathcal{B})\subseteq 2^{\mathbb{B}\mathfrak{s}(\mathcal{B})}$, then, by Theorem I.6.1:

$$\mathbb{B}\mathfrak{s}\left(\mathcal{B}_1\right)=\bigcup_{r\in\mathcal{R}} r\subseteq\mathbb{B}\mathfrak{s}(\mathcal{B}). \tag{I.33}$$

Now, we consider any two elementary-time states $\omega_1,\omega_2\in\mathbb{B}\mathfrak{s}\left(\mathcal{B}_1\right)$ such, that $\omega_2\underset{\mathcal{B}_1}{\leftarrow}\omega_1$. According to Theorem I.6.1,

$$\mathsf{tm}\left(\omega_1\right)\leq\mathsf{tm}\left(\omega_2\right), \tag{I.34}$$

moreover a trajectory $r\in\mathcal{R}$ must exist such, that $\omega_1,\omega_2\in\mathcal{R}$. Since $\mathcal{R}\subseteq\mathbb{L}l(\mathcal{B})$, we have $r\in\mathbb{L}l(\mathcal{B})$. Hence $r$ is a chain of $\mathcal{B}$. Thus, at least one of the conditions $\omega_1\underset{\mathcal{B}}{\leftarrow}\omega_2$ or $\omega_2\underset{\mathcal{B}}{\leftarrow}\omega_1$ must be satisfied. Now, we assume, that $\omega_2\underset{\mathcal{B}}{\not\leftarrow}\omega_1$. Then, we obtain $\omega_1\underset{\mathcal{B}}{\leftarrow}\omega_2$, moreover, by Property I.6.1(6), $\omega_1\neq\omega_2$. Hence, by Property I.6.1(7) we receive $\mathsf{tm}\left(\omega_2\right)<\mathsf{tm}\left(\omega_1\right)$, which contradicts to the inequality (I.34). Therefore the assumption, that $\omega_2\underset{\mathcal{B}}{\not\leftarrow}\omega_1$ is wrong. Consequently, $\omega_2\underset{\mathcal{B}}{\leftarrow}\omega_1$. From the last correlation, taking into account the inclusion (I.33), and Definition I.9.2, we get, that $\mathcal{B}_1\underset{\rightarrow}{\subsetneq}\mathcal{B}$. □

**Assertion I.9.5.** *Let $\mathcal{R}_i$ $(i \in \{1,2\})$ be systems of abstract trajectories from $\mathbb{T}$ to $M_i$, and, besides $\mathcal{R}_1 \subseteq \mathcal{R}_2$. Then*

$$\mathcal{A}t(\mathbb{T}, \mathcal{R}_1) \underset{\rightarrow}{\subseteq} \mathcal{A}t(\mathbb{T}, \mathcal{R}_2).$$

*Proof.* Let $\mathcal{R}_i$ $(i \in \{1,2\})$ be systems of abstract trajectories from $\mathbb{T}$ to $M_i$, and $\mathcal{R}_1 \subseteq \mathcal{R}_2$. Denote:

$$\mathcal{B}_i := \mathcal{A}t(\mathbb{T}, \mathcal{R}_i) \qquad (i \in \{1,2\}).$$

By Theorem I.6.1 we get:

$$\mathbb{B}\mathfrak{s}(\mathcal{B}_1) = \bigcup_{r \in \mathcal{R}_1} r \subseteq \bigcup_{r \in \mathcal{R}_2} r = \mathbb{B}\mathfrak{s}(\mathcal{B}_2). \tag{I.35}$$

Now, we chose any $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$ such, that $\omega_2 \underset{\mathcal{B}_1}{\leftarrow} \omega_1$. In accordance with Theorem I.6.1, we have $\mathsf{tm}(\omega_1) \leq \mathsf{tm}(\omega_2)$, and, besides, trajectory $r \in \mathcal{R}_1$ must exist such, that $\omega_1, \omega_2 \in r$. Since $\mathcal{R}_1 \subseteq \mathcal{R}_2$, then we get $r \in \mathcal{R}_2$. Hence, applying Theorem I.6.1, we get $\omega_2 \underset{\mathcal{B}_2}{\leftarrow} \omega_1$. And, taking into account the inclusion (I.35), by Definition I.9.2, we receive $\mathcal{B}_1 \underset{\rightarrow}{\subseteq} \mathcal{B}_2$. □

**Assertion I.9.6.** *For arbitrary chronologically affined base changeable sets $\mathcal{B}_1$ and $\mathcal{B}_2$, the following statements are equivalent:*

1. $\mathcal{B}_1 \underset{\rightarrow}{\subseteq} \mathcal{B}_2$;
2. $\mathbb{L}l(\mathcal{B}_1) \subseteq \mathbb{L}l(\mathcal{B}_2)$;
3. $\mathbb{L}d(\mathcal{B}_1) \subseteq \mathbb{L}l(\mathcal{B}_2)$;

*Proof.* **1**. First, we are going to prove the implication 1⇒2. Suppose, that $\mathcal{B}_1 \underset{\rightarrow}{\subseteq} \mathcal{B}_2$. Chose any chain $L \in \mathbb{L}l(\mathcal{B}_1)$. According to Definition I.9.2, $\mathbb{B}\mathfrak{s}(\mathcal{B}_1) \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_2)$. Hence $L \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_2)$. Therefore, we need to prove, that the binary relation $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_2}{\leftarrow}}$, defined on $L$, satisfies the following conditions:

1. $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_2}{\leftarrow}}$ is transitive on $L$;
2. for any $\omega_1, \omega_2 \in L$ at least one of the correlations $\omega_2 \underset{\mathcal{B}_2}{\leftarrow} \omega_1$ or $\omega_1 \underset{\mathcal{B}_2}{\leftarrow} \omega_2$ must be true.

Since $L \in \mathbb{L}l(\mathcal{B}_1)$, then the binary relation $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_1}{\leftarrow}}$ satisfies the conditions 1,2. By Definition I.9.2, for $\omega_1, \omega_2 \in L$ condition $\omega_2 \underset{\mathcal{B}_1}{\leftarrow} \omega_1$ leads to the correlation $\omega_2 \underset{\mathcal{B}_2}{\leftarrow} \omega_1$. Thus, the desired result will be proved if we verify, that for arbitrary $\omega_1, \omega_2 \in L$ the correlation $\omega_2 \underset{\mathcal{B}_2}{\leftarrow} \omega_1$ leads to the correlation $\omega_2 \underset{\mathcal{B}_1}{\leftarrow} \omega_1$.

Suppose, that $\omega_1, \omega_2 \in L$ and $\omega_2 \underset{\mathcal{B}_2}{\leftarrow} \omega_1$. Since $L$ is chain in $\mathbb{B}\mathfrak{s}(\mathcal{B}_1)$, at least one of the correlations $\omega_2 \underset{\mathcal{B}_1}{\leftarrow} \omega_1$ or $\omega_1 \underset{\mathcal{B}_1}{\leftarrow} \omega_2$ must be true. Assume, that $\omega_2 \underset{\mathcal{B}_1}{\not\leftarrow} \omega_1$. Then, since, by Property I.6.1(6), the binary relation $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_1}{\leftarrow}}$ is reflexive, we have $\omega_1 \neq \omega_2$. Thus, we have got $\omega_1 \neq \omega_2$ and $\omega_1 \underset{\mathcal{B}_1}{\leftarrow} \omega_2$. Hence, according to Property I.6.1(7), we receive $\mathsf{tm}(\omega_2) < \mathsf{tm}(\omega_1)$. From the other hand, since $\omega_2 \underset{\mathcal{B}_2}{\leftarrow} \omega_1$, then, by Property I.6.1(7), the inequality $\mathsf{tm}(\omega_1) \leq \mathsf{tm}(\omega_2)$ must be true. The obtained contradiction shows, that the assumption $\omega_2 \underset{\mathcal{B}_1}{\not\leftarrow} \omega_1$ is incorrect. Thus, we have seen, that $\omega_2 \underset{\mathcal{B}_1}{\leftarrow} \omega_1$, and that it was necessary to prove.

**2**. Let, $\mathbb{L}l(\mathcal{B}_1) \subseteq \mathbb{L}l(\mathcal{B}_2)$. Taking into account the fact, that any fate line of arbitrary base changeable set forms its chain, we obtain, $\mathbb{L}d(\mathcal{B}_1) \subseteq \mathbb{L}l(\mathcal{B}_1) \subseteq \mathbb{L}l(\mathcal{B}_2)$.

**3**. Now we are going to prove the implication 3⇒1. Suppose, that $\mathbb{L}d(\mathcal{B}_1) \subseteq \mathbb{L}l(\mathcal{B}_2)$. Then, according to Theorem I.7.1 and Assertion I.9.4, we get, $\mathcal{B}_1 = \mathcal{A}t(\mathbb{T}\mathbf{m}(\mathcal{B}_1), \mathbb{L}d(\mathcal{B}_1)) = \mathcal{A}t(\mathbb{T}\mathbf{m}(\mathcal{B}_2), \mathbb{L}d(\mathcal{B}_1)) \underset{\rightarrow}{\subseteq} \mathcal{B}_2$. □

**Assertion I.9.7.** *The super-evolutional inclusion possesses the following properties:*

1. $\mathcal{B} \underset{\rightarrow}{\sqsubseteq} \mathcal{B}$ *for an arbitrary base changeable set* $\mathcal{B}$*;*
2. *If* $\mathcal{B}_1 \underset{\rightarrow}{\sqsubseteq} \mathcal{B}_2$ *and* $\mathcal{B}_2 \underset{\rightarrow}{\sqsubseteq} \mathcal{B}_1$ *then* $\mathcal{B}_1 = \mathcal{B}_2$*;*
3. *If* $\mathcal{B}_1 \underset{\rightarrow}{\sqsubseteq} \mathcal{B}_2$ *and* $\mathcal{B}_2 \underset{\rightarrow}{\sqsubseteq} \mathcal{B}_3$ *then* $\mathcal{B}_1 \underset{\rightarrow}{\sqsubseteq} \mathcal{B}_3$.

*Proof.* First property is trivial. Second property is a consequence of assertions I.9.2 and I.9.3. Now, we are to prove the third property. If $\mathcal{B}_1 \underset{\rightarrow}{\sqsubseteq} \mathcal{B}_2$ and $\mathcal{B}_2 \underset{\rightarrow}{\sqsubseteq} \mathcal{B}_3$ then, according to Definition I.9.3, the base changeable sets $\mathcal{B}_1$ are $\mathcal{B}_3$ chronologically affined. Moreover, by Definition I.9.3, $\mathbb{L}d(\mathcal{B}_1) \subseteq \mathbb{L}d(\mathcal{B}_2) \subseteq \mathbb{L}d(\mathcal{B}_3)$. Thus, $\mathcal{B}_1 \underset{\rightarrow}{\sqsubseteq} \mathcal{B}_3$. □

**Definition I.9.4.** *Indexed family* $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ $(\mathcal{A} \neq \emptyset)$ *of base changeable sets will be named* ***chronologically affined*** *if and only if any two base changeable sets* $\mathcal{B}_\alpha$,$\mathcal{B}_\beta$ *(where* $\alpha, \beta \in \mathcal{A}$*) are chronologically affined.*

**Definition I.9.5.** *Let* $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ $(\mathcal{A} \neq \emptyset)$ *be any indexed family of of base changeable sets. Base changeable set* $\mathcal{B}$ *will be named by* ***evolutional union*** *of the family* $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ *if and only if:*

**(EU$_1$)** $\mathcal{B}_\alpha \underset{\rightarrow}{\subseteq} \mathcal{B}$ *for an arbitrary* $\alpha \in \mathcal{A}$.

**(EU$_2$)** *If* $\mathcal{B}_\alpha \underset{\rightarrow}{\subseteq} \mathcal{B}'$ *for any* $\alpha \in \mathcal{A}$*, then* $\mathcal{B} \underset{\rightarrow}{\subseteq} \mathcal{B}'$.

**Assertion I.9.8.** *Any indexed family* $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ $(\mathcal{A} \neq \emptyset)$ *of base changeable sets may have no more than one evolutional union.*

*Proof.* Indeed, let $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ be the evolutional union of the family $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ of base changeable sets. Then, by Definition I.9.5, we have $\mathcal{B} \underset{\rightarrow}{\subseteq} \widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}} \underset{\rightarrow}{\subseteq} \mathcal{B}$. Thus, in accordance with Assertion I.9.3, we receive $\mathcal{B} = \widetilde{\mathcal{B}}$. □

Taking into account Assertion I.9.8 (about the uniqueness of evolutional union), we will denote the evolutional union $\mathcal{B}$ of the family $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ of base changeable sets by the following way:

$$\mathcal{B} = \overset{\leftarrow}{\bigcup_{\alpha\in\mathcal{A}}} \mathcal{B}_\alpha.$$

In particular, in the case $\mathcal{A} = \{1, ...n\}$ $(n \in \mathbb{N})$, we use the following denotation:

$$\mathcal{B}_1 \overset{\leftarrow}{\cup} \cdots \overset{\leftarrow}{\cup} \mathcal{B}_n := \overset{\leftarrow}{\bigcup_{k=1}^{n}} \mathcal{B}_k := \overset{\leftarrow}{\bigcup_{\alpha\in\mathcal{A}}} \mathcal{B}_\alpha.$$

*Remark* I.9.1. From the definitions I.9.5 and I.9.2 it follows, that, in the case, where $\mathcal{B} = \overset{\leftarrow}{\bigcup_{\alpha\in\mathcal{A}}} \mathcal{B}_\alpha$, the family $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ of base changeable sets must be chronologically affined, moreover $\mathbb{T}\mathbf{m}(\mathcal{B}) = \mathbb{T}\mathbf{m}(\mathcal{B}_\alpha)$ $(\forall \alpha \in \mathcal{A})$.

Let $\mathbb{T} = (\mathbf{T}, \leq)$ be any be any linearly ordered set and $\mathcal{A}$ be any non-empty family of indexes. Suppose, that for any index $\alpha \in \mathcal{A}$ the system of abstract trajectories $\mathcal{R}_\alpha$ from $\mathbb{T}$ to $M_\alpha$ is defined. In this case we name the family $(\mathcal{R}_\alpha)_{\alpha\in\mathcal{A}}$ of systems of abstract trajectories as $\mathbb{T}$-***chronologically affined***. Then, the set $\bigcup_{\alpha\in\mathcal{A}} \mathcal{R}_\alpha$ is the system of abstract trajectories from $\mathbb{T}$ to $\bigcup_{\alpha\in\mathcal{A}} M_\alpha$. Hence, by Theorem I.6.1, the base changeable set $\mathcal{A}t\left(\mathbb{T}, \bigcup_{\alpha\in\mathcal{A}} \mathcal{R}_\alpha\right)$ must exist.

Further, according to this Theorem, we have $\mathbb{T}\mathbf{m}\left(\mathcal{A}t\left(\mathbb{T},\mathcal{R}_\alpha\right)\right)=\mathbb{T}$ (for an arbitrary $\alpha\in\mathcal{A}$). Therefore, by definitions I.9.1 and I.9.4, the indexed family $\left(\mathcal{A}t\left(\mathbb{T},\mathcal{R}_\alpha\right)\right)_{\alpha\in\mathcal{A}}$ of base changeable sets is chronologically affined.

**Assertion I.9.9.** *Let $\left(\mathcal{R}_\alpha\right)_{\alpha\in\mathcal{A}}$ be $\mathbb{T}$-chronologically affined family of systems of abstract trajectories. Then, there exist the evolutional union $\overleftarrow{\bigcup}_{\alpha\in\mathcal{A}}\mathcal{A}t\left(\mathbb{T},\mathcal{R}_\alpha\right)$, and besides:*

$$\overleftarrow{\bigcup_{\alpha\in\mathcal{A}}}\mathcal{A}t\left(\mathbb{T},\mathcal{R}_\alpha\right)=\mathcal{A}t\left(\mathbb{T},\bigcup_{\alpha\in\mathcal{A}}\mathcal{R}_\alpha\right)$$

*Proof.* Let $\mathcal{R}_\alpha$ be a system of abstract trajectories from $\mathbb{T}$ to $M_\alpha$ for any index $\alpha\in\mathcal{A}$. Denote:

$$\mathcal{B}_\alpha:=\mathcal{A}t\left(\mathbb{T},\mathcal{R}_\alpha\right)\quad(\alpha\in\mathcal{A}),\qquad\mathcal{B}:=\mathcal{A}t\left(\mathbb{T},\bigcup_{\alpha\in\mathcal{A}}\mathcal{R}_\alpha\right).$$

**a)** Since $\mathcal{R}_\alpha\subseteq\bigcup_{\beta\in\mathcal{A}}\mathcal{R}_\beta$ (for an arbitrary $\alpha\in\mathcal{A}$), then, according to Assertion I.9.5, we have:

$$\mathcal{B}_\alpha\underset{\rightarrow}{\subseteq}\mathcal{B}\quad(\forall\alpha\in\mathcal{A}).$$

**b)** Suppose, that $\mathcal{B}_\alpha\underset{\rightarrow}{\subseteq}\mathcal{B}'$ $(\forall\alpha\in\mathcal{A})$. Then, using assertions I.7.6 and I.9.6, for any index $\alpha\in\mathcal{A}$ we obtain:

$$\mathcal{R}_\alpha\subseteq\mathbb{L}l\left(\mathcal{A}t\left(\mathbb{T},\mathcal{R}_\alpha\right)\right)=\mathbb{L}l\left(\mathcal{B}_\alpha\right)\subseteq\mathbb{L}l\left(\mathcal{B}'\right).$$

Consequently, $\bigcup_{\alpha\in\mathcal{A}}\mathcal{R}_\alpha\subseteq\mathbb{L}l\left(\mathcal{B}'\right)$. Hence, in accordance with Assertion I.9.4, we receive, $\mathcal{B}=\mathcal{A}t\left(\mathbb{T},\bigcup_{\alpha\in\mathcal{A}}\mathcal{R}_\alpha\right)\underset{\rightarrow}{\subseteq}\mathcal{B}'$.

Now, the equality $\mathcal{B}=\overleftarrow{\bigcup\limits_{\alpha\in\mathcal{A}}}\mathcal{B}_\alpha$ follows from the items a) and b), by means of Definition I.9.5. □

**Corollary I.9.1.** *For any chronologically affined family of base changeable sets $\left(\mathcal{B}_\alpha\right)_{\alpha\in\mathcal{A}}$ $(\mathcal{A}\neq\emptyset)$ the evolutional union $\overleftarrow{\bigcup}_{\alpha\in\mathcal{A}}\mathcal{B}_\alpha$ exists, moreover:*

$$\overleftarrow{\bigcup_{\alpha\in\mathcal{A}}}\mathcal{B}_\alpha=\mathcal{A}t\left(\mathbb{T},\bigcup_{\alpha\in\mathcal{A}}\mathbb{L}d\left(\mathcal{B}_\alpha\right)\right),$$

*where $\mathbb{T}=\mathbb{T}\mathbf{m}\left(\mathcal{B}_\alpha\right)$ $(\alpha\in\mathcal{A})$.*

*Proof.* Chose any fixed index $\alpha_0\in\mathcal{A}$. Denote, $\mathbb{T}:=\mathbb{T}\mathbf{m}\left(\mathcal{B}_{\alpha_0}\right)$. Since $\left(\mathcal{B}_\alpha\right)_{\alpha\in\mathcal{A}}$ is chronologically affined family of base changeable sets, then $\mathbb{T}\mathbf{m}\left(\mathcal{B}_\alpha\right)=\mathbb{T}$ (for an arbitrary $\alpha\in\mathcal{A}$). According to Assertion I.7.5 (item 4)), for an arbitrary $\alpha\in\mathcal{A}$ the set $\mathbb{L}d\left(\mathcal{B}_\alpha\right)$ is a system of abstract trajectories from $\mathbb{T}=\mathbb{T}\mathbf{m}\left(\mathcal{B}_\alpha\right)$ to $\mathfrak{Bs}\left(\mathcal{B}_\alpha\right)$. Therefore, $\left(\mathbb{L}d\left(\mathcal{B}_\alpha\right)\right)_{\alpha\in\mathcal{A}}$ is $\mathbb{T}$-chronologically affined family of systems of abstract trajectories. And, according to Theorem I.7.1, $\mathcal{B}_\alpha=\mathcal{A}t\left(\mathbb{T}\mathbf{m}\left(\mathcal{B}_\alpha\right),\mathbb{L}d\left(\mathcal{B}_\alpha\right)\right)=\mathcal{A}t\left(\mathbb{T},\mathbb{L}d\left(\mathcal{B}_\alpha\right)\right)$ (for any $\alpha\in\mathcal{A}$). Hence, by Assertion I.9.9, the evolutional union $\overleftarrow{\bigcup}_{\alpha\in\mathcal{A}}\mathcal{B}_\alpha=\overleftarrow{\bigcup}_{\alpha\in\mathcal{A}}\mathcal{A}t\left(\mathbb{T},\mathbb{L}d\left(\mathcal{B}_\alpha\right)\right)$ must exist, moreover:

$$\overleftarrow{\bigcup_{\alpha\in\mathcal{A}}}\mathcal{B}_\alpha=\overleftarrow{\bigcup_{\alpha\in\mathcal{A}}}\mathcal{A}t\left(\mathbb{T},\mathbb{L}d\left(\mathcal{B}_\alpha\right)\right)=\mathcal{A}t\left(\mathbb{T},\bigcup_{\alpha\in\mathcal{A}}\mathbb{L}d\left(\mathcal{B}_\alpha\right)\right).$$

□

**Corollary I.9.2.** *If $\mathcal{B}=\overleftarrow{\bigcup}_{\alpha\in\mathcal{A}}\mathcal{B}_\alpha$ then:*

$$\mathbb{B}\mathfrak{s}(\mathcal{B})=\bigcup_{\alpha\in\mathcal{A}}\mathbb{B}\mathfrak{s}\left(\mathcal{B}_\alpha\right);\qquad\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}}=\bigcup_{\alpha\in\mathcal{A}}\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_\alpha}{\leftarrow}}.$$

*Proof.* The desired result follows from Corollary I.9.1 and Theorem I.6.1. Note, that for proving the equality $\mathbb{Bs}(\mathcal{B}) = \bigcup_{\alpha\in\mathcal{A}} \mathbb{Bs}(\mathcal{B}_\alpha)$ it is useful Formula (I.30), which follows from Theorem I.6.1 and holds for any base changeable set. □

**Denotation I.9.1.** *In this paper* **card** $(\mathcal{A})$ *means the* ***cardinality*** *of the set* $\mathcal{A}$.

**Assertion I.9.10** (on properties of evolutional union)**.** *Let* $(\mathcal{B}_i)_{i\in\{1,2,3\}}$ *and* $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ $(\mathcal{A}\neq\emptyset)$ *be two chronologically affined families of base changeable sets. The operation of evolutional union possesses the following properties:*

1. $\mathcal{B}_1 \overleftarrow{\cup} \mathcal{B}_2 = \mathcal{B}_2 \overleftarrow{\cup} \mathcal{B}_1$.
2. *If* $\mathcal{A} = \{\alpha_0\}$, *then* $\overleftarrow{\bigcup}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha = \mathcal{B}_{\alpha_0}$.
3. *If the set of indexes* $\mathcal{A}$ *is divided into disjoint union of non-empty index sets* $\mathcal{A}_\gamma$ $(\gamma\in\mathcal{G})$, *(that is* $\mathcal{A} = \bigsqcup_{\gamma\in\mathcal{G}} \mathcal{A}_\gamma$*) then*
$$\overleftarrow{\bigcup_{\alpha\in\mathcal{A}}} \mathcal{B}_\alpha = \overleftarrow{\bigcup_{\gamma\in\mathcal{G}}} \left( \overleftarrow{\bigcup_{\alpha\in\mathcal{A}_\gamma}} \mathcal{B}_\alpha \right).$$
*In particular, in the case* **card** $(\mathcal{A}) \geq 2$, *for an arbitrary* $\alpha_0\in\mathcal{A}$ *we have the following equality:*
$$\overleftarrow{\bigcup_{\alpha\in\mathcal{A}}} \mathcal{B}_\alpha = \mathcal{B}_{\alpha_0} \overleftarrow{\cup} \left( \overleftarrow{\bigcup_{\alpha\in\mathcal{A}\setminus\{\alpha_0\}}} \mathcal{B}_\alpha \right), \qquad \text{(I.36)}$$
*and in the case* $\mathcal{A} = \{1,2,3\}$ *we obtain the equality:*
$$\left(\mathcal{B}_1 \overleftarrow{\cup} \mathcal{B}_2\right) \overleftarrow{\cup} \mathcal{B}_3 = \mathcal{B}_1 \overleftarrow{\cup} \left(\mathcal{B}_2 \overleftarrow{\cup} \mathcal{B}_3\right) = \mathcal{B}_1 \overleftarrow{\cup} \mathcal{B}_2 \overleftarrow{\cup} \mathcal{B}_3. \qquad \text{(I.37)}$$
4. *If for some base changeable set* $\mathcal{B}'$, *we have* $\mathcal{B}_\alpha \underset{\to}{\subseteq} \mathcal{B}'$ *(for any* $\alpha\in\mathcal{A}$*), then* $\overleftarrow{\bigcup}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha \underset{\to}{\subseteq} \mathcal{B}'$.
5. *If for some* $\alpha_0\in\mathcal{A}$ *the inclusion* $\mathcal{B}_\alpha \underset{\to}{\subseteq} \mathcal{B}_{\alpha_0}$ *is performed for all* $\alpha\in\mathcal{A}$, *then we have* $\overleftarrow{\bigcup}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha = \mathcal{B}_{\alpha_0}$. *In particular* $\mathcal{B} \overleftarrow{\cup} \mathcal{B} = \mathcal{B}$ *for any base changeable set* $\mathcal{B}$.

*Proof.* **1.** By definition we have, $\mathcal{B}_1 \overleftarrow{\cup} \mathcal{B}_2 = \overleftarrow{\bigcup}_{i\in\{1,2\}} \mathcal{B}_i = \mathcal{B}_2 \overleftarrow{\cup} \mathcal{B}_1$.

**2.** The second property easily follows from Definition I.9.5.

**3.** Consider any fixed index $\alpha_1\in\mathcal{A}$. Denote, $\mathbb{T} := \mathbb{T}\mathbf{m}(\mathcal{B}_{\alpha_1})$. Since $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ is chronologically affined family of base changeable sets, then we have, $\mathbb{T}\mathbf{m}(\mathcal{B}_\alpha) = \mathbb{T}$ $(\forall\alpha\in\mathcal{A})$. According to Remark I.9.1, the evolutional unions $\overleftarrow{\bigcup}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha$, $\overleftarrow{\bigcup}_{\alpha\in\mathcal{A}_\gamma} \mathcal{B}_\alpha$ $(\forall\gamma\in\mathcal{G})$ and $\overleftarrow{\bigcup_{\gamma\in\mathcal{G}}} \left( \overleftarrow{\bigcup_{\alpha\in\mathcal{A}_\gamma}} \mathcal{B}_\alpha \right)$ are correctly defined. Then, applying Corollary I.9.1, Assertion I.9.9 and Theorem I.7.1, we receive:

$$\overleftarrow{\bigcup_{\alpha\in\mathcal{A}}} \mathcal{B}_\alpha = \mathcal{A}t\left(\mathbb{T}, \bigcup_{\alpha\in\mathcal{A}} \mathbb{L}d(\mathcal{B}_\alpha)\right) = \mathcal{A}t\left(\mathbb{T}, \bigcup_{\gamma\in\mathcal{G}} \bigcup_{\alpha\in\mathcal{A}_\gamma} \mathbb{L}d(\mathcal{B}_\alpha)\right) =$$
$$= \overleftarrow{\bigcup_{\gamma\in\mathcal{G}}} \mathcal{A}t\left(\mathbb{T}, \bigcup_{\alpha\in\mathcal{A}_\gamma} \mathbb{L}d(\mathcal{B}_\alpha)\right) = \overleftarrow{\bigcup_{\gamma\in\mathcal{G}}} \left( \overleftarrow{\bigcup_{\alpha\in\mathcal{A}_\gamma}} \mathcal{A}t\left(\mathbb{T}, \mathbb{L}d(\mathcal{B}_\alpha)\right) \right) =$$

$$= \overleftarrow{\bigcup_{\gamma\in\mathcal{G}}} \left( \overleftarrow{\bigcup_{\alpha\in\mathcal{A}_\gamma}} \mathcal{B}_\alpha \right).$$

In particular, in the case $\mathbf{card}\,(\mathcal{A}) \geq 2$, using item 2 of this Assertion, for any fixed index $\alpha_0 \in \mathcal{A}$, we obtain:

$$\overleftarrow{\bigcup_{\alpha\in\mathcal{A}}} \mathcal{B}_\alpha = \overleftarrow{\bigcup_{\alpha\in\{\alpha_0\}\sqcup(\mathcal{A}\setminus\{\alpha_0\})}} \mathcal{B}_\alpha = \left( \overleftarrow{\bigcup_{\alpha\in\{\alpha_0\}}} \mathcal{B}_\alpha \right) \overleftarrow{\cup} \overleftarrow{\bigcup_{\alpha\in(\mathcal{A}\setminus\{\alpha_0\})}} \mathcal{B}_\alpha$$
$$= \mathcal{B}_{\alpha_0} \overleftarrow{\cup} \left( \overleftarrow{\bigcup_{\alpha\in\mathcal{A}\setminus\{\alpha_0\}}} \mathcal{B}_\alpha \right),$$

that is, we have got the equality (I.36). The equality (I.37) follows from the equality (I.36) in particular case $\mathcal{A} = \{1,2,3\}$, where the commutativity of the evolutional union operation is taken into account.

**4.** The fourth item of this Assertion readily follows from Definition I.9.5.

**5.** Let $\mathcal{B}_\alpha \underset{\to}{\subseteq} \mathcal{B}_{\alpha_0}$ $(\forall\, \alpha \in \mathcal{A})$ for some fixed $\alpha_0 \in \mathcal{A}$. Then, in accordance with the previous item of this Assertion, we get, $\overleftarrow{\bigcup}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha \underset{\to}{\subseteq} \mathcal{B}_{\alpha_0}$. From the other hand, according to Definition I.9.5, we have, $\mathcal{B}_{\alpha_0} \underset{\to}{\subseteq} \overleftarrow{\bigcup}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha$. Thus, by Assertion I.9.3 (item 2), we obtain, $\overleftarrow{\bigcup}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha = \mathcal{B}_{\alpha_0}$. □

Let $(\mathcal{B}_{\alpha\beta})_{\alpha\in\mathbf{A},\,\beta\in\mathbf{B}}$ $(\mathbf{A},\mathbf{B} \neq \emptyset)$ be any two-parametric indexed family of base changeable sets. The family $(\mathcal{B}_{\alpha\beta})_{\alpha\in\mathbf{A},\,\beta\in\mathbf{B}}$ will be named as **chronologically affined**, if and only if base changeable sets $\mathcal{B}_{\alpha_1\beta_1}$,$\mathcal{B}_{\alpha_2\beta_2}$ are chronologically affined for arbitrary indexes $\alpha_1, \alpha_2 \in \mathbf{A}$, $\beta_1, \beta_2 \in \mathbf{B}$. Let $(\mathcal{B}_{\alpha\beta})_{\alpha\in\mathbf{A},\,\beta\in\mathbf{B}}$ $(\mathbf{A},\mathbf{B} \neq \emptyset)$ be chronologically affined family of base changeable sets. Then for arbitrary fixed $\alpha_0 \in \mathbf{A}$, $\beta_0 \in \mathbf{B}$, the one-parametric families of base changeable sets $(\mathcal{B}_{\alpha_0\beta})_{\beta\in\mathbf{B}}$ and $(\mathcal{B}_{\alpha\beta_0})_{\alpha\in\mathbf{A}}$ are chronologically affined. Hence according to Corollary I.9.1, the evolutional unions $\mathcal{U}_{\alpha_0,*} = \overleftarrow{\bigcup}_{\beta\in\mathbf{B}} \mathcal{B}_{\alpha_0\beta}$ and $\mathcal{U}_{*,\beta_0} = \overleftarrow{\bigcup}_{\alpha\in\mathbf{A}} \mathcal{B}_{\alpha\beta_0}$ must exist. Besides this, according to Remark I.9.1, the base changeable sets $\mathcal{U}_{\alpha_0,*}$ and $\mathcal{U}_{*,\beta_0}$ are chronologically affined with the base changeable set $\mathcal{B}_{\alpha_0,\beta_0}$. Hence, taking into account the chronological affinity of the family $(\mathcal{B}_{\alpha\beta})_{\alpha\in\mathbf{A},\beta\in\mathbf{B}}$, we see, that the families of base changeable sets $(\mathcal{U}_{\alpha,*})_{\alpha\in\mathbf{A}}$ and $(\mathcal{U}_{*,\beta})_{\beta\in\mathbf{B}}$ are chronologically affined also. This means, that we can define the double evolutional unions $\overleftarrow{\bigcup}_{\alpha\in\mathbf{A}} \mathcal{U}_{\alpha,*} = \overleftarrow{\bigcup}_{\alpha\in\mathbf{A}} \overleftarrow{\bigcup}_{\beta\in\mathbf{B}} \mathcal{B}_{\alpha\beta}$ and $\overleftarrow{\bigcup}_{\beta\in\mathbf{B}} \mathcal{U}_{*,\beta} = \overleftarrow{\bigcup}_{\beta\in\mathbf{B}} \overleftarrow{\bigcup}_{\alpha\in\mathbf{A}} \mathcal{B}_{\alpha\beta}$. Now, we aim to prove, that double evolutional union does not depend on the order of application of evolutional union operations. Indeed, let us consider any fixed indexes $\alpha_0 \in \mathbf{A}$, $\beta_0 \in \mathbf{B}$. Denote, $\mathbb{T} := \mathbb{T}\mathbf{m}\,(\mathcal{B}_{\alpha_0\beta_0})$. Then, applying Theorem I.7.1 and Assertion I.9.9, we receive:

$$\overleftarrow{\bigcup_{\alpha\in\mathbf{A}}} \overleftarrow{\bigcup_{\beta\in\mathbf{B}}} \mathcal{B}_{\alpha\beta} = \overleftarrow{\bigcup_{\alpha\in\mathbf{A}}} \overleftarrow{\bigcup_{\beta\in\mathbf{B}}} \mathcal{A}t\,(\mathbb{T}, \mathbb{L}d\,(\mathcal{B}_{\alpha\beta})) = \mathcal{A}t \left( \mathbb{T}, \bigcup_{\alpha\in\mathbf{A}} \bigcup_{\beta\in\mathbf{B}} \mathbb{L}d\,(\mathcal{B}_{\alpha\beta}) \right) =$$
$$= \mathcal{A}t \left( \mathbb{T}, \bigcup_{\beta\in\mathbf{B}} \bigcup_{\alpha\in\mathbf{A}} \mathbb{L}d\,(\mathcal{B}_{\alpha\beta}) \right) = \overleftarrow{\bigcup_{\beta\in\mathbf{B}}} \overleftarrow{\bigcup_{\alpha\in\mathbf{A}}} \mathcal{A}t\,(\mathbb{T}, \mathbb{L}d\,(\mathcal{B}_{\alpha\beta})) =$$
$$= \overleftarrow{\bigcup_{\beta\in\mathbf{B}}} \overleftarrow{\bigcup_{\alpha\in\mathbf{A}}} \mathcal{B}_{\alpha\beta}.$$

Taking into account the last fact, we will use the following denotations for double evolutional union:

$$\overleftarrow{\bigcup_{\alpha\in\mathbf{A},\,\beta\in\mathbf{B}}} \mathcal{B}_{\alpha\beta} := \overleftarrow{\bigcup_{\alpha\in\mathbf{A}}} \overleftarrow{\bigcup_{\beta\in\mathbf{B}}} \mathcal{B}_{\alpha\beta} = \overleftarrow{\bigcup_{\beta\in\mathbf{B}}} \overleftarrow{\bigcup_{\alpha\in\mathbf{A}}} \mathcal{B}_{\alpha\beta}.$$

By a similar way the notion of chronological affinity can be introduced for many-parametric indexed family of base changeable sets $(\mathcal{B}_{\alpha_1\ldots\alpha_n})_{\alpha_1\in\mathbf{A}_1,\ldots,\alpha_n\in\mathbf{A}_n}$, where $n\in\mathbb{N}$, $\mathbf{A}_i\neq\emptyset$, $i\in\{1,\cdots,n\}$. The base changeable set:

$$\overleftarrow{\bigcup_{\alpha_1\in\mathbf{A}_1,\ldots,\alpha_n\in\mathbf{A}_n}}\mathcal{B}_{\alpha_1\ldots\alpha_n}=\overleftarrow{\bigcup_{\alpha_1\in\mathbf{A}_1}}\cdots\overleftarrow{\bigcup_{\alpha_n\in\mathbf{A}_n}}\mathcal{B}_{\alpha_1\ldots\alpha_n}$$

will be named by evolutional union of the family $(\mathcal{B}_{\alpha_1\ldots\alpha_n})_{\alpha_1\in\mathbf{A}_1,\ldots,\alpha_n\in\mathbf{A}_n}$. Similarly to the case of two-parametric family it can be proved, that the result in the right-hand side of the last equality does not depend of the order of placing of evolutional union signs.

**Definition I.9.6.** *Let, $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ $(\mathcal{A}\neq\emptyset)$ be any chronologically affined family of base changeable sets. Base changeable set $\mathcal{B}$ will be named by* ***super-evolutional union*** *of the family $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$, if and only if the following conditions are performed:*

**(sEU$_1$)** *$\mathcal{B}_\alpha\underset{\rightarrow}{\sqsubset}\mathcal{B}$ $(\forall\alpha\in\mathcal{A})$.*

**(sEU$_2$)** *If $\mathcal{B}_\alpha\underset{\rightarrow}{\sqsubset}\mathcal{B}'$ $(\forall\alpha\in\mathcal{A})$, then $\mathcal{B}\underset{\rightarrow}{\subseteq}\mathcal{B}'$.*

The next Corollary follows from Definition I.9.6 and Assertion I.9.3 (item 2).

**Corollary I.9.3.** *Any indexed family $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ $(\mathcal{A}\neq\emptyset)$ of base changeable sets may have no more than one super-evolutional union.*

Super-evolutional union $\mathcal{B}$ of the family $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ of base changeable sets will be denoted by the following way:

$$\mathcal{B}=\overleftarrow{\bigvee_{\alpha\in\mathcal{A}}}\mathcal{B}_\alpha.$$

In particular, in the case $\mathcal{A}=\{1,\ldots n\}$ $(n\in\mathbb{N})$, we use the denotation $\overleftarrow{\bigvee}_{k=1}^{n}\mathcal{B}_k$, or or, simply, $\mathcal{B}_1\overleftarrow{\vee}\cdots\overleftarrow{\vee}\mathcal{B}_n$:

$$\overleftarrow{\bigvee_{k=1}^{n}}\mathcal{B}_k:=\mathcal{B}_1\overleftarrow{\vee}\cdots\overleftarrow{\vee}\mathcal{B}_n:=\overleftarrow{\bigvee_{\alpha\in\mathcal{A}}}\mathcal{B}_\alpha.$$

The next assertion may be interpreted as some analog of the theorem, confirming, that any bounded set of real numbers always have the least upper bound.

**Assertion I.9.11.** *Suppose, that for chronologically affined family of base changeable sets $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ $(\mathcal{A}\neq\emptyset)$ there exists the base changeable set $\widetilde{\mathcal{B}}$, such, that for any index $\alpha\in\mathcal{A}$ it is true the inclusion $\mathcal{B}_\alpha\underset{\rightarrow}{\sqsubset}\widetilde{\mathcal{B}}$. Then the super-evolutional union $\overleftarrow{\bigvee}_{\alpha\in\mathcal{A}}\mathcal{B}_\alpha$ exists, moreover, the following equality is true:*

$$\overleftarrow{\bigvee_{\alpha\in\mathcal{A}}}\mathcal{B}_\alpha=\overleftarrow{\bigcup_{\alpha\in\mathcal{A}}}\mathcal{B}_\alpha.$$

*Proof.* Denote:

$$\mathcal{B}:=\overleftarrow{\bigcup_{\alpha\in\mathcal{A}}}\mathcal{B}_\alpha.$$

Now, we aim to prove, that:

$$\forall\alpha\in\mathcal{A}\quad(\mathbb{L}d\,(\mathcal{B}_\alpha)\subseteq\mathbb{L}d(\mathcal{B}))\,. \tag{I.38}$$

Let us assume the contrary. Then there exist index $\beta\in\mathcal{A}$ and fate line $L\in\mathbb{L}d\,(\mathcal{B}_\beta)$ such, that $L\notin\mathbb{L}d(\mathcal{B})$.

By Definition I.9.5, we have $\mathcal{B}_\beta \underset{\rightarrow}{\subseteq} \overleftarrow{\bigcup}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha = \mathcal{B}$. Hence, according to Assertion I.9.6, we get $L \in \mathbb{L}l(\mathcal{B})$. Therefore, since $L \notin \mathbb{L}d(\mathcal{B})$, the chain $L_1 \in \mathbb{L}l(\mathcal{B})$ must exist such, that $L \subset L_1$. Since $\mathcal{B}_\alpha \underset{\rightarrow}{\sqsubseteq} \widetilde{\mathcal{B}}$ $(\forall\, \alpha \in \mathcal{A})$, then, by Assertion I.9.2, for an arbitrary $\alpha \in \mathcal{A}$ we have $\mathcal{B}_\alpha \underset{\rightarrow}{\subseteq} \widetilde{\mathcal{B}}$. Hence, according to Assertion I.9.10 (item 4), we get $\mathcal{B} = \overleftarrow{\bigcup}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha \underset{\rightarrow}{\subseteq} \widetilde{\mathcal{B}}$. Taking into account, that $\mathcal{B} \underset{\rightarrow}{\subseteq} \widetilde{\mathcal{B}}$ and $L_1 \in \mathbb{L}l(\mathcal{B})$, applying Assertion I.9.6, we obtain $L_1 \in \mathbb{L}l\left(\widetilde{\mathcal{B}}\right)$.

Thus, we have proved, that the chain $L_1 \in \mathbb{L}l\left(\widetilde{\mathcal{B}}\right)$ exists such, that $L \subset L_1$. This means, that $L \notin \mathbb{L}d\left(\widetilde{\mathcal{B}}\right)$. From the other hand, since $\mathcal{B}_\beta \underset{\rightarrow}{\sqsubseteq} \widetilde{\mathcal{B}}$ and $L \in \mathbb{L}d\left(\mathcal{B}_\beta\right)$, then, by Definition I.9.3, the correlation $L \in \mathbb{L}d\left(\widetilde{\mathcal{B}}\right)$ must be performed.

The obtained contradiction proves the correlation (I.38). By Definition I.9.3, from the correlation (I.38) it follows, that

$$\forall \alpha \in \mathcal{A} \quad \left(\mathcal{B}_\alpha \underset{\rightarrow}{\sqsubseteq} \mathcal{B}\right).$$

Hence, the base changeable set $\mathcal{B}$ satisfies Condition ($\mathbf{sEU}_1$) of Definition I.9.6.

Therefore, it remains to prove, that Condition ($\mathbf{sEU}_2$) of Definition I.9.6 also is satisfied for $\mathcal{B}$. Suppose, that for some base changeable set $\mathcal{B}'$ the correlation $\mathcal{B}_\alpha \underset{\rightarrow}{\sqsubseteq} \mathcal{B}'$ is true for all $\alpha \in \mathcal{A}$. Then, according to Assertion I.9.2, we have, $\mathcal{B}_\alpha \underset{\rightarrow}{\subseteq} \mathcal{B}'$ $(\forall \alpha \in \mathcal{A})$. Hence, by Assertion I.9.10 (item 4), we obtain, $\mathcal{B} = \overleftarrow{\bigcup\limits_{\alpha\in\mathcal{A}}} \mathcal{B}_\alpha \underset{\rightarrow}{\subseteq} \mathcal{B}'$. □

**Corollary I.9.4.** *If the super-evolutional union* $\overleftarrow{\bigvee}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha$ *exists, then it coincides with corresponding evolutional union, that is:*

$$\overleftarrow{\bigvee_{\alpha\in\mathcal{A}}} \mathcal{B}_\alpha = \overleftarrow{\bigcup_{\alpha\in\mathcal{A}}} \mathcal{B}_\alpha.$$

*Proof.* Indeed, suppose, that super-evolutional union $\overleftarrow{\bigvee}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha$ exists for the chronologically affined family of base changeable sets $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ $(\mathcal{A} \neq \emptyset)$. Then the base changeable set $\widetilde{\mathcal{B}} = \overleftarrow{\bigvee}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha$ satisfies the conditions of Assertion I.9.11. □

In the following example it will be shown, that, unlike evolutional union, the super-evolutional union of chronologically affined family of base changeable sets sometimes may not exist.

***Example*** I.9.2. Let the systems of abstract trajectories $\mathcal{R}_0 = \{r_0\}$, $\mathcal{R}_1 = \{r_1\}$ be the same as in Example I.9.1. In Example I.9.1 it had been shown, that, for base changeable sets

$$\mathcal{B}_0 := \mathcal{A}t\left(\mathbb{R}_{ord}, \mathcal{R}_0\right); \qquad \mathcal{B}_1 := \mathcal{A}t\left(\mathbb{R}_{ord}, \mathcal{R}_1\right)$$

the evolutional inclusion $\mathcal{B}_0 \underset{\rightarrow}{\subseteq} \mathcal{B}_1$ holds. Hence, by Assertion I.9.10, item 5, $\mathcal{B}_0 \overleftarrow{\cup} \mathcal{B}_1 = \mathcal{B}_1$. From the other hand, the evolutional union $\mathcal{B}_0 \overleftarrow{\vee} \mathcal{B}_1$ doesn't exist. Indeed, assume the contrary. Then, according to Corollary I.9.4, we have, $\mathcal{B}_0 \overleftarrow{\vee} \mathcal{B}_1 = \mathcal{B}_0 \overleftarrow{\cup} \mathcal{B}_1 = \mathcal{B}_1$. But, in Example I.9.1 it had been shown, that $\mathcal{B}_0 \underset{\rightarrow}{\not\sqsubseteq} \mathcal{B}_1$. Thus, by Definition I.9.6, $\mathcal{B}_1$ can not be super-evolutional union of $\mathcal{B}_0$ and $\mathcal{B}_1$. The obtained contradiction proves, that super-evolutional union $\mathcal{B}_0 \overleftarrow{\vee} \mathcal{B}_1$ does not exist.

**Definition I.9.7.** *An chronologically affined family of base changeable sets $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ ($\mathcal{A}\neq\emptyset$) will be named as* ***evolutionarily saturated****, if and only if the following inclusion holds:*

$$\bigcup_{\alpha\in\mathcal{A}} \mathbb{L}d\left(\mathcal{B}_\alpha\right) \subseteq \mathbb{L}d\left(\overleftarrow{\bigcup_{\alpha\in\mathcal{A}}} \mathcal{B}_\alpha\right). \tag{I.39}$$

*Remark* I.9.2. From definitions I.9.7 and I.9.3 it follows, that chronologically affined family of base changeable sets $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ ($\mathcal{A}\neq\emptyset$) is evolutionarily saturated if and only if $\mathcal{B}_\beta \underset{\rightarrow}{\sqsubseteq} \overleftarrow{\bigcup_{\alpha\in\mathcal{A}}} \mathcal{B}_\alpha$ ($\forall\beta\in\mathcal{A}$).

**Assertion I.9.12.** *The super-evolutional union $\overleftarrow{\bigvee}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha$ of chronologically affined family of base changeable sets $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ ($\mathcal{A}\neq\emptyset$) exists if and only if this family is evolutionarily saturated.*

*Proof.* Suppose, that the super-evolutional union $\mathcal{B} = \overleftarrow{\bigvee}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha$ exists. Then, by Definition I.9.6, $\mathcal{B}_\alpha \underset{\rightarrow}{\sqsubseteq} \mathcal{B}$ ($\forall\alpha\in\mathcal{A}$). Hence, by Definition I.9.3, we get $\mathbb{L}d\left(\mathcal{B}_\alpha\right) \subseteq \mathbb{L}d(\mathcal{B})$ ($\forall\alpha\in\mathcal{A}$), ie $\bigcup_{\alpha\in\mathcal{A}} \mathbb{L}d\left(\mathcal{B}_\alpha\right) \subseteq \mathbb{L}d(\mathcal{B})$. But, according to Corollary I.9.4, we have $\mathcal{B} = \overleftarrow{\bigvee}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha = \overleftarrow{\bigcup}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha$. Therefore, $\bigcup_{\alpha\in\mathcal{A}} \mathbb{L}d\left(\mathcal{B}_\alpha\right) \subseteq \mathbb{L}d\left(\overleftarrow{\bigcup}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha\right)$.

Inversely, suppose, that the chronologically affined family of base changeable sets $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ ($\mathcal{A}\neq\emptyset$) is evolutionarily saturated, that is the equality (I.39) holds. Denote, $\mathcal{B} = \overleftarrow{\bigcup}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha$. According to (I.39), we have, $\mathbb{L}d\left(\mathcal{B}_\alpha\right) \subseteq \mathbb{L}d(\mathcal{B})$ ($\forall\alpha\in\mathcal{A}$). Hence, by Definition I.9.3 we get, $\mathcal{B}_\alpha \underset{\rightarrow}{\sqsubseteq} \mathcal{B}$ ($\forall\alpha\in\mathcal{A}$). Therefore, in accordance with Assertion I.9.11, the super-evolutional union $\overleftarrow{\bigvee}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha$ exists. □

**Lemma I.9.1** (on properties of evolutional saturation)**.** *Let $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ ($\mathcal{A}\neq\emptyset$) be any chronologically affined indexed family of base changeable sets.*

1. *If there exists base changeable set $\widetilde{\mathcal{B}}$, such, that for any index $\alpha\in\mathcal{A}$ the inclusion $\mathcal{B}_\alpha \underset{\rightarrow}{\sqsubseteq} \widetilde{\mathcal{B}}$, is performed, then the family $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ is evolutionarily saturated.*
2. *If $\mathcal{B}_\alpha = \mathcal{B}$ ($\forall\alpha\in\mathcal{A}$), then the family $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ is evolutionarily saturated.*
3. *If $\mathbb{B}\mathfrak{s}\left(\mathcal{B}_\alpha\right) \cap \mathbb{B}\mathfrak{s}\left(\mathcal{B}_\beta\right) = \emptyset$ for $\mathcal{B}_\alpha \neq \mathcal{B}_\beta$, then the family $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ is evolutionarily saturated.*
4. *If the family $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ is evolutionarily saturated and $\mathcal{A}_1 \subseteq \mathcal{A}$, $\mathcal{A}_1 \neq \emptyset$, then the subfamily $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}_1}$ is evolutionarily saturated also.*

*Proof.* 1. Suppose, that $\mathcal{B}_\alpha \underset{\rightarrow}{\sqsubseteq} \widetilde{\mathcal{B}}$ ($\forall\alpha\in\mathcal{A}$). Then, according to Assertion I.9.11, the super-evolutional union $\overleftarrow{\bigvee}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha$ exists. Hence, in accordance with Assertion I.9.12, the family $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ is evolutionarily saturated.

2. If $\mathcal{B}_\alpha = \mathcal{B}$ ($\forall\alpha\in\mathcal{A}$), then, by Assertion I.9.7, we have $\mathcal{B}_\alpha \underset{\rightarrow}{\sqsubseteq} \mathcal{B}$ ($\forall\alpha\in\mathcal{A}$). Hence, according to the previous item, the family $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ is evolutionarily saturated.

3. Suppose, that $\mathbb{B}\mathfrak{s}\left(\mathcal{B}_\alpha\right) \cap \mathbb{B}\mathfrak{s}\left(\mathcal{B}_\beta\right) = \emptyset$ for $\mathcal{B}_\alpha \neq \mathcal{B}_\beta$. Denote, $\mathcal{B} := \overleftarrow{\bigcup}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha$. Let us chose any $L \in \bigcup_{\alpha\in\mathcal{A}} \mathbb{L}d\left(\mathcal{B}_\alpha\right)$. Then, there exists the index $\alpha_0\in\mathcal{A}$ such, that $L \in \mathbb{L}d\left(\mathcal{B}_{\alpha_0}\right)$. Since, by Definition I.9.5, $\mathcal{B}_{\alpha_0} \underset{\rightarrow}{\subseteq} \overleftarrow{\bigcup}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha = \mathcal{B}$, then, by Assertion I.9.6, we have $L \in \mathbb{L}l(\mathcal{B})$. Now, we aim to prove, that $L \in \mathbb{L}d(\mathcal{B})$. Suppose the contrary. Then there exists a chain $L_1 \in \mathbb{L}l(\mathcal{B})$ ($L_1 \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$) such, that $L \subset L_1$. Let us consider any elementary-time state $\omega \in L_1$. Since $L$ is a fate line of $\mathcal{B}_{\alpha_0}$, then, according to Assertion I.7.3 and Remark I.6.1, we have $L \neq \emptyset$. Hence at least one elementary-time state $\omega_0 \in L$ exists. Since $L \subset L_1$, then $\omega_0 \in L_1$. Since $\omega, \omega_0 \in L_1$, where $L_1$ is a chain of $\mathcal{B}$, then at least one of the conditions $\omega_0 \underset{\mathcal{B}}{\leftarrow} \omega$ or $\omega \underset{\mathcal{B}}{\leftarrow} \omega_0$

must be satisfied. Hence, by Corollary I.9.2, the index $\alpha_1 \in \mathcal{A}$ must exist such, that $\omega_0 \underset{\mathcal{B}_{\alpha_1}}{\leftarrow} \omega$ or $\omega \underset{\mathcal{B}_{\alpha_1}}{\leftarrow} \omega_0$. But, since the relation $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_{\alpha_1}}{\leftarrow}}$ is defined on the set $\mathbb{B}\mathfrak{s}(\mathcal{B}_{\alpha_1})$, the both correlations ($\omega_0 \underset{\mathcal{B}_{\alpha_1}}{\leftarrow} \omega$ or $\omega \underset{\mathcal{B}_{\alpha_1}}{\leftarrow} \omega_0$), lead to the correlation $\omega, \omega_0 \in \mathbb{B}\mathfrak{s}(\mathcal{B}_{\alpha_1})$. Hence, we have, $\omega_0 \in \mathbb{B}\mathfrak{s}(\mathcal{B}_{\alpha_0}) \cap \mathbb{B}\mathfrak{s}(\mathcal{B}_{\alpha_1})$. And, taking into account the fact, that $\mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha) \cap \mathbb{B}\mathfrak{s}(\mathcal{B}_\beta) = \emptyset$ for $\mathcal{B}_\alpha \neq \mathcal{B}_\beta$, we receive the equality $\mathcal{B}_{\alpha_0} = \mathcal{B}_{\alpha_1}$. Hence, $\omega, \omega_0 \in \mathbb{B}\mathfrak{s}(\mathcal{B}_{\alpha_0})$. Thus, any element $\omega \in L_1$ belongs to $\mathbb{B}\mathfrak{s}(\mathcal{B}_{\alpha_0})$. Consequently, $L_1 \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_{\alpha_0})$. Now, we are going to prove, that for arbitrary $\omega_1, \omega_2 \in L_1$, the correlation $\omega_2 \underset{\mathcal{B}}{\leftarrow} \omega_1$ holds if and only if $\omega_2 \underset{\mathcal{B}_{\alpha_0}}{\leftarrow} \omega_1$. If $\omega_1, \omega_2 \in L_1$ and $\omega_2 \underset{\mathcal{B}_{\alpha_0}}{\leftarrow} \omega_1$, then, by Corollary I.9.2, we have $\omega_2 \underset{\mathcal{B}}{\leftarrow} \omega_1$. Inversely, suppose, that $\omega_2 \underset{\mathcal{B}}{\leftarrow} \omega_1$ (where $\omega_1, \omega_2 \in L_1$). Since $\mathcal{B} = \overset{\leftarrow}{\bigcup}_{\alpha \in \mathcal{A}} \mathcal{B}_\alpha$, then, by Corollary I.9.2, an index $\alpha_1 \in \mathcal{A}$ exists such, that $\omega_2 \underset{\mathcal{B}_{\alpha_1}}{\leftarrow} \omega_1$. Since the relation $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_{\alpha_1}}{\leftarrow}}$ is defined on the set $\mathbb{B}\mathfrak{s}(\mathcal{B}_{\alpha_1})$, then $\omega, \omega_0 \in \mathbb{B}\mathfrak{s}(\mathcal{B}_{\alpha_1})$. And, taking into account that the condition $\mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha) \cap \mathbb{B}\mathfrak{s}(\mathcal{B}_\beta) = \emptyset$ must hold for $\mathcal{B}_\alpha \neq \mathcal{B}_\beta$, we get $\mathcal{B}_{\alpha_0} = \mathcal{B}_{\alpha_1}$. Hence, we obtain $\omega_2 \underset{\mathcal{B}_{\alpha_0}}{\leftarrow} \omega_1$, which was necessary to prove. Thus, the binary relations $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}}$ and $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_{\alpha_0}}{\leftarrow}}$ are coinciding on the set $L_1$. Hence, since $L_1$ is the chain in $\mathcal{B}$ (with regard to the relation $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}}$), then $L_1$ also forms the chain in $\mathcal{B}_{\alpha_0}$ (with regard to the relation $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_{\alpha_0}}{\leftarrow}}$). Thus the assumption, that $L \notin \mathbb{L}d(\mathcal{B})$ leads to the existence of chain $L_1 \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_{\alpha_0})$ in $\mathcal{B}_{\alpha_0}$ such, that $L \subset L_1$, which contradicts to the fact, that $L \in \mathbb{L}d(\mathcal{B}_{\alpha_0})$. Consequently, $L \in \mathbb{L}d(\mathcal{B})$. Therefore, any fate line $L \in \bigcup_{\alpha \in \mathcal{A}} \mathbb{L}d(\mathcal{B}_\alpha)$ belongs to $\mathbb{L}d(\mathcal{B})$. This means, that the family of base changeable sets $(\mathcal{B}_\alpha)_{\alpha \in \mathcal{A}}$ is evolutionarily saturated (by Definition I.9.7).

4. Suppose, that the family $(\mathcal{B}_\alpha)_{\alpha \in \mathcal{A}}$ is evolutionarily saturated. Then, by Assertion I.9.12, the super-evolutional union $\mathcal{B} = \overset{\leftarrow}{\bigvee}_{\alpha \in \mathcal{A}} \mathcal{B}_\alpha$ must exist. Hence, by Definition I.9.6, for any index $\alpha \in \mathcal{A}_1 \subseteq \mathcal{A}$ we have, $\mathcal{B}_\alpha \underset{\rightarrow}{\sqsubset} \mathcal{B}$. Consequently, according to the first item of this Lemma, the subfamily $(\mathcal{B}_\alpha)_{\alpha \in \mathcal{A}_1}$ is evolutionarily saturated. □

From Assertion I.9.10 and Definition I.9.6, taking into account Corollary I.9.4, Assertion I.9.12 and Lemma I.9.1, we obtain the following assertion.

**Assertion I.9.13** (on properties of super-evolutional union). *Let $(\mathcal{B}_i)_{i \in \{1,2,3\}}$ and $(\mathcal{B}_\alpha)_{\alpha \in \mathcal{A}}$ ($\mathcal{A} \neq \emptyset$) be two evolutionarily saturated families of base changeable sets. The operation of super-evolutional union possesses the following properties:*

1. $\mathcal{B}_1 \overset{\leftarrow}{\vee} \mathcal{B}_2 = \mathcal{B}_2 \overset{\leftarrow}{\vee} \mathcal{B}_1$.
2. *If* $\mathcal{A} = \{\alpha_0\}$, *then* $\overset{\leftarrow}{\bigvee}_{\alpha \in \mathcal{A}} \mathcal{B}_\alpha = \mathcal{B}_{\alpha_0}$.
3. *If the set of indexes $\mathcal{A}$ is divided into disjoint union of non-empty index sets $\mathcal{A}_\gamma$ ($\gamma \in \mathcal{G}$), (that is $\mathcal{A} = \bigsqcup_{\gamma \in \mathcal{G}} \mathcal{A}_\gamma$) then:*

$$\overset{\leftarrow}{\bigvee_{\alpha \in \mathcal{A}}} \mathcal{B}_\alpha = \overset{\leftarrow}{\bigcup_{\gamma \in \mathcal{G}}} \left( \overset{\leftarrow}{\bigvee_{\alpha \in \mathcal{A}_\gamma}} \mathcal{B}_\alpha \right). \tag{I.40}$$

*In particular, in the case* **card** $(\mathcal{A}) \geq 2$, *for an arbitrary $\alpha_0 \in \mathcal{A}$ we have the following equality:*

$$\overset{\leftarrow}{\bigvee_{\alpha \in \mathcal{A}}} \mathcal{B}_\alpha = \mathcal{B}_{\alpha_0} \overset{\leftarrow}{\cup} \left( \overset{\leftarrow}{\bigvee_{\alpha \in \mathcal{A} \setminus \{\alpha_0\}}} \mathcal{B}_\alpha \right), \tag{I.41}$$

*and in the case $\mathcal{A} = \{1, 2, 3\}$ we obtain the equality:*

$$\left(\mathcal{B}_1 \overleftarrow{\vee} \mathcal{B}_2\right) \overleftarrow{\cup} \mathcal{B}_3 = \mathcal{B}_1 \overleftarrow{\cup} \left(\mathcal{B}_2 \overleftarrow{\vee} \mathcal{B}_3\right) = \mathcal{B}_1 \overleftarrow{\vee} \mathcal{B}_2 \overleftarrow{\vee} \mathcal{B}_3. \tag{I.42}$$

4. *If for some base changeable set $\mathcal{B}'$, we have $\mathcal{B}_\alpha \underset{\to}{\subseteq} \mathcal{B}'$ (for any $\alpha \in \mathcal{A}$), then $\overleftarrow{\bigvee}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha \underset{\to}{\subseteq} \mathcal{B}'$.*
5. *If for some $\alpha_0 \in \mathcal{A}$ the inclusion $\mathcal{B}_\alpha \underset{\to}{\subseteq} \mathcal{B}_{\alpha_0}$ is performed for all $\alpha \in \mathcal{A}$, then we have $\overleftarrow{\bigvee}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha = \mathcal{B}_{\alpha_0}$. In particular $\mathcal{B} \overleftarrow{\vee} \mathcal{B} = \mathcal{B}$ for any base changeable set $\mathcal{B}$.*

It turns out, that in Item 3 of Assertion I.9.13 (more precisely, in the equalities (I.40),(I.41) and (I.42)) the sign of the evolutional union can not be replaced by the sign of super-evolutional union. Moreover, in Item 4 the evolutional inclusion can not be replaced by the super-evolutional inclusion. The next example shows that, despite the fact that any subfamily of evolutionarily saturated family $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ of base changeable sets itself is evolutionarily saturated, the family of kind $\left(\mathcal{B}_{\alpha_0}, \left(\overleftarrow{\bigvee}_{\alpha\in\mathcal{A}\setminus\{\alpha_0\}} \mathcal{B}_\alpha\right)\right)$ for $\alpha_0 \in \mathcal{A}$ may be not evolutionarily saturated (that is in the general case super-evolutional union $\mathcal{B}_{\alpha_0} \overleftarrow{\vee} \left(\overleftarrow{\bigvee}_{\alpha\in\mathcal{A}\setminus\{\alpha_0\}} \mathcal{B}_\alpha\right)$ may do not exist, while the super-evolutional union $\overleftarrow{\bigvee}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha$ exists).

***Example*** I.9.3. Let us consider the linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$, where $\mathbf{T} = \{0, 1, 2, 3\}$ and $\leq$ is the standard linear order on the set of natural numbers. Now, we define the trajectories $r_i$ $(i \in \{1, \cdots, 4\})$ from $\mathbb{T}$ to the set $M = \{0, 1, 2\}$ by means of the following tables.

| $t$ | $r_1(t)$ |
|---|---|
| 0 | 1 |
| 1 | 0 |
| 2 | 0 |
| 3 | 0 |

Table 1.

| $t$ | $r_2(t)$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 0 |
| 3 | 1 |

Table 2.

| $t$ | $r_3(t)$ |
|---|---|
| 0 | 0 |
| 1 | 0 |
| 2 | 1 |
| 3 | 2 |

Table 3.

| $t$ | $r_4(t)$ |
|---|---|
| 0 | 0 |
| 1 | 0 |
| 2 | 0 |
| 3 | 1 |

Table 4.

Any singleton set of kind $\mathcal{R}_i = \{r_i\}$, $(i \in \{1, \cdots, 4\})$ is the system of abstract trajectories from $\mathbb{T}$ to the set $M_i$, where $M_1 = M_2 = M_4 = \{0, 1\}$, $M_3 = M = \{0, 1, 2\}$. Denote:

$$\mathcal{B}_i := \mathcal{A}t\,(\mathbb{T}, \mathcal{R}_i) = \mathcal{A}t\,(\mathbb{T}, \{r_i\}) \qquad (i \in \{1, \cdots, 4\}).$$

The family $(\mathcal{B}_i)_{i=1}^4$ of base changeable sets is chronologically affined. And we are going to prove, that this family is evolutionarily saturated. Denote, $\mathcal{B} := \overleftarrow{\bigcup}_{i=1}^4 \mathcal{B}_i$. According to Assertion I.9.9, we obtain:

$$\mathcal{B} = \overleftarrow{\bigcup_{i=1}^{4}} \mathcal{B}_i = \overleftarrow{\bigcup_{i=1}^{4}} \mathcal{A}t\,(\mathbb{T}, \{r_i\}) = \mathcal{A}t\,(\mathbb{T}, \{r_1, r_2, r_3, r_4\})\,. \tag{I.43}$$

In accordance with Definition I.9.5 (item $(\mathbf{EU}_1)$) we have:

$$\mathcal{B}_i \underset{\to}{\subseteq} \overleftarrow{\bigcup_{j=1}^{4}} \mathcal{B}_j = \mathcal{B} \qquad (i \in \{1, \ldots, 4\}). \tag{I.44}$$

Now we need to prove the inclusion:

$$\bigcup_{i=1}^{4} \mathbb{L}d\,(\mathcal{B}_i) \subseteq \mathbb{L}d\,(\mathcal{B})\,. \tag{I.45}$$

Since any system of abstract trajectories $\mathcal{R}_i$ $(i \in \{1, \ldots, 4\})$ consists of only one trajectory $r_i$, then all $\mathcal{R}_i$ $(i \in \{1, \ldots, 4\})$ are systems of individual trajectories. Hence, by Theorem I.7.2,

$$\mathbb{L}d\left(\mathcal{B}_i\right) = \mathbb{L}d\left(\mathcal{A}t\left(\mathbb{T}, \mathcal{R}_i\right)\right) = \mathcal{R}_i = \{r_i\} \quad (i \in \{1, \ldots, 4\}). \tag{I.46}$$

Taking into account (I.43) and Assertion I.7.6, we have:

$$r_i \in \mathbb{L}l\left(\mathcal{A}t\left(\mathbb{T}, \{r_1, r_2, r_3, r_4\}\right)\right) = \mathbb{L}l(\mathcal{B}) \quad (i \in \{1, \ldots, 4\}).$$

Since any trajectory $r_i$ is defined on all set $\mathbf{T} = \{0, 1, 2, 3\}$, where according to equality (I.43), $\mathbf{T} = \mathbf{Tm}(\mathcal{B})$, then it can not be "expanded" in $\mathcal{B}$ by means of including into its domain new time points $t \in \mathbf{Tm}(\mathcal{B})$. Consequently $r_i \in \mathbb{L}d(\mathcal{B})$ $(i \in \{1, \ldots, 4\})$. Hence, $\bigcup_{i=1}^4 \mathbb{L}d\left(\mathcal{B}_i\right) = \bigcup_{i=1}^4 \{r_i\} \subseteq \mathbb{L}d(\mathcal{B})$, and the inclusion (I.45) has been proved now.

Therefore, by Definition I.9.7, the family $(\mathcal{B}_i)_{i=1}^4 = \left(\mathcal{A}t\left(\mathbb{T}, \{r_i\}\right)\right)_{i=1}^4$ of base changeable sets is evolutionarily saturated. Consequently, according to Assertion I.9.12, the super-evolutional union $\overleftarrow{\bigvee}_{i=1}^4 \mathcal{B}_i$, exists, moreover by Corollary I.9.4, we get $\overleftarrow{\bigvee}_{i=1}^4 \mathcal{B}_i = \overleftarrow{\bigcup}_{i=1}^4 \mathcal{B}_i = \mathcal{B}$. Hence:

$$\mathcal{B}_i \underset{\rightarrow}{\sqsubseteq} \mathcal{B} \quad (i \in \{1, \ldots, 4\}). \tag{I.47}$$

Let us denote:

$$\mathcal{B}_0 := \overleftarrow{\bigvee_{i=1}^{3}} \mathcal{B}_i = \mathcal{B}_1 \overleftarrow{\vee} \mathcal{B}_2 \overleftarrow{\vee} \mathcal{B}_3,$$

and let us prove, that $\mathcal{B}_0 \underset{\rightarrow}{\not\sqsubseteq} \mathcal{B}$.

According to Theorem I.6.1, for an arbitrary $i \in \{1, \ldots, 4\}$ the following equalities are true:

$$\mathbb{B}\mathfrak{s}\left(\mathcal{B}_i\right) = \mathbb{B}\mathfrak{s}\left(\mathcal{A}t\left(\mathbb{T}, \{r_i\}\right)\right) = r_i; \tag{I.48}$$

$$\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_i}{\leftarrow}} = \left\{(\omega_2, \omega_1) \in r_i^{\times 2} \mid \left(\mathsf{tm}\left(\omega_1\right) \leq \mathsf{tm}\left(\omega_2\right)\right)\right\}. \tag{I.49}$$

Denote:

$$\begin{aligned}
&\mathbf{w}_1 := (0, 0), \quad \mathbf{w}_2 := (1, 0), \quad \mathbf{w}_3 := (2, 0), \quad \mathbf{w}_4 := (3, 0), \\
&\mathbf{w}_5 := (0, 1), \quad \mathbf{w}_6 := (1, 1), \quad \mathbf{w}_7 := (3, 1), \quad \mathbf{w}_8 := (2, 1), \\
&\mathbf{w}_9 := (3, 2).
\end{aligned}$$

$$\mathbf{W} := \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_9\}.$$

Then, taking into account the equalities (I.48),(I.49), we obtain:

$$\begin{aligned}
&\mathbb{B}\mathfrak{s}\left(\mathcal{B}_1\right) = r_1 = \{\mathbf{w}_5, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}; \; \mathbb{B}\mathfrak{s}\left(\mathcal{B}_2\right) = \{\mathbf{w}_1, \mathbf{w}_6, \mathbf{w}_3, \mathbf{w}_7\}; \\
&\mathbb{B}\mathfrak{s}\left(\mathcal{B}_3\right) = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_8, \mathbf{w}_9\}; \qquad \mathbb{B}\mathfrak{s}\left(\mathcal{B}_4\right) = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_7\};
\end{aligned} \tag{I.50}$$

$$\begin{aligned}
\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_1}{\leftarrow}} &= \mathsf{diag}\left(\mathbb{B}\mathfrak{s}\left(\mathcal{B}_1\right)\right) \cup \left\{(\mathbf{w}_2, \mathbf{w}_5), (\mathbf{w}_3, \mathbf{w}_5), (\mathbf{w}_3, \mathbf{w}_2)\right\} \cup \\
&\quad \cup \left\{(\mathbf{w}_4, \mathbf{w}_5), (\mathbf{w}_4, \mathbf{w}_2), (\mathbf{w}_4, \mathbf{w}_3)\right\} \\
&\text{(where } \mathsf{diag}(\mathsf{K}) = \{(\omega, \omega) \mid \omega \in \mathsf{K}\}, \text{ for any set } \mathsf{K}); \\
\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_2}{\leftarrow}} &= \mathsf{diag}\left(\mathbb{B}\mathfrak{s}\left(\mathcal{B}_2\right)\right) \cup \left\{(\mathbf{w}_6, \mathbf{w}_1), (\mathbf{w}_3, \mathbf{w}_1), (\mathbf{w}_3, \mathbf{w}_6)\right\} \cup \\
&\quad \cup \left\{(\mathbf{w}_7, \mathbf{w}_1), (\mathbf{w}_7, \mathbf{w}_6), (\mathbf{w}_7, \mathbf{w}_3)\right\}; \\
\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_3}{\leftarrow}} &= \mathsf{diag}\left(\mathbb{B}\mathfrak{s}\left(\mathcal{B}_3\right)\right) \cup \left\{(\mathbf{w}_2, \mathbf{w}_1), (\mathbf{w}_8, \mathbf{w}_1), (\mathbf{w}_8, \mathbf{w}_2)\right\} \cup \\
&\quad \cup \left\{(\mathbf{w}_9, \mathbf{w}_1), (\mathbf{w}_9, \mathbf{w}_2), (\mathbf{w}_9, \mathbf{w}_8)\right\}; \\
\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_4}{\leftarrow}} &= \mathsf{diag}\left(\mathbb{B}\mathfrak{s}\left(\mathcal{B}_4\right)\right) \cup \left\{(\mathbf{w}_2, \mathbf{w}_1), (\mathbf{w}_3, \mathbf{w}_1), (\mathbf{w}_3, \mathbf{w}_2)\right\} \cup \\
&\quad \cup \left\{(\mathbf{w}_7, \mathbf{w}_1), (\mathbf{w}_7, \mathbf{w}_2), (\mathbf{w}_7, \mathbf{w}_3)\right\}.
\end{aligned} \tag{I.51}$$

Consequently, according to Corollary I.9.4, Corollary I.9.2 and Property I.6.1(9), we get:

$$\mathbb{B}\mathfrak{s}\left(\mathcal{B}_0\right) = \mathbb{B}\mathfrak{s}\left(\overleftarrow{\bigcup_{i=1}^{3}}\mathcal{B}_i\right) = \bigcup_{i=1}^{3}\mathbb{B}\mathfrak{s}\left(\mathcal{B}_i\right) = \{\mathbf{w}_1, \ldots, \mathbf{w}_9\} = \mathbf{W}; \tag{I.52}$$

$$\mathfrak{B}\mathfrak{s}\left(\mathcal{B}_0\right) = \{\mathsf{bs}\left(\omega\right) \mid \omega \in \mathbb{B}\mathfrak{s}\left(\mathcal{B}_0\right)\} = \{0,1,2\} = M;$$

$$\begin{aligned} \underset{\mathcal{B}_0}{\overset{\mathbb{B}\mathfrak{s}}{\longleftarrow}} = \underset{\overleftarrow{\bigcup}_{i=1}^{3}\mathcal{B}_i}{\overset{\mathbb{B}\mathfrak{s}}{\longleftarrow}} = \bigcup_{i=1}^{3}\left(\underset{\mathcal{B}_i}{\overset{\mathbb{B}\mathfrak{s}}{\leftarrow}}\right) = \mathbf{diag}(\mathbf{W})\cup \\ \cup\{\left(\mathbf{w}_2,\mathbf{w}_5\right),\left(\mathbf{w}_3,\mathbf{w}_5\right),\left(\mathbf{w}_3,\mathbf{w}_2\right),\left(\mathbf{w}_4,\mathbf{w}_5\right),\left(\mathbf{w}_4,\mathbf{w}_2\right),\left(\mathbf{w}_4,\mathbf{w}_3\right), \\ \left(\mathbf{w}_6,\mathbf{w}_1\right),\left(\mathbf{w}_3,\mathbf{w}_1\right),\left(\mathbf{w}_3,\mathbf{w}_6\right),\left(\mathbf{w}_7,\mathbf{w}_1\right),\left(\mathbf{w}_7,\mathbf{w}_6\right),\left(\mathbf{w}_7,\mathbf{w}_3\right), \\ \left(\mathbf{w}_2,\mathbf{w}_1\right),\left(\mathbf{w}_8,\mathbf{w}_1\right),\left(\mathbf{w}_8,\mathbf{w}_2\right),\left(\mathbf{w}_9,\mathbf{w}_1\right),\left(\mathbf{w}_9,\mathbf{w}_2\right),\left(\mathbf{w}_9,\mathbf{w}_8\right)\} \end{aligned} \tag{I.53}$$

Now, we consider the set $L_0 = \{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\} \subseteq \mathbb{B}\mathfrak{s}\left(\mathcal{B}_0\right)$. From the correlations (I.53) it follows, that for $\mathbf{w}_i,\mathbf{w}_j \in L_0$ ($i,j \in \{1,2,3\}$) the condition $\mathbf{w}_j \underset{\mathcal{B}_0}{\leftarrow} \mathbf{w}_i$ holds if and only if $i \leq j$. Consequently (since $\leq$ is the standard linear order for natural numbers) the set $L_0 = \{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\}$ is a chain of $\mathcal{B}_0$. Our next aim is to prove, that $L_0$ is a fate line of $\mathcal{B}_0$. Assume the contrary. Then the chain $L_1$ such, that $L_0 \subset L_1$ must exist in the set $\mathbb{B}\mathfrak{s}\left(\mathcal{B}_0\right)$. According to Assertion I.7.5, $L_0$ and $L_1$ are abstract trajectories from $\mathbf{Tm}\left(\mathcal{B}_0\right) = \mathbf{T} = \{0,1,2,3\}$ to $\mathfrak{B}\mathfrak{s}\left(\mathcal{B}_0\right) = \{0,1,2\}$. Since $\mathfrak{D}\left(L_0\right) = \{0,1,2\}$, the strict inclusion $L_0 \subset L_1$ is possible only under condition $\mathfrak{D}\left(L_1\right) = \{0,1,2,3\} = \mathbf{T}$. Hence, only the next three cases are possible: $L_1(3) = 0$, $L_1(3) = 1$, $L_1(3) = 2$. But, from the other hand:

**Case 1** ($L_1(3) = 0$) is impossible, because in this case $\mathbf{w}_1,\mathbf{w}_4 \in L_1$ ($\mathbf{w}_1 \in L_0 \subset L_1$), where, according to (I.53), $\mathbf{w}_4 \underset{\mathcal{B}_0}{\not\leftarrow} \mathbf{w}_1$ and $\mathbf{w}_1 \underset{\mathcal{B}_0}{\not\leftarrow} \mathbf{w}_4$.

**Case 2** ($L_1(3) = 1$) is impossible, because in this case $\mathbf{w}_2,\mathbf{w}_7 \in L_1$ ($\mathbf{w}_2 \in L_0 \subset L_1$), where, according to (I.53), $\mathbf{w}_2 \underset{\mathcal{B}_0}{\not\leftarrow} \mathbf{w}_7$ and $\mathbf{w}_7 \underset{\mathcal{B}_0}{\not\leftarrow} \mathbf{w}_2$.

**Case 3** ($L_1(3) = 2$) is impossible, because in this case $\mathbf{w}_3,\mathbf{w}_9 \in L_1$ ($\mathbf{w}_3 \in L_0 \subset L_1$), where, according to (I.53), $\mathbf{w}_3 \underset{\mathcal{B}_0}{\not\leftarrow} \mathbf{w}_9$ and $\mathbf{w}_9 \underset{\mathcal{B}_0}{\not\leftarrow} \mathbf{w}_3$.

Hence, any of considered cases is impossible. Therefore, the assumption, made above, is wrong. This means, that is $L_0 \in \mathbb{L}d\left(\mathcal{B}_0\right)$. But, $L_0 \notin \mathbb{L}d(\mathcal{B})$, because, according to (I.46) and (I.45), we have $r_4 \in \{r_4\} = \mathbb{L}d\left(\mathcal{B}_4\right) \subseteq \mathbb{L}d(\mathcal{B})$, while $r_4 \supset L_0$.

Thus, $L_0 \in \mathbb{L}d\left(\mathcal{B}_0\right)$ and $L_0 \notin \mathbb{L}d(\mathcal{B})$. This means, that $\overleftarrow{\bigvee}_{i=1}^{3}\mathcal{B}_i = \mathcal{B}_0 \not\underset{\rightarrow}{\sqsubseteq} \mathcal{B}$. Hence, in the item 4 of Assertion I.9.13, the sign "$\underset{\rightarrow}{\subseteq}$" can not be replaced by the sign "$\underset{\rightarrow}{\sqsubseteq}$" (because, according to (I.47), $\mathcal{B}_i \underset{\rightarrow}{\sqsubseteq} \mathcal{B}$ ($i \in \{1,2,3\}$), but $\overleftarrow{\bigvee}_{i=1}^{3}\mathcal{B}_i \not\underset{\rightarrow}{\sqsubseteq} \mathcal{B}$). Besides this, the family $\left(\overleftarrow{\bigvee}_{i=1}^{3}\mathcal{B}_i,\, \mathcal{B}_4\right)$ of two base changeable sets is not evolutionarily saturated, due to Remark I.9.2. That is why, the super-evolutional union $\left(\overleftarrow{\bigvee}_{i=1}^{3}\mathcal{B}_i\right) \overleftarrow{\vee}\, \mathcal{B}_4$ does not exist, while the super-evolutional union $\overleftarrow{\bigvee}_{i=1}^{4}\mathcal{B}_i$, exists. And, in accordance with Assertion I.9.13, we have, $\overleftarrow{\bigvee}_{i=1}^{4}\mathcal{B}_i = \left(\overleftarrow{\bigvee}_{i=1}^{3}\mathcal{B}_i\right) \overleftarrow{\cup} \mathcal{B}_4$.

## 9.3 On Existence of Evolutional Extensions of Base Changeable Sets.

**Theorem I.9.1.** *Let $\mathcal{B}$ be a base changeable set and $\mathcal{R}$ be a system of abstract trajectories from $\mathbb{Tm}(\mathcal{B})$ to $M$.*

*Then the base changeable set* $\widetilde{\mathcal{B}} = \mathcal{B} \overset{\leftarrow}{\cup} \mathcal{A}t\left(\mathbb{T}\mathbf{m}(\mathcal{B}), \mathcal{R}\right)$ *is an evolutional extension of* $\mathcal{B}$ *such, that* $\mathcal{R} \subseteq \mathbb{L}l\left(\widetilde{\mathcal{B}}\right)$.

*Proof.* To verify the correctness of this Theorem it is sufficient to use Assertion I.7.6, Definition I.9.5 and Assertion I.9.6. □

**Definition I.9.8.** *System of abstract trajectories* $\mathcal{R}$ *from* $\mathbb{T}$ *to* $M$ *will be named by:*

- ***Evolutionarily saturated****, if and only if* $\mathcal{R} \subseteq \mathbb{L}d\left(\mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)\right)$.
- ***Evolutionarily saturated relatively*** *a base changeable set* $\mathcal{B}$*, if and only if:*
  **1)** $\mathbb{T}\mathbf{m}(\mathcal{B}) = \mathbb{T}$*;*
  **2)** $\mathbb{L}d(\mathcal{B}) \cup \mathcal{R} \subseteq \mathbb{L}d\left(\mathcal{B} \overset{\leftarrow}{\cup} \mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)\right)$.

**Assertion I.9.14.**

1. *If the system of abstract trajectories* $\mathcal{R}$ *is evolutionarily saturated relatively a base changeable set* $\mathcal{B}$*, then it is evolutionarily saturated.*
2. *If the system of abstract trajectories* $\mathcal{R}$ *from* $\mathbb{T}$ *to* $M$ *is evolutionarily saturated and, while* $\left(\bigcup_{r\in\mathcal{R}} r\right) \cap \mathbb{B}\mathfrak{s}(\mathcal{B}) = \emptyset$ *where* $\mathcal{B}$ *is the base changeable set such, that* $\mathbb{T}\mathbf{m}(\mathcal{B}) = \mathbb{T}$*, then* $\mathcal{R}$ *is evolutionarily saturated relatively* $\mathcal{B}$.

*Proof.* **1.** Let the system of abstract trajectories $\mathcal{R}$ from $\mathbb{T}$ to $M$ be evolutionarily saturated relatively the base changeable set $\mathcal{B}$. Then, by Definition I.9.8, we have $\mathbb{T}\mathbf{m}(\mathcal{B}) = \mathbb{T}$.

According to Assertion I.7.6, any trajectory $r \in \mathcal{R}$ belongs to $\mathbb{L}l\left(\mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)\right)$. Assume, that $r$ is not fate line of $\mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)$. Then there exists a chain $L \in \mathbb{L}l\left(\mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)\right)$ such, that $r \subset L$. Since, by Definition I.9.5, $\mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right) \underset{\to}{\subseteq} \mathcal{B} \overset{\leftarrow}{\cup} \mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)$, then by Assertion I.9.6, we get $L \in \mathbb{L}l\left(\mathcal{B} \overset{\leftarrow}{\cup} \mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)\right)$. Therefore, there exists the chain $L \in \mathbb{L}l\left(\mathcal{B} \overset{\leftarrow}{\cup} \mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)\right)$ such, that $r \subset L$ in the base changeable set $\mathcal{B} \overset{\leftarrow}{\cup} \mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)$. Consequently $r \notin \mathbb{L}d\left(\mathcal{B} \overset{\leftarrow}{\cup} \mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)\right)$. But, the system of abstract trajectories $\mathcal{R}$ is evolutionarily saturated relatively $\mathcal{B}$. Hence, from the other hand, by Definition I.9.8, the correlation $r \in \mathbb{L}d\left(\mathcal{B} \overset{\leftarrow}{\cup} \mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)\right)$ must be fulfilled for the trajectory $r \in \mathcal{R}$. The contradiction, obtained above, shows, that $r \in \mathbb{L}d\left(\mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)\right)$ $(\forall r \in \mathcal{R})$. Thus, the system of abstract trajectories $\mathcal{R}$ is evolutionarily saturated.

**2**. Let the system of abstract trajectories $\mathcal{R}$ from $\mathbb{T}$ to $M$ be evolutionarily saturated with the additional condition $\left(\bigcup_{r\in\mathcal{R}} r\right) \cap \mathbb{B}\mathfrak{s}(\mathcal{B}) = \emptyset$, where $\mathbb{T}\mathbf{m}(\mathcal{B}) = \mathbb{T}$. Then, by Theorem I.6.1, we get $\mathbb{B}\mathfrak{s}\left(\mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)\right) \cap \mathbb{B}\mathfrak{s}(\mathcal{B}) = \left(\bigcup_{r\in\mathcal{R}} r\right) \cap \mathbb{B}\mathfrak{s}(\mathcal{B}) = \emptyset$. Hence, according to Lemma I.9.1 (item 3), the family of two base changeable sets $(\mathcal{B},\ \mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right))$ is evolutionarily saturated. Therefore, by Definition I.9.7, $\mathbb{L}d\left(\mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)\right) \cup \mathbb{L}d(\mathcal{B}) \subseteq \mathbb{L}d\left(\mathcal{B} \overset{\leftarrow}{\cup} \mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)\right)$. Since, the system of abstract trajectories $\mathcal{R}$ is evolutionarily saturated, then, by Definition I.9.8, we have $\mathcal{R} \subseteq \mathbb{L}d\left(\mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)\right)$. Consequently, $\mathcal{R} \cup \mathbb{L}d(\mathcal{B}) \subseteq \mathbb{L}d\left(\mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)\right) \cup \mathbb{L}d(\mathcal{B}) \subseteq \mathbb{L}d\left(\mathcal{B} \overset{\leftarrow}{\cup} \mathcal{A}t\left(\mathbb{T}, \mathcal{R}\right)\right)$. This, by Definition I.9.8, means. that the system of abstract trajectories $\mathcal{R}$ is evolutionarily saturated relatively $\mathcal{B}$. □

**Theorem I.9.2.** *Let* $\mathcal{B}$ *be a base changeable set and* $\mathcal{R}$ *be a system of abstract trajectories from* $\mathbb{T}\mathbf{m}(\mathcal{B})$ *to* $M$*, evolutionarily saturated relatively* $\mathcal{B}$.

*Then the base changeable set* $\widetilde{\mathcal{B}} = \mathcal{B} \overset{\leftarrow}{\cup} \mathcal{A}t\left(\mathbb{T}\mathbf{m}(\mathcal{B}), \mathcal{R}\right)$ *is a super-evolutional extension of* $\mathcal{B}$ *such, that* $\mathcal{R} \subseteq \mathbb{L}d\left(\widetilde{\mathcal{B}}\right)$.

*Proof.* Denote, $\widetilde{\mathcal{B}} = \mathcal{B} \overset{\leftarrow}{\cup} \mathcal{A}t\left(\mathbb{T}\mathbf{m}(\mathcal{B}), \mathcal{R}\right)$. According to Assertion I.9.9:

$$\widetilde{\mathcal{B}} = \mathcal{B} \overset{\leftarrow}{\cup} \mathcal{A}t\left(\mathbb{T}\mathbf{m}(\mathcal{B}), \bigcup_{r\in\mathcal{R}} \{r\}\right) = \mathcal{B} \overset{\leftarrow}{\cup} \overset{\longleftarrow}{\bigcup_{r\in\mathcal{R}}} \mathcal{B}_r, \qquad \text{(I.54)}$$
$$\text{where} \quad \mathcal{B}_r = \mathcal{A}t\left(\mathbb{T}\mathbf{m}(\mathcal{B}), \{r\}\right) \quad (r \in \mathcal{R}).$$

Since for an arbitrary $r \in \mathcal{R}$ the one-trajectory system $\mathcal{R}_r = \{r\}$ is the system of individual trajectories (in the sense of Definition I.7.4), then, according to Theorem I.7.2, we have $\mathbb{L}d\left(\mathcal{B}_r\right) = \{r\}$ $(\forall\, r \in \mathcal{R})$. Hence, taking into account Definition I.9.8, and Equality (I.54), we obtain:

$$\mathbb{L}d(\mathcal{B}) \cup \bigcup_{r\in\mathcal{R}} \mathbb{L}d\left(\mathcal{B}_r\right) = \mathbb{L}d(\mathcal{B}) \cup \bigcup_{r\in\mathcal{R}} \{r\} = \mathbb{L}d(\mathcal{B}) \cup \mathcal{R} \subseteq$$
$$\subseteq \mathbb{L}d\left(\mathcal{B} \overset{\leftarrow}{\cup} \mathcal{A}t\left(\mathbb{T}\mathbf{m}(\mathcal{B}), \mathcal{R}\right)\right) = \mathbb{L}d\left(\widetilde{\mathcal{B}}\right) = \mathbb{L}d\left(\mathcal{B} \overset{\leftarrow}{\cup} \overset{\longleftarrow}{\bigcup_{r\in\mathcal{R}}} \mathcal{B}_r\right). \qquad \text{(I.55)}$$

That is why, by Assertion I.9.12, the super-evolutional union $\overset{\leftarrow}{\bigvee}_{\alpha\in\mathcal{A}} \mathcal{B}_\alpha$ exists, where $\mathcal{A} = \mathcal{R} \bigsqcup \{\alpha_0\}$, $\mathcal{B}_{\alpha_0} = \mathcal{B}$ and $\alpha_0$ is any index, satisfying $\alpha_0 \notin \mathcal{R}$ (for example we can chose any index $\alpha_0$ from the nonempty set $2^{\mathcal{R}} \setminus \mathcal{R}$). According to Corollary I.9.4 and Equality (I.54) we get:

$$\overset{\longleftarrow}{\bigvee_{\alpha\in\mathcal{A}}} \mathcal{B}_\alpha = \overset{\longleftarrow}{\bigcup_{\alpha\in\mathcal{A}}} \mathcal{B}_\alpha = \mathcal{B}_{\alpha_0} \overset{\leftarrow}{\cup} \overset{\longleftarrow}{\bigcup_{r\in\mathcal{R}}} \mathcal{B}_r = \mathcal{B} \overset{\leftarrow}{\cup} \overset{\longleftarrow}{\bigcup_{r\in\mathcal{R}}} \mathcal{B}_r = \widetilde{\mathcal{B}}.$$

Consequently, by Definition I.9.6, we have $\mathcal{B} = \mathcal{B}_{\alpha_0} \underset{\rightarrow}{\sqsubseteq} \widetilde{\mathcal{B}}$. Therefore $\widetilde{\mathcal{B}}$ is the super-evolutional extension of $\mathcal{B}$. Moreover, according to (I.55), we obtain, $\mathcal{R} \subseteq \mathbb{L}d(\mathcal{B}) \cup \mathcal{R} \subseteq \mathbb{L}d\left(\widetilde{\mathcal{B}}\right)$. □

Main results of this Section were published in [9].

# 10 Multi-figurativeness and Unification of Perception. General Definition of Changeable Set

## 10.1 General Definition of Changeable Set

Base changeable sets can be treated as mathematical abstractions of physical processes models (in macro level) in the case, when the observations are conducted from one, fixed point (one, fixed frame of reference). But, real, physical nature is multi-figurative, because in physics (in particular in special relativity theory) "picture of the world" can significantly vary, according to the frame of reference. Therefore, we obtain not one but many base changeable sets (connected with everyone frame of reference of the physical model under consideration). Any of these base changeable sets can be interpreted as individual image (or area of perception) of the physical reality. Also it can be naturally assumed, that there is a natural unification between any two areas of perception (that is frames of reference), this means, that it must be defined some rule, which specifies how the object or process from one frame of reference must be looked out in other frame. More precisely, using certain rules, we identify some object or process from one frame of reference with the other object or process from other frame, saying that it is the same object, but visible from another frame of reference. In the classical mechanics such "unification of perception" is defined by the Galilean group of transformations, and in the special relativity theory (for inertial reference frames) this unification is determined by the group of Lorentz-Poincare. It should be noted that in the both cases the unification of perception is made not at the level of objects and processes, but at the level of elementary-time states tied to certain points in 4-dimensional space-time. This means that in the both cases there is assumed, that any elementary-time state, "visible" from some frame of reference, is "visible" from another frames. On author opinion, this assumption is too strong for abstract theory. Moreover in relativity theory for non-inertial reference frames the last assumption is not true. That is why, in the definition below, the unification of perception is made on the level of objects and processes. We recall, that in Section 8 it had been introduced the concept of changeable system (subset of the set $\mathbb{Bs}(\mathcal{B})$, generated by base changeable set $\mathcal{B}$) as an abstract analog of the notion of physical object or process.

**Definition I.10.1.** *Let $\overleftarrow{\mathcal{B}} = (\mathcal{B}_\alpha \mid \alpha \in \mathcal{A})$ be any indexed family of base changeable sets (where $\mathcal{A} \neq \emptyset$ is some set of indexes). System of mappings $\overleftarrow{\mathfrak{U}} = (\mathfrak{U}_{\beta\alpha} \mid \alpha, \beta \in \mathcal{A})$ of kind:*

$$\mathfrak{U}_{\beta\alpha} : 2^{\mathbb{Bs}(\mathcal{B}_\alpha)} \longmapsto 2^{\mathbb{Bs}(\mathcal{B}_\beta)} \qquad (\alpha, \beta \in \mathcal{A})$$

*is referred to as **unification of perception** on $\overleftarrow{\mathcal{B}}$ if and only if the following conditions are satisfied:*

1. *$\mathfrak{U}_{\alpha\alpha} A \equiv A$ for any $\alpha \in \mathcal{A}$ and $A \subseteq \mathbb{Bs}(\mathcal{B}_\alpha)$.*
   (Here and further we denote by $\mathfrak{U}_{\beta\alpha} A$ the action of the mapping $\mathfrak{U}_{\beta\alpha}$ to the set $A \subseteq \mathbb{Bs}(\mathcal{B}_\alpha)$, that is $\mathfrak{U}_{\beta\alpha} A := \mathfrak{U}_{\beta\alpha}(A)$.)
2. *Any mapping $\mathfrak{U}_{\beta\alpha}$ is a monotonous mapping of sets, ie for any $\alpha, \beta \in \mathcal{A}$ and $A, B \subseteq \mathbb{Bs}(\mathcal{B}_\alpha)$ the condition $A \subseteq B$ assures $\mathfrak{U}_{\beta\alpha} A \subseteq \mathfrak{U}_{\beta\alpha} B$.*
3. *For any $\alpha, \beta, \gamma \in \mathcal{A}$ and $A \subseteq \mathbb{Bs}(\mathcal{B}_\alpha)$ the following inclusion holds:*

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha} A \subseteq \mathfrak{U}_{\gamma\alpha} A. \tag{I.56}$$

*In this case the mappings $\mathfrak{U}_{\beta\alpha}$ $(\alpha, \beta \in \mathcal{A})$ we name by **unification mappings**, and the triple of kind:*

$$\mathcal{Z} = \left(\mathcal{A}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathfrak{U}}\right)$$

*we name by **changeable set**.*

The first condition of Definition I.10.1 is quite obvious. The second condition is dictated by the natural desire "to see" a subsystem of a given changeable system in a given frame of reference (area of perception) as the subsystem of "the same" changeable system in other frame. In the case of classical mechanics or special relativity theory for inertial reference frames the third condition of Definition I.10.1 may be transformed to the following (stronger) condition:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha} A = \mathfrak{U}_{\gamma\alpha} A \quad (\alpha, \beta, \gamma \in \mathcal{A},\ A \subseteq \mathbb{Bs}(\mathcal{B}_\alpha)) \tag{I.57}$$

The replacement of the equal sign by the sign inclusion is caused by the permission to "distort the picture of reality" during "transition" to other frame of reference in the case of the our abstract theory. We suppose, that during this "transition" some elementary-time states may turn out to be "invisible" in other frame of reference. Further this idea will be explained more detailed (see the Section 12, in particular, Theorem I.12.1).

## 10.2 Remarks on the Terminology and Denotations

Let $\mathcal{Z} = \left(\mathcal{A}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathfrak{U}}\right)$ be a changeable set, where $\overleftarrow{\mathcal{B}} = (\mathcal{B}_\alpha \mid \alpha \in \mathcal{A})$ is an indexed family of base changeable sets and $\overleftarrow{\mathfrak{U}} = (\mathfrak{U}_{\beta\alpha} \mid \alpha, \beta \in \mathcal{A})$ is an unification of perception on $\overleftarrow{\mathcal{B}}$. Later we will use the following terms and notations:

1) The set $\mathcal{A}$ will be named the ***index set*** of the changeable set $\mathcal{Z}$, and it will be denoted by $\mathcal{I}nd\,(\mathcal{Z})$.

2) For any index $\alpha \in \mathcal{I}nd\,(\mathcal{Z})$ the pair $(\alpha, \mathcal{B}_\alpha)$ will be named by ***reference frame*** [7] of the changeable set $\mathcal{Z}$.

3) The set of all reference frames of $\mathcal{Z}$ will be denoted by $\mathcal{L}k\,(\mathcal{Z})$: [8]

$$\mathcal{L}k\,(\mathcal{Z}) := \{(\alpha, \mathcal{B}_\alpha) \mid \alpha \in \mathcal{I}nd\,(\mathcal{Z})\}\,.$$

Typically, reference frames will be denoted by small Gothic letters ($\mathfrak{l}, \mathfrak{m}, \mathfrak{k}, \mathfrak{p}$ and so on).

4) For $\mathfrak{l} = (\alpha, \mathcal{B}_\alpha) \in \mathcal{L}k\,(\mathcal{Z})$ we introduce the following denotations:

$$\mathtt{ind}\,(\mathfrak{l}) := \alpha; \quad \mathfrak{l}^\wedge := \mathcal{B}_\alpha.$$

Thus, for any reference frame $\mathfrak{l} \in \mathcal{L}k\,(\mathcal{Z})$ the object $\mathfrak{l}^\wedge$ is a base changeable set.

Further, when it does not cause confusion, for any reference frame $\mathfrak{l} \in \mathcal{L}k\,(\mathcal{Z})$ in denotations:

$$\begin{gathered}\mathfrak{B}\mathfrak{s}\,(\mathfrak{l}^\wedge)\,,\ \mathbb{B}\mathfrak{s}\,(\mathfrak{l}^\wedge)\,,\ \mathbf{Tm}\,(\mathfrak{l}^\wedge)\,,\ \mathbb{T}\mathbf{m}\,(\mathfrak{l}^\wedge)\ \leq_{\mathfrak{l}^\wedge},\ <_{\mathfrak{l}^\wedge}, \\ \geq_{\mathfrak{l}^\wedge},\ >_{\mathfrak{l}^\wedge},\ \psi_{\mathfrak{l}^\wedge},\ \underset{\mathfrak{l}^\wedge}{\leftarrow},\ \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathfrak{l}^\wedge}{\leftarrow}},\ \mathbb{L}l\,(\mathfrak{l}^\wedge)\,,\ \mathbb{L}d\,(\mathfrak{l}^\wedge)\end{gathered} \tag{I.58}$$

the symbol " $^\wedge$" will be omitted, and the following denotations will be used instead:

$$\begin{gathered}\mathfrak{B}\mathfrak{s}\,(\mathfrak{l})\,,\ \mathbb{B}\mathfrak{s}\,(\mathfrak{l})\,,\ \mathbf{Tm}\,(\mathfrak{l})\,,\ \mathbb{T}\mathbf{m}\,(\mathfrak{l})\,,\ \leq_{\mathfrak{l}},\ <_{\mathfrak{l}}, \\ \geq_{\mathfrak{l}},\ >_{\mathfrak{l}},\ \psi_{\mathfrak{l}},\ \underset{\mathfrak{l}}{\leftarrow},\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathfrak{l}}{\leftarrow}},\ \mathbb{L}l\,(\mathfrak{l})\,,\ \mathbb{L}d\,(\mathfrak{l})\,.\end{gathered} \tag{I.59}$$

5) For any reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k\,(\mathcal{Z})$ the mapping $\mathfrak{U}_{\mathtt{ind}(\mathfrak{m}),\mathtt{ind}(\mathfrak{l})}$ will be denoted by $\langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathcal{Z}\rangle$ or by $\langle \mathfrak{l} \to \mathfrak{m}, \mathcal{Z}\rangle$. Hence:

$$\langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathcal{Z}\rangle = \langle \mathfrak{l} \to \mathfrak{m}, \mathcal{Z}\rangle = \mathfrak{U}_{\mathtt{ind}(\mathfrak{m}),\mathtt{ind}(\mathfrak{l})}.$$

In the case, when the base changeable $\mathcal{Z}$ set is known, the symbol $\mathcal{Z}$ in the above notations will be omitted, and the denotations "$\langle \mathfrak{m} \leftarrow \mathfrak{l}\rangle$, $\langle \mathfrak{l} \to \mathfrak{m}\rangle$" will be used instead. Moreover, in the case, when it does not cause confusion in the notations "$\leq_{\mathfrak{l}}$, $<_{\mathfrak{l}}$, $\geq_{\mathfrak{l}}$, $>_{\mathfrak{l}}$, $\underset{\mathfrak{l}}{\leftarrow}$, $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathfrak{l}^\wedge}{\leftarrow}}$, $\psi_{\mathfrak{l}}$" the symbol "$\mathfrak{l}$" will be omitted, and the denotations "$\leq$, $<$, $\geq$, $>$, $\leftarrow$, $\overset{\mathbb{B}\mathfrak{s}}{\leftarrow}$, $\psi$" will be used instead. Moreover, for elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$ we usually use the denotations $\omega_2 \leftarrow \omega_1$ or $\omega_2 \underset{\mathfrak{l}}{\leftarrow} \omega_1$ instead of the denotations $\omega_2 \overset{\mathbb{B}\mathfrak{s}}{\leftarrow} \omega_1$ or $\omega_2 \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathfrak{l}}{\leftarrow}} \omega_1$ correspondingly (in the cases, when it does not cause confusion).

*Remark* I.10.1. From Definition of changeable set (Definition I.10.1) it directly follows, that for any reference frame $\mathfrak{l} \in \mathcal{L}k\,(\mathcal{Z})$ of any changeable set $\mathcal{Z}$, Properties I.6.1 are holding, where *we use all abbreviated variants of notations, described in Subsection 6.3* (but, with replacement of the symbol "$\mathcal{B}$" by the symbol "$\mathfrak{l}$" and the term "base changeable set" by the term "reference frame").

### 10.3 Elementary Properties of Changeable Sets

Using Definition I.10.1 and notations, introduced in Subsection 10.2, we can write the following **basic properties of changeable sets**.

[7] Note, that the terms "***area of perception***" or "***lik***" may be considered as synonymous to the term "reference frame". In order to standardize terminology, in this paper we use only the term "reference frame". In earlier papers [1,3,4,8] usually it was used the term "area of perception" in the case of general changeable sets (the term "reference frame" was used only for the cases of kinematic changeable sets and universal kinematics).

[8] The designation "$\mathcal{L}k\,(\mathcal{Z})$" originates from the word "lik", which is synonymous with the term "reference frame" (see footnote 7).

**Properties I.10.1.** *In the properties 1-8 symbol $\mathcal{Z}$ denotes any changeable set and $\mathfrak{l},\mathfrak{m},\mathfrak{p}\in\mathcal{L}k(\mathcal{Z})$ are any reference frames of $\mathcal{Z}$.*

1. *The sets $\mathcal{L}k(\mathcal{Z})$ and $\mathcal{I}nd(\mathcal{Z})$ always are nonempty, moreover $\mathcal{I}nd(\mathcal{Z})=\{\mathtt{ind}(\mathfrak{l})\mid\mathfrak{l}\in\mathcal{L}k(\mathcal{Z})\}$;*
2. $\mathfrak{l}=(\mathtt{ind}(\mathfrak{l}),\mathfrak{l}^{\wedge})$;
3. $\mathfrak{l}^{\wedge}=\left(\left(\left(\mathfrak{Bs}(l),\underset{\mathfrak{l}}{\leftarrow}\right),(\mathbf{Tm}(\mathfrak{l}),\leq_{\mathfrak{l}}),\psi_{\mathfrak{l}}\right),\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathfrak{l}}{\leftarrow}}\right)$ *is a base changeable set.*
4. $\langle\mathfrak{l}\leftarrow\mathfrak{l}\rangle A=\langle\mathfrak{l}\rightarrow\mathfrak{l}\rangle A=A,\ A\subseteq\mathbb{B}\mathfrak{s}(\mathfrak{l})$;
5. *For arbitrary $\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathcal{Z})$ the unification mapping $\langle\mathfrak{m}\leftarrow\mathfrak{l},\ \mathcal{Z}\rangle$ is the mapping from $2^{\mathbb{B}\mathfrak{s}(\mathfrak{l})}$ into $2^{\mathbb{B}\mathfrak{s}(\mathfrak{m})}$;*
6. $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A=\langle\mathfrak{l}\rightarrow\mathfrak{m}\rangle A,\ A\subseteq\mathbb{B}\mathfrak{s}(\mathfrak{l})$;
7. *If $A\subseteq B\subseteq\mathbb{B}\mathfrak{s}(\mathfrak{l})$, then $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A\subseteq\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle B$;*
8. $\langle\mathfrak{p}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A\subseteq\langle\mathfrak{p}\leftarrow\mathfrak{l}\rangle A$, *where* $A\subseteq\mathbb{B}\mathfrak{s}(\mathfrak{l})$.

Usually in future we will use Properties I.10.1 instead of using Definition I.10.1 directly. The following three assertions are elementary corollaries of Properties I.10.1 and Definition I.10.1. In these assertions the symbol $\mathcal{Z}$ denotes any changeable set.

**Assertion I.10.1.** *Let, $\mathcal{Z}_1$, $\mathcal{Z}_2$ be arbitrary changeable sets, moreover:*

1. $\mathcal{L}k(\mathcal{Z}_1)=\mathcal{L}k(\mathcal{Z}_2)$.
2. *For arbitrary reference frames $\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathcal{Z}_1)=\mathcal{L}k(\mathcal{Z}_2)$ it is true the equality: $\langle\mathfrak{m}\leftarrow\mathfrak{l},\mathcal{Z}_1\rangle=\langle\mathfrak{m}\leftarrow\mathfrak{l},\mathcal{Z}_2\rangle$.*

*Then, $\mathcal{Z}_1=\mathcal{Z}_2$.*

*Proof.* This assertion follows directly from Definition I.10.1 and denotations, introduced in Subsection 10.2. □

**Assertion I.10.2.** *For any $\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathcal{Z})$ the following equality is true:*

$$\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\emptyset=\emptyset.$$

*Proof.* Denote $B:=\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\emptyset\subseteq\mathbb{B}\mathfrak{s}(\mathfrak{m})$. By Properties I.10.1 (8 and 4) we obtain:

$$\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle B=\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\emptyset\subseteq\langle\mathfrak{l}\leftarrow\mathfrak{l}\rangle\emptyset=\emptyset.$$

Therefore, $\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle B=\emptyset$. Since $\emptyset\subseteq B$, then, by Property I.10.1(7), we get:

$$\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle\emptyset\subseteq\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle B=\emptyset,$$

that is $\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle\emptyset=\emptyset$. Hence, by Properties I.10.1 (4 and 8), we obtain:

$$\emptyset=\langle\mathfrak{m}\leftarrow\mathfrak{m}\rangle\emptyset\supseteq\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle\emptyset=\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\emptyset=B.$$

□

**Assertion I.10.3.** *For any $\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathcal{Z})$ and any family of changeable systems $(A_\alpha|\alpha\in\mathcal{A})$ ($A_\alpha\subseteq\mathbb{B}\mathfrak{s}(\mathfrak{l})$ for each $\alpha\in\mathcal{A}$) the following inclusions take place:*

1) $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\left(\bigcap_{\alpha\in\mathcal{A}}A_\alpha\right)\subseteq\bigcap_{\alpha\in\mathcal{A}}\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A_\alpha$;
2) $\bigcap_{\alpha\in\mathcal{A}}A_\alpha\supseteq\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle\left(\bigcap_{\alpha\in\mathcal{A}}\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A_\alpha\right)$;
3) $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\left(\bigcup_{\alpha\in\mathcal{A}}A_\alpha\right)\supseteq\bigcup_{\alpha\in\mathcal{A}}\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A_\alpha$.

Note, that the set of indexes $\mathcal{A}$ in the last assertion is an arbitrary, and, in general, it does not coincide with the set of indexes in Definition I.10.1.

*Proof.* 1) Denote $A := \bigcap_{\alpha\in\mathcal{A}} A_\alpha$. Taking into account, that $A \subseteq A_\alpha$, $\alpha \in \mathcal{A}$ and using Property I.10.1(7) we obtain:

$$\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A \subseteq \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A_\alpha, \quad \alpha \in \mathcal{A}.$$

Thus, $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A \subseteq \bigcap_{\alpha\in\mathcal{A}} \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A_\alpha$.

2) Denote: $Q := \bigcap\limits_{\alpha\in\mathcal{A}} \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A_\alpha$. Then $Q \subseteq \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A_\alpha$, $\alpha \in \mathcal{A}$. Hence, by Properties I.10.1(7,8 and 4) we obtain:

$$\langle \mathfrak{l} \leftarrow \mathfrak{m} \rangle Q \subseteq \langle \mathfrak{l} \leftarrow \mathfrak{m} \rangle \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A_\alpha \subseteq \langle \mathfrak{l} \leftarrow \mathfrak{l} \rangle A_\alpha = A_\alpha, \quad \alpha \in \mathcal{A}.$$

Hence, $\langle \mathfrak{l} \leftarrow \mathfrak{m} \rangle Q \subseteq \bigcap_{\alpha\in\mathcal{A}} A_\alpha$, that was necessary to prove.

3) Denote: $A := \bigcup_{\alpha\in\mathcal{A}} A_\alpha$. Taking into account, that $A_\alpha \subseteq A$, $\alpha \in \mathcal{A}$ and using Property I.10.1(7) we obtain

$$\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A_\alpha \subseteq \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A, \quad \alpha \in \mathcal{A}.$$

Hence, $\bigcup_{\alpha\in\mathcal{A}} \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A_\alpha \subseteq \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A$. □

Main results of this Section were anonced in [1] and published in [8, Section 3].

# 11 Examples of Changeable Sets

## 11.1 Precisely Visible Changeable Set, Generated by Systems of Base Changeable Sets and Mappings

***Example*** I.11.1. Let $\overleftarrow{\mathcal{B}} = (\mathcal{B}_\alpha \mid \alpha \in \mathcal{A})$ be any non-empty ($\mathcal{A} \neq \emptyset$) indexed family of base changeable sets such, that $\mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)$ and $\mathbb{B}\mathfrak{s}(\mathcal{B}_\beta)$ are equipotent for any $\alpha, \beta \in \mathcal{A}$, that is $\mathbf{card}(\mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)) = \mathbf{card}(\mathbb{B}\mathfrak{s}(\mathcal{B}_\beta))$, $\alpha, \beta \in \mathcal{A}$, where $\mathbf{card}(M)$ is the *cardinality* of the set $M$. Let us consider any indexed family of bijections (one-to-one correspondences) $\overleftarrow{W} = (W_{\beta\alpha} \mid \alpha, \beta \in \mathcal{A})$ of kind $W_{\beta\alpha} : \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha) \mapsto \mathbb{B}\mathfrak{s}(\mathcal{B}_\beta)$, satisfying the following "pseudo-group" conditions [9]:

$$\left.\begin{aligned} W_{\alpha\alpha}(\omega) &= \omega, && \alpha \in \mathcal{A},\ \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha); \\ W_{\gamma\beta}(W_{\beta\alpha}(\omega)) &= W_{\gamma\alpha}(\omega), && \alpha, \beta, \gamma \in \mathcal{A},\ \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha). \end{aligned}\right\} \tag{I.60}$$

Let us put:

$$\mathfrak{U}_{\beta\alpha} A := W_{\beta\alpha}(A) = \{W_{\beta\alpha}(\omega) \mid \omega \in A\}, \quad A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha).$$

It is easy to see, that the family of mappings $\overleftarrow{\mathfrak{U}} = (\mathfrak{U}_{\beta\alpha} \mid \alpha, \beta \in \mathcal{A})$ satisfies all conditions of Definition I.10.1, moreover, the third condition of this Definition can be replaced by more strong condition (I.57). Thus the triple:

$$\mathcal{Z}\mathsf{pv}\left(\overleftarrow{\mathcal{B}}, \overleftarrow{W}\right) = \left(\mathcal{A}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathfrak{U}}\right)$$

[9] The family of bijections, satisfying conditions (I.60) can be easily constructed by the following way.

Since $\mathcal{A} \neq \emptyset$, we can chose any (fixed) index $\alpha_0 \in \mathcal{A}$. Also chose any family of bijections $\overleftarrow{\mathcal{W}} = (\mathcal{W}_\alpha \mid \alpha \in \mathcal{A})$ of kind $\mathcal{W}_\alpha : \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha) \mapsto \mathbb{B}\mathfrak{s}(\mathcal{B}_{\alpha_0})$ (such family of bijections necessarily must exist, because of $\mathbf{card}(\mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)) = \mathbf{card}(\mathbb{B}\mathfrak{s}(\mathcal{B}_\beta))$, $\alpha, \beta \in \mathcal{A}$). Denote:

$$W_{\beta\alpha}(\omega) := \mathcal{W}_\beta^{[-1]}(\mathcal{W}_\alpha(\omega)), \qquad \alpha, \beta \in \mathcal{A},\ \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha).$$

where $\mathcal{W}_\beta^{[-1]}$ is the mapping, inverse to $\mathcal{W}_\beta$. It is easy to verify, that the family of bijections $(W_{\beta\alpha} \mid \alpha, \beta \in \mathcal{A})$ satisfies conditions (I.60).

is a changeable set. The changeable set $\mathcal{Z}\mathsf{pv}\left(\overleftarrow{\mathcal{B}},\overleftarrow{W}\right)$ will be named a ***precisely visible changeable set***, generated by the system of base changeable sets $\overleftarrow{\mathcal{B}}$ and the system of mappings $\overleftarrow{W}$.

Using the results of Example I.11.1 and denotations, introduced in Subsection 10.2, we obtain the following properties of changeable set of kind $\mathcal{Z}\mathsf{pv}\left(\overleftarrow{\mathcal{B}},\overleftarrow{W}\right)$.

**Properties I.11.1.** *Let* $\mathcal{Z} = \mathcal{Z}\mathsf{pv}\left(\overleftarrow{\mathcal{B}},\overleftarrow{W}\right)$ *be a precisely visible changeable set, generated by system of base changeable sets* $\overleftarrow{\mathcal{B}} = (\mathcal{B}_\alpha \mid \alpha\in\mathcal{A})$ *and system of mappings* $\overleftarrow{W} = (W_{\beta\alpha}\mid \alpha,\beta\in\mathcal{A})$. *Then:*

1. $\mathcal{L}k(\mathcal{Z}) = \{(\alpha,\mathcal{B}_\alpha)\mid \alpha\in\mathcal{A}\}$;
2. $\mathcal{I}nd(\mathcal{Z}) = \mathcal{A}$;
3. *For any reference frame* $\mathfrak{l} = (\alpha,\mathcal{B}_\alpha)\in\mathcal{L}k(\mathcal{Z})$ $(\alpha\in\mathcal{A})$ *the following equalities hold:*

$$\begin{aligned} &\mathbb{B}\mathfrak{s}(\mathfrak{l}) = \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha); \quad &&\mathfrak{B}\mathfrak{s}(\mathfrak{l}) = \mathfrak{B}\mathfrak{s}(\mathcal{B}_\alpha);\\ &\mathbb{T}\mathbf{m}(\mathfrak{l}) = \mathbb{T}\mathbf{m}(\mathcal{B}_\alpha); \quad &&\mathbf{Tm}(\mathfrak{l}) = \mathbf{Tm}(\mathcal{B}_\alpha);\\ &\leq_{\mathfrak{l}} = \leq_{\mathcal{B}_\alpha}; &&\\ &\underset{\mathfrak{l}}{\leftarrow} = \underset{\mathcal{B}_\alpha}{\leftarrow}; &&\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathfrak{l}}{\leftarrow}} = \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}_\alpha}{\leftarrow}}. \end{aligned}$$

4. *For any reference frames* $\mathfrak{l} = (\alpha,\mathcal{B}_\alpha)\in\mathcal{L}k(\mathcal{Z})$, $\mathfrak{m} = (\beta,\mathcal{B}_\beta)\in\mathcal{L}k(\mathcal{Z})$ $(\alpha,\beta\in\mathcal{A})$ *and any set* $A\subseteq\mathbb{B}\mathfrak{s}(\mathfrak{l}) = \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)$ *the following equality holds:*

$$\langle\mathfrak{m}\leftarrow\mathfrak{l},\mathcal{Z}\rangle A = W_{\beta\alpha}(A) = \{W_{\beta\alpha}(\omega)\mid\omega\in A\}.$$

## 11.2 Changeable Sets, Generated by Multi-Image of Base Changeable Set

Multi-images of base changeable sets may be considered as examples of changeable sets. To construct multi-images of base changeable sets we need introduce some new definitions and prove theorem on multi-image for changeable sets.

**Definition I.11.1.** *The ordered triple* $(\mathbb{T},\mathcal{X},U)$ *will be referred to as* ***evolution projector*** *for base changeable set* $\mathcal{B}$ *if and only if:*

1. $\mathbb{T} = (\mathbf{T},\leq)$ *is linearly ordered set;*
2. $\mathcal{X}$ *is any set;*
3. $U$ *is a mapping from* $\mathbb{B}\mathfrak{s}(\mathcal{B})$ *into* $\mathbf{T}\times\mathcal{X}$ $(U:\mathbb{B}\mathfrak{s}(\mathcal{B})\mapsto\mathbf{T}\times\mathcal{X})$.

**Theorem I.11.1.** *Let* $(\mathbb{T},\mathcal{X},U)$ *be any evolution projector for base changeable set* $\mathcal{B}$. *Then there exist only one base changeable set* $U[\mathcal{B},\mathbb{T}]$, *satisfying the following conditions:*

1. $\mathbb{T}\mathbf{m}(U[\mathcal{B},\mathbb{T}]) = \mathbb{T}$;
2. $\mathbb{B}\mathfrak{s}(U[\mathcal{B},\mathbb{T}]) = U(\mathbb{B}\mathfrak{s}(\mathcal{B})) = \{U(\omega)\mid\omega\in\mathbb{B}\mathfrak{s}(\mathcal{B})\}$;
3. *Let* $\widetilde{\omega}_1,\widetilde{\omega}_2\in\mathbb{B}\mathfrak{s}(U[\mathcal{B},\mathbb{T}])$ *and* $\mathsf{tm}(\widetilde{\omega}_1)\neq\mathsf{tm}(\widetilde{\omega}_2)$. *Then* $\widetilde{\omega}_1$ *and* $\widetilde{\omega}_2$ *are united by fate in* $U[\mathcal{B},\mathbb{T}]$ *if and only if, there exist united by fate in* $\mathcal{B}$ *elementary-time states* $\omega_1,\omega_2\in\mathbb{B}\mathfrak{s}(\mathcal{B})$ *such, that* $\widetilde{\omega}_1 = U(\omega_1)$, $\widetilde{\omega}_2 = U(\omega_2)$.

*Proof.* **Proof of existence.**

1. Let $(\mathbb{T},\mathcal{X},U)$ be an evolution projector for base changeable set $\mathcal{B}$ (where $\mathbb{T} = (\mathbf{T},\leq)$). Let us define the binary relation $\twoheadleftarrow$ on the set $U(\mathbb{B}\mathfrak{s}(\mathcal{B})) = \{U(\omega)\mid\omega\in\mathbb{B}\mathfrak{s}(\mathcal{B})\}\subseteq\mathbf{T}\times X$. Namely, for any $\widetilde{\omega}_1,\widetilde{\omega}_2\in U(\mathbb{B}\mathfrak{s}(\mathcal{B}))$ we consider, that $\widetilde{\omega}_2\twoheadleftarrow\widetilde{\omega}_1$ if and only if at least one of the following conditions is performed:

**U[B]1)** $\widetilde{\omega}_1 = \widetilde{\omega}_2$;

**U[B]2)** $\mathsf{tm}(\widetilde{\omega}_1) < \mathsf{tm}(\widetilde{\omega}_2)$ and there exist united by fate in $\mathcal{B}$ elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ such, that $\widetilde{\omega}_i = U(\omega_i) \quad (i = 1, 2)$.

From Conditions U[B]1), U[B]2) it follows, that the relation $\Leftarrow$ satisfies Conditions 1,2 of Theorem I.6.2. Hence, by Theorem I.6.2, only one base changeable set $\mathcal{B}_1$, exists, satisfying the following conditions:

$$\mathbb{T}\mathbf{m}(\mathcal{B}_1) = \mathbb{T}; \qquad \mathbb{B}\mathfrak{s}(\mathcal{B}_1) = U(\mathbb{B}\mathfrak{s}(\mathcal{B})); \qquad \underset{\mathcal{B}_1}{\overset{\mathbb{B}\mathfrak{s}}{\longleftarrow}} = \Leftarrow . \tag{I.61}$$

Denote:

$$U[\mathcal{B}, \mathbb{T}] := \mathcal{B}_1.$$

From first two conditions (I.61) it follows, that the base changeable set $U[\mathcal{B}, \mathbb{T}]$ satisfies conditions 1,2 of this Theorem. From the third condition (I.61), taking into account Assertion I.7.3, we obtain that third condition of this theotem for $U[\mathcal{B}, \mathbb{T}]$ also is satisfied.

**Proof of uniqueness**.

Suppose, that the base changeable set $\mathcal{B}_2$ also satisfies Conditions 1,2,3 of this Theorem, that is:

**1′.** $\mathbb{B}\mathfrak{s}(\mathcal{B}_2) = U(\mathbb{B}\mathfrak{s}(\mathcal{B}))$;

**2′.** $\mathbb{T}\mathbf{m}(\mathcal{B}_2) = \mathbb{T}$;

**3′.** If $\widetilde{\omega}_1, \widetilde{\omega}_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B}_2)$ and $\mathsf{tm}(\widetilde{\omega}_1) \neq \mathsf{tm}(\widetilde{\omega}_2)$, then $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ are united by fate in $\mathcal{B}_2$ if and only if, there exist united by fate in $\mathcal{B}$ elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ such, that $\widetilde{\omega}_1 = U(\omega_1)$, $\widetilde{\omega}_2 = U(\omega_2)$.

Then, according to conditions 1′,2′ and conditions 1,2 of this Theorem for $U[\mathcal{B}, \mathbb{T}]$, we have $\mathbb{B}\mathfrak{s}(\mathcal{B}_2) = \mathbb{B}\mathfrak{s}(U[\mathcal{B}, \mathbb{T}])$, $\mathbb{T}\mathbf{m}(\mathcal{B}_2) = \mathbb{T}\mathbf{m}(U[\mathcal{B}, \mathbb{T}])$. Moreover from Condition 3′ and third condition of this Theorem, taking into account Property I.6.1(7) and Assertion I.7.3, we obtain the equality $\underset{\mathcal{B}_2}{\overset{\mathbb{B}\mathfrak{s}}{\leftarrow}} = \underset{U[\mathcal{B},\mathbb{T}]}{\overset{\mathbb{B}\mathfrak{s}}{\longleftarrow}}$. Hence, by Corollary I.6.1, we obtain, $\mathcal{B}_2 = U[\mathcal{B}, \mathbb{T}]$. □

**Definition I.11.2.** *The base changeable set $U[\mathcal{B}, \mathbb{T}]$, which satisfies the conditions 1,2,3 of Theorem I.11.1 will be named by the* ***image of the base changeable set*** *$\mathcal{B}$ relatively the* ***transforming mapping*** *$U$ and the time scale $\mathbb{T}$.*

*Remark* I.11.1. According to conditions U[B]1), U[B]2) in the proof of Theorem I.11.1 for any elementary-time states $\widetilde{\omega}_1$, $\widetilde{\omega}_2 \in \mathbb{B}\mathfrak{s}(U[\mathcal{B}, \mathbb{T}])$ the relation $\widetilde{\omega}_2 \underset{U[\mathcal{B},\mathbb{T}]}{\longleftarrow} \widetilde{\omega}_1$ is true if and only if $\widetilde{\omega}_1 = \widetilde{\omega}_2$ or $\mathsf{tm}(\widetilde{\omega}_1) < \mathsf{tm}(\widetilde{\omega}_2)$ and there exists united by fate in $\mathcal{B}$ elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ such, that $\widetilde{\omega}_1 = U(\omega_1)$, $\widetilde{\omega}_2 = U(\omega_2)$.

*Remark* I.11.2. In the case, when $\mathbb{T} = \mathbb{T}\mathbf{m}(\mathcal{B})$ we use the denotation $U[\mathcal{B}]$ instead of the denotation $U[\mathcal{B}, \mathbb{T}]$:

$$U[\mathcal{B}] := U[\mathcal{B}, \mathbb{T}\mathbf{m}(\mathcal{B})].$$

*Remark* I.11.3. Let $\mathcal{B}$ be any base changeable set and $\mathbb{I}_{\mathbb{B}\mathfrak{s}(\mathcal{B})} : \mathbb{B}\mathfrak{s}(\mathcal{B}) \mapsto \mathbf{Tm}(\mathcal{B}) \times \mathfrak{B}\mathfrak{s}(\mathcal{B})$ be the mapping, given by the formula: $\mathbb{I}_{\mathbb{B}\mathfrak{s}(\mathcal{B})}(\omega) = \omega \quad (\omega \in \mathbb{B}\mathfrak{s}(\mathcal{B}))$. Then the triple $(\mathbb{T}\mathbf{m}(\mathcal{B}), \mathfrak{B}\mathfrak{s}(\mathcal{B}), \mathbb{I}_{\mathbb{B}\mathfrak{s}(\mathcal{B})})$, is, apparently, evolution projector for $\mathcal{B}$. Moreover, if we substitute $\mathbb{T}\mathbf{m}(\mathcal{B})$ and $\mathcal{B}$ into Theorem I.11.1 instead of $\mathbb{T}$ and $U[\mathcal{B}, \mathbb{T}]$ (correspondingly), we can see, that all conditions of this Theorem are satisfied. Hence for the identity mapping $\mathbb{I}_{\mathbb{B}\mathfrak{s}(\mathcal{B})}$ (on $\mathbb{B}\mathfrak{s}(\mathcal{B})$), we obtain:

$$\mathbb{I}_{\mathbb{B}\mathfrak{s}(\mathcal{B})}[\mathcal{B}] = \mathcal{B}.$$

**Definition I.11.3.**

1. *The evolution projector* $(\mathbb{T}, \mathcal{X}, U)$ *(where* $\mathbb{T} = (\mathbf{T}, \leq)$*) for base changeable set* $\mathcal{B}$ *will be named as* ***injective***[10] *if and only if the mapping* $U$ *is injection from* $\mathbb{B}\mathfrak{s}(\mathcal{B})$ *to* $\mathbf{T} \times \mathcal{X}$ *(that is bijection from* $\mathbb{B}\mathfrak{s}(\mathcal{B})$ *onto the set* $\mathfrak{R}(U) \subseteq \mathbf{T} \times \mathcal{X}$*)*[11].

2. *Any indexed family* $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha) \mid \alpha \in \mathcal{A})$ *(where* $\mathcal{A} \neq \emptyset$*) of injective evolution projectors for base changeable set we name by* ***evolution multi-projector*** *for* $\mathcal{B}$.

**Theorem I.11.2** (on multi-image for changeable sets)**.** *Let,* $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha) \mid \alpha \in \mathcal{A})$ *be evolution multi-projector for base changeable set* $\mathcal{B}$*. Then only one changeable set* $\mathcal{Z}$ *exists, satisfying the following conditions:*

1. $\mathcal{L}k(\mathcal{Z}) = \{(\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]) \mid \alpha \in \mathcal{A}\}$.

2. *For any reference frames* $\mathfrak{l} = (\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]) \in \mathcal{L}k(\mathcal{Z})$, $\mathfrak{m} = (\beta, U_\beta[\mathcal{B}, \mathbb{T}_\beta]) \in \mathcal{L}k(\mathcal{Z})$ *(*$\alpha, \beta \in \mathcal{A}$*) and any set* $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l}) = U_\alpha(\mathbb{B}\mathfrak{s}(\mathcal{B}))$ *the following equality holds:*

$$\langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathcal{Z} \rangle A = U_\beta\left(U_\alpha^{[-1]}(A)\right) = \left\{U_\beta\left(U_\alpha^{[-1]}(\omega)\right) \mid \omega \in A\right\},$$

*where* $U_\alpha^{[-1]}$ *is the mapping,* ***inverse*** *to* $U_\alpha$.

*Remark* I.11.4. Suppose, that a changeable set $\mathcal{Z}$ satisfies condition 1 of Theorem I.11.2. Then for any reference frame $\mathfrak{l} = (\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]) \in \mathcal{L}k(\mathcal{Z})$, according to Property I.10.1(2), we have, $\mathtt{ind}(\mathfrak{l}) = \alpha$, $\mathfrak{l}^\wedge = U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]$, and hence, $\mathbb{B}\mathfrak{s}(\mathfrak{l}) = \mathbb{B}\mathfrak{s}(\mathfrak{l}^\wedge) = \mathbb{B}\mathfrak{s}(U_\alpha[\mathcal{B}, \mathbb{T}_\alpha])$. Therefore, by Theorem I.11.1, $\mathbb{B}\mathfrak{s}(\mathfrak{l}) = U_\alpha(\mathbb{B}\mathfrak{s}(\mathcal{B}))$. Thus, the condition 2 of Theorem I.11.2 is correctly formulated.

*Proof of Theorem I.11.2.* Let, $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha) \mid \alpha \in \mathcal{A})$ be evolution multi-projector for base changeable set $\mathcal{B}$.

By Definition I.11.3, for any $\alpha \in \mathcal{A}$ the triple $(\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha)$ is an injective evolution projector for $\mathcal{B}$. In accordance with Theorem I.11.1, we put:

$$\mathcal{B}_\alpha := U_\alpha[\mathcal{B}, \mathbb{T}_\alpha] \quad (\alpha \in \mathcal{A}).$$

Since $(\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha)$ is an injective evolution projector, then, by Definition I.11.3, the mapping $U_\alpha$ is one-to-one correspondence between $\mathbb{B}\mathfrak{s}(\mathcal{B})$ and $\mathfrak{R}(U)$. Hence, the inverse mapping $U_\alpha^{[-1]}$ exists (for all $\alpha \in \mathcal{A}$).

For any indexes $\alpha, \beta \in \mathcal{A}$ and any elementary-time state $\omega \in \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)$ we denote:

$$W_{\beta\alpha}(\omega) := U_\beta\left(U_\alpha^{[-1]}(\omega)\right) \tag{I.62}$$

(note, that, by Theorem I.11.1, $\mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha) = U_\alpha(\mathbb{B}\mathfrak{s}(\mathcal{B}))$). Hence, $W_{\beta\alpha}$ is the mapping from $\mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)$ into $\mathbb{B}\mathfrak{s}(\mathcal{B}_\beta) = U_\beta(\mathbb{B}\mathfrak{s}(\mathcal{B}))$.

It is easy to verify, that the family of mappings $\overleftarrow{W} = (W_{\beta\alpha} \mid \alpha, \beta \in \mathcal{A})$ possesses the properties (I.60). Therefore, using results of Subsection 11.1, we may denote:

$$\mathcal{Z} := \mathcal{Z}\mathsf{pv}\left(\overleftarrow{\mathcal{B}}, \overleftarrow{W}\right), \quad \text{where} \quad \overleftarrow{\mathcal{B}} = (\mathcal{B}_\alpha \mid \alpha \in \mathcal{A}). \tag{I.63}$$

Herewith, according to Property I.11.1(1), we obtain:

$$\mathcal{L}k(\mathcal{Z}) = \{(\alpha, \mathcal{B}_\alpha) \mid \alpha \in \mathcal{A}\} = \{(\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]) \mid \alpha \in \mathcal{A}\}, \tag{I.64}$$

and for arbitrary reference frames $\mathfrak{l} = (\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]) \in \mathcal{L}k(\mathcal{Z})$, $\mathfrak{m} = (\beta, U_\beta[\mathcal{B}, \mathbb{T}_\beta]) \in \mathcal{L}k(\mathcal{Z})$ (where $\alpha, \beta \in \mathcal{A}$) and for any set $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l}) = \mathbb{B}\mathfrak{s}(U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]) = U_\alpha(\mathbb{B}\mathfrak{s}(\mathcal{B}))$, by Property I.11.1(4) we obtain:

$$\langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathcal{Z} \rangle A = W_{\beta\alpha}(A) = U_\beta\left(U_\alpha^{[-1]}(A)\right). \tag{I.65}$$

[10] In previous works we used the term "***bijective*** evolution projector" instead of "injective...". But in the present paper we have made some clarifications in terminology.

[11] Here $\mathfrak{R}(U)$ means the ***range*** of (arbitrary) mapping $U$.

From (I.64) and (I.65) it follows, that the changeable set $\mathcal{Z}$ satisfies conditions 1,2 of Theorem I.11.2.

Suppose, that the changeable set $\mathcal{Z}_1$ also satisfies conditions 1,2 of Theorem I.11.2. Then, by the condition 1, $\mathcal{L}k(\mathcal{Z}) = \mathcal{L}k(\mathcal{Z}_1)$. Also, by the condition 2, for arbitrary reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{Z}) = \mathcal{L}k(\mathcal{Z}_1)$ it is true the equality: $\langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathcal{Z}\rangle = \langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathcal{Z}_1\rangle$. Hence, by Assertion I.10.1, we get $\mathcal{Z} = \mathcal{Z}_1$. Thus, changeable set, satisfying the conditions 1,2 of Theorem I.11.2 is unique. □

**Definition I.11.4.** *Let $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha) \mid \alpha \in \mathcal{A})$ be an evolution multi-projector for base changeable set $\mathcal{B}$. Changeable set $\mathcal{Z}$, satisfying conditions 1,2 of Theorem I.11.2 will be referred to as* ***evolution multi-image*** *of base changeable set $\mathcal{B}$ relatively the evolution multi-projector $\mathfrak{P}$. This evolution multi-image will be denoted by $\mathcal{Z}\mathsf{im}[\mathfrak{P}, \mathcal{B}]$:*

$$\mathcal{Z}\mathsf{im}[\mathfrak{P}, \mathcal{B}] := \mathcal{Z}.$$

*Remark* I.11.5. From Equality (I.63) in the proof of Theorem I.11.2 it follows that any changeable set of kind $\mathcal{Z}\mathsf{im}[\mathfrak{P}, \mathcal{B}]$ ($\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha) \mid \alpha \in \mathcal{A})$) may be represented in the form:

$$\mathcal{Z}\mathsf{im}[\mathfrak{P}, \mathcal{B}] = \mathcal{Z}\mathsf{pv}\left(\overleftarrow{\mathcal{B}}_{\mathfrak{P}}, \overleftarrow{W}_{\mathfrak{P}}\right), \qquad \text{where} \tag{I.66}$$

$$\overleftarrow{\mathcal{B}}_{\mathfrak{P}} = \left(\mathcal{B}_\alpha^{(\mathfrak{P})} \mid \alpha \in \mathcal{A}\right); \quad \overleftarrow{W}_{\mathfrak{P}} = \left(W_{\beta\alpha}^{(\mathfrak{P})} \mid \alpha, \beta \in \mathcal{A}\right), \quad \text{and}$$
$$\mathcal{B}_\alpha^{(\mathfrak{P})} = U_\alpha[\mathcal{B}, \mathbb{T}_\alpha] \quad (\alpha \in \mathcal{A});$$
$$W_{\beta\alpha}^{(\mathfrak{P})}(\omega) = U_\beta\left(U_\alpha^{[-1]}(\omega)\right) \quad (\alpha, \beta \in \mathcal{A})$$

---

Let $\mathcal{Z}$ be any changeable set and $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ be any reference frame of $\mathcal{Z}$. We say, that elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$ are ***united by fate*** in the reference frame $\mathfrak{l}$ of changeable set $\mathcal{Z}$, if and only if thay are united by fate in the base changeable set $\mathfrak{l}^\wedge$.

From theorems I.11.2 and I.11.1, taking into account Property I.6.1(9), Property I.10.1(1) and Remark I.10.1, we immediately deduce the following properties of multi-image for base changeable set.

**Properties I.11.2.** *Let $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha) \mid \alpha \in \mathcal{A})$, where $\mathbb{T}_\alpha = (\mathbf{T}_\alpha, \leq_\alpha)$ ($\alpha \in \mathcal{A}$) be an evolution multi-projector for base changeable set $\mathcal{B}$ and $\mathcal{Z} = \mathcal{Z}\mathsf{im}[\mathfrak{P}, \mathcal{B}]$. Then:*

1. $\mathcal{L}k(\mathcal{Z}) = \{(\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]) \mid \alpha \in \mathcal{A}\}$.
2. $\mathcal{I}nd(\mathcal{Z}) = \mathcal{A}$.
3. *For any reference frame $\mathfrak{l} = (\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha])$ the following equalities hold:*

$$\mathbb{B}\mathfrak{s}(\mathfrak{l}) = U_\alpha(\mathbb{B}\mathfrak{s}(\mathcal{B})) = \{U_\alpha(\omega) \mid \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})\};$$
$$\mathfrak{B}\mathfrak{s}(\mathfrak{l}) = \{\mathsf{bs}(U_\alpha(\omega)) \mid \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})\};$$
$$\mathbb{T}\mathbf{m}(\mathfrak{l}) = \mathbb{T}_\alpha; \quad \mathbf{Tm}(\mathfrak{l}) = \mathbf{T}_\alpha; \quad \leq_{\mathfrak{l}} = \leq_\alpha.$$

4. *Let, $\mathfrak{l} = (\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]) \in \mathcal{L}k(\mathcal{Z})$, where $\alpha \in \mathcal{A}$. Suppose, that $\widetilde{\omega}_1, \widetilde{\omega}_2 \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$ and $\mathsf{tm}(\widetilde{\omega}_1) \neq \mathsf{tm}(\widetilde{\omega}_2)$. Then $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ are united by fate in $\mathfrak{l}$ if and only if there exist united by fate in $\mathcal{B}$ elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ such, that $\widetilde{\omega}_1 = U_\alpha(\omega_1)$, $\widetilde{\omega}_2 = U_\alpha(\omega_2)$.*

***Example*** I.11.2. Let $\mathcal{B}$ be a base changeable set, and $X$ — an arbitrary set such, that $\mathfrak{B}\mathfrak{s}(\mathcal{B}) \subseteq X$. And let $\mathbb{U}$ be any set of bijections (one-to-one correspondences) of kind:

$$U : \mathbf{Tm}(\mathcal{B}) \times X \longmapsto \mathbf{Tm}(\mathcal{B}) \times X \qquad (U \in \mathbb{U})$$

Such set of bijections $\mathbb{U}$ is named by ***transforming set of bijections*** relatively the base changeable set $\mathcal{B}$ on $X$.

By Definition I.11.1, any mapping $U \in \mathbb{U}$ generates the evolution projector, $\left(\mathbb{T}\mathbf{m}(\mathcal{B}), X, U_{\restriction \mathbb{B}\mathfrak{s}(\mathcal{B})}\right)$, where $U_{\restriction \mathbb{B}\mathfrak{s}(\mathcal{B})}$ is the restriction of the mapping $U$ onto the set $\mathbb{B}\mathfrak{s}(\mathcal{B}) \subseteq \mathbb{T}\mathbf{m}(\mathcal{B}) \times X$. Henceforth, where it does not cause confusion, we identify the mapping $U_{\restriction \mathbb{B}\mathfrak{s}(\mathcal{B})}$ with the mapping $U$. Under this identification, we can consider, that $\left(\mathbb{T}\mathbf{m}(\mathcal{B}), X, U_{\restriction \mathbb{B}\mathfrak{s}(\mathcal{B})}\right) = (\mathbb{T}\mathbf{m}(\mathcal{B}), X, U))$. Hence, the indexed family:

$$\mathfrak{P}_{\mathcal{B}}[\mathbb{U}] = ((\mathbb{T}\mathbf{m}(\mathcal{B}), X, U) \mid U \in \mathbb{U})$$

is evolution multi-projector for $\mathcal{B}$. In this particular case we obtain the changeable set:

$$\mathcal{Z}\mathsf{im}\,(\mathbb{U}, \mathcal{B}) = \mathcal{Z}\mathsf{im}\,[\mathfrak{P}_{\mathcal{B}}[\mathbb{U}], \mathcal{B}]\,. \tag{I.67}$$

**Definition I.11.5.** *Changeable set* $\mathcal{Z}\mathsf{im}\,(\mathbb{U}, \mathcal{B})$ *will be named* **multi-figurative image** *of the base changeable set* $\mathcal{B}$ *relatively the transforming set of mappins* $\mathbb{U}$.

***Example*** I.11.3. Let $\mathcal{B}$ be a base changeable set such, that

$$\mathfrak{B}\mathfrak{s}(\mathcal{B}) \subseteq \mathbb{R}^3, \quad \mathbb{T}\mathbf{m}(\mathcal{B}) = \mathbb{R}_{ord} = (\mathbb{R}, \leq)\,,$$

where $\leq$ is the standard linear order relation on the real numbers. Such base changeable set $\mathcal{B}$ must exist, because, for example, we may denote $\mathcal{B} := \mathcal{A}t\,(\mathbb{R}_{ord}, \mathcal{R})$, where $\mathcal{R}$ is a system of abstract trajectories from $\mathbb{R}_{ord}$ to the subset $M \subseteq \mathbb{R}^3$. Let us consider Poincare group $\mathbb{U} = P(1,3,c)$, defined on the 4-dimensional space-time $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3 \supseteq \mathbb{T}\mathbf{m}(\mathcal{B}) \times \mathfrak{B}\mathfrak{s}(\mathcal{B})$, that is the group of affine transformations of the space $\mathbb{R}^4$, which are satisfying the following conditions:

1. Any transformation $\boldsymbol{P} \in P(1,3,c)$ leaves unchanged values of the Lorentz-Minkowski pseudo-distance on $\mathbb{R}^4$:

$$\begin{aligned}
\mathsf{M}_c\,(\boldsymbol{P}\mathrm{w}_1 - \boldsymbol{P}\mathrm{w}_2) &= \mathsf{M}_c\,(\mathrm{w}_1 - \mathrm{w}_2)\,, \quad (\forall\, \mathrm{w}_1, \mathrm{w}_2 \in \mathbb{R}^4), \quad \text{where;} \\
\mathsf{M}_c\,(\mathrm{w}) &= \sum_{j=1}^{3} w_i^2 - c^2 w_0^2 \quad \text{and} \\
\mathrm{w} - \widetilde{\mathrm{w}} &= (w_0 - \widetilde{w}_0,\ w_1 - \widetilde{w}_1,\ w_2 - \widetilde{w}_2,\ w_3 - \widetilde{w}_3) \\
&\quad \left(\mathrm{w} = (w_0, w_1, w_2, w_3) \in \mathbb{R}^4,\ \widetilde{\mathrm{w}} = (\widetilde{w}_0, \widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3) \in \mathbb{R}^4\right).
\end{aligned}$$

   Here the number $c$ means any fixed positive real constant, which has the physical content of the speed of light in vacuum.

2. Any transformation $\boldsymbol{P} \in P(1,3,c)$ has positive direction of time, that is $\boldsymbol{P}\mathrm{w}_2 - \boldsymbol{P}\mathrm{w}_1 \in \mathcal{M}_{c,+}\,(\mathbb{R}^3)$ for any $\mathrm{w}_1, \mathrm{w}_2 \in \mathbb{R}^4$ such, that $\mathrm{w}_2 - \mathrm{w}_1 \in \mathcal{M}_{c,+}\,(\mathbb{R}^3)$, where

$$\mathcal{M}_{c,+}\left(\mathbb{R}^3\right) = \left\{\mathrm{w} = (w_0, w_1, w_2, w_3) \in \mathbb{R}^4 \mid w_0 > 0,\ \mathsf{M}_c\,(\mathrm{w}) < 0\right\}$$

   (Cf. [45]).

Poincare group $\mathbb{U} = P(1,3,c)$ is transforming set of bijections relatively the base changeable set $\mathcal{B}$ on $\mathbb{R}^3$. Hence, we obtain the changeable set $\mathcal{Z}\mathsf{im}\,(P(1,3), \mathcal{B})$, which represents a mathematically strict model of the cinematics of special relativity theory in the inertial frames of reference. Note that this model does not formally prohibit the existence of tachyon transformations, because elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B}) \subseteq \mathbb{R} \times \mathbb{R}^3$ may exist such, that $\omega_2 \leftarrow \omega_1$ and $\mathsf{M}_c\,(\omega_1; \omega_2) > 0$.

### 11.3 Other Examples of Changeable Sets

In all previous examples the unification mappings $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle$ between reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$ of a changeable set $\mathcal{Z}$ are defined by means of bijections (one-to-one correspondences) between the sets of elementary-time states $\mathbb{B}\mathfrak{s}(\mathfrak{l})$ and $\mathbb{B}\mathfrak{s}(\mathfrak{m})$ (that is for any $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$ the unification mapping $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A$ can be represented in the form:

$$\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A = \bigcup_{\omega \in A} \{\mathcal{W}_{\mathfrak{m},\mathfrak{l}}(\omega)\},$$

where the mapping $\mathcal{W}_{\mathfrak{m},\mathfrak{l}} : \mathbb{B}\mathfrak{s}(\mathfrak{l}) \mapsto \mathbb{B}\mathfrak{s}(\mathfrak{m})$ is bijection between $\mathbb{B}\mathfrak{s}(\mathfrak{l})$ and $\mathbb{B}\mathfrak{s}(\mathfrak{m})$). In all these examples the third condition of Definition I.10.1 may be replaced by more strong condition (I.57). But really the conditions of Definition I.10.1 are enough general. The last thesis will be confirmed by the following examples.

***Example*** I.11.4. Let $\overleftarrow{\mathcal{B}} = (\mathcal{B}_\alpha \mid \alpha \in \mathcal{A})$ be any indexed family of base changeable sets. Denote:

$$\mathfrak{U}_{\beta\alpha} A := \begin{cases} A, & \alpha = \beta \\ \emptyset, & \alpha \neq \beta \end{cases}, \qquad \alpha, \beta \in \mathcal{A},\ A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha).$$

It is easy to verify, that the family of mappings $\overleftarrow{\mathfrak{U}} = (\mathfrak{U}_{\beta\alpha} \mid \alpha, \beta \in \mathcal{A})$ satisfies all conditions of Definition I.10.1. Therefore, the triple

$$\mathcal{Z}\mathsf{nv}\left(\overleftarrow{\mathcal{B}}\right) = \left(\mathcal{A}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathfrak{U}}\right)$$

is a changeable set.

The changeable set $\mathcal{Z}\mathsf{nv}\left(\overleftarrow{\mathcal{B}}\right)$ will be named the ***fully invisible changeable set***, generated by the system of base changeable sets $\overleftarrow{\mathcal{B}}$.

Note, that any base changeable set $\mathcal{B}$ can be identified with the changeable set of kind $\mathcal{Z}\mathsf{nv}\left(\overleftarrow{\mathcal{B}}\right)$, where $\mathcal{A} = \{1\}$, $\mathcal{B}_1 = \mathcal{B}$ and $\overleftarrow{\mathcal{B}} = (\mathcal{B}_\alpha \mid \alpha \in \mathcal{A}) = (\mathcal{B}_1)$.

***Example*** I.11.5. Let, $\overleftarrow{\mathcal{B}} = (\mathcal{B}_1, \mathcal{B}_2) = (\mathcal{B}_\alpha \mid \alpha \in \mathcal{A})$ (where $\mathcal{A} = \{1, 2\}$) be a family of two base changeable sets. Choose any elementary-time state $\omega \in \mathbb{B}\mathfrak{s}(\mathcal{B}_2)$. According to Property I.6.1(2), $\mathbb{B}\mathfrak{s}(\mathcal{B}_2) \neq \emptyset$. Therefore elementary-time state $\omega \in \mathbb{B}\mathfrak{s}(\mathcal{B}_2)$ must exist. Denote:

$$\mathfrak{U}_{11} A := A, \quad A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_1); \qquad \mathfrak{U}_{22} A := A, \quad A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_2);$$

$$\mathfrak{U}_{21} A := \begin{cases} \emptyset, & A \neq \mathbb{B}\mathfrak{s}(\mathcal{B}_1) \\ \{\omega\}, & A = \mathbb{B}\mathfrak{s}(\mathcal{B}_1) \end{cases}, \qquad A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_1);$$

$$\mathfrak{U}_{12} A := \begin{cases} \emptyset, & \omega \notin A \\ \mathbb{B}\mathfrak{s}(\mathcal{B}_1), & \omega \in A \end{cases}, \qquad A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_2);$$

1. Since $\mathfrak{U}_{11}, \mathfrak{U}_{22}$ are identity mappings of sets, the first condition of Definition I.10.1 is performed by a trivial way. For the same reason the second condition of this Definition also is satisfied in the case $\alpha = \beta$.

2. Suppose, that $\alpha, \beta \in \mathcal{A} = \{1, 2\}$, $A, B \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)$, $A \subseteq B$. According to remark, made in the previous item, it is enough to consider only the case $\alpha \neq \beta$. Thus, we have the next two subcases.

2.a) $\alpha = 1$, $\beta = 2$. In the case $A \neq \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$ we obtain $\mathfrak{U}_{\beta\alpha} A = \mathfrak{U}_{21} A = \emptyset \subseteq \mathfrak{U}_{\beta\alpha} B$, and in the case $A = \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$, since $A \subseteq B$ we have $B = \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$, and, therefore, $\mathfrak{U}_{\beta\alpha} A = \mathfrak{U}_{\beta\alpha} B$.

2.b) $\alpha = 2$, $\beta = 1$. In the case $\omega \notin A$ we obtain $\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{12}A = \emptyset \subseteq \mathfrak{U}_{\beta\alpha}B$. In the case $\omega \in A$ from the condition $A \subseteq B$ it follows, that $\omega \in B$, so $\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{12}A = \mathbb{B}\mathfrak{s}(\mathcal{B}_1) = \mathfrak{U}_{12}B = \mathfrak{U}_{\beta\alpha}B$.

3. Let $\alpha, \beta, \gamma \in \mathcal{A} = \{1, 2\}$, $A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha)$. We consider the following cases.

3.a) $\alpha = \beta$. In this case $\mathfrak{U}_{\beta\alpha}A = A$. Consequently:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{\gamma\beta}A = \mathfrak{U}_{\gamma\alpha}A.$$

3.b) $\beta = \gamma$. In this case $\mathfrak{U}_{\gamma\beta}S = S$, $S \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_\beta)$. Hence:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{\gamma\alpha}A.$$

3.c) $\alpha \neq \beta \neq \gamma$. Since the set $\mathcal{A}$ is two-element, this case can be divided into the following two subcases:

3.c.1) Let $\alpha = 1$, $\beta = 2$, $\gamma = 1$. Then in the case $A \neq \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$ we obtain:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{12}\mathfrak{U}_{21}A = \mathfrak{U}_{12}\emptyset = \emptyset \subseteq \mathfrak{U}_{\gamma\alpha}A,$$

and in the case $A = \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$ we calculate:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{12}\mathfrak{U}_{21}A = \mathfrak{U}_{12}\{\omega\} = \mathbb{B}\mathfrak{s}(\mathcal{B}_1) = A = \mathfrak{U}_{\gamma\alpha}A.$$

3.c.2) Let, $\alpha = 2$, $\beta = 1$, $\gamma = 2$. Then in the case $\omega \notin A$ we have:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{21}\mathfrak{U}_{12}A = \mathfrak{U}_{21}\emptyset = \emptyset \subseteq \mathfrak{U}_{\gamma\alpha}A,$$

and in the case $\omega \in A$ we obtain:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{21}\mathfrak{U}_{12}A = \mathfrak{U}_{21}\mathbb{B}\mathfrak{s}(\mathcal{B}_1) = \{\omega\} \subseteq A = \mathfrak{U}_{\gamma\alpha}A.$$

Consequently, the triple:

$$\mathcal{Z}_1 = \left(\mathcal{A}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathfrak{U}}\right),$$

where $\overleftarrow{\mathfrak{U}} = (\mathfrak{U}_{\beta\alpha} \mid \alpha, \beta \in \mathcal{A})$ is a changeable set.

***Example*** I.11.6. Let $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\omega$ be the same as in Example I.11.5. Also, similarly to previous Example I.11.5, $\mathfrak{U}_{11}$ and $\mathfrak{U}_{22}$ are the identical mappings of the sets. Now, we denote:

$$\mathfrak{U}_{21}A := \begin{cases} \emptyset, & A = \emptyset \\ \{\omega\}, & A \neq \emptyset \end{cases}, \qquad A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_1);$$
$$\mathfrak{U}_{12}A := \emptyset, \qquad A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_2).$$

1,2. Since, $\mathfrak{U}_{11}$ and $\mathfrak{U}_{22}$, are the identical mappings of the sets, the first condition of Definition I.10.1 is satisfied by a trivial way. The second condition of this Definition also is easy to verify.

3. In the cases $\alpha = \beta = \gamma$, $\alpha \neq \beta = \gamma$, $\alpha = \beta \neq \gamma$ verification of the third condition of Definition I.10.1 is the same, as in Example I.11.5. Thus it remains to consider the case $\alpha \neq \beta \neq \gamma$. Like the previous example we divide this case into the following two subcases:

3.1) Let, $\alpha = 1$, $\beta = 2$, $\gamma = 1$. Then:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{12}\mathfrak{U}_{21}A = \emptyset \subseteq \mathfrak{U}_{\gamma\alpha}A.$$

3.2) Let, $\alpha = 2$, $\beta = 1$, $\gamma = 2$. Then:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{21}\mathfrak{U}_{12}A = \mathfrak{U}_{21}\emptyset = \emptyset \subseteq \mathfrak{U}_{\gamma\alpha}A.$$

Thus, the triple:

$$\mathcal{Z}_2 = \left(\mathcal{A}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathfrak{U}}\right),$$

is a changeable set.

Main results of this Section were published in [8, Subsection 3.4]. Theorem I.11.2 (in the present form) is published in [13].

# 12 Visibility in Changeable Sets

## 12.1 Gradations of Visibility

**Definition I.12.1.** *Let $\mathcal{Z}$ be any changeable set, and $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$ be any reference frames of $\mathcal{Z}$. We say, that a changeable system $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$ of the reference frame $\mathfrak{l}$ is:*

1. ***visible*** *(partially visible) from the reference frame $\mathfrak{m}$, if and only if $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A \neq \emptyset$;*
2. ***normally visible*** *from the reference frame $\mathfrak{m}$, if and only if $A \neq \emptyset$ and arbitrary nonempty subsystem $B \subseteq A$ of the changeable system $A$ is visible from $\mathfrak{m}$ (that is $\forall B : \emptyset \neq B \subseteq A$ $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle B \neq \emptyset$);*
3. ***precisely visible*** *from $\mathfrak{m}$, if and only if:*
   (a) *$A$ is normally visible from $\mathfrak{m}$;*
   (b) *for any family $\{A_\alpha \,|\, \alpha \in \mathcal{A}\} \subseteq 2^A$ of changeable subsystems $A$ such, that $\bigsqcup_{\alpha\in\mathcal{A}} A_\alpha = A$ the following equality holds*
   $$\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A = \bigsqcup_{\alpha\in\mathcal{A}} \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A_\alpha,$$
   *where $\bigsqcup_{\alpha\in\mathcal{A}} A_\alpha$ denotes the disjoint union of the family of sets $\{A_\alpha \,|\, \alpha \in \mathcal{A}\}$, that is the union $\bigcup_{\alpha\in\mathcal{A}} A_\alpha$, with additional condition $A_\alpha \cap A_\beta = \emptyset$, $\alpha \neq \beta$.*
4. ***invisible*** *from the reference frame $\mathfrak{m}$, if and only if $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A = \emptyset$;*

*Remark* I.12.1. It is apparently, that the precise visibility of the changeable system $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$ ($\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$) from the reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$ involves the normal visibility of $A$ from $\mathfrak{m}$, and the normal visibility of any changeable system $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$ from $\mathfrak{m}$ involves it's visibility (partial visibility) from $\mathfrak{m}$.

**Assertion I.12.1.** *For any changeable set $\mathcal{Z}$ the following properties of visibility of changeable systems are true:*

1. *Empty changeable system $\emptyset \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$ always is invisible from any reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$.*
2. *Any nonempty changeable system $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$, $A \neq \emptyset$ always is precisely visible from its own reference frame $\mathfrak{l}$.*
3. *If a changeable system $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$ (where $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$) includes a subsystem $B \subseteq A$, which is visible from reference frames $\mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$, then the changeable system $A$ also is visible from $\mathfrak{m}$.*
4. *If a changeable system $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$ is normally visible (precisely visible) from reference frame $\mathfrak{m}$, then any nonempty subsystem $B \subseteq A$, $B \neq \emptyset$ of changeable system $A$ also is normally visible (precisely visible) from $\mathfrak{m}$.*

*Proof.* Statements 1,2,3 of this Assertion follow from Assertion I.10.2 and Properties I.10.1 of changeable sets. Statement 4 for the case of normal visibility is trivial. Thus, it remains to prove Statement 4 for the case of precise visibility. Let a changeable system $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$ be precisely visible from the reference frame $\mathfrak{m}$. Consider any changeable system $B$ such, that $\emptyset \neq B \subseteq A$. Since precise visibility involves the normal visibility, $B$ is normally visible from $\mathfrak{m}$. Suppose, that $B = \bigsqcup_{\alpha\in\mathcal{A}} B_\alpha$. Using the equalities:

$$A = B \sqcup (A \setminus B); \qquad A = \bigsqcup_{\alpha\in\mathcal{A}} B_\alpha \sqcup (A \setminus B),$$

and taking into account precise visibility of the changeable system $A$ from $\mathfrak{m}$, we obtain:

$$\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A = \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle B \sqcup \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle (A \setminus B);$$
$$\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A = \bigsqcup_{\alpha \in \mathcal{A}} \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle B_\alpha \sqcup \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle (A \setminus B).$$

Consequently, $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle B \sqcup \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle (A \setminus B) = \bigsqcup_{\alpha \in \mathcal{A}} \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle B_\alpha \sqcup \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle (A \setminus B)$. Hence:

$$\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle B = \bigsqcup_{\alpha \in \mathcal{A}} \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle B_\alpha.$$

Thus, $B$ is precisely visible from $\mathfrak{m}$. □

**Definition I.12.2.** *We say, that a reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ is:*

1. ***visible** (partially visible) from the reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$ (denotation is $\mathfrak{l} \succ \mathfrak{m}(\mathcal{Z})$), if and only if there exists at least one visible from the $\mathfrak{m}$ changeable system $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$ (that is $\exists A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})\ \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A \neq \emptyset$).*
2. ***normally visible** from the reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$ (denotation is $\mathfrak{l} \succ! \mathfrak{m}(\mathcal{Z})$), if and only if any nonempty changeable system $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$ ($A \neq \emptyset$) is normally visible from the $\mathfrak{m}$.*
3. ***precisely visible** from $\mathfrak{m}$ (denotation is $\mathfrak{l} \succ!! \mathfrak{m}(\mathcal{Z})$), if and only if any nonempty changeable system $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$ ($A \neq \emptyset$) is precisely visible from the reference frame $\mathfrak{m}$.*
4. ***invisible** from the reference frame $\mathfrak{m}$, if and only if any changeable system $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$ is invisible from the $\mathfrak{m}$.*

In the case, when the changeable set $\mathcal{Z}$ is known in advance in the denotations $\mathfrak{l} \succ \mathfrak{m}(\mathcal{Z})$, $\mathfrak{l} \succ! \mathfrak{m}(\mathcal{Z})$, $\mathfrak{l} \succ!! \mathfrak{m}(\mathcal{Z})$ the sequence of symbols "$(\mathcal{Z})$" will be omitted, and the denotations $\mathfrak{l} \succ \mathfrak{m}$, $\mathfrak{l} \succ! \mathfrak{m}$, $\mathfrak{l} \succ!! \mathfrak{m}$ will be used instead.

*Remark* I.12.2. From Remark I.12.1 it follows, that for the reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$ the next propositions are true

- if $\mathfrak{l} \succ!! \mathfrak{m}$, then $\mathfrak{l} \succ! \mathfrak{m}$;
- if $\mathfrak{l} \succ! \mathfrak{m}$, then $\mathfrak{l} \succ \mathfrak{m}$.

Thus, precise visibility involves the normal visibility and normal visibility involves visibility (partial visibility). Example I.11.5 shows, that visibility does not involve the normal visibility. Indeed, we may consider the case, when $\mathbf{card}(\mathbb{B}\mathfrak{s}(\mathcal{B}_1)) \geq 2$. In this case for the reference frames $\mathfrak{l}_1 = (1, \mathcal{B}_1)$, $\mathfrak{l}_2 = (2, \mathcal{B}_2)$ we have, that the changeable system $\mathbb{B}\mathfrak{s}(\mathfrak{l}_1) = \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$ is visible from $\mathfrak{l}_2$, but it is not normally visible from $\mathfrak{l}_2$, because any subset $A \subset \mathbb{B}\mathfrak{s}(\mathfrak{l}_1) = \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$ ($A \neq \mathbb{B}\mathfrak{s}(\mathcal{B}_1)$) is invisible from $\mathfrak{l}_2$. Thus, in the case $\mathbf{card}(\mathbb{B}\mathfrak{s}(\mathcal{B}_1)) \geq 2$ we obtain $\mathfrak{l}_1 \succ \mathfrak{l}_2$, but **not** $\mathfrak{l}_1 \succ! \mathfrak{l}_2$.

Example I.11.6 shows, that normal visibility does not involve the precise visibility. In this Example any nonempty changeable system $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l}_1)$ ($\mathfrak{l}_1 = (1, \mathcal{B}_1)$) is normally visible from the reference frame $\mathfrak{l}_2 = (2, \mathcal{B}_2)$. But, in the case $\mathbf{card}(A) \geq 2$ the changeable system $A$ is not precisely visible from $\mathfrak{l}_2$, because in this case there exist nonempty sets $A_1, A_2 \subseteq A$ such, that $A_1 \sqcup A_2 = A$, but the images of these sets ($\langle \mathfrak{l}_2 \leftarrow \mathfrak{l}_1 \rangle A_1 = \mathfrak{U}_{21} A_1 = \{\omega\}$, $\langle \mathfrak{l}_2 \leftarrow \mathfrak{l}_1 \rangle A_2 = \mathfrak{U}_{21} A_2 = \{\omega\}$) are not disjoint. Thus, in the case $\mathbf{card}(\mathbb{B}\mathfrak{s}(\mathcal{B}_1)) \geq 2$ we have $\mathfrak{l}_1 \succ! \mathfrak{l}_2$, but **not** $\mathfrak{l}_1 \succ!! \mathfrak{l}_2$.

Further it will be proved that in examples I.11.1, I.11.2 and I.11.3 any reference frame of the changeable sets $\mathcal{Z}\mathsf{pv}\left(\overleftarrow{\mathcal{B}}, \overleftarrow{W}\right)$ and $\mathcal{Z}\mathsf{im}(\mathbb{U}, \mathcal{B})$ is precisely visible from another frame (see assertions I.12.6 and I.12.7 below).

The next three assertions immediately follow from definitions I.12.2, I.12.1 and Assertion I.12.1.

**Assertion I.12.2.** *For any changeable set $\mathcal{Z}$ the next propositions are equivalent:*

**(Vi1)** *Reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ is visible from reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$ $(\mathfrak{l} \succ \mathfrak{m})$.*

**(Vi2)** *The set $\mathbb{B}\mathfrak{s}(\mathfrak{l})$ of all elementary-time states of $\mathfrak{l}$ is visible from $\mathfrak{m}$.*

**Assertion I.12.3.** *For an arbitrary changeable set $\mathcal{Z}$ the following propositions are equivalent:*

**(nVi1)** *Reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ is normally visible from reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$ $(\mathfrak{l} \succ! \, \mathfrak{m})$.*

**(nVi2)** *The set $\mathbb{B}\mathfrak{s}(\mathfrak{l})$ of all elementary-time states of $\mathfrak{l}$ is normally visible from $\mathfrak{m}$.*

**(nVi3)** *Any nonempty changeable system $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$ is visible from $\mathfrak{m}$ $(\forall \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$ $(A \neq \emptyset \Rightarrow \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A \neq \emptyset))$.*

**Assertion I.12.4.** *Let $\mathcal{Z}$ — be an arbitrary changeable set. Then:*

*1. Any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ is precisely visible from itself (that is $\forall \mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ $\mathfrak{l} \succ!! \, \mathfrak{l}$).*

*2. The following propositions are equivalent:*

**(pVi1)** *Reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ is precisely visible from reference frame $\mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$ $(\mathfrak{l} \succ!! \, \mathfrak{m})$.*

**(pVi2)** *The set $\mathbb{B}\mathfrak{s}(\mathfrak{l})$ of all elementary-time states of $\mathfrak{l}$ is precisely visible from $\mathfrak{m}$.*

**Assertion I.12.5.** *For any changeable set $\mathcal{Z}$ the binary relation $\succ!$ quasi order*[12] *on the set $\mathcal{L}k(\mathcal{Z})$ of all reference frames of $\mathcal{Z}$.*

*Proof.* Reflexivity of the relation $\succ!$ follows from the first item of Assertion I.12.4 and from Remark I.12.2. Thus, we only need to prove the transitivity of the relation $\succ!$.

Suppose, that $\mathfrak{l} \succ! \, \mathfrak{m}$ and $\mathfrak{m} \succ! \, \mathfrak{p}$, where $\mathfrak{l}, \mathfrak{m}, \mathfrak{p} \in \mathcal{L}k(\mathcal{Z})$. Then, using Assertion I.12.3 (equivalence between (nVi1) and (nVi3)), for any nonempty changeable system $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$ we obtain, $\langle \mathfrak{p} \leftarrow \mathfrak{l} \rangle A \supseteq \langle \mathfrak{p} \leftarrow \mathfrak{m} \rangle \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A \neq \emptyset$, thus, by Assertion I.12.3, $\mathfrak{l} \succ! \, \mathfrak{p}$. □

*Remark* I.12.3. First item of Assertion I.12.4 together with Remark I.12.2 also bring about the reflexivity of relations $\succ!!$ and $\succ$ on the set $\mathcal{L}k(\mathcal{Z})$ (for any changeable set $\mathcal{Z}$). But these relations, in general, are not transitive. And the next examples explain the last statement.

***Example*** I.12.1. Let $\mathcal{B}$ be any base changeable set. We consider the family $\overleftarrow{\mathcal{B}} = (\mathcal{B}_\alpha \mid \alpha \in \mathbb{N})$ of base changeable sets, which is defined as follows:

$$\mathcal{B}_\alpha := \mathcal{B}, \qquad \alpha \in \mathbb{N}.$$

For $\alpha, \beta \in \mathbb{N}$ we define the mappings $\mathfrak{U}_{\beta\alpha} : \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha) \mapsto \mathbb{B}\mathfrak{s}(\mathcal{B}_\beta)$ by the following way:

$$\mathfrak{U}_{\beta\alpha} A := \begin{cases} A, & \beta \in \{\alpha, \alpha+1\}; \\ \mathbb{B}\mathfrak{s}(\mathcal{B}), & \beta > \alpha+1, \ A \neq \emptyset; \\ \emptyset, & \beta > \alpha+1, \ A = \emptyset; \\ \emptyset, & \beta < \alpha, \end{cases} \qquad (A \in \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha) = \mathbb{B}\mathfrak{s}(\mathcal{B}), \ n \in \mathbb{N}) \qquad \text{(I.68)}$$

(where the symbols $<, >$ denote the usual order on the set of natural numbers).

We shell prove, that the system of mappings $\overleftarrow{\mathfrak{U}} = (\mathfrak{U}_{\beta\alpha} \mid \alpha, \beta \in \mathbb{N})$ is unification of perception.

[12] About quasi order relation see footnote 2.

The first two conditions of Definition I.10.1 for the system of mappings $\overleftarrow{\mathfrak{U}}$ are performed by a trivial way. Thus, we need to verify the third condition of this Definition. Let $\alpha, \beta, \gamma \in \mathbb{N}$ and $A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha) = \mathbb{B}\mathfrak{s}(\mathcal{B})$. Then in the case $\alpha \leq \beta \leq \gamma$, by (I.68), we obtain:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \begin{cases} \emptyset, & A = \emptyset; \\ A, & A \neq \emptyset,\ \beta \in \{\alpha, \alpha+1\},\ \gamma \in \{\beta, \beta+1\}; \\ \mathbb{B}\mathfrak{s}(\mathcal{B}), & A \neq \emptyset, \text{ and } (\beta > \alpha + 1 \text{ or } \gamma > \beta + 1). \end{cases} \tag{I.69}$$

Since $\mathfrak{U}_{\gamma\alpha}A \in \{A, \mathbb{B}\mathfrak{s}(\mathcal{B})\}$ for $\alpha \leq \gamma$, in the first two cases of the formula (I.69) the inclusion $\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A \subseteq \mathfrak{U}_{\gamma\alpha}A$ holds. In the third case of the formula (I.69) we have $\gamma > \alpha + 1$, and hence, $\mathfrak{U}_{\gamma\alpha}A = \mathbb{B}\mathfrak{s}(\mathcal{B})$. Thus, in this case, the last inclusion also is performed. If the condition $\alpha \leq \beta \leq \gamma$ is not satisfied, we have $\alpha > \beta$ or $\beta > \gamma$. Therefore, by the formula (I.68), we have, $\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \emptyset$. Consequently, in this case we also have the inclusion $\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A \subseteq \mathfrak{U}_{\gamma\alpha}A$. Thus, all conditions of Definition I.10.1 are satisfied.

Hence, the triple $\mathcal{Z} = \left(\mathbb{N}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathfrak{U}}\right)$ is a changeable set. According to denotation system, accepted in Subsection 10.2, for this changeable set $\mathcal{Z}$ we have:

$$\mathcal{L}k(\mathcal{Z}) = \{\mathfrak{l}_n \mid n \in \mathbb{N}\}, \text{ where}$$
$$\mathfrak{l}_n = (n, \mathcal{B}_n) = (n, \mathcal{B}),\ n \in \mathbb{N},$$

and for $\mathfrak{l}_n, \mathfrak{l}_m \in \mathcal{L}k(\mathcal{Z})$ the equality $\langle \mathfrak{l}_n \leftarrow \mathfrak{l}_m \rangle = \mathfrak{U}_{nm}$ holds. Thus, by (I.68):

$$\langle \mathfrak{l}_{n+1} \leftarrow \mathfrak{l}_n \rangle A = A, \qquad A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l}_n) = \mathbb{B}\mathfrak{s}(\mathcal{B}),\ n \in \mathbb{N};$$
$$\langle \mathfrak{l}_{n+2} \leftarrow \mathfrak{l}_n \rangle A = \begin{cases} \mathbb{B}\mathfrak{s}(\mathcal{B}), & A \neq \emptyset \\ \emptyset, & A = \emptyset \end{cases}, \quad A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l}_n) = \mathbb{B}\mathfrak{s}(\mathcal{B}),\ n \in \mathbb{N};$$

The last equalities show, that $\mathfrak{l}_n \succ!!\ \mathfrak{l}_{n+1}$ $(n \in \mathbb{N})$. But, in the case $\mathbf{card}(\mathbb{B}\mathfrak{s}(\mathcal{B})) \geq 2$, $\mathfrak{l}_n$ is normally visible, but not precisely visible from $\mathfrak{l}_{n+2}$. Thus, in the case $\mathbf{card}(\mathbb{B}\mathfrak{s}(\mathcal{B})) \geq 2$ for any $n \in \mathbb{N}$ we have $\mathfrak{l}_n \succ!!\ \mathfrak{l}_{n+1}$, $\mathfrak{l}_{n+1} \succ!!\ \mathfrak{l}_{n+2}$, although the correlation $\mathfrak{l}_n \succ!!\ \mathfrak{l}_{n+2}$ is not true.

***Example*** I.12.2. Let base changeable set $\mathcal{B}$ be such, that the set $\mathbb{B}\mathfrak{s}(\mathcal{B})$ is infinite. Then there exists the sequence $(\omega_n)_{n=1}^{\infty} \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$ of elementary-time states such, that $\omega_n \neq \omega_m$, $m \neq n$. Denote:

$$\mathcal{B}_\alpha := \mathcal{B}, \quad \alpha \in \mathbb{N}; \qquad \overleftarrow{\mathcal{B}} := (\mathcal{B}_\alpha \mid \alpha \in \mathbb{N});$$

$$\mathfrak{U}_{\beta\alpha}A := \begin{cases} A, & \beta = \alpha \\ \{\omega_\beta\}, & \beta = \alpha + 1,\ \omega_\beta \in A \\ \emptyset, & \beta = \alpha + 1,\ \omega_\beta \notin A \\ \emptyset, & \beta \notin \{\alpha, \alpha + 1\}. \end{cases} \qquad (A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B}_\alpha) = \mathbb{B}\mathfrak{s}(\mathcal{B}),\ n \in \mathbb{N}) \tag{I.70}$$

We shell prove, that the system of mappings $\overleftarrow{\mathfrak{U}} = (\mathfrak{U}_{\beta\alpha} \mid \alpha, \beta \in \mathbb{N})$ is unification of perception. The first two conditions of Definition I.10.1 for the system of mappings $\overleftarrow{\mathfrak{U}}$ are performed by a trivial way. Thus, we need to verify the third condition of this Definition. Let $\alpha, \beta, \gamma \in \mathbb{N}$. It should be noted, that from (I.70) it follows, that $\mathfrak{U}_{\beta\alpha}\emptyset = \emptyset$ for any $\alpha, \beta \in \mathbb{N}$. Thus, according to (I.70), if one of the conditions $\alpha \leq \beta$ or $\beta \leq \gamma$ is not performed, then we have $\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \emptyset \subseteq \mathfrak{U}_{\gamma\alpha}A$, $A \subseteq \mathbb{B}\mathfrak{s}(\mathcal{B})$. Hence, we shell consider the case $\alpha \leq \beta \leq \gamma$. In the case, when $\alpha = \beta$ or $\beta = \gamma$, similarly to Example I.11.5 (items 3.a),3.b)), we obtain $\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{\gamma\alpha}A$.

Thus, it remains to consider only the case $\alpha < \beta < \gamma$. In the cases $\beta > \alpha + 1$ or $\gamma > \beta + 1$, by (I.70), we obtain $\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \emptyset \subseteq \mathfrak{U}_{\gamma\alpha}A$, $A \in \mathbb{Bs}(\mathcal{B})$. Hence, it remains only the case $\beta = \alpha + 1$ and $\gamma = \beta + 1$. If $\omega_\beta \notin A$, then, by (I.70), $\mathfrak{U}_{\beta\alpha}A = \emptyset$, and we have, $\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \emptyset \subseteq \mathfrak{U}_{\gamma\alpha}A$. And in the case $\omega_\beta \in A$, we obtain $\omega_\gamma = \omega_{\beta+1} \notin \{\omega_\beta\}$. Thus, in this case:

$$\mathfrak{U}_{\gamma\beta}\mathfrak{U}_{\beta\alpha}A = \mathfrak{U}_{\gamma\beta}\{\omega_\beta\} = \emptyset \subseteq \mathfrak{U}_{\gamma\alpha}A.$$

Consequently, the triple $\mathcal{Z} = \left(\mathbb{N}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathfrak{U}}\right)$ is a changeable set, satisfying:

$$\mathcal{L}k\,(\mathcal{Z}) = \{\mathfrak{l}_n \mid n \in \mathbb{N}\}, \text{ where } \mathfrak{l}_n = (n, \mathcal{B}_n) = (n, \mathcal{B}),\ n \in \mathbb{N},$$
$$\langle \mathfrak{l}_n \leftarrow \mathfrak{l}_m \rangle = \mathfrak{U}_{nm}, \quad m, n \in \mathbb{N}\ (\mathfrak{l}_n, \mathfrak{l}_m \in \mathcal{L}k\,(\mathcal{Z})).$$

From (I.70) it follows, that any $n \in \mathbb{N}$ $\langle \mathfrak{l}_{n+2} \leftarrow \mathfrak{l}_n \rangle A = \mathfrak{U}_{n+2,n}A = \emptyset$, $A \subseteq \mathbb{Bs}(\mathcal{B}) = \mathbb{Bs}\,(\mathfrak{l}_n)$, but, under the condition, $\omega_{n+1}, \omega_{n+2} \in A$ we have $\langle \mathfrak{l}_{n+1} \leftarrow \mathfrak{l}_n \rangle A = \{\omega_{n+1}\} \neq \emptyset$, $\langle \mathfrak{l}_{n+2} \leftarrow \mathfrak{l}_{n+1} \rangle A = \{\omega_{n+2}\} \neq \emptyset$. Therefore, $\mathfrak{l}_n \succ \mathfrak{l}_{n+1}$, $\mathfrak{l}_{n+1} \succ \mathfrak{l}_{n+2}$, although the reference frame $\mathfrak{l}_n$ invisible from $\mathfrak{l}_{n+2}$ ($\mathfrak{l}_n \not\succ \mathfrak{l}_{n+2}$).

Now we turn to the investigation of visibility of reference frames in the changeable sets of kind $\mathcal{Z}\mathsf{pv}\left(\overleftarrow{\mathcal{B}}, \overleftarrow{W}\right)$ and $\mathcal{Z}\mathsf{im}\,[\mathfrak{P}, \mathcal{B}]$. We are going to prove, that in these changeable sets any reference frame is precisely visible from each another.

**Assertion I.12.6.** *Let* $\overleftarrow{\mathcal{B}} = (\mathcal{B}_\alpha \mid \alpha \in \mathcal{A})$ $(\mathcal{A} \neq \emptyset)$ *be indexed family of base changeable sets such, that* $\mathbf{card}\,(\mathbb{Bs}\,(\mathcal{B}_\alpha)) = \mathbf{card}\,(\mathbb{Bs}\,(\mathcal{B}_\beta))$ *(for any* $\alpha, \beta \in \mathcal{A}$*) and* $\overleftarrow{W} = (W_{\beta\alpha} \mid \alpha, \beta \in \mathcal{A})$ *be indexed family of bijections of kind* $W_{\beta\alpha} : \mathbb{Bs}\,(\mathcal{B}_\alpha) \mapsto \mathbb{Bs}\,(\mathcal{B}_\beta)$*, satisfying conditions* (I.60) *and*

$$\mathcal{Z} = \mathcal{Z}\mathsf{pv}\left(\overleftarrow{\mathcal{B}}, \overleftarrow{W}\right).$$

*Then the correlation* $\mathfrak{l}_1 \succ !!\ \mathfrak{l}_2$ *is performed for any reference frames* $\mathfrak{l}_1, \mathfrak{l}_2 \in \mathcal{L}k\,(\mathcal{Z})$.

*Proof.* Consider any reference frames $\mathfrak{l}_1, \mathfrak{l}_2 \in \mathcal{L}k\,(\mathcal{Z})$, where $\mathcal{Z} = \mathcal{Z}\mathsf{pv}\left(\overleftarrow{\mathcal{B}}, \overleftarrow{W}\right)$. According to Property I.11.1(1) reference frames $\mathfrak{l}_1, \mathfrak{l}_2$ can be represented in the form:

$$\mathfrak{l}_1 = (\alpha, \mathcal{B}_\alpha), \quad \mathfrak{l}_2 = (\beta, \mathcal{B}_\beta),$$

where $\alpha, \beta \in \mathcal{A}$. And, in accordance with Properties I.11.1(4,3), unification mapping between $\mathfrak{l}_1$ and $\mathfrak{l}_2$ is represented in the form:

$$\langle \mathfrak{l}_2 \leftarrow \mathfrak{l}_1 \rangle A = W_{\beta\alpha}(A) = \{W_{\beta\alpha}(\omega) \mid \omega \in A\} \quad (\forall A \subseteq \mathbb{Bs}\,(\mathfrak{l}_1) = \mathbb{Bs}\,(\mathcal{B}_\alpha)),$$

where $W_{\beta\alpha} : \mathbb{Bs}\,(\mathcal{B}_\alpha) \mapsto \mathbb{Bs}\,(\mathcal{B}_\beta)$ is an bijection between $\mathbb{Bs}\,(\mathcal{B}_\alpha) = \mathbb{Bs}\,(\mathfrak{l}_1)$ and $\mathbb{Bs}\,(\mathcal{B}_\beta) = \mathbb{Bs}\,(\mathfrak{l}_2)$. Hence, any non-empty changeable system $\forall A \subseteq \mathbb{Bs}\,(\mathfrak{l}_1)$ is visible from $\mathfrak{l}_2$. Hence, by Definition I.12.1 (item 2) $\mathbb{Bs}\,(\mathfrak{l}_1)$ is normally visible from $\mathfrak{l}_2$. Since the mapping $W_{\beta\alpha}$ is an bijection between $\mathbb{Bs}\,(\mathfrak{l}_1)$ and $\mathbb{Bs}\,(\mathfrak{l}_2)$, for any disjoint system of changeable systems $(A_\beta \mid \beta \in \mathscr{B})$ $(A_\beta \subseteq \mathbb{Bs}\,(\mathfrak{l}_1)$, $\beta \in \mathscr{B}$ and $A_\beta \cap A_\gamma = \emptyset$ for $\beta \neq \gamma)$, we have $\langle \mathfrak{l}_2 \leftarrow \mathfrak{l}_1 \rangle \left(\bigsqcup_{\beta \in \mathscr{B}} A_\beta\right) = \bigsqcup_{\beta \in \mathscr{B}} \langle \mathfrak{l}_2 \leftarrow \mathfrak{l}_1 \rangle A_\beta$. Therefore, by Definition I.12.1 (item 3), $\mathbb{Bs}\,(\mathfrak{l}_1)$, is precisely visible from $\mathfrak{l}_2$. Thus, by Assertion I.12.4, $\mathfrak{l}_1 \succ !!\ \mathfrak{l}_2$. □

**Assertion I.12.7.** *Let* $\mathfrak{P}$ *be an evolution multi-projector for base changeable set* $\mathcal{B}$ *and*

$$\mathcal{Z} = \mathcal{Z}\mathsf{im}\,[\mathfrak{P}, \mathcal{B}].$$

*Then the correlation* $\mathfrak{l}_1 \succ !!\ \mathfrak{l}_2$ *is performed for any reference frames* $\mathfrak{l}_1, \mathfrak{l}_2 \in \mathcal{L}k\,(\mathcal{Z})$.

*Proof.* Assertion I.12.7 follows from Assertion I.12.6 and Equality (I.66) in Remark I.11.5. □

**Corollary I.12.1.** *Let $\mathbb{U}$ be transforming set of bijections relatively the base changeable set $\mathcal{B}$ on $X$ and*

$$\mathcal{Z} = \mathcal{Z}\mathsf{im}\,(\mathbb{U}, \mathcal{B}).$$

*Then the correlation $\mathfrak{l}_1 \succ!! \, \mathfrak{l}_2$ is performed for any reference frames $\mathfrak{l}_1, \mathfrak{l}_2 \in \mathcal{L}k\,(\mathcal{Z})$.*

*Proof.* Corollary I.12.1 follows from Assertion I.12.7 and Equality (I.67). □

From Corollary I.12.1 it follows that in changeable set, considered in Example I.11.3, any reference frame is precisely visible from each another.

**Definition I.12.3.** *We say, that a changeable set $\mathcal{Z}$ is* ***visible (normally visible, precisely visible)*** *if and only if for any $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k\,(\mathcal{Z})$ it satisfied the condition $\mathfrak{l} \succ \mathfrak{m}$ ($\mathfrak{l} \succ! \, \mathfrak{m}$, $\mathfrak{l} \succ!! \, \mathfrak{m}$) correspondingly.*

From Remark I.12.2 it follows, that any normally visible changeable set is visible. Example I.11.5 shows, that the inverse assertion is not true. Indeed, we may consider the case, when in this Example $\mathbf{card}\,(\mathbb{B}\mathfrak{s}\,(\mathcal{B}_1)) \geq 2$. As it has been shown in Remark I.12.2, in this case for the reference frames $\mathfrak{l}_1 = (1, \mathcal{B}_1)$, $\mathfrak{l}_2 = (2, \mathcal{B}_2)$ we have, $\mathfrak{l}_1 \succ \mathfrak{l}_2$, but **not** $\mathfrak{l}_1 \succ! \, \mathfrak{l}_2$. Since in this Example $\omega \in \mathbb{B}\mathfrak{s}\,(\mathcal{B}_2)$, we obtain $\langle \mathfrak{l}_1 \leftarrow \mathfrak{l}_2 \rangle \, \mathbb{B}\mathfrak{s}\,(\mathcal{B}_2) = \mathfrak{U}_{12}\mathbb{B}\mathfrak{s}\,(\mathcal{B}_2) = \mathbb{B}\mathfrak{s}\,(\mathcal{B}_1) \neq \emptyset$. Hence, $\mathfrak{l}_2 \succ \mathfrak{l}_1$. Thus $\mathfrak{l}_1 \succ \mathfrak{l}_2$, $\mathfrak{l}_2 \succ \mathfrak{l}_1$, but **not** $\mathfrak{l}_1 \succ! \, \mathfrak{l}_2$. And, taking into account, that $\mathcal{L}k\,(\mathcal{Z}_1) = \{\mathfrak{l}_1, \mathfrak{l}_2\}$, we obtain, that the changeable set $\mathcal{Z}_1$ in Example I.11.5 is visible, but not normally visible. In the subsection 12.2 (Corollary I.12.5) it will be shown, that the changeable set $\mathcal{Z}$ is precisely visible if and only if it is normally visible.

Using the notion of precisely visible changeable set, introduced in Definition I.12.3, we obtain the following three corollaries from Assertions I.12.6, I.12.7 and Corollary I.12.1.

**Corollary I.12.2.** *Let $\overleftarrow{\mathcal{B}} = (\mathcal{B}_\alpha \mid \alpha \in \mathcal{A})$ $(\mathcal{A} \neq \emptyset)$ be indexed family of base changeable sets such, that $\mathbf{card}\,(\mathbb{B}\mathfrak{s}\,(\mathcal{B}_\alpha)) = \mathbf{card}\,(\mathbb{B}\mathfrak{s}\,(\mathcal{B}_\beta))$ (for any $\alpha, \beta \in \mathcal{A}$) and $\overleftarrow{W} = (W_{\beta\alpha} | \, \alpha, \beta \in \mathcal{A})$ be indexed family of bijections of kind $W_{\beta\alpha} : \mathbb{B}\mathfrak{s}\,(\mathcal{B}_\alpha) \mapsto \mathbb{B}\mathfrak{s}\,(\mathcal{B}_\beta)$, satisfying conditions* (I.60). *Then the changeable set $\mathcal{Z} = \mathcal{Z}\mathsf{pv}\left(\overleftarrow{\mathcal{B}}, \overleftarrow{W}\right)$ is precisely visible.*

**Corollary I.12.3.** *Let $\mathfrak{P}$ be be an evolution multi-projector for base changeable set $\mathcal{B}$. Then the changeable set $\mathcal{Z} = \mathcal{Z}\mathsf{im}\,[\mathfrak{P}, \mathcal{B}]$ is precisely visible.*

**Corollary I.12.4.** *Let $\mathbb{U}$ be transforming set of bijections relatively the base changeable set $\mathcal{B}$ on $X$. Then the changeable set $\mathcal{Z} = \mathcal{Z}\mathsf{im}\,(\mathbb{U}, \mathcal{B})$ is precisely visible.*

## 12.2 Visibility Classes

**Assertion I.12.8.** *For any reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k\,(\mathcal{Z})$ of any changeable set $\mathcal{Z}$ the following propositions are equivalent:*

**(I)** $\mathfrak{l} \succ! \, \mathfrak{m}$ *and* $\mathfrak{m} \succ! \, \mathfrak{l}$;

**(II)** $\mathfrak{l} \succ!! \, \mathfrak{m}$ *and* $\mathfrak{m} \succ!! \, \mathfrak{l}$.

*Proof.* Since precise visibility always involves normal visibility, it is enough only to prove the implication (I)⇒(II). Hence, suppose, that $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k\,(\mathcal{Z})$, $\mathfrak{l} \succ! \, \mathfrak{m}$ and $\mathfrak{m} \succ! \, \mathfrak{l}$.

1) First we shall prove, that for any $A, B \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$, the equality $A \cap B = \emptyset$ is true if and only if $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle \, A \cap \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle \, B = \emptyset$. Suppose, that $A \cap B = \emptyset$. Then, according to second item of Assertion I.10.3, $\emptyset = A \cap B \supseteq \langle \mathfrak{l} \leftarrow \mathfrak{m} \rangle \left( \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle \, A \cap \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle \, B \right)$. Since $\mathfrak{m} \succ! \, \mathfrak{l}$ and $\langle \mathfrak{l} \leftarrow \mathfrak{m} \rangle \left( \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle \, A \cap \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle \, B \right) = \emptyset$, then, by the definition of normal visibility, $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle \, A \cap \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle \, B = \emptyset$, what is necessary to prove. Conversely, let $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle \, A \cap \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle \, B = \emptyset$.

Then, by first item of Assertion I.10.3, $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle (A \cap B) \subseteq \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A \cap \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle B = \emptyset$. Since $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle (A \cap B) = \emptyset$ and $\mathfrak{l} \succ! \mathfrak{m}$, then, by the definition of normal visibility, $A \cap B = \emptyset$.

2) Let, $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$ and $A = \bigsqcup_{\alpha \in \mathcal{A}} A_\alpha$ (where $A_\alpha \subseteq A$, $\alpha \in \mathcal{A}$; $A_\alpha \cap A_\beta = \emptyset$, $\alpha \neq \beta$). By Item 3) of Assertion I.10.3, $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A \supseteq \bigcup_{\alpha \in \mathcal{A}} \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A_\alpha$. Since the family of sets $(A_\alpha \mid \alpha \in \mathcal{A})$ is disjoint, by first item of this proof, the family of sets $(\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A_\alpha \mid \alpha \in \mathcal{A})$ also is disjoint, that is $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A \supseteq \bigsqcup_{\alpha \in \mathcal{A}} \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A_\alpha$. Assume, that the last inclusion is strict (ie $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A \neq \bigsqcup_{\alpha \in \mathcal{A}} \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A_\alpha$). Then the set $\tilde{B} = (\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A) \setminus \left( \bigsqcup_{\alpha \in \mathcal{A}} \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A_\alpha \right)$ is nonempty. Hence, by definition of normal visibility, the set $B = \langle \mathfrak{l} \leftarrow \mathfrak{m} \rangle \tilde{B}$ also is nonempty. Since $\tilde{B} \subseteq \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A$, by Properties I.10.1, we have, $B = \langle \mathfrak{l} \leftarrow \mathfrak{m} \rangle \tilde{B} \subseteq \langle \mathfrak{l} \leftarrow \mathfrak{m} \rangle \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A \subseteq \langle \mathfrak{l} \leftarrow \mathfrak{l} \rangle A = A$. Since the set $\tilde{B} = (\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A) \setminus \left( \bigsqcup_{\alpha \in \mathcal{A}} \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A_\alpha \right)$ is disjoint with with any of the sets $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A_\alpha$ $(\alpha \in \mathcal{A})$, the set $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle B = \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle \langle \mathfrak{l} \leftarrow \mathfrak{m} \rangle \tilde{B} \subseteq \langle \mathfrak{m} \leftarrow \mathfrak{m} \rangle \tilde{B} = \tilde{B}$ also is disjoint with with any of $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A_\alpha$ $(\alpha \in \mathcal{A})$ (ie $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle B \cap \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A_\alpha = \emptyset$, $\alpha \in \mathcal{A}$). Hence, by the first item of this proof, $B \cap A_\alpha = \emptyset$, $\alpha \in \mathcal{A}$. Thus, we can conclude, that there exist the **nonempty** set $B \subseteq A$ such, that $B \cap A_\alpha = \emptyset$, $\alpha \in \mathcal{A}$, which contradicts the equality $A = \bigsqcup_{\alpha \in \mathcal{A}} A_\alpha$. Thus, the assumption above is wrong, and, consequently, we obtain $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A = \bigsqcup_{\alpha \in \mathcal{A}} \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A_\alpha$.

Thus, any set $A \subseteq \mathbb{B}\mathfrak{s}(l)$ is precisely visible from the reference frame $\mathfrak{m}$, ie $\mathfrak{l} \succ!! \mathfrak{m}$. Similarly, we obtain, that $\mathfrak{m} \succ!! \mathfrak{l}$. □

The next corollary immediately follows from Assertion I.12.8.

**Corollary I.12.5.** *Changeable set $\mathcal{Z}$ is precisely visible if and only if it is normally visible.*

Taking into account Corollary I.12.5, the notion "normally visible changeable set" will be not used henceforth.

**Definition I.12.4.** *We say, that reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$ are* ***equivalent respectively the precise visibility*** *(or, abbreviated,* ***precisely-equivalent****) if and only if it is satisfied the condition (II) (or, equivalently, the condition (I)) of Assertion I.12.8.*

The fact, that reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$ are precisely-equivalent will be denoted by the following way:

$$\mathfrak{l} \equiv! \mathfrak{m}\,(\mathcal{Z}).$$

And in the case, when changeable set $\mathcal{Z}$ known in advance we shall use the denotation $\mathfrak{l} \equiv! \mathfrak{m}$ instead.

**Assertion I.12.9.** *Relation $\equiv!$ is relation of equivalence on the set $\mathcal{L}k(\mathcal{Z})$.*

*Proof.* For $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$ condition $\mathfrak{l} \equiv! \mathfrak{m}$ is equivalent to the condition (I) of Assertion I.12.8. Thus, since (by Assertion I.12.5) the relation $\succ!$ is quasi order on $\mathcal{L}k(\mathcal{Z})$, the desired result follows from [40, page. 21]. □

**Definition I.12.5.** *Equivalence classes, generated by the relation $\equiv!$ will be referred to as* ***precise visibility classes*** *of the changeable set $\mathcal{Z}$.*

Thus, for any changeable set, the set of all its reference frames can be splited on the precise visibility classes. Within an arbitrary precise visibility class any reference frame is precisely visible from other. It is evident, that changeable set $\mathcal{Z}$ is precisely visible if and only if $\mathcal{L}k(\mathcal{Z})$ contains only one precise visibility class.

It turns out, that, using the relation of visibility "$\succ$", we can divide the set $\mathcal{L}k(\mathcal{Z})$ by equivalence classes also.

**Definition I.12.6.** *Let $\mathcal{Z}$ be a changeable set.*

**(a)** *We say, that reference frames* $\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathcal{Z})$ *are* ***directly connected by visibility*** *(denotation is* $\mathfrak{l}\prec\!\succ\mathfrak{m}\,(\mathcal{Z})$*, or* $\mathfrak{l}\prec\!\succ\mathfrak{m}$ *in the case, when changeable set* $\mathcal{Z}$ *known in advance) if and only if at least one of the following conditions is satisfied:*

$$\mathfrak{l}\succ\mathfrak{m}\quad or\quad \mathfrak{m}\succ\mathfrak{l}.$$

**(b)** *We say, that reference frames* $\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathcal{Z})$ *are* ***connected by visibility*** *(denotation is* $\mathfrak{l}\widehat{\equiv}\mathfrak{m}\,(\mathcal{Z})$*, or* $\mathfrak{l}\widehat{\equiv}\mathfrak{m}$ *in the case, when changeable set* $\mathcal{Z}$ *known in advance) if and only if there exists a sequence* $\mathfrak{l}_0,\mathfrak{l}_1,\cdots,\mathfrak{l}_\nu\in\mathcal{L}k(\mathcal{Z})$ $(\nu\in\mathbb{N})$ *such, that:*

$$\mathfrak{l}_0=\mathfrak{l},\quad \mathfrak{l}_\nu=\mathfrak{m},\ and\quad \mathfrak{l}_i\prec\!\succ\mathfrak{l}_{i-1}\ (\forall\, i\in\overline{1,\nu}).$$

**Assertion I.12.10.** *Relation* $\widehat{\equiv}$ *is relation of equivalence on the set* $\mathcal{L}k(\mathcal{Z})$*.*

*Proof.* Since the relation of visibility, according to Remark I.12.3, is reflexive, the relation $\prec\!\succ$ is reflexive and symmetric on $\mathcal{L}k(\mathcal{Z})$. The relation $\widehat{\equiv}$ is transitive closure of the relation $\prec\!\succ$ in the sense of [43, page 69], [44, page. 32]. Thus, by [43, assertions 5.8, 5.9 and theorem 5.8], $\widehat{\equiv}$ is equivalence relation on $\mathcal{L}k(\mathcal{Z})$. □

**Definition I.12.7.** *Equivalence classes in the set* $\mathcal{L}k(\mathcal{Z})$*, generated by the relation* $\widehat{\equiv}$ *will be named by* ***visibility classes*** *of the changeable set* $\mathcal{Z}$*.*

But it may occur, that in the changeable set only one visibility class exist.

**Definition I.12.8.** *We say, that a changeable set* $\mathcal{Z}$ *is* ***connected visible*** *if and only if for any* $\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathcal{Z})$ *it is true the correlation* $\mathfrak{l}\widehat{\equiv}\mathfrak{m}$*.*

It is evident, that any visible changeable set is connected visible. Analyzing the examples I.12.1 and I.12.2 it is easy to verify that the inverse proposition, in general, is false.

So, we see, that in the case, when a changeable set $\mathcal{Z}$ is not connected visible the set of all it's reference frames is splitted by “parallel worlds” (visibility classes) and any visibility class is “fully invisible” from other visibility classes. As formal example of changeable set with many visibility classes it can be considered the changeable set $\mathcal{Z}\mathsf{nv}\left(\overleftarrow{\mathcal{B}}\right)$ (see Example I.11.4) with $\mathbf{card}(\overleftarrow{\mathcal{B}})\geq 2$. In the changeable set $\mathcal{Z}\mathsf{nv}\left(\overleftarrow{\mathcal{B}}\right)$ any reference frame forms the separated visibility class.

Precise visibility classes also can be interpreted as “parallel worlds”. But these “parallel worlds” may be partially visible from other “parallel worlds”.

### 12.3 Precisely Visible Changeable Sets

In the classical mechanics and special relativity theory (for inertial reference frames) it is supposed, that any elementary-time state (or “physical event”) is visible in any frame of reference. Hence, the precisely visible changeable sets are to be important for physics. In this subsection we investigate precisely visible changeable sets in more details. The changeable sets of kind $\mathcal{Z}\mathsf{pv}\left(\overleftarrow{\mathcal{B}},\overleftarrow{W}\right)$, $\mathcal{Z}\mathsf{im}\,[\mathfrak{P},\mathcal{B}]$ and $\mathcal{Z}\mathsf{im}\,(\mathbb{U},\mathcal{B})$, introduced in examples I.11.1,I.11.2, I.11.3 and Definition I.11.4, evidently are precisely visible.

*Remark* I.12.4. It should be noted, that by Assertion I.12.8 and definition of the relation $\equiv!$, *for any changeable set* $\mathcal{Z}$ *the following propositions are equivalent:*

**(I)** $\mathcal{Z}$ *is precisely visible changeable set*;

**(II)** *for any* $\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathcal{Z})$ *it is performed the condition* $\mathfrak{l}\succ!!\,\mathfrak{m}$;

**(III)** *for any* $\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathcal{Z})$ *it is performed the condition* $\mathfrak{l}\succ!\,\mathfrak{m}$*;*

**(IV)** *for any* $\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathcal{Z})$ *it is performed the condition* $\mathfrak{l}\equiv!\,\mathfrak{m}$*.*

Note also that in the first item of the proof of Assertion I.12.8 it was proved, the following lemma.

**Lemma I.12.1.** *Let $\mathcal{Z}$ be a precisely visible changeable set. Then for any $\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathcal{Z})$ and $A,B\subseteq\mathbb{Bs}(\mathfrak{l})$ the equality $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A\cap\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle B=\emptyset$ is true if and only if $A\cap B=\emptyset$.*

**Theorem I.12.1.** *Changeable set $\mathcal{Z}$ is precisely visible if and only if for any $\mathfrak{l},\mathfrak{m},\mathfrak{p}\in\mathcal{L}k(\mathcal{Z})$ the followind equality is true:*

$$\langle\mathfrak{p}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle=\langle\mathfrak{p}\leftarrow\mathfrak{l}\rangle. \tag{I.71}$$

*Proof.* **Sufficiency**. Suppose, that for any $\mathfrak{l},\mathfrak{m},\mathfrak{p}\in\mathcal{L}k(\mathcal{Z})$ the equality (I.71) holds. Chose any reference frames $\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathcal{Z})$ and any changeable system $A\subseteq\mathbb{Bs}(\mathfrak{l})$ such, that $A\neq\emptyset$. Then, by (I.71),

$$A=\langle\mathfrak{l}\leftarrow\mathfrak{l}\rangle A=\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A.$$

Therefore, by Assertion I.10.2, $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A\neq\emptyset$. Thus, by Assertion I.12.3, $\mathfrak{l}\succ!\,\mathfrak{m}$ (for any reference frames $\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathcal{Z})$). Hence, by Remark I.12.4, the changeable set $\mathcal{Z}$ is precisely visible.

**Necessity**. Conversely, suppose, that the changeable set $\mathcal{Z}$ is precisely visible. Consider any reference frames $\mathfrak{l},\mathfrak{m},\mathfrak{p}\in\mathcal{L}k(\mathcal{Z})$ and any changeable system $A\subseteq\mathbb{Bs}(\mathfrak{l})$. By Property I.10.1(8) $\langle\mathfrak{p}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A\subseteq\langle\mathfrak{p}\leftarrow\mathfrak{l}\rangle A$. Denote:

$$B_1:=\langle\mathfrak{p}\leftarrow\mathfrak{l}\rangle A\setminus\langle\mathfrak{p}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A.$$

Then, $B_1\subseteq\langle\mathfrak{p}\leftarrow\mathfrak{l}\rangle A$ and $B_1\cap\langle\mathfrak{p}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A=\emptyset$. Denote $B:=\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{p}\rangle B_1$. Using Properties I.10.1 we obtain:

$$\begin{aligned}B=\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{p}\rangle B_1\subseteq\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{p}\rangle\langle\mathfrak{p}\leftarrow\mathfrak{l}\rangle A\subseteq\\ \subseteq\langle\mathfrak{l}\leftarrow\mathfrak{l}\rangle A=A;\\ \langle\mathfrak{p}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle B=\langle\mathfrak{p}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{p}\rangle B_1\subseteq\\ \subseteq\langle\mathfrak{p}\leftarrow\mathfrak{p}\rangle B_1=B_1.\end{aligned}$$

Hence, since $B_1\cap\langle\mathfrak{p}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A=\emptyset$, we have $\langle\mathfrak{p}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle B\cap\langle\mathfrak{p}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A=\emptyset$. Consequently, using Lemma I.12.1, we obtain $B\cap A=\emptyset$. Since $B\subseteq A$ and $B\cap A=\emptyset$, we obtain $B=\emptyset$. Thus, $\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{p}\rangle B_1=B=\emptyset$. Therefore, taking into account, that, by Remark I.12.4 $\mathfrak{p}\succ!\,\mathfrak{m}$ and $\mathfrak{m}\succ!\,\mathfrak{l}$, we obtain (by definition of normal visibility) $B_1=\emptyset$. □

Note, that, for the changeable set $\mathcal{Z}=\left(\mathcal{A},\overleftarrow{\mathcal{B}},\overleftarrow{\mathfrak{U}}\right)$ from Definition I.10.1, the condition (I.71) is equivalent to the condition (I.57).

**Assertion I.12.11.** *Let $\mathcal{Z}$ be a precisely visible changeable set. Then for any reference frames $\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathcal{Z})$, any family of changeable systems $(A_\alpha|\alpha\in\mathcal{A})$ ($A_\alpha\subseteq\mathbb{Bs}(\mathfrak{l})$, $\alpha\in\mathcal{A}$) and any changeable systems $A,B\in\mathbb{Bs}(\mathfrak{l})$ the following assertions are true:*

1. $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\left(\bigcap_{\alpha\in\mathcal{A}}A_\alpha\right)=\bigcap_{\alpha\in\mathcal{A}}\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A_\alpha$;
2. $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle(A\setminus B)=\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A\setminus\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle B$;
3. $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\mathbb{Bs}(\mathfrak{l})=\mathbb{Bs}(\mathfrak{m})$;
4. $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\left(\bigcup_{\alpha\in\mathcal{A}}A_\alpha\right)=\bigcup_{\alpha\in\mathcal{A}}\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A_\alpha$;
5. *If a changeable system $A\subseteq\mathbb{Bs}(\mathfrak{l})$ is a singleton (i.e. $\mathbf{card}(A)=1$), then the changeable system $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A$ also is a singleton.*

*Proof.* 1) Using Assertion I.10.3, item 2), Properties I.10.1 and Theorem I.12.1 (equality (I.71)) we obtain:

$$\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\left(\bigcap_{\alpha\in\mathcal{A}}A_\alpha\right)\supseteq\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle\left(\bigcap_{\alpha\in\mathcal{A}}\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A_\alpha\right)=$$
$$=\langle\mathfrak{m}\leftarrow\mathfrak{m}\rangle\left(\bigcap_{\alpha\in\mathcal{A}}\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A_\alpha\right)=\bigcap_{\alpha\in\mathcal{A}}\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A_\alpha.$$

Hence, $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\left(\bigcap_{\alpha\in\mathcal{A}}A_\alpha\right)\supseteq\bigcap_{\alpha\in\mathcal{A}}\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A_\alpha$. The inverse inclusion is contained in Assertion I.10.3, item 1).

2) Since $A\setminus B\subseteq A$, then by Property I.10.1(7) we have, $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle(A\setminus B)\subseteq\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A$. Since $(A\setminus B)\cap B=\emptyset$, then, by Lemma I.12.1, $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle(A\setminus B)\cap\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle B=\emptyset$. Hence:

$$\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle(A\setminus B)\subseteq\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A\setminus\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle B. \tag{I.72}$$

Using the correlation (I.72) to the sets $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A$, $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle B$, with unification mapping $\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle$, applying the formula (I.71) and Properties I.10.1 we obtain:

$$\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle(\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A\setminus\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle B)\subseteq$$
$$\subseteq\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A\setminus\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle B=\langle\mathfrak{l}\leftarrow\mathfrak{l}\rangle A\setminus\langle\mathfrak{l}\leftarrow\mathfrak{l}\rangle B=A\setminus B.$$

Hence, by Property I.10.1(7) $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle(\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A\setminus\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle B)\subseteq\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle(A\setminus B)$. And applying the formula (I.71) and Property I.10.1(4), we obtain the inverse inclusion to (I.72).

3) By definition of unification mapping,

$$\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\mathbb{B}\mathfrak{s}(\mathfrak{l})\subseteq\mathbb{B}\mathfrak{s}(\mathfrak{m}). \tag{I.73}$$

Similarly, $\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle\mathbb{B}\mathfrak{s}(\mathfrak{m})\subseteq\mathbb{B}\mathfrak{s}(\mathfrak{l})$. Applying to the last inclusion unification mapping $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle$, and using Properties I.10.1 as well as correlation (I.71) we obtain the inverse inclusion to (I.73).

4) Note, that: $\bigcup_{\alpha\in\mathcal{A}}A_\alpha=\mathbb{B}\mathfrak{s}(\mathfrak{l})\setminus\left(\bigcap_{\alpha\in\mathcal{A}}(\mathbb{B}\mathfrak{s}(\mathfrak{l})\setminus A_\alpha)\right)$. Hence, using items 1, 2, and 3 of this Assertion we obtain:

$$\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\left(\bigcup_{\alpha\in\mathcal{A}}A_\alpha\right)=(\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\mathbb{B}\mathfrak{s}(\mathfrak{l}))\setminus\left(\bigcap_{\alpha\in\mathcal{A}}(\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\mathbb{B}\mathfrak{s}(\mathfrak{l})\setminus\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A_\alpha)\right)=$$
$$=\mathbb{B}\mathfrak{s}(\mathfrak{m})\setminus\left(\bigcap_{\alpha\in\mathcal{A}}(\mathbb{B}\mathfrak{s}(\mathfrak{m})\setminus\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A_\alpha)\right)=\bigcup_{\alpha\in\mathcal{A}}\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A_\alpha.$$

5) Let $A\subseteq\mathbb{B}\mathfrak{s}(\mathfrak{l})$, and $A=\{\omega\}$ is a singleton. By Remark I.12.4, $\mathfrak{l}\succ!\,\mathfrak{m}$ and, since $A\neq\emptyset$, by definition of normal visibility, we have $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A\neq\emptyset$. Suppose, that the set $B=\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A$ contains more, than one element. Then, there exist sets $B_1,B_2\subseteq B$ such, that $B_1,B_2\neq\emptyset$ and $B=B_1\sqcup B_2$. Denote: $A_1:=\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle B_1$, $A_2:=\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle B_2$. Since $B_1,B_2\neq\emptyset$, then, by the definition of normal visibility, $A_1,A_2\neq\emptyset$. Since $B=B_1\sqcup B_2$, then, by the definition of precise visibility, $\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle B=\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle B_1\sqcup\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle B_2=A_1\sqcup A_2$. Hence, taking into account, that $B=\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A$ and using the equality (I.71), we obtain:

$$A_1\sqcup A_2=\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle B=\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A=A.$$

Thus, we see, that the set $A$ can be divided into two nonempty disjoint sets, which contradicts the fact, that the set $A$ is a singleton. Therefore, the set $\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle A$ is nonempty, and it can not contain more, than one element, hence, it is a singleton. □

**Definition I.12.9.** *Let $\mathcal{Z}$ be a precisely visible changeable set, $\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathcal{Z})$ and $\omega\in\mathbb{B}\mathfrak{s}(\mathfrak{l})$. Elementary-time state $\omega'\in\mathbb{B}\mathfrak{s}(\mathfrak{m})$ such, that $\{\omega'\}=\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle\{\omega\}$ will be referred to as* ***visible***

***image*** *of elementary-time state $\omega \in \mathbb{Bs}(l)$ in the reference frame $\mathfrak{m}$ and it will be denoted by $\langle ! \,\mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \omega$:*

$$\omega' = \langle ! \,\mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \omega.$$

By Assertion I.12.11, item 5, any elementary-time state $\omega \in \mathbb{Bs}(\mathfrak{l})$ always has a visible image $\omega' = \langle ! \,\mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \omega$ in a precisely visible changeable set. Hence, by Definition I.12.9, for any elementary-time state $\omega \in \mathbb{Bs}(\mathfrak{l})$ in the reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ of precisely visible changeable set $\mathcal{Z}$ the following equality holds:

$$\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle \{\omega\} = \{\langle ! \,\mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \omega\} \quad (\mathfrak{m} \in \mathcal{L}k(\mathcal{Z})) \tag{I.74}$$

Using the equality $A = \bigsqcup_{\omega \in A} \{\omega\}$, definition of precise visibility and equality (I.74) we obtain the following theorem.

**Theorem I.12.2.** *For any nonempty changeable system $A \subseteq \mathbb{Bs}(\mathfrak{l})$ in reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ of precisely visible changeable set $\mathcal{Z}$ the following equality is true:*

$$\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle\, A = \bigsqcup_{\omega \in A} \{\langle ! \,\mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \omega\} = \{\langle ! \,\mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \omega \mid \omega \in A\} \quad (\mathfrak{m} \in \mathcal{L}k(\mathcal{Z})). \tag{I.75}$$

**Corollary I.12.6.** *Let $\mathcal{Z}$ be a precisely visible changeable set and $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$ be any its reference frames.*

*Then for any changeable system $A \subseteq \mathbb{Bs}(\mathfrak{l})$ the sets $A$ and $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle\, A$ are equipotent. In the case $A \neq \emptyset$ the mapping:*

$$f(\omega) = \langle ! \,\mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \omega, \qquad \omega \in \mathbb{Bs}(\mathfrak{l}) \tag{I.76}$$

*is bijection between the sets $A$ and $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle\, A$.*

*In particular the sets $\mathbb{Bs}(\mathfrak{l})$ and $\mathbb{Bs}(\mathfrak{m}) = \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \mathbb{Bs}(\mathfrak{l})$ are equipotent and the mapping* (I.76) *is bijection between these sets.*

*Proof.* In the case $A = \emptyset$ the statement of the Corollary follows from Assertion I.10.2. In the case $A \neq \emptyset$ from Theorem I.12.2 (pay attention to the sign of disjoint union in equality (I.75)) it follows, that the mapping (I.76) is bijection between the sets $A$ and $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle\, A$. And from Assertion I.12.11 (item 3)) it follows, that $\mathbb{Bs}(\mathfrak{m}) = \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \mathbb{Bs}(\mathfrak{l})$ Hence, the sets $\mathbb{Bs}(\mathfrak{l})$ and $\mathbb{Bs}(\mathfrak{m})$ are equipotent and the mapping (I.76) is bijection between these sets. □

Using Property I.10.1(4), as well as theorems I.12.2 and I.12.1 we receive the following properties of precise unification mappings in precisely visible changeable sets.

**Properties I.12.1.** *Let $\mathcal{Z}$ be any precisely visible changeable set, and $\mathfrak{l}, \mathfrak{m}, \mathfrak{p} \in \mathcal{L}k(\mathcal{Z})$ be arbitrary reference frames of $\mathcal{Z}$. Then:*

1. $\forall\, \omega \in \mathbb{Bs}(\mathfrak{l}) \quad \langle ! \,\mathfrak{l} \leftarrow \mathfrak{l} \rangle\, \omega = \omega$;
2. $\forall\, A \subseteq \mathbb{Bs}(\mathfrak{l}) \quad \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle\, A = \{\langle ! \,\mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \omega \mid \omega \in A\}$;
3. $\forall\, \omega \in \mathbb{Bs}(\mathfrak{l}) \quad \langle ! \,\mathfrak{p} \leftarrow \mathfrak{m} \rangle \langle ! \,\mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \omega = \langle ! \,\mathfrak{p} \leftarrow \mathfrak{l} \rangle\, \omega$.

From corollaries I.12.3 and I.12.4 it follows, that the changeable sets of kind $\mathcal{Z}\mathsf{im}\,[\mathfrak{P}, \mathcal{B}]$ and $\mathcal{Z}\mathsf{im}\,(\mathbb{U}, \mathcal{B})$ are precisely visible. Therefore, we deliver the following corollary of Theorem I.11.2:

**Corollary I.12.7.** *If $\mathcal{Z} = \mathcal{Z}\mathsf{im}\,[\mathfrak{P}, \mathcal{B}]$, where $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha) \mid \alpha \in \mathcal{A})$. Then for any reference frames $\mathfrak{l} = (\alpha, U_\alpha\,[\mathcal{B}, \mathbb{T}_\alpha]) \in \mathcal{L}k(\mathcal{Z})$, $\mathfrak{m} = (\beta, U_\beta\,[\mathcal{B}, \mathbb{T}_\beta]) \in \mathcal{L}k(\mathcal{Z})$ $(\alpha, \beta \in \mathcal{A})$ the following equality is performed:*

$$\langle ! \,\mathfrak{m} \leftarrow \mathfrak{l},\, \mathcal{Z} \rangle\, \omega = U_\beta \left(U_\alpha^{[-1]}(\omega)\right) \quad (\omega \in \mathbb{Bs}(\mathfrak{l}) = U_\alpha\,(\mathbb{Bs}(\mathcal{B})))\,.$$

Main results of this Section were published in [4].

# Part II
# Kinematic Changeable Sets and their Properties

## 13 Introduction to Second Part

Due to the OPERA experiments conducted within 2011-2012 years [46], quite a lot physical works appeared, in which authors are trying to modify the special relativity theory to agree its conclusions with the hypothesis of existence of objects moving at velocity, greater than the velocity of light. Despite the fact that the superluminal results of OPERA experiments (2011-2012) were not confirmed later, the problem of constructing a theory of super-light movement, posed in the papers [33, 34], remains actual within more than 50 last years [35]. At the present time existence of a few kinematic theories of tachyon motion generates the problem of construction new mathematical structures, which would allow to simulate of evolution of physical systems in a framework of different laws of kinematics. Under the lack of experimental verification of conclusions for tachyon kinematics theories, such mathematical structures may at least guarantee the correctness of receiving these conclusions in accordance with the postulates of these theories. This part of the paper is devoted to building of these mathematical structures. Investigations in this direction may be also interesting for astrophysics, because there exists the hypothesis, that in large scale of the Universe, physical laws (in particular, the laws of kinematics) may be different from the laws, acting in the neighborhood of our solar System.

On a physical level, the problem of investigation of kinematics with arbitrary space-time coordinate transforms for inertial reference frames, was presented in the [47] for the case, when the space of geometric variables is three-dimensional and Euclidean. The particular case of coordinate transforms, considered in [47] are the (three-dimensional) classical Lorentz transforms as well as generalized Lorentz transforms in the sense of E. Recami and V. Olkhovsky [36–38, 51, 52] (for reference frames moving at a velocity greater than the velocity light). In the papers [6, 7] the general definition of linear coordinate transforms and generalized Lorentz transforms is given for the case, where the space of geometric variables is any real Hilbert space.

It should be noted, that mathematical apparatus of the papers [6,7,36–38,47] is not based on the theory of changeable sets, which greatly reduces its generality. In particular, mathematical apparatus of these papers allows only studying of universal coordinate transforms (that is coordinate transforms, which are uniquely determined by the geometrically-time position of the considered object). The present part of the paper is based on the general theory of changeable sets, developed in the previous part. In this part the definitions of the actual and universal coordinate transform in kinematic changeable sets are given. We prove, that in classical Galilean and Lorentz-Poincare kinematics the universal coordinate transform always exists. Also we construct the class of kinematics, in which every particle in every time point can have its own "velocity of light" and prove, that, in these kinematics, universal coordinate transform does not exist.

# 14 Changeable Sets and Kinematics.

## 14.1 Mathematical Objects for Constructing of Geometric Environments of Changeable Sets.

This subsection is purely technical in nature. In this subsection we don't introduce any essentially new notions. But we try to include the most frequently used mathematical spaces, which at least somehow related to geometry, into single mathematical structure, which will be convenient for further construction of abstract kinematics.

**Definition II.14.1.** *The ordered triple $\mathbb{L} = (\mathbb{K}, \oplus, \otimes)$ will be named by **linear structure** over non-empty set $\mathfrak{X}$ if and only if:*

1. $\mathbb{K} = (\mathbf{K}, +_{\mathbb{K}}, \times_{\mathbb{K}})$ *is a field.*
2. $\oplus : \mathfrak{X} \times \mathfrak{X} \mapsto \mathfrak{X}$ *is a binary operation over* $\mathfrak{X}$*;*
3. $\otimes : \mathbf{K} \times \mathfrak{X} \mapsto \mathfrak{X}$ *is a binary operation, acting from* $\mathbf{K} \times \mathfrak{X}$ *into* $\mathfrak{X}$*.*
4. *The ordered triple* $(\mathfrak{X}, \oplus, \otimes)$ *is a linear space over the field* $\mathbb{K}$*.*

*In the case, when $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, the linear structure $\mathbb{L}$ will be named as **numerical linear structure** over $\mathfrak{X}$.*

Let $\mathbb{L} = (\mathbb{K}, \oplus, \otimes)$ be a linear structure over $\mathfrak{X}$. In this case the linear space over the field $\mathbb{K}$, generated by $\mathbb{L}$ will be denoted by $\mathfrak{Lp}(\mathfrak{X}, \mathbb{L})$ $\quad (\mathfrak{Lp}(\mathfrak{X}, \mathbb{L}) = (\mathfrak{X}, \oplus, \otimes))$.

Next definition is based on the conception, that the majority of the most frequently used mathematical objects (including functions, relations, algebraic operations, ordered pairs or compositions) are sets.

**Definition II.14.2.** *An ordered composition of six sets $\mathfrak{Q} = (\mathfrak{X}, \mathcal{T}, \mathbb{L}, \rho, \|\cdot\|, (\cdot, \cdot))$ will be named by **coordinate space**, if and only if the following conditions are satisfied:*

1. $\mathfrak{X} \neq \emptyset$.
2. $\mathcal{T} \cup \mathbb{L} \neq \emptyset$.
3. *If* $\mathcal{T} \neq \emptyset$*, then* $\mathcal{T}$ *is topology over* $\mathfrak{X}$*.*
4. *If* $\mathbb{L} \neq \emptyset$*, then* $\mathbb{L}$ *is numerical linear structure over* $\mathfrak{X}$*.*
5. *If* $\mathbb{L} \neq \emptyset$ *and* $\mathcal{T} \neq \emptyset$*, then the pair* $(\mathfrak{Lp}(\mathfrak{X}, \mathbb{L}), \mathcal{T})$ *is a linear topological space.*
6. *If* $\rho \neq \emptyset$*, then:*
   - **6.1)** $\rho$ *is the metrics over* $\mathfrak{X}$*;*
   - **6.2)** $\mathcal{T} \neq \emptyset$ *and the topology* $\mathcal{T}$ *is generated by the metrics* $\rho$*.*
7. *If* $\|\cdot\| \neq \emptyset$*, then:*
   - **7.1)** $\mathbb{L} \neq \emptyset$ *and* $\|\cdot\|$ *is the norm on the linear space* $\mathfrak{Lp}(\mathfrak{X}, \mathbb{L})$*;*
   - **7.2)** $\rho \neq \emptyset$ *and the metrics* $\rho$ *is generated by the norm* $\|\cdot\|$*.*
8. *If* $(\cdot, \cdot) \neq \emptyset$*, then:*
   - **8.1)** $\|\cdot\| \neq \emptyset$ *(and hence, according to* **7.1)**, $\mathbb{L} \neq \emptyset$*);*
   - **8.2)** $(\cdot, \cdot)$ *is the inner product on the linear space* $\mathfrak{Lp}(\mathfrak{X}, \mathbb{L})$*;*
   - **8.3)** *the norm* $\|\cdot\|$ *is generated by the inner product* $(\cdot, \cdot)$*.*

**Notes on denotations.** Let $\mathfrak{Q} = (\mathfrak{X}, \mathcal{T}, \mathbb{L}, \rho, \|\cdot\|, (\cdot,\cdot))$ be a coordinate space, where in the case $\mathbb{L} \neq \emptyset$ we have, that $\mathbb{L} = (\mathbb{K}, \oplus, \otimes)$ is a numerical linear structure over $\mathfrak{X}$. Further we will use the following denotations:

1. $\mathbf{Zk}(\mathfrak{Q}) := \mathfrak{X}$ (the set $\mathbf{Zk}(\mathfrak{Q})$ will be named the ***set of coordinate values*** of $\mathfrak{Q}$).
2. $\mathcal{T}p(\mathfrak{Q}) := \mathcal{T}$ ($\mathcal{T}p(\mathfrak{Q})$ will be referred to as ***topology*** of $\mathfrak{Q}$).
3. $\mathbb{L}s(\mathfrak{Q}) := \mathbb{L}$ ($\mathbb{L}s(\mathfrak{Q})$ will be named the ***linear structure*** of $\mathfrak{Q}$).
4. $\mathfrak{Ps}(\mathfrak{Q}) := \begin{cases} \mathbb{K}, & \mathbb{L}s(\mathfrak{Q}) \neq \emptyset \\ \emptyset, & \mathbb{L}s(\mathfrak{Q}) = \emptyset \end{cases}$ ($\mathfrak{Ps}(\mathfrak{Q})$ will be referred to as ***field of scalars*** of $\mathfrak{Q}$).
5. For the elements $x_1, \dots, x_n \in \mathbf{Zk}(\mathfrak{Q})$, $\lambda_1, \dots, \lambda_n \in \mathfrak{Ps}(\mathfrak{Q})$ $(n \in \mathbb{N})$ we will use the denotation, $(\lambda_1 x_1 + \cdots + \lambda_n x_n)_{\mathfrak{Q}} := \lambda_1 \otimes x_1 \oplus \cdots \oplus \lambda_n \otimes x_n$.
6. $\mathbf{di}_{\mathfrak{Q}} := \rho$ ($\mathbf{di}_{\mathfrak{Q}}$ will be named the ***distance*** on $\mathfrak{Q}$).
7. $\|\cdot\|_{\mathfrak{Q}} := \|\cdot\|$ ($\|\cdot\|_{\mathfrak{Q}}$ will be named the ***norm*** on $\mathfrak{Q}$).
8. $(\cdot,\cdot)_{\mathfrak{Q}} := (\cdot,\cdot)$ ($(\cdot,\cdot)_{\mathfrak{Q}}$ will be referred to as ***inner product*** on $\mathfrak{Q}$).

Elements of kind $x \in \mathbf{Zk}(\mathfrak{Q})$ will be named as ***coordinates*** of the coordinate space $\mathfrak{Q}$, also, in the case $\mathbb{L}s(\mathfrak{Q}) \neq \emptyset$ we will name these elements as ***vectors (vector coordinates)*** of $\mathfrak{Q}$. Where it does not cause confusion the symbol "$\mathfrak{Q}$" in the denotations $(\lambda_1 x_1 + \cdots + \lambda_n x_n)_{\mathfrak{Q}}$, $\mathbf{di}_{\mathfrak{Q}}$, $\|\cdot\|_{\mathfrak{Q}}$, $(\cdot,\cdot)_{\mathfrak{Q}}$ will be released, and we will use the abbreviated denotations $\lambda_1 x_1 + \cdots + \lambda_n x_n$, $\mathbf{di}$, $\|\cdot\|$, $(\cdot,\cdot)$ correspondingly.

### 14.2 Kinematic Changeable Sets.

**Definition II.14.3.** *1. The pair $\mathcal{G}_0 = (\mathfrak{Q}, k)$ we name by **geometric environment** of base changeable set $\mathcal{B}$, if and only if:*

**a)** *$\mathfrak{Q}$ is a coordinate space;*

**b)** *$k : \mathfrak{Bs}(\mathcal{B}) \mapsto \mathbf{Zk}(\mathfrak{Q})$ is a mapping from $\mathfrak{Bs}(\mathcal{B})$ into $\mathbf{Zk}(\mathfrak{Q})$.*

*In this case the pair $\mathfrak{C}^{\mathfrak{b}} = (\mathcal{B}, \mathcal{G}_0) = (\mathcal{B}, (\mathfrak{Q}, k))$ we name by **base kinematic changeable set**, or, abbreviated, by **base kinematic** set.*

*2. Let $\mathcal{Z}$ be any changeable set. An indexed family of pairs $\mathcal{G} = ((\mathfrak{Q}_{\mathfrak{l}}, k_{\mathfrak{l}}) \mid \mathfrak{l} \in \mathcal{L}k(\mathcal{Z}))$ will be named by **geometric environment** of the changeable set $\mathcal{Z}$, if and only if for any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ the ordered pair $(\mathfrak{Q}_{\mathfrak{l}}, k_{\mathfrak{l}})$ is geometric environment of the base changeable set $\mathfrak{l}^{\wedge}$, generated by the reference frame $\mathfrak{l}$, i.e. if and only if the pair $(\mathfrak{l}^{\wedge}, (\mathfrak{Q}_{\mathfrak{l}}, k_{\mathfrak{l}}))$ is a base kinematic changeable set for an arbitrary $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$.*

*In this case we name the pair $\mathfrak{C} = (\mathcal{Z}, \mathcal{G})$ by **kinematic changeable set**, or, abbreviated, by **kinematic set**.*

Note, that in this paper we consider only kinematic sets with constant (unchanging over time) geometry. These kinematic sets are sufficient for construction of abstract kinematics in inertial reference frames. If we make a some modification of Definition II.14.3, we will be able to define also kinematic sets with variable (over time) geometry (i.e., in principle, this is, possible to do).

#### 14.2.1 System of Denotations for Base Kinematic Sets.

Let, $\mathfrak{C}^\mathfrak{b} = (\mathcal{B}, \mathcal{G}_0)$ be any base kinematic set (where $\mathcal{G}_0 = (\mathfrak{Q}, k)$). Henceforth we use the following system of denotations.

a) Denotations, induced from the theory of base changeable sets:

$$\begin{array}{lll} \mathfrak{Bs}\left(\mathfrak{C}^\mathfrak{b}\right) := \mathfrak{Bs}(\mathcal{B}); & \mathbb{B}\mathfrak{s}\left(\mathfrak{C}^\mathfrak{b}\right) := \mathbb{B}\mathfrak{s}(\mathcal{B}); & \underset{\mathfrak{C}^\mathfrak{b}}{\leftarrow} := \underset{\mathcal{B}}{\leftarrow}; \\ \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathfrak{C}^\mathfrak{b}}{\leftarrow}} := \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}}; & \mathbb{L}l\left(\mathfrak{C}^\mathfrak{b}\right) := \mathbb{L}l(\mathcal{B}); & \mathbb{L}d\left(\mathfrak{C}^\mathfrak{b}\right) := \mathbb{L}d(\mathcal{B}) \\ \mathbf{Tm}\left(\mathfrak{C}^\mathfrak{b}\right) := \mathbf{Tm}(\mathcal{B}); & \mathbb{T}\mathbf{m}\left(\mathfrak{C}^\mathfrak{b}\right) := \mathbb{T}\mathbf{m}(\mathcal{B}); & \leq_{\mathfrak{C}^\mathfrak{b}} := \leq_{\mathcal{B}}; \\ <_{\mathfrak{C}^\mathfrak{b}} := <_{\mathcal{B}}; & \geq_{\mathfrak{C}^\mathfrak{b}} := \geq_{\mathcal{B}}; & >_{\mathfrak{C}^\mathfrak{b}} := >_{\mathcal{B}}. \end{array}$$

b) Denotations, induced from the denotations for coordinate spaces:

$$\begin{array}{lll} \mathbf{Zk}\left(\mathfrak{C}^\mathfrak{b}\right) := \mathbf{Zk}\left(\mathfrak{Q}\right); & \mathcal{T}p\left(\mathfrak{C}^\mathfrak{b}\right) := \mathcal{T}p\left(\mathfrak{Q}\right); & \mathbb{L}s\left(\mathfrak{C}^\mathfrak{b}\right) := \mathbb{L}s\left(\mathfrak{Q}\right); \\ \mathfrak{Ps}\left(\mathfrak{C}^\mathfrak{b}\right) := \mathfrak{Ps}\left(\mathfrak{Q}\right); & \mathbf{di}_{\mathfrak{C}^\mathfrak{b}} := \mathbf{di}_{\mathfrak{Q}}; & \|\cdot\|_{\mathfrak{C}^\mathfrak{b}} := \|\cdot\|_{\mathfrak{Q}}; \\ (\cdot,\cdot)_{\mathfrak{C}^\mathfrak{b}} := (\cdot,\cdot)_{\mathfrak{Q}}. & & \end{array}$$

Also in the case $\mathbb{L}s\left(\mathfrak{C}^\mathfrak{b}\right) \neq \emptyset$ for arbitrary $a_1, \ldots, a_n \in \mathbf{Zk}\left(\mathfrak{C}^\mathfrak{b}\right)$, $\lambda_1, \ldots \lambda_n \in \mathfrak{Ps}\left(\mathfrak{C}^\mathfrak{b}\right)$ we use the denotation, $(\lambda_1 a_1 + \cdots + \lambda_n a_n)_{\mathfrak{C}^\mathfrak{b}} := (\lambda_1 a_1 + \cdots + \lambda_n a_n)_{\mathcal{Q}}$.

c) Own designations for base kinematic sets:

$$\mathsf{BE}\left(\mathfrak{C}^\mathfrak{b}\right) := \mathcal{B}; \qquad \mathsf{BG}\left(\mathfrak{C}^\mathfrak{b}\right) := \mathfrak{Q}; \qquad \mathfrak{q}_{\mathfrak{C}^\mathfrak{b}}(x) := k(x) \ \left(x \in \mathfrak{Bs}\left(\mathfrak{C}^\mathfrak{b}\right)\right).$$

Note, that for any elementary state $x \in \mathfrak{Bs}\left(\mathfrak{C}^\mathfrak{b}\right)$ the function $\mathfrak{q}_{\mathfrak{C}^\mathfrak{b}}(\cdot)$ puts in accordance its coordinate $\mathfrak{q}_{\mathfrak{C}^\mathfrak{b}}(x) \in \mathbf{Zk}\left(\mathfrak{C}^\mathfrak{b}\right)$.

d) Abbreviated version of denotations

- We use all abbreviated variants of denotations, described in Subsection 6.3 (but, with the replacement of the symbol "$\mathcal{B}$" by the symbol "$\mathfrak{C}^\mathfrak{b}$" and the term "base changeable set" by the term "base kinematic set".
- In the cases, when the base kinematic set $\mathfrak{C}^\mathfrak{b}$ is known in advance, we will use the denotations $\mathbf{di}$, $\|\cdot\|$, $(\cdot,\cdot)$, $\mathfrak{q}(x)$ instead of the denotations $\mathbf{di}_{\mathfrak{C}^\mathfrak{b}}$, $\|\cdot\|_{\mathfrak{C}^\mathfrak{b}}$, $(\cdot,\cdot)_{\mathfrak{C}^\mathfrak{b}}$, $\mathfrak{q}_{\mathfrak{C}^\mathfrak{b}}(x)$ (correspondingly).

#### 14.2.2 System of Denotations for Kinematic Sets.

Let, $\mathfrak{C} = (\mathcal{Z}, \mathcal{G})$, where $\mathcal{G} = ((\mathfrak{Q}_\mathfrak{l}, k_\mathfrak{l}) \mid \mathfrak{l} \in \mathcal{L}k(\mathcal{Z}))$ be any kinematic set.

**a)** The changeable set $\mathbb{BE}(\mathfrak{C}) := \mathcal{Z}$ will be named the ***evolution base*** of the kinematic set $\mathfrak{C}$.

**b)** The sets $\mathcal{I}nd(\mathfrak{C}) := \mathcal{I}nd(\mathcal{Z}) = \mathcal{I}nd(\mathbb{BE}(\mathfrak{C}))$; $\mathcal{L}k(\mathfrak{C}) := \mathcal{L}k(\mathcal{Z}) = \mathcal{L}k(\mathbb{BE}(\mathfrak{C}))$ will be named by the set of ***indexes*** and the the set of all ***reference frames*** of kinematic set $\mathfrak{C}$ (correspondingly).

**c)** For any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C}) = \mathcal{L}k(\mathcal{Z})$ we keep all denotations, introduced for reference frames of changeable sets (it concerns the denotations: $\mathtt{ind}(\mathfrak{l})$, $\mathfrak{Bs}(\mathfrak{l})$, $\underset{\mathfrak{l}}{\leftarrow}$, $\mathbb{B}\mathfrak{s}(\mathfrak{l})$, $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathfrak{l}}{\leftarrow}}$, $\mathbf{Tm}(\mathfrak{l})$, $\mathbb{T}\mathbf{m}(\mathfrak{l})$, $\mathbb{L}l(\mathfrak{l})$, $\mathbb{L}d(\mathfrak{l})$, $\leq_\mathfrak{l}$, $<_\mathfrak{l}$, $\geq_\mathfrak{l}$, $>_\mathfrak{l}$).

**d)** For arbitrary reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathfrak{C})$ it is induced the denotation for unification mapping:

$$\langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathfrak{C} \rangle := \langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathcal{Z} \rangle.$$

In particular in the case, when the changeable set $\mathcal{Z}$ is precisely visible (in this case we say, that the kinematic set $\mathfrak{C}$ is ***precisely visible***), we introduce the denotation:

$$\langle !\, \mathfrak{m} \leftarrow \mathfrak{l}, \mathfrak{C} \rangle := \langle !\, \mathfrak{m} \leftarrow \mathfrak{l}, \mathcal{Z} \rangle.$$

**e)** For any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C})$ we introduce the denotation

$$\mathfrak{C} \upharpoonright \mathfrak{l} := \left(\mathfrak{l}^{\wedge}, (\mathfrak{Q}_{\mathfrak{l}}, k_{\mathfrak{l}})\right).$$

By Definition II.14.3, the pair $\mathfrak{C} \upharpoonright \mathfrak{l}$ is a base kinematic set (for arbitrary reference frame $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C})$). The base kinematic set $\mathfrak{C} \upharpoonright \mathfrak{l}$ will be named the ***image of kinematic set*** $\mathfrak{C}$ in the reference frame $\mathfrak{l}$.

**f)** For any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C})$ we introduce the following denotations:

$$\begin{aligned}
&\mathbf{Zk}(\mathfrak{l};\,\mathfrak{C}) := \mathbf{Zk}(\mathfrak{C} \upharpoonright \mathfrak{l}) = \mathbf{Zk}(\mathfrak{Q}_{\mathfrak{l}}); && \mathbb{L}s(\mathfrak{l};\,\mathfrak{C}) := \mathbb{L}s(\mathfrak{C} \upharpoonright \mathfrak{l}) = \mathbb{L}s(\mathfrak{Q}_{\mathfrak{l}});\\
&\mathcal{T}p(\mathfrak{l};\,\mathfrak{C}) := \mathcal{T}p(\mathfrak{C} \upharpoonright \mathfrak{l}) = \mathcal{T}p(\mathfrak{Q}_{\mathfrak{l}}); && \mathfrak{Ps}(\mathfrak{l};\,\mathfrak{C}) := \mathfrak{Ps}(\mathfrak{C} \upharpoonright \mathfrak{l}) = \mathfrak{Ps}(\mathfrak{Q}_{\mathfrak{l}});\\
&\|\cdot\|_{\mathfrak{l},\mathfrak{C}} := \|\cdot\|_{\mathfrak{C}\upharpoonright\mathfrak{l}} = \|\cdot\|_{\mathfrak{Q}_{\mathfrak{l}}}; && \mathbf{di}_{\mathfrak{l}}(\cdot;\,\mathfrak{C}) := \mathbf{di}_{\mathfrak{C}\upharpoonright\mathfrak{l}} = \mathbf{di}_{\mathfrak{Q}_{\mathfrak{l}}};\\
&(\cdot,\cdot)_{\mathfrak{l},\mathfrak{C}} := (\cdot,\cdot)_{\mathfrak{C}\upharpoonright\mathfrak{l}} = (\cdot,\cdot)_{\mathfrak{Q}_{\mathfrak{l}}}; && \mathsf{BE}(\mathfrak{l}) := \mathsf{BE}(\mathfrak{C} \upharpoonright \mathfrak{l}) = \mathfrak{l}^{\wedge};\\
& && \mathsf{BG}(\mathfrak{l};\,\mathfrak{C}) := \mathsf{BG}(\mathfrak{C} \upharpoonright \mathfrak{l}) = \mathfrak{Q}_{\mathfrak{l}}.
\end{aligned}$$

Also for reference frames $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C})$ such, that $\mathbb{L}s(\mathfrak{l}) \neq \emptyset$ and for arbitrary $a_1, \dots, a_n \in \mathbf{Zk}(\mathfrak{l};\,\mathfrak{C})$, $\lambda_1, \dots \lambda_n \in \mathfrak{Ps}(\mathfrak{l};\,\mathfrak{C})$ we will use the denotation, $(\lambda_1 a_1 + \cdots + \lambda_n a_n)_{\mathfrak{l},\mathfrak{C}} := (\lambda_1 a_1 + \cdots + \lambda_n a_n)_{\mathcal{Q}_l}$.

**g)** For any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C})$ we use the following denotation:

$$\mathfrak{q}_{\mathfrak{l}}(x;\,\mathfrak{C}) := \mathfrak{q}_{\mathfrak{C}\upharpoonright\mathfrak{l}}(x) = k_{\mathfrak{l}}(x),\ x \in \mathfrak{Bs}(\mathfrak{l}).$$

**h)** Abbreviated versions of denotations:

- In the cases, when the kinematic set $\mathfrak{C}$ is known in advance, we will use the denotations $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle$, $\langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle$, $\mathbf{Zk}(\mathfrak{l})$, $\mathbb{L}s(\mathfrak{l})$, $\mathbf{di}_{\mathfrak{l}}$, $(\cdot,\cdot)_{\mathfrak{l}}$, $\mathcal{T}p(\mathfrak{l})$, $\mathfrak{Ps}(\mathfrak{l})$, $\|\cdot\|_{\mathfrak{l}}$, $\mathsf{BG}(\mathfrak{l})$, $\mathfrak{q}_{\mathfrak{l}}(x)$ instead of the denotations $\langle \mathfrak{m} \leftarrow \mathfrak{l},\,\mathfrak{C} \rangle$, $\langle !\, \mathfrak{m} \leftarrow \mathfrak{l},\,\mathfrak{C} \rangle$, $\mathbf{Zk}(\mathfrak{l};\,\mathfrak{C})$, $\mathbb{L}s(\mathfrak{l};\,\mathfrak{C})$, $\mathbf{di}_{\mathfrak{l}}(\cdot;\,\mathfrak{C})$, $(\cdot,\cdot)_{\mathfrak{l},\mathfrak{C}}$, $\mathcal{T}p(\mathfrak{l};\,\mathfrak{C})$, $\mathfrak{Ps}(\mathfrak{l};\,\mathfrak{C})$, $\|\cdot\|_{\mathfrak{l},\mathfrak{C}}$, $\mathsf{BG}(\mathfrak{l};\,\mathfrak{C})$, $\mathfrak{q}_{\mathfrak{l}}(x;\,\mathfrak{C})$ (correspondingly).
- In the cases, when the reference frame $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C})$ is known in advance, we will use the denotations $\mathbf{di}$, $\|\cdot\|$, $(\cdot,\cdot)$, $\mathfrak{q}(x)$, $\lambda_1 a_1 + \cdots + \lambda_n a_n$ instead of the denotations $\mathbf{di}_{\mathfrak{l}}$, $\|\cdot\|_{\mathfrak{l}}$, $(\cdot,\cdot)_{\mathfrak{l}}$, $\mathfrak{q}_{\mathfrak{l}}(x)$, $(\lambda_1 a_1 + \cdots + \lambda_n a_n)_{\mathfrak{l},\mathfrak{C}}$ (correspondingly). Also we use all abbreviated variants of denotations, introduced for reference frames of changeable sets and described in Subsection 10.2 (see text under item 5)).

**Assertion II.14.1.** *Let $\mathfrak{C}_1$, $\mathfrak{C}_2$ be arbitrary kinematic sets, and besides:*

1. $\mathcal{L}k(\mathfrak{C}_1) = \mathcal{L}k(\mathfrak{C}_2)$.
2. *For any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C}_1) = \mathcal{L}k(\mathfrak{C}_2)$ it is true the equality, $\mathfrak{C}_1 \upharpoonright \mathfrak{l} = \mathfrak{C}_2 \upharpoonright \mathfrak{l}$.*
3. *For arbitrary reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathfrak{C}_1) = \mathcal{L}k(\mathfrak{C}_2)$ it holds the equality, $\langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathfrak{C}_1 \rangle = \langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathfrak{C}_2 \rangle$.*

*Then, $\mathfrak{C}_1 = \mathfrak{C}_2$.*

*Proof.* Let, $\mathfrak{C}_1 = (\mathcal{Z}_1, \mathcal{G}_1)$, $\mathfrak{C}_2 = (\mathcal{Z}_2, \mathcal{G}_2)$, where $\mathcal{G}_1 = \left(\left(\mathfrak{Q}_{\mathfrak{l}}^{(1)}, k_{\mathfrak{l}}^{(1)}\right) \mid \mathfrak{l} \in \mathcal{L}k(\mathcal{Z}_1)\right)$, $\mathcal{G}_2 = \left(\left(\mathfrak{Q}_{\mathfrak{l}}^{(2)}, k_{\mathfrak{l}}^{(2)}\right) \mid \mathfrak{l} \in \mathcal{L}k(\mathcal{Z}_2)\right)$ be the kinematic sets, satisfying the conditions of Assertion II.14.1. Then, under these assumptions, the changeable sets $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are satisfying the conditions of Assertion I.10.1. Hence, $\mathcal{Z}_1 = \mathcal{Z}_2$.

By the condition of Assertion, which we are to prove, for any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z}_1) = \mathcal{L}k(\mathcal{Z}_2)$ it holds the equality $\mathfrak{C}_1 \upharpoonright \mathfrak{l} = \mathfrak{C}_2 \upharpoonright \mathfrak{l}$. Hence, by the denotations, accepted in the subsection 14.2.2, we have, $\left(\mathfrak{l}^{\wedge}, \left(\mathfrak{Q}_{\mathfrak{l}}^{(1)}, k_{\mathfrak{l}}^{(1)}\right)\right) = \mathfrak{C}_1 \upharpoonright \mathfrak{l} = \mathfrak{C}_2 \upharpoonright \mathfrak{l} = \left(\mathfrak{l}^{\wedge}, \left(\mathfrak{Q}_{\mathfrak{l}}^{(2)}, k_{\mathfrak{l}}^{(2)}\right)\right)$. Therefore, $\left(\mathfrak{Q}_{\mathfrak{l}}^{(1)}, k_{\mathfrak{l}}^{(1)}\right) = \left(\mathfrak{Q}_{\mathfrak{l}}^{(2)}, k_{\mathfrak{l}}^{(2)}\right)$ $(\forall \mathfrak{l} \in \mathcal{L}k(\mathcal{Z}_1) = \mathcal{L}k(\mathcal{Z}_2))$. Consequently:

$$\begin{aligned}\mathcal{G}_1 &= \left(\left(\mathfrak{Q}_{\mathfrak{l}}^{(1)}, k_{\mathfrak{l}}^{(1)}\right) \mid \mathfrak{l} \in \mathcal{L}k(\mathcal{Z}_1)\right) \\ &= \left(\left(\mathfrak{Q}_{\mathfrak{l}}^{(2)}, k_{\mathfrak{l}}^{(2)}\right) \mid \mathfrak{l} \in \mathcal{L}k(\mathcal{Z}_2)\right) = \mathcal{G}_2.\end{aligned}$$

Thus, $\mathfrak{C}_1 = (\mathcal{Z}_1, \mathcal{G}_1) = (\mathcal{Z}_2, \mathcal{G}_2) = \mathfrak{C}_2$. □

**Corollary II.14.1.** *Let $\mathfrak{C}_1$, $\mathfrak{C}_2$ be arbitrary kinematic sets, and besides:*

1. $\mathcal{L}k(\mathfrak{C}_1) = \mathcal{L}k(\mathfrak{C}_2)$.
2. *For any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C}_1) = \mathcal{L}k(\mathfrak{C}_2)$ they hold the equalities:*

$$\begin{aligned}&\mathsf{BG}(\mathfrak{l};\ \mathfrak{C}_1) = \mathsf{BG}(\mathfrak{l};\ \mathfrak{C}_2) \\ &\mathfrak{q}_{\mathfrak{l}}(x,\ \mathfrak{C}_1) = \mathfrak{q}_{\mathfrak{l}}(x,\ \mathfrak{C}_2) \quad (\forall x \in \mathfrak{Bs}(\mathfrak{l})).\end{aligned}$$

3. *For arbitrary reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathfrak{C}_1) = \mathcal{L}k(\mathfrak{C}_2)$ it is true the equality, $\langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathfrak{C}_1 \rangle = \langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathfrak{C}_2 \rangle$.*

*Then, $\mathfrak{C}_1 = \mathfrak{C}_2$.*

*Proof.* Let, $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be the kinematic sets, satisfying the conditions of the Corollary. Then, by the system of denotations, accepted in the subsection 14.2.2, for any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C}_1) = \mathcal{L}k(\mathfrak{C}_2)$, we obtain:

$$\begin{aligned}\mathfrak{C}_1 \upharpoonright \mathfrak{l} &= (\mathsf{BE}(\mathfrak{l}), (\mathsf{BG}(\mathfrak{l}, \mathfrak{C}_1), \mathfrak{q}_{\mathfrak{l}}(\cdot;\ \mathfrak{C}_1))) = \\ &= (\mathsf{BE}(\mathfrak{l}), (\mathsf{BG}(\mathfrak{l}, \mathfrak{C}_2), \mathfrak{q}_{\mathfrak{l}}(\cdot;\ \mathfrak{C}_2))) = \mathfrak{C}_2 \upharpoonright \mathfrak{l}.\end{aligned}$$

Thus, by Assertion II.14.1, we have, $\mathfrak{C}_1 = \mathfrak{C}_2$. □

*Remark* II.14.1. From the system of denotations, accepted in the subsection 14.2, it follows, that for any kinematic set $\mathfrak{C}$, Properties I.10.1 and Corollary I.12.6 are kept to be true, and in the case, when the kinematic set $\mathfrak{C}$ is precisely visible, Properties I.12.1 also remain true (but everywhere in these properties we should replace the symbol $\mathcal{Z}$ by the symbol $\mathfrak{C}$ and the term "changeable set" by the term "kinematic set").

Main results of this Section were anonced in [11] and published in [10, sections 3,4,5].

## 15 Coordinate Transforms in Kinematic Sets.

Let, $\mathfrak{C}$ be any kinematic set. For any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C})$ we introduce the following denotations:

$$\begin{aligned}\mathbb{M}k(\mathfrak{l};\mathfrak{C}) &:= \mathbf{Tm}(\mathfrak{l}) \times \mathbf{Zk}(\mathfrak{l}). \\ \mathbf{Q}^{\langle\mathfrak{l}\rangle}(\omega;\mathfrak{C}) &:= (\mathsf{tm}(\omega), \mathfrak{q}_{\mathfrak{l}}(\mathsf{bs}(\omega))) \in \mathbb{M}k(\mathfrak{l};\mathfrak{C}), \quad \omega \in \mathbb{B}\mathfrak{s}(\mathfrak{l}).\end{aligned}$$

The set $\mathbb{M}k(\mathfrak{l};\mathfrak{C})$ we name by the ***Minkowski set*** of reference frame $\mathfrak{l}$ in kinematic set $\mathfrak{C}$. The value $\mathbf{Q}^{\langle\mathfrak{l}\rangle}(\omega;\mathfrak{C})$ will be named by the ***Minkowski coordinates*** of the elementary-time state $\omega \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$ ***in the reference frame*** $\mathfrak{l}$.

In the cases, when the kinematic set $\mathfrak{C}$ is known in advance, we use the denotations $\mathbb{M}k(\mathfrak{l})$, $\mathbf{Q}^{\langle\mathfrak{l}\rangle}(\omega)$ instead of the denotations $\mathbb{M}k(\mathfrak{l};\mathfrak{C})$, $\mathbf{Q}^{\langle\mathfrak{l}\rangle}(\omega;\mathfrak{C})$ (correspondingly).

**Definition II.15.1.** *Let $\mathfrak{C}$ be any precisely visible kinematic set and $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathfrak{C})$ be arbitrary reference frames of $\mathfrak{C}$.*

1. *The mapping $\mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}(\cdot\,; \mathfrak{C}) : \mathbb{B}\mathfrak{s}(\mathfrak{l}) \mapsto \mathbb{M}k(\mathfrak{m})$, represented by the formula:*
$$\mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}(\omega; \mathfrak{C}) = \mathbf{Q}^{\langle \mathfrak{m} \rangle}(\langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \omega), \quad \omega \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$$
*we name* ***actual coordinate transform*** *from $\mathfrak{l}$ to $\mathfrak{m}$.*
*Hence, for any $\omega \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$ the value $\mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}(\omega; \mathfrak{C})$ coincides with Minkowski coordinates of the elementary-time state $\omega$ in the (another) reference frame $\mathfrak{m} \in \mathcal{L}k(\mathfrak{C})$.*

2. *We name the mapping $\widetilde{Q} : \mathbb{M}k(\mathfrak{l}) \mapsto \mathbb{M}k(\mathfrak{m})$ by* ***universal coordinate transform*** *from $\mathfrak{l}$ to $\mathfrak{m}$ if and only if:*
   - *$\widetilde{Q}$ is bijection (one-to-one mapping) between $\mathbb{M}k(\mathfrak{l})$ and $\mathbb{M}k(\mathfrak{m})$.*
   - *For any elementary-time state $\omega \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$ the following equality is performed:*
$$\mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}(\omega; \mathfrak{C}) = \widetilde{Q}\left(\mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega)\right).$$

3. *We say, that reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathfrak{C})$* ***allow universal coordinate transform****, if and only if at least one universal coordinate transform $\widetilde{Q} : \mathbb{M}k(\mathfrak{l}) \mapsto \mathbb{M}k(\mathfrak{m})$ from $\mathfrak{l}$ to $\mathfrak{m}$ exists.*
*In the case, where reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathfrak{C})$ allow universal coordinate transform, we use the denotation:*
$$\mathfrak{l} \underset{\mathfrak{C}}{\rightleftarrows} \mathfrak{m},$$
*In the case, when the kinematic set $\mathfrak{C}$ is known in advance, we use the abbreviated denotation $\mathfrak{l} \rightleftarrows \mathfrak{m}$.*

4. *Indexed family of mappings $\left(\widetilde{Q}_{\mathfrak{m},\mathfrak{l}}\right)_{\mathfrak{l},\mathfrak{m} \in \mathcal{L}k(\mathfrak{C})}$ is named by* ***universal coordinate transform for the kinematic set*** *$\mathfrak{C}$ if and only if:*
   - *For arbitrary $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathfrak{C})$ the mapping $\widetilde{Q}_{\mathfrak{m},\mathfrak{l}}$ is universal coordinate transform from $\mathfrak{l}$ to $\mathfrak{m}$.*
   - *For any $\mathfrak{l}, \mathfrak{m}, \mathfrak{p} \in \mathcal{L}k(\mathfrak{C})$ and $\mathrm{w} \in \mathbb{M}k(\mathfrak{l})$ the following equalities are true:*
$$\widetilde{Q}_{\mathfrak{l},\mathfrak{l}}(\mathrm{w}) = \mathrm{w}; \qquad \widetilde{Q}_{\mathfrak{p},\mathfrak{m}}\left(\widetilde{Q}_{\mathfrak{m},\mathfrak{l}}(\mathrm{w})\right) = \widetilde{Q}_{\mathfrak{p},\mathfrak{l}}(\mathrm{w}). \tag{II.1}$$

5. *We say, that the kinematic set $\mathfrak{C}$* ***allows universal coordinate transform****, if and only if there exists at least one universal coordinate transform $\left(\widetilde{Q}_{\mathfrak{m},\mathfrak{l}}\right)_{\mathfrak{l},\mathfrak{m} \in \mathcal{L}k(\mathfrak{C})}$ for $\mathfrak{C}$.*

*Remark* II.15.1. In the cases, when the kinematic set $\mathfrak{C}$ is known in advance, we use the abbreviated denotation $\mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}(\omega)$ instead of the denotation $\mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}(\omega; \mathfrak{C})$.

**Assertion II.15.1.** *Let $\mathfrak{C}$ be any precisely visible kinematic set. Then:*

1. *For an arbitrary $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C})$ the identity mapping on $\mathbb{M}k(\mathfrak{l})$:*
$$\mathbb{I}_{[\mathfrak{l}]}(\mathrm{w}) := \mathrm{w}, \qquad \mathrm{w} \in \mathbb{M}k(\mathfrak{l})$$
*is universal coordinate transform from $\mathfrak{l}$ to $\mathfrak{l}$.*

2. *If $\widetilde{Q}$ is universal coordinate transform from $\mathfrak{l}$ to $\mathfrak{m}$ ($\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathfrak{C})$), then $\widetilde{Q}^{[-1]}$ is universal coordinate transform from $\mathfrak{m}$ to $\mathfrak{l}$ (the mapping $\widetilde{Q}^{[-1]}$, inverse to $\widetilde{Q}$, exists, because, according to Definition II.15.1 (item 2), $\widetilde{Q}$ is bijection from $\mathbb{M}k(\mathfrak{l})$ onto $\mathbb{M}k(\mathfrak{m})$).*

3. *If* $\widetilde{Q}^{(\mathfrak{m},\mathfrak{l})}$ *is universal coordinate transform from* $\mathfrak{l}$ *to* $\mathfrak{m}$*, and* $\widetilde{Q}^{(\mathfrak{p},\mathfrak{m})}$ *is universal coordinate transform from* $\mathfrak{m}$ *to* $\mathfrak{p}$ *(*$\mathfrak{l},\mathfrak{m},\mathfrak{p}\in\mathcal{L}k(\mathfrak{C})$*), then the composition of the mappings* $\widetilde{Q}^{(\mathfrak{p},\mathfrak{m})}$ *and* $\widetilde{Q}^{(\mathfrak{m},\mathfrak{l})}$*, that is the mapping:*
$$\widetilde{Q}^{(\mathfrak{p},\mathfrak{l})}(\mathrm{w})=\widetilde{Q}^{(\mathfrak{p},\mathfrak{m})}\left(\widetilde{Q}^{(\mathfrak{m},\mathfrak{l})}(\mathrm{w})\right),\quad \mathrm{w}\in\mathbb{M}k(\mathfrak{l}).$$
*is universal coordinate transform from* $\mathfrak{l}$ *to* $\mathfrak{p}$.

4. *The binary relation* $\rightleftarrows$ *is equivalence relation on the set* $\mathcal{L}k(\mathfrak{C})$ *of all reference frames of* $\mathfrak{C}$.

*Proof.* 1. Consider any $\mathfrak{l}\in\mathcal{L}k(\mathfrak{C})$. It is evident, that $\mathbb{I}_{[\mathfrak{l}]}$ is bijection from $\mathbb{M}k(\mathfrak{l})$ to $\mathbb{M}k(\mathfrak{l})$. Using Definition II.15.1 (item 1) and Property I.12.1(1), for any elementary-time state $\omega\in\mathbb{B}\mathfrak{s}(\mathfrak{l})$ we obtain:
$$\begin{aligned}\mathbf{Q}^{\langle\mathfrak{l}\leftarrow\mathfrak{l}\rangle}(\omega)&=\mathbf{Q}^{\langle\mathfrak{l}\rangle}(\langle!\,\mathfrak{l}\leftarrow\mathfrak{l}\rangle\,\omega)=\\&=\mathbf{Q}^{\langle\mathfrak{l}\rangle}(\omega)=\mathbb{I}_{[\mathfrak{l}]}\left(\mathbf{Q}^{\langle\mathfrak{l}\rangle}(\omega)\right).\end{aligned}$$
Therefore, by Definition II.15.1 (item 2), $\mathbb{I}_{[\mathfrak{l}]}$ is universal coordinate transform from $\mathfrak{l}$ to $\mathfrak{l}$.

2. Let $\widetilde{Q}$ be universal coordinate transform from $\mathfrak{l}$ to $\mathfrak{m}$ ($\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathfrak{C})$). Then, for any $\omega\in\mathbb{B}\mathfrak{s}(\mathfrak{l})$, according to Definition II.15.1 (items 1 and 2), we have:
$$\begin{aligned}\mathbf{Q}^{\langle\mathfrak{m}\rangle}\left(\langle!\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega\right)&=\mathbf{Q}^{\langle\mathfrak{m}\leftarrow\mathfrak{l}\rangle}(\omega)=\\&=\widetilde{Q}\left(\mathbf{Q}^{\langle\mathfrak{l}\rangle}(\omega)\right).\end{aligned}$$
Hence:
$$\mathbf{Q}^{\langle\mathfrak{l}\rangle}(\omega)=\widetilde{Q}^{[-1]}\left(\mathbf{Q}^{\langle\mathfrak{m}\rangle}\left(\langle!\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega\right)\right).$$
Therefore, for any $\omega_1\in\mathbb{B}\mathfrak{s}(\mathfrak{m})$ ($\langle!\,\mathfrak{l}\leftarrow\mathfrak{m}\rangle\,\omega_1\in\mathbb{B}\mathfrak{s}(\mathfrak{l})$), in accordance with Properties I.12.1(1,3) we obtain:
$$\begin{aligned}&\mathbf{Q}^{\langle\mathfrak{l}\rangle}\left(\langle!\,\mathfrak{l}\leftarrow\mathfrak{m}\rangle\,\omega_1\right)=\\&=\widetilde{Q}^{[-1]}\left(\mathbf{Q}^{\langle\mathfrak{m}\rangle}\left(\langle!\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\langle!\,\mathfrak{l}\leftarrow\mathfrak{m}\rangle\,\omega_1\right)\right)=\\&=\widetilde{Q}^{[-1]}\left(\mathbf{Q}^{\langle\mathfrak{m}\rangle}\left(\langle!\,\mathfrak{m}\leftarrow\mathfrak{m}\rangle\,\omega_1\right)\right)=\\&=\widetilde{Q}^{[-1]}\left(\mathbf{Q}^{\langle\mathfrak{m}\rangle}\left(\omega_1\right)\right).\end{aligned}$$
That is, by Definition II.15.1 (item 1):
$$\begin{aligned}\mathbf{Q}^{\langle\mathfrak{l}\leftarrow\mathfrak{m}\rangle}\left(\omega_1\right)=\widetilde{Q}^{[-1]}&\left(\mathbf{Q}^{\langle\mathfrak{m}\rangle}\left(\omega_1\right)\right)\\&(\forall\omega_1\in\mathbb{B}\mathfrak{s}(\mathfrak{m})).\end{aligned}$$
Thus, by Definition II.15.1 (item 2), $\widetilde{Q}^{[-1]}$ is universal coordinate transform from $\mathfrak{m}$ to $\mathfrak{l}$.

3. Let $\widetilde{Q}^{(\mathfrak{m},\mathfrak{l})}$ be universal coordinate transform from $\mathfrak{l}$ to $\mathfrak{m}$, and $\widetilde{Q}^{(\mathfrak{p},\mathfrak{m})}$ be universal coordinate transform from $\mathfrak{m}$ to $\mathfrak{p}$ ($\mathfrak{l},\mathfrak{m},\mathfrak{p}\in\mathcal{L}k(\mathfrak{C})$). Denote: $\widetilde{Q}^{(\mathfrak{p},\mathfrak{l})}(\mathrm{w}):=\widetilde{Q}^{(\mathfrak{p},\mathfrak{m})}\left(\widetilde{Q}^{(\mathfrak{m},\mathfrak{l})}(\mathrm{w})\right)$, $\mathrm{w}\in\mathbb{M}k(\mathfrak{l})$. It is clear, that the mapping $\widetilde{Q}^{(\mathfrak{p},\mathfrak{l})}$ is bijection between $\mathbb{M}k(\mathfrak{l})$ and $\mathbb{M}k(\mathfrak{p})$. At the same time, using Definition II.15.1 (items 1,2) and applying Properties I.12.1, for any $\omega\in\mathbb{B}\mathfrak{s}(\mathfrak{l})$ we deduce:
$$\begin{aligned}\mathbf{Q}^{\langle\mathfrak{p}\leftarrow\mathfrak{l}\rangle}(\omega)&=\mathbf{Q}^{\langle\mathfrak{p}\rangle}\left(\langle!\,\mathfrak{p}\leftarrow\mathfrak{l}\rangle\,\omega\right)=\\&=\mathbf{Q}^{\langle\mathfrak{p}\rangle}\left(\langle!\,\mathfrak{p}\leftarrow\mathfrak{m}\rangle\,\langle!\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega\right)=\\&=\mathbf{Q}^{\langle\mathfrak{p}\leftarrow\mathfrak{m}\rangle}\left(\langle!\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega\right)=\\&=\widetilde{Q}^{(\mathfrak{p},\mathfrak{m})}\left(\mathbf{Q}^{\langle\mathfrak{m}\rangle}\left(\langle!\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega\right)\right)=\end{aligned}$$

$$= \widetilde{Q}^{(\mathfrak{p},\mathfrak{m})}\left(\mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}(\omega)\right) =$$
$$= \widetilde{Q}^{(\mathfrak{p},\mathfrak{m})}\left(\widetilde{Q}^{(\mathfrak{m},\mathfrak{l})}\left(\mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega)\right)\right) = \widetilde{Q}^{(\mathfrak{p},\mathfrak{l})}\left(\mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega)\right).$$

Consequently, by Definition II.15.1 (item 2), $\widetilde{Q}^{(\mathfrak{p},\mathfrak{l})}$ is universal coordinate transform from $\mathfrak{l}$ to $\mathfrak{p}$.

4. Item 4 of Assertion II.15.1 immediately follows from the items 1,2 and 3. □

**Assertion II.15.2.** *For an arbitrary precisely visible kinematic set $\mathfrak{C}$ the following propositions are equivalent:*

1. *$\mathfrak{C}$ allows universal coordinate transform.*
2. *For arbitrary reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathfrak{C})$ it is true the correlation $\mathfrak{l} \rightleftarrows \mathfrak{m}$ (that is arbitrary two reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathfrak{C})$ allow universal coordinate transform).*
3. *There exists a reference frame $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C})$ such, that for any reference frame $\mathfrak{m} \in \mathcal{L}k(\mathfrak{C})$ it is true the correlation $\mathfrak{l} \rightleftarrows \mathfrak{m}$.*

*Proof.* 1. The implication 1 $\Longrightarrow$ 2 follows from Definition II.15.1 (items 3 and 4).

2. According to Property I.10.1(1), the set $\mathcal{L}k(\mathfrak{C})$ is nonempty. Therefore, to verify the truth of the implication 2 $\Longrightarrow$ 3 it is sufficient to chose any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C})$.

3. Consequently, it remains to prove the implication 3 $\Longrightarrow$ 1. Suppose, there exists a reference frame $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C})$ such, that for any reference frame $\mathfrak{m} \in \mathcal{L}k(\mathfrak{C})$ the correlation $\mathfrak{l} \rightleftarrows \mathfrak{m}$ is performed. Hence, for any reference frame $\mathfrak{m} \in \mathcal{L}k(\mathfrak{C})$, there exists an universal coordinate transform $Q^{(\mathfrak{m},\mathfrak{l})} : \mathbb{M}k(\mathfrak{l}) \mapsto \mathbb{M}k(\mathfrak{m})$. For arbitrary reference frames $\mathfrak{m}, \mathfrak{p} \in \mathcal{L}k(\mathfrak{C})$ we denote:

$$\widetilde{Q}_{\mathfrak{p},\mathfrak{m}}(\mathrm{w}) := Q^{(\mathfrak{p},\mathfrak{l})}\left(\left(Q^{(\mathfrak{m},\mathfrak{l})}\right)^{[-1]}(\mathrm{w})\right), \qquad \mathrm{w} \in \mathbb{M}k(\mathfrak{m}). \tag{II.2}$$

In accordance with Assertion II.15.1 (items 2 and 3), the mapping $\widetilde{Q}_{\mathfrak{p},\mathfrak{m}} : \mathbb{M}k(\mathfrak{m}) \mapsto \mathbb{M}k(\mathfrak{p})$ is universal coordinate transform from $\mathfrak{m}$ to $\mathfrak{p}$ (for arbitrary $\mathfrak{m}, \mathfrak{p} \in \mathcal{L}k(\mathfrak{C})$). Moreover, by the equality (II.2), for arbitrary $\mathfrak{m}, \mathfrak{p}, \mathfrak{k} \in \mathcal{L}k(\mathfrak{C})$ and $\mathrm{w} \in \mathbb{M}k(\mathfrak{m})$ we obtain:

$$\widetilde{Q}_{\mathfrak{m},\mathfrak{m}}(\mathrm{w}) = Q^{(\mathfrak{m},\mathfrak{l})}\left(\left(Q^{(\mathfrak{m},\mathfrak{l})}\right)^{[-1]}(\mathrm{w})\right) = \mathrm{w};$$
$$\widetilde{Q}_{\mathfrak{k},\mathfrak{p}}\left(\widetilde{Q}_{\mathfrak{p},\mathfrak{m}}(\mathrm{w})\right) = Q^{(\mathfrak{k},\mathfrak{l})}\left(\left(Q^{(\mathfrak{p},\mathfrak{l})}\right)^{[-1]}\left(Q^{(\mathfrak{p},\mathfrak{l})}\left(\left(Q^{(\mathfrak{m},\mathfrak{l})}\right)^{[-1]}(\mathrm{w})\right)\right)\right) =$$
$$= Q^{(\mathfrak{k},\mathfrak{l})}\left(\left(Q^{(\mathfrak{m},\mathfrak{l})}\right)^{[-1]}(\mathrm{w})\right) = \widetilde{Q}_{\mathfrak{k},\mathfrak{m}}(\mathrm{w}).$$

Thus, according to Definition II.15.1 (item 4), the family of mappings $\left(\widetilde{Q}_{\mathfrak{p},\mathfrak{m}}\right)_{\mathfrak{m},\mathfrak{p}\in\mathcal{L}k(\mathfrak{C})}$ is universal coordinate transform for the kinematic set $\mathfrak{C}$. Hence, by Definition II.15.1 (item 5), kinematic set $\mathfrak{C}$ allows universal coordinate transform. □

Examples of kinematic sets, which allow universal coordinate transform will be presented in Section 19. In Section 20 it will be proved the existence of kinematic sets, which do not allow universal coordinate transform. Therefore (by Assertion II.15.2) there exist kinematic set $\mathfrak{C}$, in which some reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathfrak{C})$ do not allow universal coordinate transform. The next aim is to prove necessary and sufficient condition for existence of universal coordinate transform between reference frames of precisely visible kinematic set. Below we introduce the necessary notions to do this.

Let, $\mathfrak{C}^{\mathfrak{b}}$ be any base kinematic set. For any subset $A \subseteq \mathbb{B}\mathfrak{s}\left(\mathfrak{C}^{\mathfrak{b}}\right)$ we introduce the denotations:

$$\mathfrak{trj}_{\mathfrak{C}^{\mathfrak{b}}}[A] := \mathbf{Q}^{\left\langle \mathfrak{C}^{\mathfrak{b}} \right\rangle}(A) =$$

$$= \left\{ \mathbf{Q}^{\langle \mathfrak{C}^{\mathfrak{b}} \rangle}(\omega) \mid \omega \in A \right\} \subseteq \mathbb{M}k\left(\mathfrak{C}^{\mathfrak{b}}\right). \tag{II.3}$$

The set $\mathfrak{trj}_{\mathfrak{C}^{\mathfrak{b}}}[A]$ will be named by the ***trajectory*** of the subset $A \subseteq \mathbb{B}\mathfrak{s}\left(\mathfrak{C}^{\mathfrak{b}}\right)$. For any base kinematic set $\mathfrak{C}^{\mathfrak{b}}$ we denote:

$$\begin{aligned} \mathbb{T}\mathbf{rj}\left(\mathfrak{C}^{\mathfrak{b}}\right) &:= \mathfrak{trj}_{\mathfrak{C}^{\mathfrak{b}}}\left[\mathbb{B}\mathfrak{s}\left(\mathfrak{C}^{\mathfrak{b}}\right)\right] = \\ &= \left\{ \mathbf{Q}^{\langle \mathfrak{C}^{\mathfrak{b}} \rangle}(\omega) \mid \omega \in \mathbb{B}\mathfrak{s}\left(\mathfrak{C}^{\mathfrak{b}}\right) \right\} \subseteq \mathbb{M}k\left(\mathfrak{C}^{\mathfrak{b}}\right), \\ \overline{\mathbb{T}\mathbf{rj}}\left(\mathfrak{C}^{\mathfrak{b}}\right) &:= \mathbb{M}k\left(\mathfrak{C}^{\mathfrak{b}}\right) \setminus \mathbb{T}\mathbf{rj}\left(\mathfrak{C}^{\mathfrak{b}}\right). \end{aligned}$$

The set $\mathbb{T}\mathbf{rj}\left(\mathfrak{C}^{\mathfrak{b}}\right)$ will be named by the ***(general) trajectory*** of base kinematic set $\mathfrak{C}^{\mathfrak{b}}$, and the set $\overline{\mathbb{T}\mathbf{rj}}\left(\mathfrak{C}^{\mathfrak{b}}\right)$ will be named as ***complement of (general) trajectory*** of $\mathfrak{C}^{\mathfrak{b}}$. Respectively, for any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathfrak{C})$ of any kinematic set $\mathfrak{C}$ we can define the trajectory of any subset $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$, as well as (general) trajectory and complement of (general) trajectory for the reference frame $\mathfrak{l}$:

$$\left.\begin{aligned} \mathfrak{trj}_{\mathfrak{l}}[A;\, \mathfrak{C}] &:= \mathfrak{trj}_{\mathfrak{C}\restriction\mathfrak{l}}[A] = \left\{ \mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega) \mid \omega \in A \right\}; \\ \mathbb{T}\mathbf{rj}(\mathfrak{l};\, \mathfrak{C}) &:= \mathbb{T}\mathbf{rj}(\mathfrak{C} \restriction \mathfrak{l}) = \\ &= \left\{ \mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega) \mid \omega \in \mathbb{B}\mathfrak{s}(\mathfrak{l}) \right\}; \\ \overline{\mathbb{T}\mathbf{rj}}(\mathfrak{l};\, \mathfrak{C}) &:= \overline{\mathbb{T}\mathbf{rj}}(\mathfrak{C} \restriction \mathfrak{l}) = \mathbb{M}k(\mathfrak{l}) \setminus \mathbb{T}\mathbf{rj}(\mathfrak{l};\, \mathfrak{C}) \end{aligned}\right\} \tag{II.4}$$

(In the cases, when the kinematic set $\mathfrak{C}$ is known in advance, we use the abbreviated denotations $\mathfrak{trj}_{\mathfrak{l}}[A]$, $\mathbb{T}\mathbf{rj}(\mathfrak{l})$, $\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{l})$ instead of the denotations $\mathfrak{trj}_{\mathfrak{l}}[A;\, \mathfrak{C}]$, $\mathbb{T}\mathbf{rj}(\mathfrak{l};\, \mathfrak{C})$, $\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{l};\, \mathfrak{C})$ (correspondingly).)

The set $\mathbb{T}\mathbf{rj}(\mathfrak{l})$ will be named by the ***(general) trajectory*** for the reference frame $\mathfrak{l}$, and the set $\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{l})$ will be named by ***complement of (general) trajectory*** of the reference frame $\mathfrak{l}$ in the kinematic set $\mathfrak{C}$.

**Theorem II.15.1.** *Let $\mathfrak{C}$ be a precisely visible kinematic set and $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathfrak{C})$ be any fixed reference frames of $\mathfrak{C}$.*

*The reference frames $\mathfrak{l}, \mathfrak{m}$ allow universal coordinate transform (i.e. $\mathfrak{l} \rightleftarrows \mathfrak{m}$) if and only if the following conditions are satisfied:*

1. $\mathbf{card}\left(\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{l})\right) = \mathbf{card}\left(\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{m})\right)$, *where* $\mathbf{card}(\mathcal{M})$ *means the* ***cardinality*** *of a set* $\mathcal{M}$.
2. *For arbitrary elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$ the equality $\mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}(\omega_1) = \mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}(\omega_2)$ is performed if and only if $\mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega_1) = \mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega_2)$.*

*Proof.* **1.** Suppose, that $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathfrak{C})$ and $\mathfrak{l} \rightleftarrows \mathfrak{m}$. Then, by Definition II.15.1, there exists the bijection $\widetilde{Q} : \mathbb{M}k(\mathfrak{l}) \mapsto \mathbb{M}k(\mathfrak{m})$ such, that for any elementary-time state $\omega \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$ the following equality holds:

$$\mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}(\omega) = \widetilde{Q}\left(\mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega)\right). \tag{II.5}$$

1.a) Using the definition of general trajectory for reference frame (see (II.4)), Properties I.12.1(1,3), Definition II.15.1 (item 1) and equality (II.5), we deduce:

$$\begin{aligned} \mathbb{T}\mathbf{rj}(\mathfrak{m}) &= \left\{ \mathbf{Q}^{\langle \mathfrak{m} \rangle}(\omega) \mid \omega \in \mathbb{B}\mathfrak{s}(\mathfrak{m}) \right\} = \\ &= \left\{ \mathbf{Q}^{\langle \mathfrak{m} \rangle}\left(\langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle \langle !\, \mathfrak{l} \leftarrow \mathfrak{m} \rangle\, \omega\right) \mid \omega \in \mathbb{B}\mathfrak{s}(\mathfrak{m}) \right\} = \\ &= \left\{ \mathbf{Q}^{\langle \mathfrak{m} \rangle}\left(\langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \omega_1\right) \mid \omega_1 \in \mathbb{B}\mathfrak{s}(\mathfrak{l}) \right\} = \\ &= \left\{ \mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}(\omega_1) \mid \omega_1 \in \mathbb{B}\mathfrak{s}(\mathfrak{l}) \right\} = \\ &= \left\{ \widetilde{Q}\left(\mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega_1)\right) \mid \omega_1 \in \mathbb{B}\mathfrak{s}(\mathfrak{l}) \right\} = \widetilde{Q}(\mathbb{T}\mathbf{rj}(\mathfrak{l})). \end{aligned}$$

According to the equalities (II.4), taking into account, that $\widetilde{Q}$ is bijection between $\mathbb{M}k(\mathfrak{l})$ and $\mathbb{M}k(\mathfrak{m})$, we obtain:

$$\begin{aligned}\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{m}) &= \mathbb{M}k(\mathfrak{m}) \setminus \mathbb{T}\mathbf{rj}(\mathfrak{m}) = \\ &= \widetilde{Q}(\mathbb{M}k(\mathfrak{l})) \setminus \widetilde{Q}(\mathbb{T}\mathbf{rj}(\mathfrak{l})) = \\ &= \widetilde{Q}(\mathbb{M}k(\mathfrak{l}) \setminus \mathbb{T}\mathbf{rj}(\mathfrak{l})) = \widetilde{Q}(\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{l})).\end{aligned}$$

Since $\widetilde{Q}$ is bijection, we have proved, that $\mathbf{card}\left(\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{m})\right) = \mathbf{card}\left(\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{l})\right)$.

1.b) Let, $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$ and $\mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega_1) = \mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega_2)$. Then, according to (II.5):

$$\begin{aligned}\mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}(\omega_1) &= \widetilde{Q}\left(\mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega_1)\right) = \\ &= \widetilde{Q}\left(\mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega_2)\right) = \mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}(\omega_2).\end{aligned}$$

Inversely, if we suppose, that $\mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}(\omega_1) = \mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}(\omega_2)$, then, by (II.5), $\widetilde{Q}\left(\mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega_1)\right) = \widetilde{Q}\left(\mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega_2)\right)$, and since $\widetilde{Q}$ is bijection, we have, $\mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega_1) = \mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega_2)$.

**2.** Conversely: suppose, that for reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathfrak{C})$ the conditions 1,2 of this Theorem are satisfied. For $\mathrm{w} = \mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega) \in \mathbb{T}\mathbf{rj}(\mathfrak{l})$, where $\omega \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$ we put:

$$\widetilde{Q}_0(\mathrm{w}) := \mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}(\omega). \tag{II.6}$$

From the definition of general trajectory for reference frame (II.4) and the second condition of this Theorem it follows, that the formula (II.6) defines the mapping $\widetilde{Q}_0 : \mathbb{T}\mathbf{rj}(\mathfrak{l}) \mapsto \mathbb{M}k(\mathfrak{m})$ by a correct way. We aim to prove, that this mapping is bijection between $\mathbb{T}\mathbf{rj}(\mathfrak{l})$ and $\mathbb{T}\mathbf{rj}(\mathfrak{m})$. According to Definition II.15.1 (item 1) and equalities (II.4), for arbitrary $\mathrm{w} = \mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega) \in \mathbb{T}\mathbf{rj}(\mathfrak{l})$ ($\omega \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$), we receive:

$$\begin{aligned}\widetilde{Q}_0(\mathrm{w}) &= \mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}(\omega) = \\ &= \mathbf{Q}^{\langle \mathfrak{m} \rangle}\left(\langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \omega\right) \in \mathbb{T}\mathbf{rj}(\mathfrak{m}).\end{aligned} \tag{II.7}$$

Moreover, using Properties I.12.1, for any $\mathrm{w}_1 = \mathbf{Q}^{\langle \mathfrak{m} \rangle}(\omega_1) \in \mathbb{T}\mathbf{rj}(\mathfrak{m})$ ($\omega_1 \in \mathbb{B}\mathfrak{s}(\mathfrak{m})$) we get:

$$\begin{aligned}\mathrm{w}_1 &= \mathbf{Q}^{\langle \mathfrak{m} \rangle}(\omega_1) = \\ &= \mathbf{Q}^{\langle \mathfrak{m} \rangle}\left(\langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle \langle !\, \mathfrak{l} \leftarrow \mathfrak{m} \rangle\, \omega_1\right) = \\ &= \mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}\left(\langle !\, \mathfrak{l} \leftarrow \mathfrak{m} \rangle\, \omega_1\right) = \widetilde{Q}_0\left(\mathbf{Q}^{\langle \mathfrak{l} \rangle}\left(\langle !\, \mathfrak{l} \leftarrow \mathfrak{m} \rangle\, \omega_1\right)\right), \\ &\quad \text{where} \quad \mathbf{Q}^{\langle \mathfrak{l} \rangle}\left(\langle !\, \mathfrak{l} \leftarrow \mathfrak{m} \rangle\, \omega_1\right) \in \mathbb{T}\mathbf{rj}(\mathfrak{l}).\end{aligned} \tag{II.8}$$

From the correlations (II.7) and (II.8) it follows, that $\widetilde{Q}_0$ is the mapping from $\mathbb{T}\mathbf{rj}(\mathfrak{l})$ on $\mathbb{T}\mathbf{rj}(\mathfrak{m})$. From the second condition of this Theorem we obtain, that for arbitrary $\mathrm{w}, \mathrm{w}' \in \mathbb{T}\mathbf{rj}(\mathfrak{l})$ such, that $\mathrm{w} \neq \mathrm{w}'$ it is true the correlation $\widetilde{Q}_0(\mathrm{w}) \neq \widetilde{Q}_0(\mathrm{w}')$. Hence, the mapping $\widetilde{Q}_0$ is a bijection between $\mathbb{T}\mathbf{rj}(\mathfrak{l})$ and $\mathbb{T}\mathbf{rj}(\mathfrak{m})$.

By the conditions of Theorem, the sets $\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{l})$ and $\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{m})$ are equipotent. Thus, there exists a bijection $\widetilde{Q}_1 : \overline{\mathbb{T}\mathbf{rj}}(\mathfrak{l}) \mapsto \overline{\mathbb{T}\mathbf{rj}}(\mathfrak{m})$ between $\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{l})$ and $\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{m})$. According to the definition of general trajectory for reference frame (see (II.4)), we have, $\mathbb{T}\mathbf{rj}(\mathfrak{l}) \sqcup \overline{\mathbb{T}\mathbf{rj}}(\mathfrak{l}) = \mathbb{M}k(\mathfrak{l})$ (where the symbol "$\sqcup$" denotes disjoint union of sets). Hence for $\omega \in \mathbb{M}k(\mathfrak{l})$ we can put:

$$\widetilde{Q}(\mathrm{w}) := \begin{cases}\widetilde{Q}_0(\mathrm{w}), & \mathrm{w} \in \mathbb{T}\mathbf{rj}(\mathfrak{l}) \\ \widetilde{Q}_1(\mathrm{w}), & \mathrm{w} \in \overline{\mathbb{T}\mathbf{rj}}(\mathfrak{l}).\end{cases} \tag{II.9}$$

Since (in accordance with the statements, proved above) $\widetilde{Q}_0$ is bijection between $\mathbb{T}\mathbf{rj}(\mathfrak{l})$ and $\mathbb{T}\mathbf{rj}(\mathfrak{m})$ as well as $\widetilde{Q}_1$ is bijection between $\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{l})$ and $\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{m})$, we must conclude, that $\widetilde{Q}$ is bijection between $\mathbb{M}k(\mathfrak{l}) = \mathbb{T}\mathbf{rj}(\mathfrak{l}) \sqcup \overline{\mathbb{T}\mathbf{rj}}(\mathfrak{l})$ and $\mathbb{M}k(\mathfrak{m}) = \mathbb{T}\mathbf{rj}(\mathfrak{m}) \sqcup \overline{\mathbb{T}\mathbf{rj}}(\mathfrak{m})$. Moreover, for any $\omega \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$, by definitions of the mappings $\widetilde{Q}$ and $\widetilde{Q}_0$, we get:

$$\widetilde{Q}\left(\mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega)\right) = \widetilde{Q}_0\left(\mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega)\right) = \mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle}(\omega).$$

Thus, by the Definition II.15.1 (item 2), $\widetilde{Q}$ is universal coordinate transform from $\mathfrak{l}$ to $\mathfrak{m}$. □

*Remark* II.15.2. Universal coordinate transform between two reference frames of kinematic set (if it exists) is not uniquely defined for the general case. Indeed, suppose, that two reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathfrak{C})$ of kinematic set $\mathfrak{C}$ satisfy the following conditions:

$$\mathfrak{l} \rightleftarrows \mathfrak{m} \quad \text{and} \quad \mathbf{card}\left(\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{l})\right) = \mathbf{card}\left(\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{m})\right) \geq 2.$$

Then there exist many bijections between $\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{l})$ and $\overline{\mathbb{T}\mathbf{rj}}(\mathfrak{m})$. So universal coordinate transform in (II.9) is not uniquely defined.

Main results of this Section are published in [10, Section 6].

In the next three sections (16, 17, 18) we prove Theorem on multi-image for kinematic sets as well as we introduce and investigate generalized Lorentz-Poincare transformations (in the sense of E. Recami and V. Olkhovsky), which are necessary to build mathematically strict model of kinematics of special relativity and its extension to the tachyon kinematics, which allows super-light motion for inertial reference frames.

## 16 Theorem on Multi-image for Kinematic Sets

**Definition II.16.1.**

1. *The ordered composition of five sets* $(\mathbb{T}, \mathcal{X}, U, \mathfrak{Q}, k)$ *is named by* ***kinematic projector*** *for base changeable set* $\mathcal{B}$ *if and only if:*

   **1.1.** $(\mathbb{T}, \mathcal{X}, U)$ *is evolution projector for* $\mathcal{B}$.

   **1.2.** $\mathfrak{Q}$ *is a coordinate space.*

   **1.3.** $k$ *is a mapping from* $\mathcal{X}$ *into* $\mathbf{Zk}(\mathfrak{Q})$.

   ►► *In the case, where* $(\mathbb{T}, \mathcal{X}, U)$ *is injective evolution projector for* $\mathcal{B}$*, the kinematic projector* $(\mathbb{T}, \mathcal{X}, U, \mathfrak{Q}, k)$ *is named by* ***injective****.*

2. *Any indexed family* $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha, \mathfrak{Q}_\alpha, k_\alpha) \mid \alpha \in \mathcal{A})$ *(where* $\mathcal{A} \neq \emptyset$*) of injective kinematic projectors for base changeable set* $\mathcal{B}$ *we name by* ***kinematic multi-projector*** *for* $\mathcal{B}$.

*Remark* II.16.1. Henceforward we will consider only injective kinematic projectors. That is why we will use the term "kinematic projector" instead of the term "injective kinematic projector".

Let $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha, \mathfrak{Q}_\alpha, k_\alpha) \mid \alpha \in \mathcal{A})$ be any kinematic multi-projector for $\mathcal{B}$. Denote:

$$\mathfrak{P}^{[e]} := ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha) \mid \alpha \in \mathcal{A}).$$

By the definitions II.16.1 and I.11.3, $\mathfrak{P}^{[e]}$ is (injective) evolution multi-projector for $\mathcal{B}$.

**Theorem II.16.1.** *Let* $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha, \mathfrak{Q}_\alpha, k_\alpha) \mid \alpha \in \mathcal{A})$ *be a kinematic multi-projector for a base changeable set* $\mathcal{B}$*. Then:*
***A)*** *Only one kinematic set* $\mathfrak{C}$ *exists, satisfying the following conditions:*

1. $\mathbb{BE}(\mathfrak{C}) = \mathcal{Z}\mathsf{im}\left[\mathfrak{P}^{[e]}, \mathcal{B}\right]$.

2. *For any reference frame* $\mathfrak{l} = (\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]) \in \mathcal{L}k(\mathfrak{C})$ *(where* $\alpha \in \mathcal{A}$*) the following equalities are performed:*

$$\textbf{2.1)}\ \mathsf{BG}(\mathfrak{l}) = \mathfrak{Q}_\alpha; \qquad \textbf{2.2)}\ \mathsf{q}_\mathfrak{l}(x) = k_\alpha(x) \quad (x \in \mathfrak{Bs}(\mathfrak{l})).$$

***B)*** *Kinematic set* $\mathfrak{C}$*, satisfying the conditions 1,2 is precisely visible.*

*Proof.* Let $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha, \mathfrak{Q}_\alpha, k_\alpha) \mid \alpha \in \mathcal{A})$ (where $\mathbb{T}_\alpha = (\mathbf{T}_\alpha, \leq_\alpha)$, $\alpha \in \mathcal{A}$) be a kinematic multi-projector for $\mathcal{B}$.

**A)** Put:

$$\mathcal{Z} := \mathcal{Z}\mathsf{im}\left[\mathfrak{P}^{[e]}, \mathcal{B}\right].$$

Then, according to Theorem I.11.2:

$$\mathcal{L}k(\mathcal{Z}) = \{(\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]) \mid \alpha \in \mathcal{A}\}.$$

Consider any fixed reference frame $\mathfrak{l} = (\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]) \in \mathcal{L}k(\mathcal{Z})$ (where $\alpha \in \mathcal{A}$). Denote:

$$\mathfrak{Q}^{(\mathfrak{l})} := \mathfrak{Q}_\alpha.$$

The ordered five-elements composition $(\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha, \mathfrak{Q}_\alpha, k_\alpha)$ is a kinematic projector. Hence, by Definition II.16.1, the triple $(\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha) = ((\mathbf{T}_\alpha, \leq_\alpha), \mathcal{X}_\alpha, U_\alpha)$ is evolution projector for $\mathcal{B}$. Consequently, by the definition of evolution projector (Definition I.11.1), $U_\alpha$ is the mapping of kind $U_\alpha : \mathbb{B}\mathfrak{s}(\mathcal{B}) \mapsto \mathbf{T}_\alpha \times \mathcal{X}_\alpha$. Therefore, by Property I.11.2(3), we obtain:

$$\mathfrak{Bs}(\mathfrak{l}) = \{\mathsf{bs}(U_\alpha(\omega)) \mid \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})\} \subseteq \mathcal{X}_\alpha.$$

For an arbitrary $x \in \mathfrak{Bs}(\mathfrak{l})$ we denote:

$$k^{(\mathfrak{l})}(x) := k_\alpha(x).$$

According to the definition of a kinematic projector (Definition II.16.1) $k_\alpha$ is the mapping from $\mathcal{X}_\alpha$ into $\mathbf{Zk}(\mathfrak{Q}_\alpha) = \mathbf{Zk}\left(\mathfrak{Q}^{(\mathfrak{l})}\right)$. Hence, $k^{(\mathfrak{l})}$ is the mapping from $\mathfrak{Bs}(\mathfrak{l})$ into $\mathbf{Zk}\left(\mathfrak{Q}^{(\mathfrak{l})}\right)$.

Hence, by the Definition II.14.3 (item 2), the pair

$$\mathfrak{C} = \left(\mathcal{Z},\ \left(\left(\mathfrak{Q}^{(\mathfrak{l})},\ k^{(\mathfrak{l})}\right) \mid \mathfrak{l} \in \mathcal{L}k(\mathcal{Z})\right)\right) \tag{II.10}$$

is a kinematic set. Herewith, taking into account the system of denotations, accepted in the subsection 14.2.2, we get:

$$\mathbb{BE}(\mathfrak{C}) = \mathcal{Z} = \mathcal{Z}\mathsf{im}\left[\mathfrak{P}^{[e]}, \mathcal{B}\right],$$

and for any reference frame $\mathfrak{l} = (\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]) \in \mathcal{L}k(\mathfrak{C})$, where $\alpha \in \mathcal{A}$ we have:

$$\mathsf{BG}(\mathfrak{l}) = \mathfrak{Q}^{(\mathfrak{l})} = \mathfrak{Q}_\alpha;$$
$$\mathsf{q}_\mathfrak{l}(x) = k^{(\mathfrak{l})}(x) = k_\alpha(x) \quad (x \in \mathfrak{Bs}(\mathfrak{l})).$$

Thus, the kinematic set $\mathfrak{C}$ satisfies conditions 1,2 of the item **A)** of Theorem II.16.1.

Now, we are going to prove, that kinematic set $\mathfrak{C}$, satisfying conditions 1,2 of the item **A)** of Theorem II.16.1 is unique. Assume, that the kinematic set $\mathfrak{C}_1$ also satisfies the conditions 1,2 of the item **A)** of Theorem II.16.1. Then, by Condition 1 of the item **A)** of Theorem II.16.1, $\mathbb{BE}(\mathfrak{C}) = \mathcal{Z} = \mathbb{BE}(\mathfrak{C}_1)$. Hence,

$$\mathcal{L}k(\mathfrak{C}) = \mathcal{L}k(\mathcal{Z}) = \mathcal{L}k(\mathfrak{C}_1),$$

moreover, for any reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathfrak{C}) = \mathcal{L}k(\mathfrak{C}_1)$ we have:

$$\langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathfrak{C} \rangle = \langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathcal{Z} \rangle = \langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathfrak{C}_1 \rangle.$$

Further, by Condition 2 of the item **A)** of Theorem II.16.1, for any reference frame $\mathfrak{l} = (\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]) \in \mathcal{L}k(\mathfrak{C}) = \mathcal{L}k(\mathfrak{C}_1)$ we deliver:

$$\mathsf{BG}(\mathfrak{l};\ \mathfrak{C}) = \mathfrak{Q}_\alpha = \mathsf{BG}(\mathfrak{l};\ \mathfrak{C}_1); \qquad \mathfrak{q}_\mathfrak{l}(x,\ \mathfrak{C}) = k_\alpha(x) = \mathfrak{q}_\mathfrak{l}(x,\ \mathfrak{C}_1) \quad (x \in \mathfrak{Bs}(\mathfrak{l})).$$

Therefore, by Corollary II.14.1, $\mathfrak{C} = \mathfrak{C}_1$.

**B)** Recall, that the notion of *precise visibility*, for kinematic sets is introduced in item d) of the subsection 14.2.2. So, since the changeable set $\mathcal{Z}$ is precisely visible (according to Corollary I.12.3), then the kinematic set $\mathfrak{C}$, represented, by the formula (II.10), also is precisely visible. □

**Definition II.16.2.** *Let, $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha, \mathfrak{Q}_\alpha, k_\alpha) \mid \alpha \in \mathcal{A})$ be a kinematic multi-projector for a base changeable set $\mathcal{B}$. The kinematic set $\mathfrak{C}$, satisfying conditions 1,2 of Theorem II.16.1 will be named by **kinematic multi-image** of base changeable set $\mathcal{B}$ relatively the kinematic multi-projector $\mathfrak{P}$. This kinematic set will be denoted via $\mathfrak{Kim}[\mathfrak{P}, \mathcal{B}]$:*

$$\mathfrak{Kim}[\mathfrak{P}, \mathcal{B}] := \mathfrak{C}.$$

Applying Properties I.11.2, Corollary I.12.7 and Theorem II.16.1, we obtain the following properties for kinematic multi-image of base changeable set.

**Properties II.16.1.** *Let, $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha, \mathfrak{Q}_\alpha, k_\alpha) \mid \alpha \in \mathcal{A})$ be a kinematic multi-projector for $\mathcal{B}$ (where $\mathbb{T}_\alpha = (\mathbf{T}_\alpha, \le_\alpha)$, $\alpha \in \mathcal{A}$). Suppose, that $\mathfrak{C} = \mathfrak{Kim}[\mathfrak{P}, \mathcal{B}]$. Then:*

1. $\mathcal{L}k(\mathfrak{C}) = \{(\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]) \mid \alpha \in \mathcal{A}\}$.
2. $\mathcal{I}nd(\mathfrak{C}) = \mathcal{A}$.
3. *For any reference frame $\mathfrak{l} = (\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha])$ the following equalities hold:*

$$\begin{aligned}
\mathbb{B}\mathfrak{s}(\mathfrak{l}) &= U_\alpha(\mathbb{B}\mathfrak{s}(\mathcal{B})) = \{U_\alpha(\omega) \mid \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})\};\\
\mathfrak{Bs}(\mathfrak{l}) &= \{\mathsf{bs}(U_\alpha(\omega)) \mid \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})\};\\
\mathbb{T}\mathbf{m}(\mathfrak{l}) &= \mathbb{T}_\alpha; \quad \mathbf{Tm}(\mathfrak{l}) = \mathbf{T}_\alpha;\\
\mathbf{Zk}(\mathfrak{l}) &= \mathbf{Zk}(\mathsf{BG}(\mathfrak{l})) = \mathbf{Zk}(\mathfrak{Q}_\alpha);\\
\mathbb{M}k(\mathfrak{l}) &= \mathbf{Tm}(\mathfrak{l}) \times \mathbf{Zk}(\mathfrak{l}) = \mathbf{T}_\alpha \times \mathbf{Zk}(\mathfrak{Q}_\alpha);\\
\mathfrak{q}_\mathfrak{l}(x) &= k_\alpha(x) \quad (x \in \mathfrak{Bs}(\mathfrak{l}));\\
\mathbf{Q}^{\langle\mathfrak{l}\rangle}(\omega) &= (\mathsf{tm}(\omega), \mathfrak{q}_\mathfrak{l}(\mathsf{bs}(\omega))) = (\mathsf{tm}(\omega), k_\alpha(\mathsf{bs}(\omega))) \qquad (\omega \in \mathbb{B}\mathfrak{s}(\mathfrak{l})).
\end{aligned}$$

4. *Let, $\mathfrak{l} = (\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]) \in \mathcal{L}k(\mathfrak{C})$, where $\alpha \in \mathcal{A}$. Suppose, that $\widetilde{\omega}_1, \widetilde{\omega}_2 \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$ and $\mathsf{tm}(\widetilde{\omega}_1) \ne \mathsf{tm}(\widetilde{\omega}_2)$. Then $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ are united by fate in $\mathfrak{l}$ if and only if there exist united by fate in $\mathcal{B}$ elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ such, that $\widetilde{\omega}_1 = U_\alpha(\omega_1)$, $\widetilde{\omega}_2 = U_\alpha(\omega_2)$.*
5. *For any reference frames $\mathfrak{l} = (\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]) \in \mathcal{L}k(\mathfrak{C})$, $\mathfrak{m} = (\beta, U_\beta[\mathcal{B}, \mathbb{T}_\beta]) \in \mathcal{L}k(\mathfrak{C})$ $(\alpha, \beta \in \mathcal{A})$ the following equality holds:*

$$\langle !\, \mathfrak{m} \leftarrow \mathfrak{l},\ \mathfrak{C}\rangle\, \omega = U_\beta\left(U_\alpha^{[-1]}(\omega)\right) \quad (\omega \in \mathbb{B}\mathfrak{s}(\mathfrak{l}) = U_\alpha(\mathbb{B}\mathfrak{s}(\mathcal{B}))).$$

Let, $\mathfrak{Q}$ be a coordinate space, $\mathcal{B}$ be a base changeable set such, that $\mathfrak{Bs}(\mathcal{B}) \subseteq \mathbf{Zk}(\mathfrak{Q})$ (such base changeable set $\mathcal{B}$ exists, because, for example, we may put $\mathcal{B} := \mathcal{A}t(\mathbb{T}, \mathcal{R})$, where $\mathcal{R}$ is a system of abstract trajectories from the linear ordered set $\mathbb{T}$ to a set $\mathbf{M} \subseteq \mathbf{Zk}(\mathfrak{Q})$, where the definition of $\mathcal{A}t(\mathbb{T}, \mathcal{R})$ can be found in Example I.6.3 (see also Theorem I.6.1)). Let $\mathbb{U}$ be any transforming set of bijections relatively the $\mathcal{B}$ on $\mathbf{Zk}(\mathfrak{Q})$ (in the sense of Example I.11.2). Then, any mapping $\mathbf{U} \in \mathbb{U}$ is the mapping of kind, $\mathbf{U} : \mathbf{Tm}(\mathcal{B}) \times \mathbf{Zk}(\mathfrak{Q}) \longleftrightarrow \mathbf{Tm}(\mathcal{B}) \times \mathbf{Zk}(\mathfrak{Q})$, where $\mathbb{B}\mathfrak{s}(\mathcal{B}) \subseteq \mathbf{Tm}(\mathcal{B}) \times \mathfrak{Bs}(\mathcal{B}) \subseteq \mathbf{Tm}(\mathcal{B}) \times \mathbf{Zk}(\mathfrak{Q})$. Hence, the set of bijections $\mathbb{U}$ generates the kinematic multi-projector $\widehat{\mathbb{U}} := \left(\left(\mathbb{T}\mathbf{m}(\mathcal{B}), \mathbf{Zk}(\mathfrak{Q}), \mathbf{U}, \mathfrak{Q}, \mathbb{I}_{\mathbf{Zk}(\mathfrak{Q})}\right) \mid \mathbf{U} \in \mathbb{U}\right)$ for $\mathcal{B}$, where $\mathbb{I}_{\mathbf{Zk}(\mathfrak{Q})}$ is the identity mapping on $\mathbf{Zk}(\mathfrak{Q})$. Denote:

$$\mathfrak{Kim}(\mathbb{U}, \mathcal{B}, \mathfrak{Q}) := \mathfrak{Kim}\left[\widehat{\mathbb{U}}, \mathcal{B}\right]. \tag{II.11}$$

**Theorem II.16.2.** *The kinematic set* $\mathfrak{C} = \mathfrak{Kim}\,(\mathbb{U}, \mathcal{B}, \mathfrak{Q})$ *allows universal coordinate transform. Moreover,* $\mathcal{L}k\,(\mathfrak{C}) = ((\mathbf{U}, \mathbf{U}\,[\mathcal{B}]) \mid \mathbf{U} \in \mathbb{U})$*, and the system of mappings* $\left(\widetilde{Q}_{\mathfrak{m},\mathfrak{l}}\right)_{\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathfrak{C})}$*:*

$$\widetilde{Q}_{\mathfrak{m},\mathfrak{l}}(\mathrm{w}) = \mathbf{V}\left(\mathbf{U}^{[-1]}(\mathrm{w})\right), \quad \mathrm{w} \in \mathbb{M}k(\mathfrak{l}) = \mathbf{Tm}(\mathcal{B}) \times \mathbf{Zk}(\mathfrak{Q}) \tag{II.12}$$
$$(\,\mathfrak{l} = (\mathbf{U}, \mathbf{U}\,[\mathcal{B}]) \in \mathcal{L}k\,(\mathfrak{C})\,, \quad \mathfrak{m} = (\mathbf{V}, \mathbf{V}\,[\mathcal{B}]) \in \mathcal{L}k\,(\mathfrak{C})\,)$$

*is universal coordinate transform for* $\mathfrak{C}$.

*Proof.* Let, $\mathfrak{Q}$ be a coordinate space and $\mathbb{U}$ be transforming set of bijections relatively the base changeable set $\mathcal{B}$ ($\mathfrak{Bs}(\mathcal{B}) \subseteq \mathbf{Zk}(\mathfrak{Q})$) on $\mathbf{Zk}(\mathfrak{Q})$. Denote $\mathfrak{C} := \mathfrak{Kim}\,(\mathbb{U}, \mathcal{B}, \mathfrak{Q})$. Then, $\mathfrak{C} = \mathfrak{Kim}\left[\widehat{\mathbb{U}}, \mathcal{B}\right]$, where $\widehat{\mathbb{U}} = \left(\left(\mathbb{T}\mathbf{m}(\mathcal{B}), \mathbf{Zk}(\mathfrak{Q}), \mathbf{U}, \mathfrak{Q}, \mathbb{I}_{\mathbf{Zk}(\mathfrak{Q})}\right) \mid \mathbf{U} \in \mathbb{U}\right)$. Hence, according to Property II.16.1(1), $\mathcal{L}k\,(\mathfrak{C}) = \{(\mathbf{U}, \mathbf{U}\,[\mathcal{B}]) \mid \mathbf{U} \in \mathbb{U}\}$. And, by Property II.16.1(3), for an arbitrary reference frame $\mathfrak{l} = (\mathbf{U}, \mathbf{U}\,[\mathcal{B}]) \in \mathcal{L}k\,(\mathfrak{C})$ we have: $\mathfrak{Bs}(\mathfrak{l}) = \{\mathsf{bs}\,(\mathbf{U}(\omega)) \mid \omega \in \mathbb{Bs}(\mathcal{B})\} \subseteq \mathbf{Zk}(\mathfrak{Q})$. Herewith, by Theorem II.16.1, $\mathfrak{q}_{\mathfrak{l}}\,(x, \mathfrak{C}) = x \quad (\forall\, x \in \mathfrak{Bs}(\mathfrak{l}))$. Hence:

$$\mathbf{Q}^{\langle \mathfrak{l}\rangle}(\omega; \mathfrak{C}) = (\mathsf{tm}\,(\omega)\,, \mathfrak{q}_{\mathfrak{l}}(\mathsf{bs}\,(\omega))) = (\mathsf{tm}\,(\omega)\,, \mathsf{bs}\,(\omega)) = \omega \quad (\mathfrak{l} \in \mathcal{L}k\,(\mathfrak{C})\,,\ \omega \in \mathbb{Bs}(\mathfrak{l}))\,.$$

Using the last equality and Property II.16.1(5), for arbitrary reference frames $\mathfrak{l} = (\mathbf{U}, \mathbf{U}\,[\mathcal{B}]) \in \mathcal{L}k\,(\mathfrak{C})$, $\mathfrak{m} = (\mathbf{V}, \mathbf{V}\,[\mathcal{B}]) \in \mathcal{L}k\,(\mathfrak{C}) \quad (\mathbf{U}, \mathbf{V} \in \mathbb{U})$ we obtain:

$$\begin{aligned}\mathbf{Q}^{\langle \mathfrak{m} \leftarrow \mathfrak{l}\rangle}(\omega; \mathfrak{C}) &= \mathbf{Q}^{\langle \mathfrak{m}\rangle}(\langle !\, \mathfrak{m} \leftarrow \mathfrak{l}\rangle\, \omega) = \langle !\, \mathfrak{m} \leftarrow \mathfrak{l}\rangle\, \omega = \\ &= \mathbf{V}\left(\mathbf{U}^{[-1]}(\omega)\right) = \mathbf{V}\left(\mathbf{U}^{[-1]}\left(\mathbf{Q}^{\langle \mathfrak{l}\rangle}(\omega)\right)\right) = \widetilde{Q}_{\mathfrak{m},\mathfrak{l}}\left(\mathbf{Q}^{\langle \mathfrak{l}\rangle}(\omega)\right).\end{aligned}$$

It is not hard to verify, that the system of mappings $\left(\widetilde{Q}_{\mathfrak{m},\mathfrak{l}}\right)_{\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathfrak{C})}$ satisfies conditions (II.1). Therefore, by Definition II.15.1 (item 4), we see, that $\left(\widetilde{Q}_{\mathfrak{m},\mathfrak{l}}\right)_{\mathfrak{l},\mathfrak{m}\in\mathcal{L}k(\mathfrak{C})}$ is universal coordinate transform for $\mathfrak{C}$. □

Main results of this Section were anonced in [11] and published in [13, Subsection 6.3].

## 17 Generalized Lorentz Transformations in the Sense of E. Recami and V. Olkhovsky for Hilbert Space

The fact that the existence of superlight motions is consistent with the kinematics of Einstein's special theory of relativity at the present time may be considered as generally known. In [48,49] this fact is proved by means of mathematical logic. It is interesting, that the last fact also can be proved by another way. In Example I.11.3 the kinematics, which permit superlight transformations, was built explicitly using the theory of changeable sets (this example was also published in [4, p. 128, example 2.3] and [3, p. 41, example 10.3]). Although the existence of tachyons can not be considered as experimentally verified fact, the theory of tachyons and superluminal motions is intensively developing more than 50 years [33–35], and it is very actual in our time. In first studies for this direction the theory of tachyons was considered in the framework of classical Lorentz transformations, and superluminal motion for the frames of reference was forbidden. Then, in the papers of E. Recami and V.S. Olkhovsky (in collaboration R. Mignani) [51, 52], the extended Lorentz transformations for reference frames moving with the velocity, greater, then the velocity of light $c$ were proposed (see also [36]). A little later, similar formulas were obtained in [53]. The ideas of E. Recami are still relevant in our time. In particular B. Cox and J. Hill in the paper [38] have rediscovered the formulas of Recami-Olkhovsky's extended Lorentz transformations and published a new and elegant way to deduce of them. Recami-Olkhovsky's extended Lorentz transformations are investigated in the paper [37]. Application of the Recami-Olkhovsky's extended Lorentz transformations to the problem of spinless tachyon localization can be found in [50]. It should be emphasized that in the papers [36–38] extended Lorentz transformations are examined only in the case, when two inertial frames are moving along the common $x$-axis. In the paper [6] the Recami's extended Lorentz transformations are obtained for arbitrary orientation of axes in the case, where the space of geometrical coordinates may be any real Hilbert space of any dimension, including infinity. In the paper [7] we investigate algebraic properties of extended Lorentz transformations in the sense of E. Recami and V. Olkhovsky for Hilbert space, introduced in [6]. Also some properties of these transformations were established in [8, 13].

This section contains results, connected with extended Lorentz transformations in the sense of E. Recami and V. Olkhovsky, which are necessary to build mathematically strict model of kinematics of special relativity and its tachyon extensions. Main results of this Section were published in [6, 8, 13].

### 17.1 Abstract Coordinate Transforms in Minkowski Space Time over Hilbert Space and their Properties

Let $(\mathfrak{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a real Hilbert space, where the $\|\cdot\|$ is the norm and $\langle\cdot,\cdot\rangle$ is the inner product over the space $\mathfrak{H}$. Further we assume automatically the condition $\dim(\mathfrak{H}) > 0$. Under this condition, the space $\mathfrak{H}$ contains at least one nonzero vector. Denote by $\mathcal{M}(\mathfrak{H})$ the Hilbert space

$$\mathcal{M}(\mathfrak{H}) := \mathbb{R} \times \mathfrak{H} = \{(t, x) \mid t \in \mathbb{R},\ x \in \mathfrak{H}\},$$

equipped by the following inner product and norm:

$$\begin{aligned} &\langle \mathrm{w}_1, \mathrm{w}_2 \rangle_{\mathcal{M}(\mathfrak{H})} = t_1 t_2 + \langle x_1, x_2 \rangle; \\ &\|\mathrm{w}_1\|_{\mathcal{M}(\mathfrak{H})} = t_1^2 + \|x_1\|^2 \qquad (\mathrm{w}_i = (t_i, x_i) \in \mathcal{M}(\mathfrak{H}),\ i \in \{1, 2\}). \end{aligned}$$

The space $\mathcal{M}(\mathfrak{H})$ we name by the ***Minkowski space*** over the Hilbert space $\mathfrak{H}$. In the space $\mathcal{M}(\mathfrak{H})$ we select the subspaces

$$\left.\begin{aligned} \mathfrak{H}_0 &= \{(t, \mathbf{0}) \mid t \in \mathbb{R}\} \\ \mathfrak{H}_1 &= \{(0, x) \mid x \in \mathfrak{H}\}, \end{aligned}\right\} \tag{II.13}$$

with $\mathbf{0}$ being zero vector. Then

$$\mathcal{M}(\mathfrak{H}) = \mathfrak{H}_0 \oplus \mathfrak{H}_1,$$

where $\oplus$ means the orthogonal sum of the subspaces. The space $\mathfrak{H}_0$ is isomorphic to the real field $\mathbb{R}$ and the space $\mathfrak{H}_1$ is isomorphic to the space $\mathfrak{H}$. Hence, the space $\mathfrak{H}$ may be identified with the subspace $\mathfrak{H}_1$ of the space $\mathcal{M}(\mathfrak{H})$, and $\mathcal{M}(\mathfrak{H})$ may be considered as the extension of the space $\mathfrak{H}$. That is why, futher we will use the same denotations for inner product and norm in the spaces $\mathfrak{H}$ and $\mathcal{M}(\mathfrak{H})$ (that is $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, without the index "$\mathcal{M}(\mathfrak{H})$" in subscript).

Denote by $\mathbf{e}_0$ the vector

$$\mathbf{e}_0 = (1, \mathbf{0}) \in \mathcal{M}(\mathfrak{H}).$$

We introduce the following orthogonal projectors by the subspaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$:

$$\left.\begin{aligned} &\widehat{\mathbf{T}}\mathrm{w} = t\mathbf{e}_0 = (t, \mathbf{0}) \in \mathfrak{H}_0, \quad \mathrm{w} = (t, x) \in \mathcal{M}(\mathfrak{H}); \\ &\mathbf{X}\mathrm{w} = (0, x) \in \mathfrak{H}_1, \qquad\quad \mathrm{w} = (t, x) \in \mathcal{M}(\mathfrak{H}) \end{aligned}\right\} \tag{II.14}$$

(recall, that an operator $P \in \mathcal{L}(\mathfrak{H})$ is named orthogonal projector if $P^2 = P^* = P$, where $P^*$ is the adjoint operator to the operator $P$). Also we denote by $\mathcal{T}$ the following linear operator

$$\mathcal{T}(\mathrm{w}) = t = \mathsf{tm}(\mathrm{w}), \quad \mathrm{w} = (t, x) \in \mathcal{M}(\mathfrak{H})$$

from $\mathcal{M}(\mathfrak{H})$ to $\mathbb{R}$. Then the following equality apparently holds:

$$\widehat{\mathbf{T}}\mathrm{w} = \mathcal{T}(\mathrm{w})\,\mathbf{e}_0, \quad \mathrm{w}\in\mathcal{M}(\mathfrak{H}). \tag{II.15}$$

And any vector $\mathrm{w}\in\mathcal{M}(\mathfrak{H})$ can be uniquely represented as

$$\mathrm{w} = t\mathbf{e}_0 + x = \mathcal{T}(\mathrm{w})\,\mathbf{e}_0 + \mathbf{X}\mathrm{w}, \tag{II.16}$$

where $x=\mathbf{X}\mathrm{w}\in\mathfrak{H}_1$, $t=\mathcal{T}(\mathrm{w})\in\mathbb{R}$.

Denote by $\mathcal{L}(\mathfrak{H})$ the space of linear continuous operators over the space $\mathfrak{H}$.

**Definition II.17.1.** *The operator $S\in\mathcal{L}(\mathcal{M}(\mathfrak{H}))$ is referred to as **linear coordinate transform operator** if and only if there exist the continuous inverse operator $S^{-1}\in\mathcal{L}(\mathcal{M}(\mathfrak{H}))$* {[13]}.

It is clear, that product (composition) of any two linear coordinate transform operators is a linear coordinate transform operator and the operator, inverse to linear coordinate transform operator again is a linear coordinate transform operator. Thus *the set of all linear coordinate transform operators is the group of operators over the space $\mathcal{M}(\mathfrak{H})$*.

**Definition II.17.2.** *The linear coordinate transform operator $S\in\mathcal{L}(\mathcal{M}(\mathfrak{H}))$ is called **v-determined** if and only if $\mathcal{T}(S^{-1}\mathbf{e}_0)\neq 0$. The vector*

$$\mathcal{V}(S) = \frac{\mathbf{X}S^{-1}\mathbf{e}_0}{\mathcal{T}(S^{-1}\mathbf{e}_0)} \in \mathfrak{H}_1$$

*is named the velocity of the v-determined linear coordinate transform operator $S$.*

The definition II.17.2 is consistent with the physical understanding of the speed of reference frame. Indeed suppose, that the v-determined linear coordinate transform operator $S\in\mathcal{L}(\mathcal{M}(\mathfrak{H}))$ maps the coordinates of any point in the fixed frame of reference $\mathfrak{l}$ to coordinates of this point in another frame $\mathfrak{l}'$, moving with constant velocity relative to the frame $\mathfrak{l}$. Consider any stationary relative the frame $\mathfrak{l}'$ point $\mathrm{w}_t' = x_0 + t\mathbf{e}_0$ (where $x_0\in\mathfrak{H}_1$ is fixed vector, and variable $t$ runs over all real axis $\mathbb{R}$). Then the point $\mathrm{w}_t'$ in the frame $\mathfrak{l}$ will look like as $\mathrm{w}_t = S^{-1}\mathrm{w}_t'$, and using (II.16) we obtain:

$$\begin{aligned}\mathrm{w}_t &= S^{-1}x_0 + tS^{-1}\mathbf{e}_0 = \mathcal{T}\left(S^{-1}x_0\right)\mathbf{e}_0 + \mathbf{X}S^{-1}x_0 + t\left(\mathcal{T}\left(S^{-1}\mathbf{e}_0\right)\mathbf{e}_0 + \mathbf{X}S^{-1}\mathbf{e}_0\right) = \\ &= \mathcal{T}\left(S^{-1}(x_0+t\mathbf{e}_0)\right)\mathbf{e}_0 + \mathbf{X}S^{-1}(x_0+t\mathbf{e}_0).\end{aligned}$$

Thus, for any $t_1,t_2\in\mathbb{R}$ such, that $t_1\neq t_2$ we deliver:

$$\frac{\mathbf{X}\mathrm{w}_{t_2} - \mathbf{X}\mathrm{w}_{t_1}}{\mathcal{T}(\mathrm{w}_{t_2}) - \mathcal{T}(\mathrm{w}_{t_1})} = \frac{\mathbf{X}S^{-1}(x_0+t_2\mathbf{e}_0 - (x_0+t_1\mathbf{e}_0))}{\mathcal{T}(S^{-1}(x_0+t_2\mathbf{e}_0) - S^{-1}(x_0+t_1\mathbf{e}_0))} = \mathcal{V}(S).$$

Thus, any stationary relative the frame $\mathfrak{l}'$ point is moving relative the frame $\mathfrak{l}$ with constant velocity $\mathcal{V}(S)$.

For any vector $V\in\mathfrak{H}_1$ we introduce the following subspaces:

$$\begin{aligned}\mathfrak{H}_1[V] &= \mathbf{span}\,\{V\};\\ \mathfrak{H}_{1\perp}[V] &= \mathfrak{H}_1\ominus\mathfrak{H}_1[V] = \{x\in\mathfrak{H}_1 \mid \langle x,V\rangle = 0\},\end{aligned}$$

where $\mathbf{span}\,M$ denotes the linear span of the set $M\subseteq\mathcal{M}(\mathfrak{H})$. The orthogonal projectors for the subspaces $\mathfrak{H}_1[V]$ and $\mathfrak{H}_{1\perp}[V]$ will be denoted by $\mathbf{X}_1[V]$, $\mathbf{X}_1^{\perp}[V]$:

$$\begin{aligned}\mathbf{X}_1[V]\,\mathrm{w} &= \begin{cases}\frac{\langle V,\mathrm{w}\rangle}{\|V\|^2}V, & V\neq\mathbf{0}\\ \mathbf{0}, & V=\mathbf{0}\end{cases}, \quad \mathrm{w}\in\mathcal{M}(\mathfrak{H});\\ \mathbf{X}_1^{\perp}[V] &= \mathbf{X} - \mathbf{X}_1[V].\end{aligned} \tag{II.17}$$

[13] In the case, where the mapping $S$ is linear (or aphine) operator, acting in some linear space, we use the conventional denotation $S^{-1}$ instead of $S^{[-1]}$ for inverce mapping.

It is not hard to verify, that for an arbitrary $V \in \mathfrak{H}_1$ the following equalities are performed:

$$\left.\begin{array}{rlrl} \widehat{\mathbf{T}}+\mathbf{X}= & \mathbb{I}; & \mathbf{X}_1[V]+\mathbf{X}_1^{\perp}[V]= & \mathbf{X}; \\ \widehat{\mathbf{T}}+\mathbf{X}_1[V]+\mathbf{X}_1^{\perp}[V]= & \mathbb{I}; & & \\ \widehat{\mathbf{T}}\mathbf{X}=\mathbf{X}\widehat{\mathbf{T}}= & \mathbb{O}; & \mathbf{X}_1[V]\mathbf{X}_1^{\perp}[V]=\mathbf{X}_1^{\perp}[V]\mathbf{X}_1[V]= & \mathbb{O}; \\ \widehat{\mathbf{T}}\mathbf{X}_1[V]=\mathbf{X}_1[V]\widehat{\mathbf{T}}= & \mathbb{O}; & \widehat{\mathbf{T}}\mathbf{X}_1^{\perp}[V]=\mathbf{X}_1^{\perp}[V]\widehat{\mathbf{T}}= & \mathbb{O}; \\ \mathbf{X}\mathbf{X}_1[V]=\mathbf{X}_1[V]\mathbf{X}= & \mathbf{X}_1[V]; & \mathbf{X}\mathbf{X}_1^{\perp}[V]=\mathbf{X}_1^{\perp}[V]\mathbf{X}= & \mathbf{X}_1^{\perp}[V], \\ \mathbf{X}_1[\lambda V]= & \mathbf{X}_1[V]; & \mathbf{X}_1^{\perp}[\lambda V]=\mathbf{X}_1^{\perp}[V] \quad (\forall\lambda\in & \mathbb{R}\setminus\{0\}), \end{array}\right\} \qquad \text{(II.18)}$$

where $\mathbb{I}=\mathbb{I}_{\mathcal{M}(\mathfrak{H})}$ and $\mathbb{O}$ are identity and zero operators in the space $\mathcal{L}(\mathcal{M}(\mathfrak{H}))$ correspondingly:

$$\mathbb{I}\mathrm{w}=\mathbb{I}_{\mathcal{M}(\mathfrak{H})}\mathrm{w}=\mathrm{w}; \qquad \mathbb{O}\mathrm{w}=\mathbf{0} \qquad (\forall \mathrm{w}\in\mathcal{M}(\mathfrak{H})).$$

**Lemma II.17.1.** *Let $S \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ be a linear coordinate transform operator such, that the both operators $S$ ans $S^{-1}$ are v-determined. Then $S$ is bijection between the subspaces $\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S)]$ and $\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S^{-1})]$. Moreover for any $\mathrm{w}=t\mathbf{e}_0+\lambda\mathcal{V}(S)\in\mathfrak{H}_0\oplus\mathfrak{H}_1[\mathcal{V}(S)]$ the following equality is true:*

$$S(t\mathbf{e}_0+\lambda\mathcal{V}(S))=\alpha_S\left((t-\lambda\beta_S)\mathbf{e}_0+(t-\lambda)\mathcal{V}\left(S^{-1}\right)\right) \qquad (\forall t,\lambda\in\mathbb{R}),$$

*where*

$$\alpha_S=\mathcal{T}(S\mathbf{e}_0), \quad \beta_S=1-\frac{1}{\mathcal{T}(S\mathbf{e}_0)\,\mathcal{T}(S^{-1}\mathbf{e}_0)}=1-\frac{1}{\alpha_S\alpha_{S^{-1}}}.$$

*Proof.* Let $S$, $S^{-1}$ be v-determined linear coordinate transform operators. Then, by definition II.17.2 and equalities (II.15), (II.18), for any $t,\lambda\in\mathbb{R}$ we obtain:

$$\begin{aligned} S(t\mathbf{e}_0+\lambda\mathcal{V}(S)) &= tS\mathbf{e}_0+\lambda S\mathcal{V}(S)=tS\mathbf{e}_0+\lambda S\frac{\mathbf{X}S^{-1}\mathbf{e}_0}{\mathcal{T}(S^{-1}\mathbf{e}_0)}= \\ &= tS\mathbf{e}_0+\frac{\lambda}{\mathcal{T}(S^{-1}\mathbf{e}_0)}S\left(S^{-1}\mathbf{e}_0-\widehat{\mathbf{T}}S^{-1}\mathbf{e}_0\right)= \\ &= tS\mathbf{e}_0+\frac{\lambda}{\mathcal{T}(S^{-1}\mathbf{e}_0)}S\left(S^{-1}\mathbf{e}_0-\mathcal{T}\left(S^{-1}\mathbf{e}_0\right)\mathbf{e}_0\right)= \\ &= tS\mathbf{e}_0+\frac{\lambda}{\mathcal{T}(S^{-1}\mathbf{e}_0)}\left(\mathbf{e}_0-\mathcal{T}\left(S^{-1}\mathbf{e}_0\right)S\mathbf{e}_0\right)= \\ &= (t-\lambda)S\mathbf{e}_0+\frac{\lambda}{\mathcal{T}(S^{-1}\mathbf{e}_0)}\mathbf{e}_0= \\ &= (t-\lambda)\left(\widehat{\mathbf{T}}S\mathbf{e}_0+\mathbf{X}S\mathbf{e}_0\right)+\frac{\lambda}{\mathcal{T}(S^{-1}\mathbf{e}_0)}\mathbf{e}_0= \\ &= (t-\lambda)\left(\mathcal{T}(S\mathbf{e}_0)\mathbf{e}_0+\mathbf{X}S\mathbf{e}_0\right)+\frac{\lambda}{\mathcal{T}(S^{-1}\mathbf{e}_0)}\mathbf{e}_0= \\ &= (t-\lambda)\mathcal{T}(S\mathbf{e}_0)\left(\mathbf{e}_0+\frac{\mathbf{X}S\mathbf{e}_0}{\mathcal{T}(S\mathbf{e}_0)}\right)+\frac{\lambda}{\mathcal{T}(S^{-1}\mathbf{e}_0)}\mathbf{e}_0= \\ &= (t-\lambda)\mathcal{T}(S\mathbf{e}_0)\left(\mathbf{e}_0+\mathcal{V}\left(S^{-1}\right)\right)+\frac{\lambda}{\mathcal{T}(S^{-1}\mathbf{e}_0)}\mathbf{e}_0= \\ &= \mathcal{T}(S\mathbf{e}_0)\left(\left(t-\lambda\left(1-\frac{1}{\mathcal{T}(S\mathbf{e}_0)\,\mathcal{T}(S^{-1}\mathbf{e}_0)}\right)\right)\mathbf{e}_0+(t-\lambda)\mathcal{V}\left(S^{-1}\right)\right)= \\ &= \alpha_S\left((t-\lambda\beta_S)\mathbf{e}_0+(t-\lambda)\mathcal{V}\left(S^{-1}\right)\right). \end{aligned}$$

Hence, the operator $S$ maps the subspace $\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S)]$ into the subspace $\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S^{-1})]$. In the case $\mathcal{V}(S) \neq \mathbf{0}$ the subspace $\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S)]$ is two-dimensional ($\dim(\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S)]) = 2$). And since $S$ is bijection on $\mathcal{M}(\mathfrak{H})$, dimension of the image $S(\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S)]) \subseteq \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S^{-1})]$ also must be equal 2. And since $\dim(\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S^{-1})]) \leq 2$, we have, that $S(\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S)]) = \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S^{-1})]$ and $\dim(\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S^{-1})]) = 2$. Thus, in the case $\mathcal{V}(S) \neq \mathbf{0}$, the lemma is proved.

Above we have proved, that if $\mathcal{V}(S) \neq \mathbf{0}$, then $\dim(\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S^{-1})]) = 2$, and, consequently, $\mathcal{V}(S^{-1}) \neq 0$. And, conversely, if $\mathcal{V}(S^{-1}) \neq 0$, then $\mathcal{V}(S) = \mathcal{V}\left((S^{-1})^{-1}\right) \neq 0$. Thus, in the case $\mathcal{V}(S) = \mathbf{0}$, we have $\mathcal{V}(S^{-1}) = \mathbf{0}$. Therefore, in this case $\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S)] = \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S^{-1})] = \mathfrak{H}_0$, and, consequently, one-dimensional subspace $\mathfrak{H}_0$ is invariant subspace of the operator $S$. And, since $S$ is one-to-one mapping, we deliver that $S(\mathfrak{H}_0) = \mathfrak{H}_0$, and, hence, $S(\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S)]) = \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S^{-1})]$. Thus, in the case $\mathcal{V}(S) = \mathbf{0}$, the lemma also is proved. □

### 17.2 General Lorentz Group in Hilbert Space

Everywhere in this paper $c$ will be a fixed positive real constant, which has the physical content of the speed of light in vacuum. Denote by $\mathsf{M}_c(\cdot)$ Lorentz-Minkowski pseudo-metric on the space $\mathcal{M}(\mathfrak{H})$:

$$\mathsf{M}_c(\mathrm{w}) = \|\mathbf{X}\mathrm{w}\|^2 - c^2\mathcal{T}^2(\mathrm{w}), \qquad \mathrm{w} \in \mathcal{M}(\mathfrak{H}). \tag{II.19}$$

Pseudo-metric (II.19) is generated by the quasi-inner product:

$$\langle\langle \mathrm{w}_1, \mathrm{w}_2 \rangle\rangle_c = \langle \mathbf{X}\mathrm{w}_1, \mathbf{X}\mathrm{w}_2 \rangle - c^2\mathcal{T}(\mathrm{w}_1)\mathcal{T}(\mathrm{w}_2), \quad \mathrm{w}_1, \mathrm{w}_2 \in \mathcal{M}(\mathfrak{H}) \tag{II.20}$$

$$\mathsf{M}_c(\mathrm{w}) = \langle\langle \mathrm{w}, \mathrm{w} \rangle\rangle_c, \qquad \mathrm{w} \in \mathcal{M}(\mathfrak{H}). \tag{II.21}$$

It is clear, that quasi-inner product $\langle\langle \mathrm{w}_1, \mathrm{w}_2 \rangle\rangle_c$ ($\mathrm{w}_1, \mathrm{w}_2 \in \mathcal{M}(\mathfrak{H})$) is bilinear form relatively the variables $\mathrm{w}_1, \mathrm{w}_2$. Hence (by (II.21)), for any $\mathrm{w}_1, \mathrm{w}_2 \in \mathcal{M}(\mathfrak{H})$ it holds the equality:

$$\langle\langle \mathrm{w}_1, \mathrm{w}_2 \rangle\rangle_c = \frac{1}{2}\left(\mathsf{M}_c(\mathrm{w}_1 + \mathrm{w}_2) - \mathsf{M}_c(\mathrm{w}_1) - \mathsf{M}_c(\mathrm{w}_2)\right). \tag{II.22}$$

**Denotation II.17.1.** *Denote by $\mathfrak{O}(\mathfrak{H}, c)$ the set of all linear coordinate transform operators over $\mathcal{M}(\mathfrak{H})$, leaving unchanged values of the functional (II.19), that is the set of all linear coordinate transform operators $L \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ such, that:*

$$\mathsf{M}_c(L\mathrm{w}) = \mathsf{M}_c(\mathrm{w}) \qquad (\forall \mathrm{w} \in \mathcal{M}(\mathfrak{H})). \tag{II.23}$$

Using the equality (II.22) it is easy to verify, that any operator $L \in \mathfrak{O}(\mathfrak{H}, c)$ leaves unchanged the values of the quasi-inner product (II.20):

$$\langle\langle L\mathrm{w}_1, L\mathrm{w}_2 \rangle\rangle_c = \langle\langle \mathrm{w}_1, \mathrm{w}_2 \rangle\rangle_c, \qquad \mathrm{w}_1, \mathrm{w}_2 \in \mathcal{M}(\mathfrak{H}) \tag{II.24}$$

It is not hard to see, that product of any two operators from $\mathfrak{O}(\mathfrak{H}, c)$ belongs to $\mathfrak{O}(\mathfrak{H}, c)$ and the mapping, inverse to any operator from $\mathfrak{O}(\mathfrak{H}, c)$ also belongs to the set $\mathfrak{O}(\mathfrak{H}, c)$. Hence:

**Assertion II.17.1.** *The set $\mathfrak{O}(\mathfrak{H}, c)$ is the group of operators over the space $\mathcal{M}(\mathfrak{H})$.*

According to [45] we name this group by the ***general Lorentz group*** over the space $\mathcal{M}(\mathfrak{H})$. Note, that generalization of the classical Lorentz group for the case of real Hilbert space was also investigated in [55–57]. In these papers a somewhat different (but logically equivalent) approach to the definition of Lorentz group over Hilbert space is proposed. Namely, in these papers the "time" dimension is not "attached" to the given Hilbert space $\mathfrak{H}$ (by means of construction the space $\mathcal{M}(\mathfrak{H})$), but this dimension is selected in the space $\mathfrak{H}$ by an arbitrary way. So, the last construction is correct only in the case $\dim(\mathfrak{H}) \geq 2$. In our approach, we, apparently, need not this restriction.

**Assertion II.17.2.** *Any general Lorentz transform operator $L \in \mathfrak{O}(\mathfrak{H}, c)$ is v-determined and $\|\mathcal{V}(L)\| < c$.*

*Proof.* Indeed,

$$\mathsf{M}_c(\mathbf{e}_0) = \|\mathbf{X}\mathbf{e}_0\|^2 - c^2\mathcal{T}^2(\mathbf{e}_0) = 0 - c^2 \cdot 1 = -c^2.$$

As it was mentioned above, $L^{-1} \in \mathfrak{O}(\mathfrak{H}, c)$ for $L \in \mathfrak{O}(\mathfrak{H}, c)$. Therefore, by (II.23),

$$\mathsf{M}_c\left(L^{-1}\mathbf{e}_0\right) = \left\|\mathbf{X}L^{-1}\mathbf{e}_0\right\|^2 - c^2\mathcal{T}^2\left(L^{-1}\mathbf{e}_0\right) = -c^2.$$

Hence, $\left|\mathcal{T}\left(L^{-1}\mathbf{e}_0\right)\right| = \frac{1}{c}\sqrt{\left\|\mathbf{X}L^{-1}\mathbf{e}_0\right\|^2 + c^2} > 0$. Thus the linear coordinate transform operator $L$ is v-determined, moreover:

$$\|\mathcal{V}(L)\| = \frac{\left\|\mathbf{X}L^{-1}\mathbf{e}_0\right\|}{\left|\mathcal{T}\left(L^{-1}\mathbf{e}_0\right)\right|} = c\frac{\left\|\mathbf{X}L^{-1}\mathbf{e}_0\right\|}{\sqrt{\left\|\mathbf{X}L^{-1}\mathbf{e}_0\right\|^2 + c^2}} < c. \qquad \text{(II.25)}$$

□

The aim of the next assertion is to emphasize some characteristic properties of the general Lorentz transforms, which may serve as a basis for another definition of the general Lorentz group. And these properties also will become the motivation for definition of the set of extended Lorentz transforms, which allow superlight speed of reference frames.

**Assertion II.17.3.** *Any linear coordinate transform operator $L \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ belongs to $\mathfrak{O}(\mathfrak{H}, c)$ if and only if the following conditions are satisfied:*

1. *Both linear coordinate transform operators $L$ and $L^{-1}$ are v-determined;*
2. $\mathsf{M}_c(L\mathrm{w}) = \mathsf{M}_c(\mathrm{w})$ $(\forall \mathrm{w} \in \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(L)])$;
3. *if* $\widehat{\mathbf{T}}\mathrm{w} = \mathbf{X}_1[\mathcal{V}(L)]\mathrm{w} = \mathbf{0}$, *then* $\widehat{\mathbf{T}}L\mathrm{w} = \mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathrm{w} = \mathbf{0}$ $(\forall \mathrm{w} \in \mathcal{M}(\mathfrak{H}))$;
4. $\left\|\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\mathrm{w}\right\| = \left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathrm{w}\right\|$, $(\forall \mathrm{w} \in \mathcal{M}(\mathfrak{H}))$.

*Proof.* **1.** Let $L \in \mathfrak{O}(\mathfrak{H}, c)$.

**1.1.** By Assertion II.17.2, $L$ is v-determined. Since $\mathfrak{O}(\mathfrak{H}, c)$ is the group of operators, $L^{-1} \in \mathfrak{O}(\mathfrak{H}, c)$, and so $L^{-1}$ also is v-determined.

**1.2.** Performance of the second condition follows from the equality (II.23).

**1.3.** Suppose, that $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ and $\widehat{\mathbf{T}}\mathrm{w} = \mathbf{X}_1[\mathcal{V}(L)]\mathrm{w} = \mathbf{0}$. Then, for any vector $w_{t,\lambda} = t\mathbf{e}_0 + \lambda\mathcal{V}(L) \in \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(L)]$ we obtain:

$$\langle\langle w_{t,\lambda}, \mathrm{w}\rangle\rangle_c = \langle \mathbf{X}w_{t,\lambda}, \mathbf{X}\mathrm{w}\rangle - c^2\mathcal{T}(w_{t,\lambda})\mathcal{T}(\mathrm{w}) = \lambda\langle\mathcal{V}(L), \mathbf{X}\mathrm{w}\rangle - c^2t\left\langle\widehat{\mathbf{T}}\mathrm{w}, \mathbf{e}_0\right\rangle =$$
$$= \lambda\langle\mathcal{V}(L), \mathbf{X}\mathrm{w}\rangle = \lambda\langle\mathbf{X}\mathcal{V}(L), \mathrm{w}\rangle = \lambda\langle\mathcal{V}(L), \mathrm{w}\rangle = 0$$

Consequently, by the equality (II.24):

$$\langle\langle Lw_{t,\lambda}, L\mathrm{w}\rangle\rangle_c = 0 \qquad (\forall t, \lambda \in \mathbb{R}).$$

Hence, using the lemma II.17.1, we deliver:

$$\left\langle\left\langle \alpha_L\left((t-\lambda\beta_L)\mathbf{e}_0 + (t-\lambda)\mathcal{V}\left(L^{-1}\right)\right), L\mathrm{w}\right\rangle\right\rangle_c = 0 \qquad (\forall t, \lambda \in \mathbb{R}),$$

where (because $L, L^{-1}$ are v-determined), $\alpha_L = \mathcal{T}(L\mathbf{e}_0) \neq 0$, $\alpha_{L^{-1}} \neq 0$, $\beta_L = 1 - \frac{1}{\alpha_L\alpha_{L^{-1}}} \neq 1$. Since $\beta_L \neq 1$, the set of pairs $\{(t-\lambda\beta_L, t-\lambda) \mid t, \lambda \in \mathbb{R}\}$ coincides with $\mathbb{R}^2$. Thus, since $\alpha_L \neq 0$, we obtain:

$$\left\langle\left\langle t\mathbf{e}_0 + \lambda\mathcal{V}\left(L^{-1}\right), L\mathrm{w}\right\rangle\right\rangle_c = 0 \qquad (\forall t, \lambda \in \mathbb{R}).$$

In particular for $t_1 = -\frac{1}{c^2}$, $\lambda_1 = 0$ and $t_2 = 0$, $\lambda_2 = 1$ we have:

$$0 = \left\langle\left\langle -\frac{1}{c^2}\mathbf{e}_0, L\mathrm{w} \right\rangle\right\rangle_c = -c^2\mathcal{T}\left(-\frac{1}{c^2}\mathbf{e}_0\right)\mathcal{T}(L\mathrm{w}) = \mathcal{T}(L\mathrm{w});$$

$$\widehat{\mathbf{T}}L\mathrm{w} = \mathcal{T}(L\mathrm{w})\,\mathbf{e}_0 = \mathbf{0}; \qquad \text{(II.26)}$$

$$0 = \left\langle\left\langle \mathcal{V}\left(L^{-1}\right), L\mathrm{w} \right\rangle\right\rangle_c = \left\langle \mathcal{V}\left(L^{-1}\right), L\mathrm{w} \right\rangle;$$

$$\mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathrm{w} = \begin{Bmatrix} \frac{\langle \mathcal{V}(L^{-1}), L\mathrm{w}\rangle}{\|\mathcal{V}(L^{-1})\|^2}\mathcal{V}\left(L^{-1}\right), & \mathcal{V}\left(L^{-1}\right) \neq \mathbf{0} \\ \mathbf{0}, & \mathcal{V}\left(L^{-1}\right) = \mathbf{0} \end{Bmatrix} = \mathbf{0}. \qquad \text{(II.27)}$$

Thus, by (II.26), (II.27), $\widehat{\mathbf{T}}L\mathrm{w} = \mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathrm{w} = \mathbf{0}$.

**1.4.** Let $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$. Then, by (II.24) and (II.18):

$$\begin{aligned} \left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w}\right\|^2 &= \left\langle\left\langle \mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w},\ \mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w} \right\rangle\right\rangle_c = \\ &= \left\langle\left\langle L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w},\ L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w} \right\rangle\right\rangle_c = \\ &= \left\langle\left\langle \left(\widehat{\mathbf{T}} + \mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right] + \mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]\right) L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w},\ L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w} \right\rangle\right\rangle_c = \\ &= \left\langle\left\langle \widehat{\mathbf{T}}L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w},\ L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w} \right\rangle\right\rangle_c + \\ &+ \left\langle\left\langle \mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right] L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w},\ L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w} \right\rangle\right\rangle_c + \\ &+ \left\langle\left\langle \mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right] L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w},\ L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w} \right\rangle\right\rangle_c. \end{aligned} \qquad \text{(II.28)}$$

By (II.18), $\widehat{\mathbf{T}}\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w} = \mathbf{X}_1\left[\mathcal{V}(L)\right]\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w} = \mathbf{0}$. So using the previous item, we conclude, that $\widehat{\mathbf{T}}L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w} = \mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w} = \mathbf{0}$. Hence, from (II.28) it follows, that:

$$\begin{aligned} \left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w}\right\|^2 &= \left\langle\left\langle \mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right] L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w},\ L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w} \right\rangle\right\rangle_c = \\ &= \left\langle \mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right] L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w},\ L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w} \right\rangle = \\ &= \left\langle \mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right] L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w},\ \mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right] L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w} \right\rangle = \\ &= \left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right] L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w}\right\|^2. \end{aligned} \qquad \text{(II.29)}$$

Note, that by (II.18), $L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w} = L\left(\mathrm{w} - \widehat{\mathbf{T}}\mathrm{w} - \mathbf{X}_1\left[\mathcal{V}(L)\right]\mathrm{w}\right) = L\mathrm{w} - Lw$, where $w = \widehat{\mathbf{T}}\mathrm{w} + \mathbf{X}_1\left[\mathcal{V}(L)\right]\mathrm{w} \in \mathfrak{H}_0 \oplus \mathfrak{H}_1\left[\mathcal{V}(L)\right]$. By lemma II.17.1, $Lw \in \mathfrak{H}_0 \oplus \mathfrak{H}_1\left[\mathcal{V}\left(L^{-1}\right)\right]$. Therefore, by (II.18), $\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]Lw = \mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]\left(\widehat{\mathbf{T}} + \mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]\right)Lw = \mathbf{0}$, and $\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w} = \mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right](L\mathrm{w} - Lw) = \mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathrm{w}$. Hence, by (II.29):

$$\left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w}\right\|^2 = \left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathrm{w}\right\|^2, \quad \mathrm{w} \in \mathcal{M}(\mathfrak{H}).$$

Thus, all conditions 1-4 for any linear coordinate transform operator $L \in \mathfrak{O}(\mathfrak{H}, c)$ are satisfied.

**2.** Suppose, that linear coordinate transform operator $L \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ satisfies the conditions 1-4. Chose any $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$. Vector w can be represented in the form

$$\begin{gathered} \mathrm{w} = \mathrm{w}_1 + \mathrm{w}_2, \quad \text{where} \\ \mathrm{w}_1 = \mathcal{T}(\mathrm{w})\,\mathbf{e}_0 + \mathbf{X}_1\left[\mathcal{V}(L)\right]\mathrm{w} \in \mathfrak{H}_0 \oplus \mathfrak{H}_1\left[\mathcal{V}(L)\right], \quad \mathrm{w}_2 = \mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\mathrm{w} \in \mathfrak{H}_{1\perp}\left[\mathcal{V}(L)\right]. \end{gathered} \qquad \text{(II.30)}$$

Note, that by (II.30) and (II.18), $\widehat{\mathbf{T}}\mathrm{w}_2 = \mathbf{X}_1\left[\mathcal{V}(L)\right]\mathrm{w}_2 = \mathbf{0}$. Therefore, by the condition 3:

$$\widehat{\mathbf{T}}L\mathrm{w}_2 = \mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathrm{w}_2 = \mathbf{0}. \qquad \text{(II.31)}$$

So:

$$\begin{aligned}\mathsf{M}_c(L\mathrm{w}) = \mathsf{M}_c(L\mathrm{w}_1 + L\mathrm{w}_2) = \|\mathbf{X}L\mathrm{w}_1 + \mathbf{X}L\mathrm{w}_2\|^2 - c^2\left(\mathcal{T}(L\mathrm{w}_1) + \mathcal{T}(L\mathrm{w}_2)\right)^2 = \\ = \left\|\mathbf{X}L\mathrm{w}_1 + \mathbf{X}L\mathrm{w}_2 + \widehat{\mathbf{T}}L\mathrm{w}_2\right\|^2 - c^2\left(\mathcal{T}(L\mathrm{w}_1) + 0\right)^2 = \\ = \|\mathbf{X}L\mathrm{w}_1 + L\mathrm{w}_2\|^2 - c^2\mathcal{T}^2(L\mathrm{w}_1).\end{aligned} \tag{II.32}$$

Since $\mathrm{w}_1 \in \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(L)]$, by lemma II.17.1, $L\mathrm{w}_1 \in \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(L^{-1})]$. Hence, by (II.31),

$$\begin{aligned}\langle \mathbf{X}L\mathrm{w}_1, L\mathrm{w}_2\rangle = \langle L\mathrm{w}_1, \mathbf{X}L\mathrm{w}_2\rangle = \left\langle L\mathrm{w}_1, \left(\widehat{\mathbf{T}} + \mathbf{X}\right)L\mathrm{w}_2\right\rangle = \langle L\mathrm{w}_1, L\mathrm{w}_2\rangle = \\ = \left\langle\left(\widehat{\mathbf{T}} + \mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]\right)L\mathrm{w}_1, L\mathrm{w}_2\right\rangle = \left\langle L\mathrm{w}_1, \left(\widehat{\mathbf{T}} + \mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]\right)L\mathrm{w}_2\right\rangle = 0.\end{aligned}$$

Thus, $\|\mathbf{X}L\mathrm{w}_1 + L\mathrm{w}_2\|^2 = \|\mathbf{X}L\mathrm{w}_1\|^2 + \|L\mathrm{w}_2\|^2$. And using the equalities (II.32),(II.31), conditions 2,4, taking into account, that $\mathrm{w}_1 \in \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(L)]$ we obtain:

$$\begin{aligned}\mathsf{M}_c(L\mathrm{w}) = \mathsf{M}_c(L\mathrm{w}_1) + \|L\mathrm{w}_2\|^2 = \mathsf{M}_c(L\mathrm{w}_1) + \left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathrm{w}_2\right\|^2 = \\ = \mathsf{M}_c(\mathrm{w}_1) + \left\|\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\mathrm{w}_2\right\|^2 = \mathsf{M}_c\left(\mathcal{T}(\mathrm{w})\mathbf{e}_0 + \mathbf{X}_1[\mathcal{V}(L)]\mathrm{w}\right) + \left\|\left(\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\right)^2\mathrm{w}\right\|^2 = \\ = \|\mathbf{X}_1[\mathcal{V}(L)]\mathrm{w}\|^2 - c^2\mathcal{T}^2(\mathrm{w}) + \left\|\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\mathrm{w}\right\|^2 = \mathsf{M}_c(\mathrm{w}) \quad (\forall \mathrm{w} \in \mathcal{M}(\mathfrak{H})).\end{aligned}$$

Consequently, $L \in \mathfrak{O}(\mathfrak{H}, c)$. □

### 17.3 Generalized Lorentz Transforms for Finite Speeds

**Denotation II.17.2.** *Denote by $\mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c)$ the set of all linear coordinate transform operators $L \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$, satisfying conditions:*

*1′.* *Both linear coordinate transform operators $L$ and $L^{-1}$ are v-determined;*

*2′.* $(\mathsf{M}_c(L\mathrm{w}))^2 = (\mathsf{M}_c(\mathrm{w}))^2$ $(\forall \mathrm{w} \in \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(L)])$;

*3′.* *if* $\widehat{\mathbf{T}}\mathrm{w} = \mathbf{X}_1[\mathcal{V}(L)]\mathrm{w} = \mathbf{0}$, *then* $\widehat{\mathbf{T}}L\mathrm{w} = \mathbf{X}_1[\mathcal{V}(L^{-1})]L\mathrm{w} = \mathbf{0}$ $(\forall \mathrm{w} \in \mathcal{M}(\mathfrak{H}))$;

*4′.* $\left\|\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\mathrm{w}\right\| = \left\|\mathbf{X}_1^{\perp}[\mathcal{V}(L^{-1})]L\mathrm{w}\right\|$, $(\forall \mathrm{w} \in \mathcal{M}(\mathfrak{H}))$.

In comparison with conditions 1-4 of Assertion II.17.3, only Condition 2 is modified. It is evidently, that the Condition 2 of Assertion II.17.3 implies Condition 2′. Thus:

$$\mathfrak{O}(\mathfrak{H}, c) \subseteq \mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c). \tag{II.33}$$

And, as it will be proved below, in Theorem II.17.1, this small modification of the second condition leads to permission of superlight speed for reference frames (that is to the possibility of $\|\mathcal{V}(L)\| > c$ for $L \in \mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c)$). This, means, that the inclusion, inverse to (II.33) can not be true.

From condition 3′ it follows, that for any operator $L \in \mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c)$

$$L\left(\mathfrak{H}_{1\perp}[\mathcal{V}(L)]\right) \subseteq \mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]. \tag{II.34}$$

Indeed, for any $\mathrm{w} \in \mathfrak{H}_{1\perp}[\mathcal{V}(L)]$ we have, $\widehat{\mathbf{T}}\mathrm{w} = \mathbf{X}_1[\mathcal{V}(L)]\mathrm{w} = \mathbf{0}$. Thus, by condition 3′, $\widehat{\mathbf{T}}L\mathrm{w} = \mathbf{X}_1[\mathcal{V}(L^{-1})]L\mathrm{w} = \mathbf{0}$, and, by equalities (II.18), $L\mathrm{w} = \left(\widehat{\mathbf{T}} + \mathbf{X}_1[\mathcal{V}(L^{-1})] + \mathbf{X}_1^{\perp}[\mathcal{V}(L^{-1})]\right)L\mathrm{w} = \mathbf{X}_1^{\perp}[\mathcal{V}(L^{-1})]L\mathrm{w} \in \mathfrak{H}_{1\perp}[\mathcal{V}(L^{-1})]$.

Denote by $\mathfrak{U}(\mathfrak{H}_1)$ the set of all unitary operators over the space $\mathfrak{H}_1$. That is the set of all linear operators $J : \mathfrak{H}_1 \mapsto \mathfrak{H}_1$ $(J \in \mathcal{L}(\mathfrak{H}_1))$, such, that:

$$\|Jx\| = \|x\| \ (\forall x \in \mathfrak{H}_1) \quad \text{and} \quad J\mathfrak{H}_1 = \mathfrak{H}_1.$$

For any operator $J \in \mathfrak{U}(\mathfrak{H}_1)$ we introduce the operator $\widetilde{J} \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$:

$$\widetilde{J}\mathrm{w} := \widehat{\mathbf{T}}\mathrm{w} + J\mathbf{X}\mathrm{w} = \mathcal{T}(\mathrm{w})\,\mathbf{e}_0 + J\mathbf{X}\mathrm{w}, \quad \mathrm{w} \in \mathcal{M}(\mathfrak{H}). \tag{II.35}$$

From (II.35) it follows, that:

$$\forall J \in \mathfrak{U}(\mathfrak{H}_1)\ \ \widetilde{J} \in \mathfrak{U}(\mathcal{M}(\mathfrak{H})), \tag{II.36}$$

where $\mathfrak{U}(\mathcal{M}(\mathfrak{H}))$ is the set of all unitary operators over the space $\mathcal{M}(\mathfrak{H})$.

**Theorem II.17.1.** *Operator $L \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ belongs to the class $\mathfrak{O}\mathfrak{T}_{\mathbf{fin}}(\mathfrak{H}, c)$ if and only if there exist number $s \in \{-1, 1\}$, vector $V \in \mathfrak{H}_1$, $\|V\| \neq c$ and operator $J \in \mathfrak{U}(\mathfrak{H}_1)$ such, that for any $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ vector $L\mathrm{w}$ can be represented by the formula:*

$$L\mathrm{w} = \frac{s\left(\mathcal{T}(\mathrm{w}) - \frac{\langle V, \mathrm{w}\rangle}{c^2}\right)}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}}\mathbf{e}_0 + J\left(\frac{s\left(\mathcal{T}(\mathrm{w})V - \mathbf{X}_1[V]\,\mathrm{w}\right)}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}} + \mathbf{X}_1^{\perp}[V]\,\mathrm{w}\right), \tag{II.37}$$

*moreover,*

$$\mathcal{V}(L) = V.$$

Note, that in the case[14] $\mathfrak{H} = \mathbb{R}^3$, $\mathcal{M}(\mathfrak{H}) = \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4$, $V = (0, v, 0, 0)$ (where $v \in \mathbb{R}$, $|v| > c$), and

$$J(0, x, y, z) = (0, -x, y, z), \quad x, y, z \in \mathbb{R}$$

we obtain transforms [36, formula (43)], [37, formula (12)] and [38, formulas (3.17)–(3.18)] as particular cases of the formula (II.37) from Therem II.17.1. Under the additional conditions $\|V\| < c$, $\dim(\mathfrak{H}) = 3$, $s = 1$ the formula (II.37) is equivalent to the formula (28b) from [54, page 43]. That is why, in this case we obtain the classical Lorentz transforms for inertial reference frame in the most general form (with arbitrary orientation of axes).

To prove Theorem II.17.1 we need the following lemma.

**Lemma II.17.2.** *If for operator $L \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ there exist number $s \in \{-1, 1\}$, vector $V \in \mathfrak{H}_1$, $\|V\| \neq c$ and operator $J \in \mathfrak{U}(\mathfrak{H}_1)$ such, that for any $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ vector $L\mathrm{w}$ can be represented by the formula (II.37), then $L$ is a linear coordinate transform operator, moreover*

$$L^{-1} = \mathbf{L}_0\left[\mathrm{sign}\,(c - \|V\|)\, s, V\right]\widetilde{J^{-1}}, \quad \textit{where}$$

$$\mathbf{L}_0[s, V]\,\mathrm{w} = \frac{s\left(\mathcal{T}(\mathrm{w}) - \frac{\langle V, \mathrm{w}\rangle}{c^2}\right)}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}}\mathbf{e}_0 + \frac{s\left(\mathcal{T}(\mathrm{w})V - \mathbf{X}_1[V]\,\mathrm{w}\right)}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}} + \mathbf{X}_1^{\perp}[V]\,\mathrm{w} \tag{II.38}$$

*and operator $\widetilde{J^{-1}}$ is determined by formula (II.35).*

*Proof.* Let the operator $L \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ satisfy the conditions of the lemma. We need to prove, that the operator $L$ have the inverse $L^{-1}$. By (II.37), operator $L$ can be represented in the form:

$$L = \widetilde{J}\mathbf{L}_0[s, V].$$

Since $\widetilde{J}$ is unitary operator over $\mathcal{M}(\mathfrak{H})$, it is sufficient to prove that the inverse operator exist for the operator $\mathbf{L}_0[s, V]$. It is obvious that

$$\widetilde{J}^{-1} = \widetilde{J^{-1}}. \tag{II.39}$$

[14] We consider $\mathbb{R}^d$ ($d \in \mathbb{N}$) as a Hilbert space with the inner product $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \sum_{j=1}^{d} x_j y_j$ ($\boldsymbol{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $\boldsymbol{y} = (y_1, \cdots, y_d) \in \mathbb{R}^d$)

Hence, the lemma will be fully proved, if we will be be able to verify the equality:

$$\mathbf{L}_0\left[s,V\right]\mathbf{L}_0\left[\text{sign }\left(c-\|V\|\right)s,V\right]=\mathbb{I} \tag{II.40}$$

(then the equallity $\mathbf{L}_0\left[\text{sign }\left(c-\|V\|\right)s,V\right]\mathbf{L}_0\left[s,V\right]=\mathbb{I}$ will be follow by applying the equality (II.40) to the operator $\mathbf{L}_0\left[s',V\right]$, where $s'=\text{sign }\left(c-\|V\|\right)s$).

In the case $V=\mathbf{0}$, using (II.38) and (II.17), we obtain:

$$\mathbf{L}_0\left[s,V\right]\mathrm{w}=s\mathcal{T}\left(\mathrm{w}\right)\mathbf{e}_0+\mathbf{X}_1^{\perp}\left[V\right]\mathrm{w}=s\mathcal{T}\left(\mathrm{w}\right)\mathbf{e}_0+\left(\mathbf{X}-\mathbf{X}_1\left[V\right]\right)\mathrm{w}=s\mathcal{T}\left(\mathrm{w}\right)\mathbf{e}_0+\mathbf{X}\mathrm{w}.$$

Thus, in this case equality (II.40) is clear.

So, one can be restricted by the case $V\neq\mathbf{0}$. Applying equalities (II.38) and (II.17) we deliver:

$$\mathbf{L}_0\left[s,V\right]\mathrm{w}=\frac{s\left(\mathcal{T}\left(\mathrm{w}\right)-\frac{\left\langle V,\mathrm{w}\right\rangle}{c^2}\right)}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}}\mathbf{e}_0+\frac{s\left(\mathcal{T}\left(\mathrm{w}\right)-\frac{\left\langle V,\mathrm{w}\right\rangle}{\|V\|^2}\right)}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}}V+\mathbf{X}_1^{\perp}\left[V\right]\mathrm{w},\quad \mathrm{w}\in\mathcal{M}\left(\mathfrak{H}\right) \tag{II.41}$$

Denote $s':=\text{sign }\left(c-\|V\|\right)s$. Then for an arbitrary $\mathrm{w}\in\mathcal{M}\left(\mathfrak{H}\right)$ we have.

$$\mathbf{L}_0\left[s,V\right]\mathbf{L}_0\left[\text{sign }\left(c-\|V\|\right)s,V\right]\mathrm{w}=\mathbf{L}_0\left[s,V\right]\widetilde{\mathrm{w}},\quad \text{where } \widetilde{\mathrm{w}}=\mathbf{L}_0\left[s',V\right]\mathrm{w}. \tag{II.42}$$

By (II.41), we obtain:

$$\left.\begin{aligned}
&\mathcal{T}\left(\widetilde{\mathrm{w}}\right)=\mathcal{T}\left(\mathbf{L}_0\left[s',V\right]\mathrm{w}\right)=\frac{s'\left(\mathcal{T}\left(\mathrm{w}\right)-\frac{\left\langle V,\mathrm{w}\right\rangle}{c^2}\right)}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}};\\
&\left\langle V,\widetilde{\mathrm{w}}\right\rangle=\frac{s'\left(\mathcal{T}\left(\mathrm{w}\right)\|V\|^2-\left\langle V,\mathrm{w}\right\rangle\right)}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}};\\
&\mathbf{X}_1^{\perp}\left[V\right]\widetilde{\mathrm{w}}=\mathbf{X}_1^{\perp}\left[V\right]\mathrm{w}.
\end{aligned}\right\} \tag{II.43}$$

Applying equality (II.41) for vector $\widetilde{\mathrm{w}}$ and using (II.43), we deduce:

$$\begin{aligned}
\mathbf{L}_0\left[s,V\right]\widetilde{\mathrm{w}}&=\frac{s\left(\mathcal{T}\left(\widetilde{\mathrm{w}}\right)-\frac{\left\langle V,\widetilde{\mathrm{w}}\right\rangle}{c^2}\right)}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}}\mathbf{e}_0+\frac{s\left(\mathcal{T}\left(\widetilde{\mathrm{w}}\right)-\frac{\left\langle V,\widetilde{\mathrm{w}}\right\rangle}{\|V\|^2}\right)}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}}V+\mathbf{X}_1^{\perp}\left[V\right]\widetilde{\mathrm{w}}=\\
&=\frac{s\left(s'\left(\mathcal{T}\left(\mathrm{w}\right)-\frac{\left\langle V,\mathrm{w}\right\rangle}{c^2}\right)-s'\left(\mathcal{T}\left(\mathrm{w}\right)\frac{\|V\|^2}{c^2}-\frac{\left\langle V,\mathrm{w}\right\rangle}{c^2}\right)\right)}{\left|1-\frac{\|V\|^2}{c^2}\right|}\mathbf{e}_0+\\
&\qquad+\frac{s\left(s'\left(\mathcal{T}\left(\mathrm{w}\right)-\frac{\left\langle V,\mathrm{w}\right\rangle}{c^2}\right)-s'\left(\mathcal{T}\left(\mathrm{w}\right)-\frac{\left\langle V,\mathrm{w}\right\rangle}{\|V\|^2}\right)\right)}{\left|1-\frac{\|V\|^2}{c^2}\right|}V+\mathbf{X}_1^{\perp}\left[V\right]\mathrm{w}=\\
&=ss'\frac{\mathcal{T}\left(\mathrm{w}\right)\left(1-\frac{\|V\|^2}{c^2}\right)}{\left|1-\frac{\|V\|^2}{c^2}\right|}\mathbf{e}_0+ss'\frac{\frac{\left\langle V,\mathrm{w}\right\rangle}{\|V\|^2}\left(1-\frac{\|V\|^2}{c^2}\right)}{\left|1-\frac{\|V\|^2}{c^2}\right|}V+\mathbf{X}_1^{\perp}\left[V\right]\mathrm{w}=\\
&=\left(\mathcal{T}\left(\mathrm{w}\right)\mathbf{e}_0+\frac{\left\langle V,\mathrm{w}\right\rangle}{\|V\|^2}V\right)+\mathbf{X}_1^{\perp}\left[V\right]\mathrm{w}=\mathrm{w}.
\end{aligned}$$

Thus, using (II.42), we obtain (II.40). □

*Proof of Theorem II.17.1.* **I.** Suppose, that $L \in \mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c)$. Then, $L$ is linear coordinate transform operator, which satisfies the conditions $1'$–$4'$. Denote:

$$V := \mathcal{V}(L). \tag{II.44}$$

First we prove the formula (II.37) in the case $V \neq 0$. By equalities (II.18),(II.15) and (II.17), for any $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ we have:

$$L\mathrm{w} = L\left(\widehat{\mathbf{T}} + \mathbf{X}_1[V] + \mathbf{X}_1^{\perp}[V]\right)\mathrm{w} = L\left(\mathcal{T}(\mathrm{w})\,\mathbf{e}_0 + \mathbf{X}_1[V]\,\mathrm{w}\right) + L\mathbf{X}_1^{\perp}[V]\,\mathrm{w} =$$
$$= L\left(\mathcal{T}(\mathrm{w})\,\mathbf{e}_0 + \frac{\langle V, \mathrm{w}\rangle}{\|V\|^2}V\right) + L\mathbf{X}_1^{\perp}[V]\,\mathrm{w}.$$

Hence, by lemma II.17.1

$$L\mathrm{w} = \alpha_L\left(\left(\mathcal{T}(\mathrm{w}) - \frac{\langle V, \mathrm{w}\rangle}{\|V\|^2}\beta_L\right)\mathbf{e}_0 + \left(\mathcal{T}(\mathrm{w}) - \frac{\langle V, \mathrm{w}\rangle}{\|V\|^2}\right)\mathcal{V}\left(L^{-1}\right)\right) +$$
$$+ L\mathbf{X}_1^{\perp}[V]\,\mathrm{w} \quad (\mathrm{w} \in \mathcal{M}(\mathfrak{H})) \tag{II.45}$$

Now, introduce the linear operator $J_1$ on the subspace $\mathfrak{H}_{1\perp}[V] = \mathfrak{H}_{1\perp}[\mathcal{V}(L)]$. Denote:

$$J_1 x := Lx, \quad x \in \mathfrak{H}_{1\perp}[V]. \tag{II.46}$$

According to the formula (II.34), operator $J_1$ maps the subspace $\mathfrak{H}_{1\perp}[V]$ into the subspace $\mathfrak{H}_{1\perp}[\mathcal{V}(L^{-1})]$. By the formula (II.34) and condition $4'$, for any $x \in \mathfrak{H}_{1\perp}[V]$ we obtain:

$$\|J_1 x\| = \|Lx\| = \left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]Lx\right\| = \left\|\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\,x\right\| = \left\|\mathbf{X}_1^{\perp}[V]\,x\right\| = \|x\|. \tag{II.47}$$

Hence, $J_1$ is isometric operator from the subspace $\mathfrak{H}_{1\perp}[V]$ to $\mathfrak{H}_{1\perp}[\mathcal{V}(L^{-1})]$. Now the aim is to prove, that operator $J_1$ is unitary operator from $\mathfrak{H}_{1\perp}[V]$ to $\mathfrak{H}_{1\perp}[\mathcal{V}(L^{-1})]$, that is

$$J_1\mathfrak{H}_{1\perp}[V] = \mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]. \tag{II.48}$$

Let us consider any vector $y \in \mathfrak{H}_{1\perp}[\mathcal{V}(L^{-1})]$. Since $L$ is linear coordinate transform operator, there exist vector $x = L^{-1}y$. By equalities (II.18) vector $x$ can be represented as:

$$x = \left(\widehat{\mathbf{T}} + \mathbf{X}_1[V]\right)x + \mathbf{X}_1^{\perp}[V]\,x, \tag{II.49}$$

where $\left(\widehat{\mathbf{T}} + \mathbf{X}_1[V]\right)x \in \mathfrak{H}_1 \oplus \mathfrak{H}_1[V] = \mathfrak{H}_1 \oplus \mathfrak{H}_1[\mathcal{V}(L)]$, $\mathbf{X}_1^{\perp}[V]\,x \in \mathfrak{H}_{1\perp}[V]$. Therefore, $Lx = L\left(\widehat{\mathbf{T}} + \mathbf{X}_1[V]\right)x + L\mathbf{X}_1^{\perp}[V]\,x$. Hence:

$$L\left(\widehat{\mathbf{T}} + \mathbf{X}_1[V]\right)x + L\mathbf{X}_1^{\perp}[V]\,x = Lx = LL^{-1}y = y \in \mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]. \tag{II.50}$$

where, by lemma II.17.1 and formula (II.34)

$$L\left(\widehat{\mathbf{T}} + \mathbf{X}_1[V]\right)x \in \mathfrak{H}_0 \oplus \mathfrak{H}_1\left[\mathcal{V}\left(L^{-1}\right)\right];$$
$$L\mathbf{X}_1^{\perp}[V]\,x \in \mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]. \tag{II.51}$$

Since $\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(L^{-1})] \oplus \mathfrak{H}_{1\perp}[\mathcal{V}(L^{-1})] = \mathcal{M}(\mathfrak{H})$, from the equalities (II.50),(II.51) we conclude, that

$$L\mathbf{X}_1^{\perp}[V]\,x = y, \quad \text{and} \quad L\left(\widehat{\mathbf{T}} + \mathbf{X}_1[V]\right)x = \mathbf{0}.$$

Since $L$ is linear coordinate transform operator, from the equality $L\left(\widehat{\mathbf{T}}+\mathbf{X}_1[V]\right)x=\mathbf{0}$ it follows, that $\left(\widehat{\mathbf{T}}+\mathbf{X}_1[V]\right)x=\mathbf{0}$. Hence, by (II.49), $x=\mathbf{X}_1^{\perp}[V]x\in\mathfrak{H}_{1\perp}[V]$, and, by definition of the operator $J_1$, we deliver:

$$J_1x=Lx=y.$$

Thus, we have proved, that for any $y\in\mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]$ there exists the element $x\in\mathfrak{H}_{1\perp}[V]$ such, that $J_1x=y$. This means, that the operator $J_1:\mathfrak{H}_{1\perp}[V]\mapsto\mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]$ truly is unitary. Applying the operator $J_1$ we can write:

$$L\mathbf{X}_1^{\perp}[V]\,\mathrm{w}=J_1\mathbf{X}_1^{\perp}[V]\,\mathrm{w},\quad \mathrm{w}\in\mathcal{M}(\mathfrak{H})\,. \tag{II.52}$$

Next, using the lemma II.17.1, for any $t,\lambda\in\mathbb{R}$ we obtain:

$$L\left(t\mathbf{e}_0+\lambda\mathcal{V}(L)\right)=\alpha_L\left(\left(t-\lambda\beta_L\right)\mathbf{e}_0+(t-\lambda)\,\mathcal{V}\left(L^{-1}\right)\right). \tag{II.53}$$

Using the formuals (II.19) and (II.53) we deliver:

$$\begin{aligned}\mathsf{M}_c\left(t\mathbf{e}_0+\lambda\mathcal{V}(L)\right)&=\lambda^2\left\|\mathcal{V}(L)\right\|^2-c^2t^2=\lambda^2\|V\|^2-c^2t^2;\\ \mathsf{M}_c\left(L\left(t\mathbf{e}_0+\lambda\mathcal{V}(L)\right)\right)&=\alpha_L^2\mathsf{M}_c\left(\left(t-\lambda\beta_L\right)\mathbf{e}_0+(t-\lambda)\,\mathcal{V}\left(L^{-1}\right)\right)=\\ &=\alpha_L^2\left((t-\lambda)^2\left\|\mathcal{V}\left(L^{-1}\right)\right\|^2-c^2\left(t-\lambda\beta_L\right)^2\right)=\alpha_L^2\left((t-\lambda)^2\gamma_L-c^2\left(t-\lambda\beta_L\right)^2\right),\end{aligned}$$

where $\gamma_L=\left\|\mathcal{V}\left(L^{-1}\right)\right\|^2$. Since $t\mathbf{e}_0+\lambda\mathcal{V}(L)\in\mathfrak{H}_0\oplus\mathfrak{H}_1\left[\mathcal{V}(L)\right]$, by the condition $2'$, $\left(\mathsf{M}_c\left(L\left(t\mathbf{e}_0+\lambda\mathcal{V}(L)\right)\right)\right)^2=\left(\mathsf{M}_c\left(t\mathbf{e}_0+\lambda\mathcal{V}(L)\right)\right)^2$, $t,\lambda\in\mathbb{R}$. Thus:

$$\begin{aligned}\left(\lambda^2\|V\|^2-c^2t^2\right)^2&=\left(\alpha_L^2\left((t-\lambda)^2\gamma_L-c^2\left(t-\lambda\beta_L\right)^2\right)\right)^2,\quad\text{hence:}\\ \lambda^2\|V\|^2-c^2t^2&=\pm\alpha_L^2\left((t-\lambda)^2\gamma_L-c^2\left(t-\lambda\beta_L\right)^2\right)\quad(t,\lambda\in\mathbb{R}).\end{aligned}$$

And after simple transformations the last formula takes the form:

$$\lambda^2\|V\|^2-c^2t^2=\pm\alpha_L^2\left(t^2\left(\gamma_L-c^2\right)-2t\lambda\left(\gamma_L-c^2\beta_L\right)+\lambda^2\left(\gamma_L-c^2\beta_L^2\right)\right)\quad(\forall\,t,\lambda\in\mathbb{R}).$$

Equating coefficients near the same powers of $\lambda$, we obtain two systems of equations:

$$\begin{cases}\alpha_L^2\left(\gamma_L-c^2\right)=-c^2\\ \gamma_L-c^2\beta_L=0\\ \alpha_L^2\left(\gamma_L-c^2\beta_L^2\right)=\|V\|^2\end{cases}\qquad\begin{cases}\alpha_L^2\left(\gamma_L-c^2\right)=c^2\\ \gamma_L-c^2\beta_L=0\\ \alpha_L^2\left(\gamma_L-c^2\beta_L^2\right)=-\|V\|^2.\end{cases}$$

By means of simple transformations, these two systems can be reduced to the form:

$$\begin{cases}\alpha_L^2\left(1-\frac{\|V\|^2}{c^2}\right)=1\\ \gamma_L=\|V\|^2\\ \beta_L=\frac{\|V\|^2}{c^2}\end{cases}\qquad\begin{cases}\alpha_L^2\left(1-\frac{\|V\|^2}{c^2}\right)=-1\\ \gamma_L=\|V\|^2\\ \beta_L=\frac{\|V\|^2}{c^2}.\end{cases}$$

The first system has (real) solutions only for $\|V\|<c$, and the second system has solutions only for $\|V\|>c$. Thus, the solutions exist only for $\|V\|\neq c$. Solving the last systems and taking into account, that $\gamma_L=\left\|\mathcal{V}\left(L^{-1}\right)\right\|^2$, in the both cases we obtain:

$$\alpha_L=\frac{s}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}};\quad\beta_L=\frac{\|V\|^2}{c^2};\quad\left\|\mathcal{V}\left(L^{-1}\right)\right\|^2=\|V\|^2\qquad(\|V\|\neq c), \tag{II.54}$$

where $s\in\{-1,1\}$.

Substituting the values of $L\mathbf{X}_1^{\perp}[V]\,\mathrm{w}$ from the formula (II.52) and the values of $\alpha_L$,$\beta_L$ from the formula (II.54) into (II.45), we deliver:

$$L\mathrm{w} = \frac{s}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}}\left(\left(\mathcal{T}(\mathrm{w}) - \frac{\langle V,\mathrm{w}\rangle}{\|V\|^2}\frac{\|V\|^2}{c^2}\right)\mathbf{e}_0 + \left(\mathcal{T}(\mathrm{w}) - \frac{\langle V,\mathrm{w}\rangle}{\|V\|^2}\right)\mathcal{V}\left(L^{-1}\right)\right) +$$
$$+J_1\mathbf{X}_1^{\perp}[V]\,\mathrm{w} =$$
$$= \frac{s\left(\mathcal{T}(\mathrm{w}) - \frac{\langle V,\mathrm{w}\rangle}{c^2}\right)}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}}\mathbf{e}_0 + \frac{s\left(\mathcal{T}(\mathrm{w})\,\mathcal{V}(L^{-1}) - \frac{\langle V,\mathrm{w}\rangle}{\|V\|^2}\mathcal{V}(L^{-1})\right)}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}} + J_1\mathbf{X}_1^{\perp}[V]\,\mathrm{w}. \qquad \text{(II.55)}$$

Introduce the following operator on the subspace $\mathfrak{H}_1$:

$$Jx := \frac{\langle V,x\rangle}{\|V\|^2}\mathcal{V}\left(L^{-1}\right) + J_1\mathbf{X}_1^{\perp}[V]\,x, \quad x\in\mathfrak{H}_1. \qquad \text{(II.56)}$$

Since $J_1$ maps subspace $\mathfrak{H}_{1\perp}[\mathcal{V}(L)]$ to subspace $\mathfrak{H}_{1\perp}[\mathcal{V}(L^{-1})]$, we have, $\left\langle \mathcal{V}(L^{-1}),\, J_1\mathbf{X}_1^{\perp}[V]\,x\right\rangle = 0$. Hence, using (II.47),(II.54) and (II.17), for $x\in\mathfrak{H}_1$ we obtain:

$$\|Jx\|^2 = \left(\frac{\langle V,x\rangle}{\|V\|^2}\left\|\mathcal{V}\left(L^{-1}\right)\right\|\right)^2 + \left\|J_1\mathbf{X}_1^{\perp}[V]\,x\right\|^2 = \left(\frac{\langle V,x\rangle}{\|V\|}\right)^2 + \left\|\mathbf{X}_1^{\perp}[V]\,x\right\|^2 =$$
$$= \left\|\frac{\langle V,x\rangle}{\|V\|^2}V\right\|^2 + \left\|\mathbf{X}_1^{\perp}[V]\,x\right\|^2 = \|\mathbf{X}_1[V]\,x\|^2 + \left\|\mathbf{X}_1^{\perp}[V]\,x\right\|^2 = \|x\|^2.$$

Thus, operator $J$ is isometric on $\mathfrak{H}_1$.

For $x=\lambda V\in\mathfrak{H}_1[V]$ by (II.56) we have $J(\lambda V) = \lambda\mathcal{V}(L^{-1})$. Hence:

$$J\mathfrak{H}_1[V] = \mathfrak{H}_1\left[\mathcal{V}\left(L^{-1}\right)\right]. \qquad \text{(II.57)}$$

And for $x\in\mathfrak{H}_{1\perp}[V]$ according to (II.56) we obtain:

$$Jx = J_1\mathbf{X}_1^{\perp}[V]\,x = J_1x \quad (x\in\mathfrak{H}_{1\perp}[V]). \qquad \text{(II.58)}$$

Hence, by (II.48):

$$J\mathfrak{H}_{1\perp}[V] = J_1\mathfrak{H}_{1\perp}[V] = \mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]. \qquad \text{(II.59)}$$

From (II.57) and (II.59) it follows, that

$$J\mathfrak{H}_1 = J\left(\mathfrak{H}_1[V]\oplus\mathfrak{H}_{1\perp}[V]\right) \supseteq \mathfrak{H}_1\left[\mathcal{V}\left(L^{-1}\right)\right]\oplus\mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right] = \mathfrak{H}_1.$$

Thus, $J\mathfrak{H}_1 = \mathfrak{H}_1$. And so operator $J$ is unitary on $\mathfrak{H}_1$, that is

$$J\in\mathfrak{U}(\mathfrak{H}_1).$$

In accordance with (II.56), $JV = \mathcal{V}(L^{-1})$. Hence, using (II.55), (II.58) and (II.17), we deliver the formula (II.37). So, for the case $V\neq\mathbf{0}$ formula (II.37) is proved.

Now consider the case $V=\mathbf{0}$, that is $\mathcal{V}(L)=\mathbf{0}$. In this case, by the formula (II.17):

$$\mathbf{X}_1[V] = \mathbf{X}_1[\mathbf{0}] = \mathbb{O}, \quad \mathbf{X}_1^{\perp}[V] = \mathbf{X}. \qquad \text{(II.60)}$$

Since, by condition $1'$, transforms $L$ and $L^{-1}$ are v-determined, by lemma II.17.1, the following equality must hold:

$$tL\mathbf{e}_0 = L\left(t\mathbf{e}_0 + \lambda\mathcal{V}(L)\right) = \alpha_L\left((t-\lambda\beta_L)\,\mathbf{e}_0 + (t-\lambda)\,\mathcal{V}\left(L^{-1}\right)\right) \quad (\forall t,\lambda\in\mathbb{R}), \qquad \text{(II.61)}$$

with $\alpha_L = \mathcal{T}(L\mathbf{e}_0) \neq 0$, $\beta_L = 1 - \frac{1}{\alpha_L \alpha_{L^{-1}}} \neq 1$. Since the left-hand side of the equality (II.61) does not depend of $\lambda$, the coefficient of the variable $\lambda$ in the right-hand side of the equality must be zero. Hence, $\beta_L \mathbf{e}_0 + \mathcal{V}(L^{-1}) = \mathbf{0}$, and so

$$\beta_L = 0, \ \mathcal{V}\left(L^{-1}\right) = \mathbf{0}. \tag{II.62}$$

Thus, the formula (II.61) takes the form $L\mathbf{e}_0 = \alpha_L \mathbf{e}_0$. And, applying the condition $2'$ to the vector $\mathbf{e}_0 \in \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(L)]$, we obtain $\alpha_L = s$, where $s \in \{-1, 1\}$. Consequently:

$$L\mathbf{e}_0 = s\mathbf{e}_0, \qquad \text{where } s \in \{-1, 1\}. \tag{II.63}$$

Using (II.60),(II.63) for any vector $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ we obtain:

$$L\mathrm{w} = L(\mathcal{T}(\mathrm{w})\mathbf{e}_0 + \mathbf{X}\mathrm{w}) = s\mathcal{T}(\mathrm{w})\mathbf{e}_0 + L\mathbf{X}\mathrm{w} = s\mathcal{T}(\mathrm{w})\mathbf{e}_0 + J\mathbf{X}_1^{\perp}[V]\mathrm{w}, \tag{II.64}$$

where

$$Jx = Lx, \qquad x \in \mathfrak{H}_1 = \mathbf{X}\mathcal{M}(\mathfrak{H}) = \mathbf{X}_1^{\perp}[V]\mathcal{M}(\mathfrak{H}) = \mathfrak{H}_{1\perp}[V]. \tag{II.65}$$

By condition $3'$ and formula (II.60), the subspace $\mathfrak{H}_1 = \left\{\mathrm{w} \in \mathcal{M}(\mathfrak{H}) \mid \widehat{\mathbf{T}}\mathrm{w} = \mathbf{0}\right\}$ is invariant for the operator $L$. Hence, the operator $J$ from (II.65) maps the subspace $\mathfrak{H}_1$ into the subspace $\mathfrak{H}_1$.

According to the formula (II.62), $\mathcal{V}(L^{-1}) = \mathbf{0}$. Consequently, by the formula (II.60) $\mathbf{X}_1^{\perp}[V] = \mathbf{X}_1^{\perp}[\mathcal{V}(L)] = \mathbf{X}_1^{\perp}[\mathcal{V}(L^{-1})] = \mathbf{X}$. So, by the condition $4'$ operator $J$ is isometric on the subspace $\mathfrak{H}_1$. Now, we have to prove, that operator $J$ is unitary. Consider any vector $y \in \mathfrak{H}_1$. Denote $x := L^{-1}y$. Then, by (II.64), $Lx = s\mathcal{T}(x)\mathbf{e}_0 + J\mathbf{X}x = y \in \mathfrak{H}_1$. Hence, $\mathcal{T}(x) = 0$ and $J\mathbf{X}x = y$. This means, that $y \in J\mathfrak{H}_1$. Therefore, we have seen, that $J\mathfrak{H}_1 = \mathfrak{H}_1$, and the operator $J$ really is unitary on the $\mathfrak{H}_1 = \mathfrak{H}_{1\perp}[V]$. Thus for the case $V = \mathbf{0}$ the formula (II.37) also is proved.

**II.** Inversely, suppose, that the operarot $L \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ can be represented in the form (II.37). Then, by the lemma II.17.2, $L$ is a linear coordinate transform operator.

1. By the formula (II.37) we deliver:

$$L\mathbf{e}_0 = \chi_V(\mathbf{e}_0 + JV), \quad \text{where } \chi_V = \frac{s}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}} \neq 0;$$

$$L(\mathbf{e}_0 + V) = \frac{s\left(1 - \frac{\|V\|^2}{c^2}\right)}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}}\mathbf{e}_0 = \frac{\operatorname{sign}(c - \|V\|)}{\chi_V}\mathbf{e}_0;$$

$$L^{-1}\mathbf{e}_0 = \frac{\chi_V(\mathbf{e}_0 + V)}{\operatorname{sign}(c - \|V\|)}.$$

Hence $\mathcal{T}(L\mathbf{e}_0) = \chi_V \neq 0$, $\mathcal{T}(L^{-1}\mathbf{e}_0) = \frac{\chi_V}{\operatorname{sign}(c - \|V\|)} \neq 0$. Thus, linear coordinate transform operators $L$ and $L^{-1}$ are v-determined, moreover:

$$\mathcal{V}(L) = \frac{\mathbf{X}L^{-1}\mathbf{e}_0}{\mathcal{T}(L^{-1}\mathbf{e}_0)} = V; \quad \mathcal{V}\left(L^{-1}\right) = \frac{\mathbf{X}L\mathbf{e}_0}{\mathcal{T}(L\mathbf{e}_0)} = JV. \tag{II.66}$$

2. In accordance with (II.37), for $\mathrm{w} = t\mathbf{e}_0 + \lambda V \in \mathfrak{H}_0 \oplus \mathfrak{H}_1[V] = \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(L)]$, we obtain:

$$L\mathrm{w} = \frac{s\left(t - \lambda\frac{\|V\|^2}{c^2}\right)}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}}\mathbf{e}_0 + \frac{s(t - \lambda)JV}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}}.$$

Hence, since $J$ is isometric operator, we obtain:

$$\left(\mathsf{M}_c\left(L\mathrm{w}\right)\right)^2=\left(\left\|\frac{s\left(t-\lambda\right)JV}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}}\right\|^2-c^2\left(\frac{s\left(t-\lambda\frac{\|V\|^2}{c^2}\right)}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}}\right)^2\right)^2=$$

$$=\left(\frac{1}{\left|1-\frac{\|V\|^2}{c^2}\right|}\left(\left(t-\lambda\right)^2\left\|JV\right\|^2-c^2\left(t-\lambda\frac{\|V\|^2}{c^2}\right)^2\right)\right)^2=$$

$$=\left(\frac{1}{1-\frac{\|V\|^2}{c^2}}\left(\left(t-\lambda\right)^2\left\|V\right\|^2-c^2\left(t-\lambda\frac{\|V\|^2}{c^2}\right)^2\right)\right)^2=$$

$$=\left(\frac{1}{1-\frac{\|V\|^2}{c^2}}\left(\left(t^2-2t\lambda+\lambda^2\right)\left\|V\right\|^2-c^2\left(t^2-2t\lambda\frac{\|V\|^2}{c^2}+\lambda^2\frac{\|V\|^4}{c^4}\right)\right)\right)^2=$$

$$=\left(\frac{1}{1-\frac{\|V\|^2}{c^2}}\left(\|V\|^2t^2-2t\lambda\|V\|^2+\lambda^2\|V\|^2-\left(c^2t^2-2t\lambda\|V\|^2+\lambda^2\frac{\|V\|^4}{c^2}\right)\right)\right)^2=$$

$$=\left(\frac{1}{1-\frac{\|V\|^2}{c^2}}\left(\|V\|^2t^2-2t\lambda\|V\|^2+\lambda^2\|V\|^2-c^2t^2+2t\lambda\|V\|^2-\lambda^2\frac{\|V\|^4}{c^2}\right)\right)^2=$$

$$=\left(\frac{1}{1-\frac{\|V\|^2}{c^2}}\left(\|V\|^2t^2+\lambda^2\|V\|^2-c^2t^2-\lambda^2\frac{\|V\|^4}{c^2}\right)\right)^2=$$

$$=\left(\frac{1}{1-\frac{\|V\|^2}{c^2}}\left(\lambda^2\|V\|^2\left(1-\frac{\|V\|^2}{c^2}\right)-\left(1-\frac{\|V\|^2}{c^2}\right)c^2t^2\right)\right)^2=$$

$$=\left(\lambda^2\|V\|^2-c^2t^2\right)^2=\left(\mathsf{M}_c\left(\mathrm{w}\right)\right)^2.$$

Thus, the condition $2'$ for the operator $L$ also is satisfied.

3. Suppose, that $\mathrm{w}\in\mathcal{M}\left(\mathfrak{H}\right)$, $\widehat{\mathbf{T}}\mathrm{w}=\mathbf{X}_1\left[\mathcal{V}\left(L\right)\right]\mathrm{w}=\mathbf{0}$. Then, $\mathcal{T}\left(\mathrm{w}\right)=0$, and (since $\mathcal{V}\left(L\right)=V$, by (II.66)), we have, $\langle V,\mathrm{w}\rangle=0$, $\mathbf{X}_1^{\perp}\left[V\right]\mathrm{w}=\left(\mathbf{X}-\mathbf{X}_1\left[\mathcal{V}\left(L\right)\right]\right)\mathrm{w}=\mathbf{X}\mathrm{w}=\mathrm{w}$. So, by (II.37):

$$L\mathrm{w}=J\mathbf{X}_1^{\perp}\left[V\right]\mathrm{w}=J\mathrm{w}.$$

And, taking into account, that $J$ is unitary operator on $\mathfrak{H}_1$, using (II.66) and (II.17) we obtain:

$$\widehat{\mathbf{T}}L\mathrm{w}=\widehat{\mathbf{T}}J\mathrm{w}=\mathbf{0};$$

$$\mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathrm{w}=\mathbf{X}_1\left[JV\right]J\mathrm{w}=\begin{cases}\frac{\langle JV,J\mathrm{w}\rangle}{\|V\|^2}JV, & JV\neq\mathbf{0}\\ \mathbf{0}, & JV=\mathbf{0}\end{cases}=$$

$$=J\left\{\begin{matrix}\frac{\langle V,\mathrm{w}\rangle}{\|V\|^2}V, & V\neq\mathbf{0}\\ \mathbf{0}, & V=\mathbf{0}\end{matrix}\right\}=J\mathbf{X}_1\left[V\right]\mathrm{w}=\mathbf{0}.$$

Hence, we have checked the condition $3'$ for the operator $L$.

4. Using the unitarity of the operator $J$ ($\langle Jx,Jy\rangle=\langle x,y\rangle$, $x,y\in\mathfrak{H}_1$) and equalities (II.17),(II.18) one obtains the following:

$$J\mathbf{X}_1\left[V\right]\mathrm{w}=\begin{cases}\frac{\langle V,\mathrm{w}\rangle}{\|V\|^2}JV, & V\neq\mathbf{0}\\ \mathbf{0}, & V=\mathbf{0}\end{cases}=\begin{cases}\frac{\langle\mathbf{X}V,\mathrm{w}\rangle}{\|V\|^2}JV, & V\neq\mathbf{0}\\ \mathbf{0}, & V=\mathbf{0}\end{cases}=$$

$$= \begin{cases} \frac{\langle V, \mathbf{X}\mathrm{w}\rangle}{\|V\|^2} JV, & V \neq \mathbf{0} \\ \mathbf{0}, & V = \mathbf{0} \end{cases} = \left.\begin{cases} \frac{\langle JV, J\mathbf{X}\mathrm{w}\rangle}{\|JV\|^2} JV, & JV \neq \mathbf{0} \\ \mathbf{0}, & JV = \mathbf{0} \end{cases}\right\} = \mathbf{X}_1[JV]\, J\mathbf{X}\mathrm{w};$$

$$J\mathbf{X}_1^{\perp}[V]\,\mathrm{w} = J(\mathbf{X} - \mathbf{X}_1[V])\,\mathrm{w} = J\mathbf{X}\mathrm{w} - \mathbf{X}_1[JV]\,J\mathbf{X}\mathrm{w} =$$
$$= \mathbf{X}J\mathbf{X}\mathrm{w} - \mathbf{X}_1[JV]\,J\mathbf{X}\mathrm{w} = \mathbf{X}_1^{\perp}[JV]\,J\mathbf{X}\mathrm{w}.$$

So, by the formula (II.37) for any $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ we deliver:

$$\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right] L\mathrm{w} = \mathbf{X}_1^{\perp}[JV]\, L\mathrm{w} =$$
$$= \mathbf{X}_1^{\perp}[JV] \left( \frac{s\left(\mathcal{T}(\mathrm{w}) - \frac{\langle V, \mathrm{w}\rangle}{c^2}\right)}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}} \mathbf{e}_0 + \frac{s\left(\mathcal{T}(\mathrm{w})\,JV - \mathbf{X}_1[JV]\,J\mathbf{X}\mathrm{w}\right)}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}} + J\mathbf{X}_1^{\perp}[V]\,\mathrm{w} \right) =$$
$$= \mathbf{X}_1^{\perp}[JV]\; J\mathbf{X}_1^{\perp}[V]\,\mathrm{w} = \mathbf{X}_1^{\perp}[JV]\; \mathbf{X}_1^{\perp}[JV]\,J\mathbf{X}\mathrm{w} =$$
$$= \mathbf{X}_1^{\perp}[JV]\,J\mathbf{X}\mathrm{w} = J\mathbf{X}_1^{\perp}[V]\,\mathrm{w};$$
$$\left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right] L\mathrm{w}\right\| = \left\|J\mathbf{X}_1^{\perp}[V]\,\mathrm{w}\right\| = \left\|\mathbf{X}_1^{\perp}[V]\,\mathrm{w}\right\|.$$

Thus, all conditions $1'$-$4'$ for the linear coordinate transform operator $L$ are satisfied. Hence $L \in \mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c)$. □

### 17.4 Generalized Lorentz Transforms for Infinite Speeds

Now we investigate the behavior of coordinate transform operatos from the class $\mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c)$, when the norm of the rate of reference frame ($\|V\|$) tends to infinity. For this purpose we denote by $\mathbf{B}_1(\mathfrak{H}_1)$ the set:

$$\mathbf{B}_1(\mathfrak{H}_1) = \{x \in \mathfrak{H}_1 \mid \|x\| = 1\}$$

and substitute:

$$V = \lambda s\mathbf{n}, \quad \text{where } \lambda > 0,\ \lambda \neq c; \quad \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1) \tag{II.67}$$

into the formula (II.37). Then we are going to take the limit while $\lambda \to \infty$.

Note, that, by two lower equalities of (II.18), we have:

$$\mathbf{X}_1[\lambda s\mathbf{n}] = \mathbf{X}_1[\mathbf{n}]; \qquad \mathbf{X}_1^{\perp}[\lambda s\mathbf{n}] = \mathbf{X}_1^{\perp}[\mathbf{n}]. \tag{II.68}$$

Hence, substitution the velocity (II.67) to the formula (II.37) lead us to the following representation for operators $L \in \mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c)$ (with $\mathcal{V}(L) \neq \mathbf{0}$):

$$L\mathrm{w} = \mathbf{W}_{\lambda,c}[s, \mathbf{n}, J]\,\mathrm{w} :=$$
$$= \frac{\left(s\mathcal{T}(\mathrm{w}) - \frac{\lambda}{c^2}\langle \mathbf{n}, \mathrm{w}\rangle\right)}{\sqrt{\left|1 - \frac{\lambda^2}{c^2}\right|}} \mathbf{e}_0 + J\left( \frac{\lambda\mathcal{T}(\mathrm{w})\,\mathbf{n} - s\mathbf{X}_1[\mathbf{n}]\,\mathrm{w}}{\sqrt{\left|1 - \frac{\lambda^2}{c^2}\right|}} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w} \right), \quad \mathrm{w} \in \mathcal{M}(\mathfrak{H}), \tag{II.69}$$

where $s \in \{-1, 1\}$, $J \in \mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$, $\lambda > 0$.

Taking in (II.69) limit while $\lambda \to \infty$, we get the following linear operators in the space $\mathcal{M}(\mathfrak{H})$:

$$\mathbf{W}_{\infty,c}[\mathbf{n}, J]\,\mathrm{w} = \lim_{\lambda \to +\infty} \mathbf{W}_{\lambda}[s, \mathbf{n}, J]\,\mathrm{w} =$$
$$= -\frac{\langle \mathbf{n}, \mathrm{w}\rangle}{c}\mathbf{e}_0 + J\left(c\mathcal{T}(\mathrm{w})\,\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right) \qquad (\mathrm{w} \in \mathcal{M}(\mathfrak{H})), \tag{II.70}$$

where limit exists in the sense of norm of the space $\mathcal{M}(\mathfrak{H})$. Note, that limit in (II.70) does not depend of the number $s$. It is not hard to verify, that $\mathbf{W}_{\infty,c}[\mathbf{n}, J] \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$.

Now we introduce the following class of linear bounded operators in the space $\mathcal{M}(\mathfrak{H})$:

$$\mathfrak{O}\mathfrak{T}_\infty(\mathfrak{H},c) := \{\mathbf{W}_{\infty,c}[\mathbf{n},J] \mid \mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ J\in\mathfrak{U}(\mathfrak{H}_1)\}. \tag{II.71}$$

**Lemma II.17.3.** *For any $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$ and $J\in\mathfrak{U}(\mathfrak{H}_1)$ the following equalities holds:*

$$\widetilde{J}\mathbf{W}_{\infty,c}[\mathbf{n},\mathbb{I}_1] = \mathbf{W}_{\infty,c}[\mathbf{n},J];\quad \mathbf{W}_{\infty,c}[\mathbf{n},\mathbb{I}_1]\widetilde{J} = \mathbf{W}_{\infty,c}\left[J^{-1}\mathbf{n},J\right], \tag{II.72}$$

*where the operator $\widetilde{J}$ is defined in (II.35), and $\mathbb{I}_1=\mathbb{I}_{\mathfrak{H}_1}$ denotes the identity operator on the subspace $\mathfrak{H}_1$.*

*Proof.* The first equality (II.72) immediately follows from (II.35) and (II.70). Hence, we prove only the second equality (II.72). Using (II.35) and (II.70) we obtain for any $\mathrm{w}\in\mathcal{M}(\mathfrak{H})$:

$$\begin{aligned}\mathbf{W}_{\infty,c}[\mathbf{n},\mathbb{I}_1]\widetilde{J}\mathrm{w} &= \mathbf{W}_{\infty,c}[\mathbf{n},\mathbb{I}_1](\mathcal{T}(\mathrm{w})\mathbf{e}_0 + J\mathbf{X}\mathrm{w}) = -\frac{\langle\mathbf{n},\mathcal{T}(\mathrm{w})\mathbf{e}_0+J\mathbf{X}\mathrm{w}\rangle}{c}\mathbf{e}_0+\\ &\quad + c\mathcal{T}(\mathcal{T}(\mathrm{w})\mathbf{e}_0+J\mathbf{X}\mathrm{w})\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}](\mathcal{T}(\mathrm{w})\mathbf{e}_0+J\mathbf{X}\mathrm{w}) =\\ &= -\frac{\langle\mathbf{n},J\mathbf{X}\mathrm{w}\rangle}{c}\mathbf{e}_0 + c\mathcal{T}(\mathrm{w})\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]J\mathbf{X}\mathrm{w} =\\ &= -\frac{\langle J^{-1}\mathbf{n},\mathrm{w}\rangle}{c}\mathbf{e}_0 + c\mathcal{T}(\mathrm{w})\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]J\mathbf{X}\mathrm{w}.\end{aligned} \tag{II.73}$$

Note, that, by definition of class $\mathfrak{O}\mathfrak{T}_\infty(\mathfrak{H},c)$, $\mathbf{n}\neq 0$. So, applying (II.17),(II.18), and using the fact that the operator $J$ maps $\mathfrak{H}_1$ into $\mathfrak{H}_1$, we obtain:

$$\begin{aligned}\mathbf{X}_1^{\perp}[\mathbf{n}]J\mathbf{X}\mathrm{w} &= (\mathbf{X}-\mathbf{X}_1[\mathbf{n}])J\mathbf{X}\mathrm{w} = \mathbf{X}J\mathbf{X}\mathrm{w} - \langle\mathbf{n},J\mathbf{X}\mathrm{w}\rangle\mathbf{n} =\\ &= \mathbf{X}J\mathbf{X}\mathrm{w} - \left\langle\mathbf{X}J^{-1}\mathbf{n},\mathrm{w}\right\rangle\mathbf{n} = J\mathbf{X}\mathrm{w} - \left\langle J^{-1}\mathbf{n},\mathrm{w}\right\rangle\mathbf{n} =\\ &= J\left(\mathbf{X}\mathrm{w} - \left\langle J^{-1}\mathbf{n},\mathrm{w}\right\rangle J^{-1}\mathbf{n}\right) = J\left(\mathbf{X}-\mathbf{X}_1\left[J^{-1}\mathbf{n}\right]\right)\mathrm{w} = J\mathbf{X}_1^{\perp}\left[J^{-1}\mathbf{n}\right]\mathrm{w}.\end{aligned}$$

Thus, according to (II.73), we deduce:

$$\begin{aligned}\mathbf{W}_{\infty,c}[\mathbf{n},\mathbb{I}_1]\widetilde{J}\mathrm{w} &= -\frac{\langle J^{-1}\mathbf{n},\mathrm{w}\rangle}{c}\mathbf{e}_0 + c\mathcal{T}(\mathrm{w})\mathbf{n} + J\mathbf{X}_1^{\perp}\left[J^{-1}\mathbf{n}\right]\mathrm{w} =\\ &= -\frac{\langle J^{-1}\mathbf{n},\mathrm{w}\rangle}{c}\mathbf{e}_0 + J\left(c\mathcal{T}(\mathrm{w})J^{-1}\mathbf{n} + \mathbf{X}_1^{\perp}\left[J^{-1}\mathbf{n}\right]\mathrm{w}\right) = \mathbf{W}_{\infty,c}\left[J^{-1}\mathbf{n},J\right]\mathrm{w}.\end{aligned}$$

□

**Lemma II.17.4.** *For any vector $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$ it is true the following equality:*

$$\mathbf{W}_{\infty,c}[\mathbf{n},\mathbb{I}_1]\mathbf{W}_{\infty,c}[-\mathbf{n},\mathbb{I}_1] = \mathbb{I}.$$

*Proof.* Consider an arbitrary vector $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$. For vector $\mathrm{w}\in\mathcal{M}(\mathfrak{H})$, using (II.70), (II.68), (II.18), we get:

$$\begin{aligned}\mathbf{W}_{\infty,c}[\mathbf{n},\mathbb{I}_1]\mathbf{W}_{\infty,c}[-\mathbf{n},\mathbb{I}_1]\mathrm{w} &= \mathbf{W}_{\infty,c}[\mathbf{n},\mathbb{I}_1]\left(-\frac{\langle-\mathbf{n},\mathrm{w}\rangle}{c}\mathbf{e}_0 - c\mathcal{T}(\mathrm{w})\mathbf{n} + \mathbf{X}_1^{\perp}[-\mathbf{n}]\mathrm{w}\right) =\\ &= \mathbf{W}_{\infty,c}[\mathbf{n},\mathbb{I}_1]\left(\frac{\langle\mathbf{n},\mathrm{w}\rangle}{c}\mathbf{e}_0 - c\mathcal{T}(\mathrm{w})\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\mathrm{w}\right) =\\ &= -\frac{\left\langle\mathbf{n},\left(\frac{\langle\mathbf{n},\mathrm{w}\rangle}{c}\mathbf{e}_0 - c\mathcal{T}(\mathrm{w})\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\mathrm{w}\right)\right\rangle}{c}\mathbf{e}_0+\\ &\quad + c\mathcal{T}\left(\frac{\langle\mathbf{n},\mathrm{w}\rangle}{c}\mathbf{e}_0 - c\mathcal{T}(\mathrm{w})\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\mathrm{w}\right)\mathbf{n}+\end{aligned}$$

$$+\mathbf{X}_1^{\perp}[\mathbf{n}]\left(\frac{\langle \mathbf{n},\mathrm{w}\rangle}{c}\mathbf{e}_0 - c\mathcal{T}(\mathrm{w})\,\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right) =$$
$$= \quad \frac{c\mathcal{T}(\mathrm{w})\,\mathbf{e}_0}{c} \;+\; c\frac{\langle \mathbf{n},\mathrm{w}\rangle}{c}\mathbf{n} \;+\; \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w} \;=\; \mathrm{w}.$$

□

From lemmas II.17.4, II.17.3 and formula (II.39), we deduce the following theorem.

**Theorem II.17.2.** *Any operator* $\mathbf{W}_{\infty,c}[\mathbf{n},J] \in \mathfrak{OT}_{\infty}(\mathfrak{H},c)$ *is a linear coordinate transform operator, moreover:*

$$(\mathbf{W}_{\infty,c}[\mathbf{n},J])^{-1} = \mathbf{W}_{\infty,c}\left[-J\mathbf{n},J^{-1}\right].$$

*Proof.* For any operator $\mathbf{W}_{\infty,c}[\mathbf{n},J] \in \mathfrak{OT}_{\infty}(\mathfrak{H},c)$ (where $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$, $J \in \mathfrak{U}(\mathfrak{H}_1)$) using lemmas II.17.4, II.17.3 and formula (II.39), we obtain:

$$(\mathbf{W}_{\infty,c}[\mathbf{n},J])^{-1} = \left(\widetilde{J}\mathbf{W}_{\infty,c}[\mathbf{n},\mathbb{I}_1]\right)^{-1} = (\mathbf{W}_{\infty,c}[\mathbf{n},\mathbb{I}_1])^{-1}\,\widetilde{J^{-1}} = \mathbf{W}_{\infty,c}[-\mathbf{n},\mathbb{I}_1]\,\widetilde{J^{-1}} =$$
$$= \mathbf{W}_{\infty,c}\left[\left(J^{-1}\right)^{-1}(-\mathbf{n}),J^{-1}\right] = \mathbf{W}_{\infty,c}\left[-J\mathbf{n},J^{-1}\right]$$

□

Operators, which belong to the class $\mathfrak{OT}_{\infty}(\mathfrak{H},c)$ will be named ***generalized Lorentz transforms for infinite speeds*** of reference frames.

*Remark* II.17.1. Note, that any generalized Lorentz transform operator $\mathbf{W}_{\infty,c}[\mathbf{n},J] \in \mathfrak{OT}_{\infty}(\mathfrak{H},c)$ (with infinite speed) is not v-determined, because, by (II.70), $\mathcal{T}(\mathbf{W}_{\infty,c}[\mathbf{n},J]\,\mathbf{e}_0) = 0$.

**Denotation II.17.3.** *Denote:*

$$\mathfrak{OT}(\mathfrak{H},c) := \mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H},c) \cup \mathfrak{OT}_{\infty}(\mathfrak{H},c).$$

*Coordinate transforms, which belong to the class* $\mathfrak{OT}(\mathfrak{H},c)$ *will be named generalized* ***tachyon Lorentz transforms (in the Sense of E. Recami and V. Olkhovsky) for Hilbert Space*** $\mathfrak{H}$.

### 17.5 General Representation for Tachyon Lorentz Transforms

The aim of this subsection is to give general representation for operators, from the class $\mathfrak{OT}(\mathfrak{H},c)$, which would be true for finite as well as for infinite velocities of reference frames.

Since any velocity vector $V \in \mathfrak{H}_1$, $\|V\| \notin \{0,c\}$ can be represented by the form (II.67), where

$$\mathbf{n} = s\frac{V}{\|V\|}, \qquad \lambda = \|V\| \qquad (\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\ \lambda > 0)$$

the formula (II.69) may be considered as general representation for operators from $\mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H},c)$, with nonzero velocity, that is any operator $L \in \mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H},c)$, such, that $\mathcal{V}(L) \neq \mathbf{0}$ can be represented in the form (II.69).

Now we consider the case $\mathcal{V}(L) = \mathbf{0}$. By the formula (II.37), we have, that any operator $L \in \mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H},c)$ with zero velocity $\mathcal{V}(L)$ can be represented in the form:

$$L\mathrm{w} = s\mathcal{T}(\mathrm{w})\,\mathbf{e}_0 + J\left(\mathbf{X}_1^{\perp}[\mathbf{0}]\,\mathrm{w}\right) = s\mathcal{T}(\mathrm{w})\,\mathbf{e}_0 + J(\mathbf{X}\mathrm{w}) \quad (\mathrm{w} \in \mathcal{M}(\mathfrak{H})) \tag{II.74}$$

From the other hand, substituting $\lambda = 0$ ($s \in \{-1,1\}$, $J \in \mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$) into the formula (II.69), we can define the following operators:

$$\mathbf{W}_0[s,\mathbf{n},J]\,\mathrm{w} := s\mathcal{T}(\mathrm{w})\,\mathbf{e}_0 + J\left(-s\mathbf{X}_1[\mathbf{n}]\,\mathrm{w} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right) =$$

$$= s\mathcal{T}(\mathrm{w})\,\mathbf{e}_0 + J\left(-s\,\mathbb{I}_{1,-s}[\mathbf{n}]\right)\mathbf{X}\mathrm{w} \qquad (\mathrm{w}\in\mathcal{M}(\mathfrak{H})), \tag{II.75}$$

$$\text{where} \quad \mathbb{I}_{1,\sigma}[\mathbf{n}]\,x = \mathbf{X}_1[\mathbf{n}]\,x + \sigma\mathbf{X}_1^{\perp}[\mathbf{n}]\,x, \quad x\in\mathfrak{H}_1,\ \sigma\in\{-1,1\}.$$

Since, $-s\mathbb{I}_{1,-s}[\mathbf{n}]\in\mathfrak{U}(\mathfrak{H}_1)$, the set of operators, which can be defined by the formula (II.75) coincides with the set of operators, which can be defined by the formula (II.74).

Hence, we have seen, that (in the both cases $\mathcal{V}(L)\neq\mathbf{0}$ and $\mathcal{V}(L)=\mathbf{0}$) it is true the following assertion:

**Assertion II.17.4.** *Operator $L\in\mathcal{L}(\mathcal{M}(\mathfrak{H}))$ belongs to the class $\mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H},c)$ if and only if it can be represented by the formula:*

$$L = \mathbf{W}_{\lambda,c}[s,\mathbf{n},J],$$

*where $\lambda\geq 0$, $s\in\{-1,1\}$, $J\in\mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$ and operator $\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]$ is defined in (II.69). Velocity of the linear coordinate transform operator $L$ is determined by the formula $\mathcal{V}(L)=\lambda s\mathbf{n}$.*

Note, that we can extend the definition of operator-valued function $\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]$, which appears in the representation (II.69) for $\lambda\in[0,\infty]\backslash\{c\}$. Indeed, let $\lambda\in[0,\infty]\backslash\{c\}$, $s\in\{-1,1\}$, $J\in\mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$. Denote:

$$\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]\,\mathrm{w} = \begin{cases} \frac{\left(s\mathcal{T}(\mathrm{w})-\frac{\lambda}{c^2}\langle\mathbf{n},\mathrm{w}\rangle\right)}{\sqrt{\left|1-\frac{\lambda^2}{c^2}\right|}}\mathbf{e}_0 + J\left(\frac{\lambda\mathcal{T}(\mathrm{w})\mathbf{n}-s\mathbf{X}_1[\mathbf{n}]\mathrm{w}}{\sqrt{\left|1-\frac{\lambda^2}{c^2}\right|}} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right), & \lambda<\infty \\ \mathbf{W}_{\infty,c}[\mathbf{n},J] = -\frac{\langle\mathbf{n},\mathrm{w}\rangle}{c}\mathbf{e}_0 + J\left(c\mathcal{T}(\mathrm{w})\,\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right), & \lambda=\infty. \end{cases} \tag{II.76}$$

Using Assertion II.17.4, formulas (II.71), (II.70) and denotation II.17.3, we obtain the following assertion.

**Assertion II.17.5.** *Operator $L\in\mathcal{L}(\mathcal{M}(\mathfrak{H}))$ belongs to the class $\mathfrak{OT}(\mathfrak{H},c)$ if and only if there exist numbers $s\in\{-1,1\}$, $\lambda\in[0,\infty]\setminus\{c\}$ vector $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$ and operator $J\in\mathfrak{U}(\mathfrak{H}_1)$ such, that operator $L$ can be represented by the form:*

$$L = \mathbf{W}_{\lambda,c}[s,\mathbf{n},J].$$

*Operator $\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]$ is v-determined if and only if $\lambda<\infty$, and in this case*

$$\mathcal{V}(L) = \lambda s\mathbf{n}.$$

At first glans Assertion II.17.5 gives general representation for operators, from the class $\mathfrak{OT}(\mathfrak{H},c)$ for finite as well as for infinite velocities of reference frames. But, in reality, the definition of operator $\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]$ in (II.76) is different for the cases $\lambda<\infty$ and $\lambda=\infty$. So, our aim is not reached yet.

Now we introduce the new parameter:

$$\theta := \frac{1-\frac{\lambda}{c}}{\sqrt{\left|1-\frac{\lambda^2}{c^2}\right|}}. \tag{II.77}$$

Using simple calculations formula (II.77) can be reduced to the form:

$$\theta = -\operatorname{sign}\left(1-\frac{2}{1+\frac{\lambda}{c}}\right)\sqrt{\left|1-\frac{2}{1+\frac{\lambda}{c}}\right|}. \tag{II.78}$$

Since function $f(\lambda) = -\mathrm{sign}\left(1 - \frac{2}{1+\frac{\lambda}{c}}\right)\sqrt{\left|1 - \frac{2}{1+\frac{\lambda}{c}}\right|}$, is decreasing on $[0,+\infty)$, it maps the interval $[0,\infty)$ into the interval $=(-1,1]$, and any value $\lambda \geq 0$ can be uniquely determined by the parameter $\theta \in (-1,1]$. Using simple calculation, one can ensure, that parameter $\lambda$ can be determined by the parameter $\theta$ by means of the formula:

$$\lambda = c\frac{1-\theta\,|\theta|}{1+\theta\,|\theta|}, \quad \theta \in (-1,1], \tag{II.79}$$

and the case $\lambda = c$ corresponds the case $\theta = 0$.

By means of substitution the value of parameter $\lambda$ from the formula (II.79) to the correlation (II.69), we obtain the following representation of the operators $L \in \mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c)$:

$$\begin{aligned} L\mathrm{w} &= \mathbf{W}_{c\frac{1-\theta|\theta|}{1+\theta|\theta|},c}[s,\mathbf{n},J]\,\mathrm{w} = \\ &= \left(s\varphi_0(\theta)\,\mathcal{T}(\mathrm{w}) - \varphi_1(\theta)\frac{\langle \mathbf{n},\mathrm{w}\rangle}{c}\right)\mathbf{e}_0 + \\ &\quad + J\left(c\varphi_1(\theta)\,\mathcal{T}(\mathrm{w})\,\mathbf{n} - s\varphi_0(\theta)\,\mathbf{X}_1[\mathbf{n}]\,\mathrm{w} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right), \end{aligned} \tag{II.80}$$

$$\left(\mathrm{w} \in \mathcal{M}(\mathfrak{H}),\ s \in \{-1,1\},\ J \in \mathfrak{U}(\mathfrak{H}_1),\ \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\ \theta \in (-1,1]\setminus\{0\}\right),$$

where

$$\varphi_0(\theta) = \frac{1+\theta\,|\theta|}{2\,|\theta|}; \quad \varphi_1(\theta) = \frac{1-\theta\,|\theta|}{2\,|\theta|} \qquad (\theta \in \mathbb{R},\ \theta \neq 0). \tag{II.81}$$

Note, that the case $\theta = 0$ must be excluded, because in this case we have $\lambda = c$, and the norm of velocity $\mathcal{V}(L)$ is equal to the speed of light $c$ (note, that in the case $\|\mathcal{V}(L)\| = c$ the transforms (II.37), and, hence, (II.80) are undefined). From the equality (II.79) it follows, that in the case $\theta \in (0,1)$ we have, $\lambda = \|\mathcal{V}(L)\| \in (0,c)$. So, in this case, the norm of the velocity of reference frame $\|\mathcal{V}(L)\|$ frame is less then the speed of light $c$. Similarly, in the case $\theta \in (-1,0)$, we have $\lambda \in (c,+\infty)$. Hence, in this case the norm of frame velocity is greater, then $c$.

It is easy to verify, that for any $\theta \in \mathbb{R}\setminus\{0\}$ the following equalities are true:

$$\left.\begin{array}{ll} \varphi_0(\theta)\,\varphi_1(\theta) = -\frac{1}{4}\left(\theta^2 - \frac{1}{\theta^2}\right); & c\frac{\varphi_1(\theta)}{\varphi_0(\theta)} = \lambda = c\frac{1-\theta\,|\theta|}{1+\theta\,|\theta|}; \\ \varphi_0(\theta) + \varphi_1(\theta) = \frac{1}{|\theta|}; & \varphi_0(\theta) - \varphi_1(\theta) = \theta; \\ \varphi_0(\theta)^2 - \varphi_1(\theta)^2 = \mathrm{sign}\,\theta; & \\ \varphi_0(-\theta) = \varphi_1(\theta); & \varphi_1(-\theta) = \varphi_0(\theta); \\ \varphi_0(\theta^{-1}) = \mathrm{sign}\,\theta\,\varphi_0(\theta); & \varphi_1(\theta^{-1}) = -\mathrm{sign}\,\theta\,\varphi_1(\theta). \end{array}\right\} \tag{II.82}$$

Denote:

$$\begin{aligned} &\mathbf{U}_{\theta,c}[s,\mathbf{n},J] := \mathbf{W}_{c\frac{1-\theta|\theta|}{1+\theta|\theta|},c}[s,\mathbf{n},J], \\ &\qquad s \in \{-1,1\},\ \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\ J \in \mathfrak{U}(\mathfrak{H}_1), \quad \theta \in (-1,1],\ \theta \neq 0. \end{aligned} \tag{II.83}$$

From (II.81) it follows, that for $\theta = -1$ the functions $\varphi_0(\theta)$ and $\varphi_1(\theta)$ also are defined:

$$\varphi_0(-1) = 0, \qquad \varphi_1(-1) = 1.$$

And substitution $\theta = -1$ to the formula (II.80) leads us to the following linear operators:

$$\mathbf{U}_{-1,c}[s,\mathbf{n},J] := \mathbf{W}_{\infty,c}[s,\mathbf{n},J] = \mathbf{W}_{\infty,c}[\mathbf{n},J], \tag{II.84}$$

which do not depend on the number $s \in \{-1, 1\}$, because terms, which contain variable $s$ are zero (where the operators $\mathbf{W}_{\infty,c}[\mathbf{n}, J]$ are defined in (II.70)).

Hence, for $\theta = -1$ we obtain the generalized Lorentz transforms for infinite speeds $\mathbf{W}_{\infty,c}[\mathbf{n}, J]$, which, by remark II.17.1, are not v-determined.

Thus, above we have proved the following theorem.

**Theorem II.17.3.** *Operator $L \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ belongs to the class $\mathfrak{OT}(\mathfrak{H}, c)$ if and only if there exist numbers $s \in \{-1, 1\}$, $\theta \in [-1, 1] \setminus \{0\}$, vector $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ and operator $J \in \mathfrak{U}(\mathfrak{H}_1)$ such, that for any $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ vector $L\mathrm{w}$ can be represented by the formula:*

$$L\mathrm{w} = \mathbf{U}_{\theta,c}[s, \mathbf{n}, J]\,\mathrm{w} = \left(s\varphi_0(\theta)\,\mathcal{T}(\mathrm{w}) - \varphi_1(\theta)\frac{\langle \mathbf{n}, \mathrm{w}\rangle}{c}\right)\mathbf{e}_0 + \\ + J\left(c\varphi_1(\theta)\,\mathcal{T}(\mathrm{w})\,\mathbf{n} - s\varphi_0(\theta)\,\mathbf{X}_1[\mathbf{n}]\,\mathrm{w} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right). \tag{II.85}$$

*Linear coordinate transform operator $L = \mathbf{U}_{\theta,c}[s, \mathbf{n}, J]$ is v-determined if and only if $\theta \neq -1$, and in this case:*

$$\mathcal{V}(L) = cs\frac{1 - \theta|\theta|}{1 + \theta|\theta|}\mathbf{n}.$$

Now, we are going to reformulate Theorem II.17.3 in more convenient (for some further considerations) form.

Note, that, paremeter $\theta$ in Theorem II.17.3 belongs to the set $[-1, 1] \setminus \{0\}$, while the functions $\varphi_0(\theta)$ and $\varphi_1(\theta)$, are defined in formula (II.81) for any $\theta \in \mathbb{R} \setminus \{0\}$. So we can extend the definition of operator family $\{\mathbf{U}_{\theta,c}[s, \mathbf{n}, J]\}$, presented in formulas (II.83) or (II.85) for the values of parameter $\theta$ benonging to the set $\mathbb{R} \setminus \{0\}$:

$$\mathbf{U}_{\theta,c}[s, \mathbf{n}, J]\,\mathrm{w} := \left(s\varphi_0(\theta)\,\mathcal{T}(\mathrm{w}) - \varphi_1(\theta)\frac{\langle \mathbf{n}, \mathrm{w}\rangle}{c}\right)\mathbf{e}_0 + \\ + J\left(c\varphi_1(\theta)\,\mathcal{T}(\mathrm{w})\,\mathbf{n} - s\varphi_0(\theta)\,\mathbf{X}_1[\mathbf{n}]\,\mathrm{w} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right) \tag{II.86}$$

$$(\theta \in \mathbb{R} \setminus \{0\},\ s \in \{-1, 1\},\ \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\ J \in \mathfrak{U}(\mathfrak{H}_1)).$$

Hence, applying two lower equalities of (II.82) and two lower equalities of (II.18), we deliver:

$$\mathbf{U}_{\theta,c}[s, \mathbf{n}, J]\,\mathrm{w} = \left(s\,\mathrm{sign}\,\theta\,\varphi_0\left(\theta^{-1}\right)\mathcal{T}(\mathrm{w}) - \left(-\mathrm{sign}\,\theta\,\varphi_1\left(\theta^{-1}\right)\right)\frac{\langle \mathbf{n}, \mathrm{w}\rangle}{c}\right)\mathbf{e}_0 + \\ + J\left(c\left(-\mathrm{sign}\,\theta\,\varphi_1\left(\theta^{-1}\right)\right)\mathcal{T}(\mathrm{w})\,\mathbf{n} - s\,\mathrm{sign}\,\theta\,\varphi_0\left(\theta^{-1}\right)\mathbf{X}_1[\mathbf{n}]\,\mathrm{w} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right) = \\ = \left((s\,\mathrm{sign}\,\theta)\,\varphi_0\left(\theta^{-1}\right)\mathcal{T}(\mathrm{w}) - \varphi_1\left(\theta^{-1}\right)\frac{\langle(-\mathrm{sign}\,\theta\,\mathbf{n}), \mathrm{w}\rangle}{c}\right)\mathbf{e}_0 + = \\ + J\left(c\varphi_1\left(\theta^{-1}\right)\mathcal{T}(\mathrm{w})\left(-\mathrm{sign}\,\theta\,\mathbf{n}\right) - (s\,\mathrm{sign}\,\theta)\,\varphi_0\left(\theta^{-1}\right)\mathbf{X}_1[\mathbf{n}]\,\mathrm{w} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right) = \\ = \mathbf{U}_{\theta^{-1},c}[s\,\mathrm{sign}\,\theta, -\mathrm{sign}\,\theta\,\mathbf{n}, J].$$

Thus:

$$\mathbf{U}_{\theta,c}[s, \mathbf{n}, J] = \mathbf{U}_{\theta^{-1},c}[s\,\mathrm{sign}\,\theta, -\mathrm{sign}\,\theta\,\mathbf{n}, J] \tag{II.87}$$

$$(s \in \{-1, 1\},\ \theta \in \mathbb{R} \setminus \{0\},\ \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\ J \in \mathfrak{U}(\mathfrak{H}_1)).$$

For $|\theta| > 1$ we have $0 < |\theta^{-1}| < 1$. Hence, taking into account the formula (II.87), we see, that substitution the values $|\theta| > 1$ does not lead outside of the class of transformations, defined by the formula (II.85) for $\theta \in [-1, 1] \setminus \{0\}$. Besides this, for $|\theta| > 1$, according to the formula (II.87) and theorem II.17.3, we receive:

$$\mathcal{V}(\mathbf{U}_{\theta,c}[s, \mathbf{n}, J]) = \mathcal{V}(\mathbf{U}_{\theta^{-1},c}[s\,\mathrm{sign}\,\theta, -\mathrm{sign}\,\theta\,\mathbf{n}, J]) =$$

$$= cs\,\mathrm{sign}\,\theta\,\frac{1-\theta^{-1}\left|\theta^{-1}\right|}{1+\theta^{-1}\left|\theta^{-1}\right|}\left(-\mathrm{sign}\,\theta\,\mathbf{n}\right) = cs\frac{1-\theta\left|\theta\right|}{1+\theta\left|\theta\right|}\mathbf{n}.$$

Thus, we obtain the following corollary of Theorem II.17.3:

**Corollary II.17.1.** *Operator $L \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ belongs to the class $\mathfrak{O}\mathfrak{T}(\mathfrak{H}, c)$ if and only if there exist numbers $s \in \{-1, 1\}$, $\theta \in \mathbb{R} \setminus \{0\}$, vector $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ and operator $J \in \mathfrak{U}(\mathfrak{H}_1)$ such, that for any $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ vector $L\mathrm{w}$ can be represented by the formula:*

$$L\mathrm{w} = \mathbf{U}_{\theta,c}[s, \mathbf{n}, J]\,\mathrm{w} = \left(s\varphi_0(\theta)\,\mathcal{T}(\mathrm{w}) - \varphi_1(\theta)\frac{\langle \mathbf{n}, \mathrm{w}\rangle}{c}\right)\mathbf{e}_0 + \\ + J\left(c\varphi_1(\theta)\,\mathcal{T}(\mathrm{w})\,\mathbf{n} - s\varphi_0(\theta)\,\mathbf{X}_1[\mathbf{n}]\,\mathrm{w} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right).$$

*Linear coordinate transform operator $L = \mathbf{U}_{\theta,c}[s, \mathbf{n}, J]$ is v-determined if and only if $\theta \neq -1$, and in this case:*

$$\mathcal{V}\left(\mathbf{U}_{\theta,c}[s, \mathbf{n}, J]\right) = cs\frac{1-\theta\left|\theta\right|}{1+\theta\left|\theta\right|}\mathbf{n}.$$

## 17.6 Representations of Some Subclasses of Generalized Lorentz Transforms

From Assertion II.17.5 and Corollary II.17.1 we obtain the following two equivalent representations of the class of operators $\mathfrak{O}\mathfrak{T}(\mathfrak{H}, c)$:

$$\mathfrak{O}\mathfrak{T}(\mathfrak{H}, c) = \left\{\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J] \mid s \in \{-1,1\},\ \lambda \in [0,\infty]\setminus\{c\},\ \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\ J \in \mathfrak{U}(\mathfrak{H}_1)\right\}; \tag{II.88}$$

$$\mathfrak{O}\mathfrak{T}(\mathfrak{H}, c) = \left\{\mathbf{U}_{\theta,c}[s, \mathbf{n}, J] \mid s \in \{-1,1\},\ \theta \in \mathbb{R}\setminus\{0\},\ \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\ J \in \mathfrak{U}(\mathfrak{H}_1)\right\}. \tag{II.89}$$

Recall, that in Subsection 17.2 we have introduced the class of operators $\mathfrak{O}(\mathfrak{H}, c)$, and in Subsection 17.3 (see (II.33)) we have seen that $\mathfrak{O}(\mathfrak{H}, c) \subseteq \mathfrak{O}\mathfrak{T}_{\mathbf{fin}}(\mathfrak{H}, c)$. Hence, class of operators $\mathfrak{O}(\mathfrak{H}, c)$ is a subclass of $\mathfrak{O}\mathfrak{T}(\mathfrak{H}, c)$. The next aim is give the representation of the class $\mathfrak{O}(\mathfrak{H}, c)$, similar to (II.88), (II.89). First of all, for this aim we should prove the following lemma.

**Lemma II.17.5.** *Operator $L \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ belongs to the class $\mathfrak{O}(\mathfrak{H}, c)$ if and only if the following two conditions are satisfied:*

1. *$L \in \mathfrak{O}\mathfrak{T}(\mathfrak{H}, c)$;*
2. *$L$ is v-determined and $\|\mathcal{V}(L)\| < c$.*

*Proof.* 1) Let $L \in \mathfrak{O}(\mathfrak{H}, c)$. Then, according to (II.33), $L \in \mathfrak{O}\mathfrak{T}_{\mathbf{fin}}(\mathfrak{H}, c)$. And, according to denotation II.17.3, $L \in \mathfrak{O}\mathfrak{T}(\mathfrak{H}, c)$. Moreover, by Assertion II.17.2, $L$ is v-determined and $\|\mathcal{V}(L)\| < c$.

2) Inversely, suppose, that $L \in \mathfrak{O}\mathfrak{T}(\mathfrak{H}, c)$ and $L$ is v-determined with

$$\|\mathcal{V}(L)\| < c. \tag{II.90}$$

Then, in accordance with Remark II.17.1, $L \notin \mathfrak{O}\mathfrak{T}_{\infty}(\mathfrak{H}, c)$. Hence, $L \in \mathfrak{O}\mathfrak{T}_{\mathbf{fin}}(\mathfrak{H}, c)$. So, according to Theorem II.17.1, there exist the number $s \in \{-1, 1\}$ vector $V \in \mathfrak{H}_1$ and operator $J \in \mathfrak{U}(\mathfrak{H}_1)$ such, that for any vector $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ the action of the operator $L$ in regard to the vector $\mathrm{w}$ can be represented in the form (II.37), where $V = \mathcal{V}(L)$, and, according to (II.90), $\|V\| < c$. Since $\|V\| < c$, then, by the formula (II.37) for any vector $\mathrm{w} = t\mathbf{e}_0 + \mu V = t\mathbf{e}_0 + \mu\mathcal{V}(L) \in \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(L)]$ we obtain:

$$L\mathrm{w} = \frac{s}{\sqrt{1-\frac{\|V\|^2}{c^2}}}\left(\left(t - \mu\frac{\|V\|^2}{c^2}\right)\mathbf{e}_0 + (t-\mu)JV\right).$$

Because $J$ is unitary operator, we have $\|JV\| = \|V\|$. Hence:

$$\begin{aligned}\mathsf{M}_c(L\mathrm{w}) &= \|\mathbf{X}L\mathrm{w}\|^2 - c^2\mathcal{T}^2(L\mathrm{w}) = \\ &= \frac{1}{1-\frac{\|V\|^2}{c^2}}\left((t-\mu)^2\|V\|^2 - c^2\left(t-\mu\frac{\|V\|^2}{c^2}\right)^2\right) = \\ &= \mu^2\|V\|^2 - c^2t^2 = \|\mathbf{X}\mathrm{w}\|^2 - c^2\mathcal{T}^2(\mathrm{w}) = \mathsf{M}_c(\mathrm{w}),\end{aligned}$$

where w is arbitrary vector from the subspace $\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(L)]$. And, since $L \in \mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c)$, according to Assertion II.17.3, and Denotation II.17.2, we have $L \in \mathfrak{O}(\mathfrak{H}, c)$. □

Applying Lemma II.17.5 and equality (II.88) we obtain the following equality:

$$\begin{aligned}\mathfrak{O}(\mathfrak{H}, c) &= \{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J] \in \mathfrak{OT}(\mathfrak{H}, c) \mid 0 \le \lambda < c\} = \\ &= \{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J] \mid s \in \{-1,1\},\ \lambda \in [0,c),\ \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\ J \in \mathfrak{U}(\mathfrak{H}_1)\},\end{aligned} \tag{II.91}$$

which gives the representation of the class of operators $\mathfrak{O}(\mathfrak{H}, c)$. Using Lemma II.17.5 and Theorem II.17.3 we obtain the following equivalent representation of $\mathfrak{O}(\mathfrak{H}, c)$:

$$\begin{aligned}\mathfrak{O}(\mathfrak{H}, c) &= \{\mathbf{U}_{\theta,c}[s,\mathbf{n},J] \in \mathfrak{OT}(\mathfrak{H}, c) \mid 0 < \theta \le 1\} = \\ &= \{\mathbf{U}_{\theta,c}[s,\mathbf{n},J] \mid s \in \{-1,1\},\ \theta \in (0,1],\ \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\ J \in \mathfrak{U}(\mathfrak{H}_1)\}.\end{aligned} \tag{II.92}$$

And Lemma II.17.5 together with equality (II.89) gives the following representation of $\mathfrak{O}(\mathfrak{H}, c)$:

$$\begin{aligned}\mathfrak{O}(\mathfrak{H}, c) &= \{\mathbf{U}_{\theta,c}[s,\mathbf{n},J] \in \mathfrak{OT}(\mathfrak{H}, c) \mid 0 < \theta < \infty\} = \\ &= \{\mathbf{U}_{\theta,c}[s,\mathbf{n},J] \mid s \in \{-1,1\},\ \theta \in (0,\infty),\ \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\ J \in \mathfrak{U}(\mathfrak{H}_1)\}.\end{aligned} \tag{II.93}$$

According to Denotation II.17.3, we have $\mathfrak{OT}(\mathfrak{H}, c) := \mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c) \cup \mathfrak{OT}_{\infty}(\mathfrak{H}, c)$. Union in the last equality is disjoint, since, by Denotation II.17.2, any linear coordinate transform operator $L \in \mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c)$ is v-determined, while any linear coordinate transform operator $L_1 \in \mathfrak{OT}_{\infty}(\mathfrak{H}, c)$ must be not v-determined (in accordance with Remark II.17.1). So:

$$\mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c) \cap \mathfrak{OT}_{\infty}(\mathfrak{H}, c) = \emptyset. \tag{II.94}$$

Hence, using Assertion II.17.5 and Theorem II.17.3, we obtain the following equalities:

$$\begin{aligned}\mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c) &= \{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J] \in \mathfrak{OT}(\mathfrak{H}, c) :\ \lambda < \infty\} = \\ &= \{\mathbf{U}_{\theta,c}[s,\mathbf{n},J] \in \mathfrak{OT}(\mathfrak{H}, c) \mid \theta \ne -1\};\end{aligned} \tag{II.95}$$

$$\begin{aligned}\mathfrak{OT}_{\infty}(\mathfrak{H}, c) &= \{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J] \in \mathfrak{OT}(\mathfrak{H}, c) :\ \lambda = \infty\} = \\ &= \{\mathbf{U}_{\theta,c}[s,\mathbf{n},J] \in \mathfrak{OT}(\mathfrak{H}, c) \mid \theta = -1\}.\end{aligned} \tag{II.96}$$

For the case $\mathfrak{H} = \mathbb{R}^3$ in the paper [45] apart from General Lorentz Group, it is introduced the full Lorentz group. According to [45], full Lorentz group is a subgroup of the general General Lorentz Group, which consists of general Lorentz transforms with positive direction of time (that is such Lorentz transforms, which leave invariant the class of positive time-like vectors). By analogy with [45], we can introduce the full Lorentz group in the general situation of real Hilbert space.

**Definition II.17.3.** *Let $\mathfrak{H}$ be a real Hilbert space. Vector* $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ *we name by:*

- ***positive****, if and only if* $\mathcal{T}(\mathrm{w}) > 0$*;*
- ***c-timelike****, if and only if* $\mathsf{M}_c(\mathrm{w}) < 0$*.*

Denote by $\mathcal{M}_{c,+}(\mathfrak{H})$ the set of all positive $c$-timelike vectors of the space $\mathcal{M}(\mathfrak{H})$:

$$\mathcal{M}_{c,+}(\mathfrak{H}) := \{\mathrm{w} \in \mathcal{M}(\mathfrak{H}) \mid \mathcal{T}(\mathrm{w}) > 0,\ \mathsf{M}_c(\mathrm{w}) < 0\}. \tag{II.97}$$

Introduce the following class of operators:

$$\mathfrak{O}_+(\mathfrak{H}, c) = \{L \in \mathfrak{O}(\mathfrak{H}, c) \mid L\mathrm{w} \in \mathcal{M}_{c,+}(\mathfrak{H})\ (\forall \mathrm{w} \in \mathcal{M}_{c,+}(\mathfrak{H}))\}. \tag{II.98}$$

**Assertion II.17.6.** $\mathfrak{O}_+(\mathfrak{H}, c)$ *is a group of operators over the Minkowski space* $\mathcal{M}(\mathfrak{H})$ *over the Hilbert space* $\mathfrak{H}$.

*Proof.* **1**. Let $L_1, L_2 \in \mathfrak{O}_+(\mathfrak{H}, c)$ and $L = L_1 L_2$. Then, according to (II.98) and Assertion II.17.1, $L \in \mathfrak{O}(\mathfrak{H}, c)$ and $L\mathrm{w} \in \mathcal{M}_{c,+}(\mathfrak{H})$ $(\forall \mathrm{w} \in \mathcal{M}_{c,+}(\mathfrak{H}))$. So, by (II.98), $L \in \mathfrak{O}_+(\mathfrak{H}, c)$.

**2**. Suppose, that $L \in \mathfrak{O}_+(\mathfrak{H}, c)$. Since (by Assertion II.17.1) $\mathfrak{O}(\mathfrak{H}, c)$ is the group of operators over the space $\mathcal{M}(\mathfrak{H})$ and $\mathfrak{O}_+(\mathfrak{H}, c) \subseteq \mathfrak{O}(\mathfrak{H}, c)$, we have $L^{-1} \in \mathfrak{O}(\mathfrak{H}, c)$. Consider any vector $\mathrm{w} \in \mathcal{M}_{c,+}(\mathfrak{H})$. By definition of $\mathcal{M}_{c,+}(\mathfrak{H})$ (see (II.97)), we have:

$$\mathcal{T}(\mathrm{w}) > 0,\ \mathsf{M}_c(\mathrm{w}) < 0.$$

Since $\mathsf{M}_c(\mathrm{w}) < 0$ and $L^{-1} \in \mathfrak{O}(\mathfrak{H}, c)$, then according to Denotation II.17.1, we have $\mathsf{M}_c(L^{-1}\mathrm{w}) < 0$. Hence:

$$\mathcal{T}\left(L^{-1}\mathrm{w}\right) = \pm c^{-1}\sqrt{-\mathsf{M}_c(L^{-1}\mathrm{w}) + \|\mathbf{X}L^{-1}\mathrm{w}\|^2} \neq 0.$$

So, one and only one of the inequalities $\mathcal{T}(L^{-1}\mathrm{w}) > 0$ or $\mathcal{T}(L^{-1}\mathrm{w}) < 0$ is performed. Assume, that $\mathcal{T}(L^{-1}\mathrm{w}) < 0$. Then the vector $\widetilde{\mathrm{w}} = -L^{-1}\mathrm{w}$ will belong to $\mathcal{M}_{c,+}(\mathfrak{H})$, while $\mathcal{T}(L\widetilde{\mathrm{w}}) = \mathcal{T}(-\mathrm{w}) = -\mathcal{T}(\mathrm{w}) < 0$. Thus $\widetilde{\mathrm{w}} \in \mathcal{M}_{c,+}(\mathfrak{H})$, while $L\widetilde{\mathrm{w}} \notin \mathcal{M}_{c,+}(\mathfrak{H})$, which is impossible, because $L \in \mathfrak{O}_+(\mathfrak{H}, c)$. This contradiction proves, that $\mathcal{T}(L^{-1}\mathrm{w}) > 0$. Hence, we have proved, that $\mathsf{M}_c(L^{-1}\mathrm{w}) < 0$ and $\mathcal{T}(L^{-1}\mathrm{w}) > 0$. Thus, by (II.97), $L^{-1}\mathrm{w} \in \mathcal{M}_{c,+}(\mathfrak{H})$ (for any vector $\mathrm{w} \in \mathcal{M}_{c,+}(\mathfrak{H})$). So, according to (II.98), $L^{-1} \in \mathfrak{O}_+(\mathfrak{H}, c)$ (for any operator $L \in \mathfrak{O}_+(\mathfrak{H}, c)$).

Thus, we have proved, that $L_1 L_2 \in \mathfrak{O}_+(\mathfrak{H}, c)$ and $L^{-1} \in \mathfrak{O}_+(\mathfrak{H}, c)$ (for any $L, L_1, L_2 \in \mathfrak{O}_+(\mathfrak{H}, c)$), what was needed to prove. □

It is not hard to verify, that in the case $\mathfrak{H} = \mathbb{R}^3$ group $\mathfrak{O}_+(\mathfrak{H}, c)$ coincides with the full Lorentz group, introduced in [45].

The next assertion gives the representation of the class $\mathfrak{O}_+(\mathfrak{H}, c)$ in the terms of operators of kind $\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J]$.

**Assertion II.17.7.** *The following equality is true:*

$$\begin{aligned} \mathfrak{O}_+(\mathfrak{H}, c) &= \{\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J] \in \mathfrak{O}(\mathfrak{H}, c) \mid s = 1\} = \\ &= \{\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J] \in \mathfrak{OT}(\mathfrak{H}, c) \mid s = 1,\ 0 \leq \lambda < c\} = \\ &= \{\mathbf{W}_{\lambda,c}[1, \mathbf{n}, J] \mid \lambda \in [0, c),\ \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\ J \in \mathfrak{U}(\mathfrak{H}_1)\}. \end{aligned} \tag{II.99}$$

*Proof.* It is sufficient to prove only the equality:

$$\mathfrak{O}_+(\mathfrak{H}, c) = \{\mathbf{W}_{\lambda,c}[1, \mathbf{n}, J] \mid \lambda \in [0, c),\ \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\ J \in \mathfrak{U}(\mathfrak{H}_1)\}, \tag{II.100}$$

because other parts of the equality (II.99) follow from equality (II.100) and equalities (II.88), (II.91).

1. Suppose, that operator $L$ can be represented by the form:

$$L = \mathbf{W}_{\lambda,c}[1, \mathbf{n}, J],$$

where $\lambda \in [0, c)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$, $J \in \mathfrak{U}(\mathfrak{H}_1)$. Chose any vector $\mathrm{w} \in \mathcal{M}_{c,+}(\mathfrak{H})$. Then, according to (II.97), we have:

$$\mathcal{T}(\mathrm{w}) > 0, \quad \mathsf{M}_c(\mathrm{w}) < 0. \tag{II.101}$$

Since $\lambda \in [0, c)$, then, according to (II.91), $L = \mathbf{W}_{\lambda,c}[1, \mathbf{n}, J] \in \mathfrak{O}(\mathfrak{H}, c)$. So, using (II.101) in accordance with Denotation II.17.1, we obtain:

$$\mathsf{M}_c(L\mathrm{w}) = \mathsf{M}_c(\mathrm{w}) < 0. \tag{II.102}$$

Next, applying (II.101) and (II.76), we get:

$$\begin{aligned}
\mathcal{T}(L\mathrm{w}) &= \mathcal{T}(\mathbf{W}_{\lambda,c}[1, \mathbf{n}, J]\,\mathrm{w}) = \frac{\mathcal{T}(\mathrm{w}) - \frac{\lambda}{c^2}\langle \mathbf{n}, \mathrm{w}\rangle}{\sqrt{\left|1 - \frac{\lambda^2}{c^2}\right|}} = \\
&= \frac{\mathcal{T}(\mathrm{w})}{\sqrt{\left|1 - \frac{\lambda^2}{c^2}\right|}} - \frac{\frac{\lambda}{c^2}}{\sqrt{\left|1 - \frac{\lambda^2}{c^2}\right|}}\langle \mathbf{X}\mathbf{n}, \mathrm{w}\rangle = \frac{\mathcal{T}(\mathrm{w})}{\sqrt{\left|1 - \frac{\lambda^2}{c^2}\right|}} - \frac{\frac{\lambda}{c^2}}{\sqrt{\left|1 - \frac{\lambda^2}{c^2}\right|}}\langle \mathbf{n}, \mathbf{X}\mathrm{w}\rangle \geq \\
&\geq \frac{\mathcal{T}(\mathrm{w})}{\sqrt{\left|1 - \frac{\lambda^2}{c^2}\right|}} - \frac{\frac{\lambda}{c^2}}{\sqrt{\left|1 - \frac{\lambda^2}{c^2}\right|}}\|\mathbf{X}\mathrm{w}\| = \frac{\left(1 - \frac{\lambda}{c}\right)\mathcal{T}(\mathrm{w})}{\sqrt{\left|1 - \frac{\lambda^2}{c^2}\right|}} - \frac{\frac{\lambda}{c^2}}{\sqrt{\left|1 - \frac{\lambda^2}{c^2}\right|}}\left(\|\mathbf{X}\mathrm{w}\| - c\mathcal{T}(\mathrm{w})\right) = \\
&= \frac{\left(1 - \frac{\lambda}{c}\right)\mathcal{T}(\mathrm{w})}{\sqrt{\left|1 - \frac{\lambda^2}{c^2}\right|}} - \frac{\frac{\lambda}{c^2}}{\sqrt{\left|1 - \frac{\lambda^2}{c^2}\right|}}\frac{\mathsf{M}_c(\mathrm{w})}{\|\mathbf{X}\mathrm{w}\| + c\mathcal{T}(\mathrm{w})} > 0.
\end{aligned} \tag{II.103}$$

From (II.102) and (II.103) it follows that $L\mathrm{w} \in \mathcal{M}_{c,+}(\mathfrak{H})$ (for any vector $\mathrm{w} \in \mathcal{M}_{c,+}(\mathfrak{H})$). Therefore, according to (II.98), we obtain $L \in \mathfrak{O}_+(\mathfrak{H}, c)$.

2. Inversely, assume, that $L \in \mathfrak{O}_+(\mathfrak{H}, c)$. Then, in accordance with (II.98), $L \in \mathfrak{O}(\mathfrak{H}, c)$ and:

$$\forall \mathrm{w} \in \mathcal{M}_{c,+}(\mathfrak{H})\ \left(L\mathrm{w} \in \mathcal{M}_{c,+}(\mathfrak{H})\right). \tag{II.104}$$

Since $L \in \mathfrak{O}(\mathfrak{H}, c)$, then, by (II.91), operator $L$ can be represented in the form:

$$L = \mathbf{W}_{\lambda,c}[s, \mathbf{n}, J], \tag{II.105}$$

where $s \in \{-1, 1\}$, $\lambda \in [0, c)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ and $J \in \mathfrak{U}(\mathfrak{H}_1)$. It is easy to see, that $\mathbf{e}_0 \in \mathcal{M}_{c,+}(\mathfrak{H})$. Hence, According to (II.104), $L\mathbf{e}_0 \in \mathcal{M}_{c,+}(\mathfrak{H})$. Therefore, by (II.97), $\mathcal{T}(L\mathbf{e}_0) > 0$. So, in accordance with (II.76), we obtain:

$$\begin{aligned}
0 &< \mathcal{T}(L\mathbf{e}_0) = \mathcal{T}(\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J]\,\mathbf{e}_0) = \\
&= \mathcal{T}\left(\frac{\left(s\mathcal{T}(\mathbf{e}_0) - \frac{\lambda}{c^2}\langle \mathbf{n}, \mathbf{e}_0\rangle\right)}{\sqrt{\left|1 - \frac{\lambda^2}{c^2}\right|}}\mathbf{e}_0 + J\left(\frac{\lambda\mathcal{T}(\mathbf{e}_0)\,\mathbf{n} - s\mathbf{X}_1[\mathbf{n}]\,\mathbf{e}_0}{\sqrt{\left|1 - \frac{\lambda^2}{c^2}\right|}} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathbf{e}_0\right)\right) = \\
&= \mathcal{T}\left(\frac{s}{\sqrt{\left|1 - \frac{\lambda^2}{c^2}\right|}}\mathbf{e}_0 + J\left(\frac{\lambda\mathbf{n}}{\sqrt{\left|1 - \frac{\lambda^2}{c^2}\right|}}\right)\right) = \frac{s}{\sqrt{\left|1 - \frac{\lambda^2}{c^2}\right|}}.
\end{aligned}$$

Last inequality proves, that $s > 0$. So, since $s \in \{-1, 1\}$, we conclude, that $s = 1$. And, according to (II.105), $L = \mathbf{W}_{\lambda,c}[1, \mathbf{n}, J]$ (where $\lambda \in [0, c)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ and $J \in \mathfrak{U}(\mathfrak{H}_1)$).

Thus equality (II.100) is completely proved. □

Let $\lambda \in [0, c)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$, and $J \in \mathfrak{U}(\mathfrak{H}_1)$. Then, according to (II.83), operator $\mathbf{W}_{\lambda,c}[1, \mathbf{n}, J]$ can be represented in the form:

$$\mathbf{W}_{\lambda,c}[1, \mathbf{n}, J] = \mathbf{U}_{\theta'_{(\lambda)},c}[1, \mathbf{n}, J], \quad \text{where} \quad \theta'_{(\lambda)} = \frac{1 - \frac{\lambda}{c}}{\sqrt{1 - \frac{\lambda^2}{c^2}}}, \quad \theta'_{(\lambda)} \in (0, 1].$$

Inversely, any operator of kind $\mathbf{U}_{\theta,c}[1, \mathbf{n}, J]$, where $\theta \in (0, 1]$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$, and $J \in \mathfrak{U}(\mathfrak{H}_1)$, according to (II.83), may be represented in the form:

$$\mathbf{U}_{\theta,c}[1, \mathbf{n}, J] = \mathbf{W}_{\lambda'_{(\theta)},c}[1, \mathbf{n}, J], \quad \text{where} \quad \lambda'_{(\theta)} = c\frac{1 - \theta^2}{1 + \theta^2}, \quad \lambda'_{(\theta)} \in [0, c).$$

Hence, the following equality is true:

$$\{\mathbf{W}_{\lambda,c}[1,\mathbf{n},J] \mid \lambda\in[0,c),\ \mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ J\in\mathfrak{U}(\mathfrak{H}_1)\} = \\ = \{\mathbf{U}_{\theta,c}[1,\mathbf{n},J] \mid \theta\in(0,1],\ \mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ J\in\mathfrak{U}(\mathfrak{H}_1)\}.$$

Using the last equality together with equalities (II.99) and (II.92), we obtain the following representation of the class $\mathfrak{O}_+(\mathfrak{H},c)$ in the terms of operators of kind $\mathbf{U}_{\theta,c}[s,\mathbf{n},J]$:

$$\begin{aligned}\mathfrak{O}_+(\mathfrak{H},c) &= \{\mathbf{U}_{\theta,c}[1,\mathbf{n},J] \mid \theta\in(0,1],\ \mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ J\in\mathfrak{U}(\mathfrak{H}_1)\} = \\ &= \{\mathbf{U}_{\theta,c}[s,\mathbf{n},J]\in\mathfrak{OT}(\mathfrak{H},c) \mid s=1,\ 0<\theta\le 1\} = \\ &= \{\mathbf{U}_{\theta,c}[s,\mathbf{n},J]\in\mathfrak{O}(\mathfrak{H},c) \mid s=1\}. \end{aligned}\tag{II.106}$$

From the equality (II.87) it follows, that $\mathbf{U}_{\theta,c}[1,\mathbf{n},J]=\mathbf{U}_{\theta^{-1},c}[1,-\mathbf{n},J]$ for any $\theta\in(0,\infty)$, $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$, and $J\in\mathfrak{U}(\mathfrak{H}_1)$. So, we can replace condition $\theta\in(0,1]$ in the formula (II.106) by the condition $\theta\in(0,\infty)$. Hence, we obtain the following equality:

$$\begin{aligned}\mathfrak{O}_+(\mathfrak{H},c) &= \{\mathbf{U}_{\theta,c}[1,\mathbf{n},J] \mid \theta\in(0,\infty),\ \mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ J\in\mathfrak{U}(\mathfrak{H}_1)\} = \\ &= \{\mathbf{U}_{\theta,c}[s,\mathbf{n},J]\in\mathfrak{OT}(\mathfrak{H},c) \mid s=1,\ \theta>0\}. \end{aligned}\tag{II.107}$$

Formula (II.99) serves as motivation for introduction of the following subclass of operators from $\mathfrak{OT}(\mathfrak{H},c)$:

$$\begin{aligned}\mathfrak{OT}_+(\mathfrak{H},c) &:= \{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]\in\mathfrak{OT}(\mathfrak{H},c) \mid s=1\} = \\ &= \{\mathbf{W}_{\lambda,c}[1,\mathbf{n},J] \mid \lambda\in[0,\infty]\setminus\{c\},\ \mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ J\in\mathfrak{U}(\mathfrak{H}_1)\}. \end{aligned}\tag{II.108}$$

Using (II.83), (II.84) and (II.87), we can obtain the following representation of the class $\mathfrak{OT}_+(\mathfrak{H},c)$ in the terms of operators of kind $\mathbf{U}_{\theta,c}[s,\mathbf{n},J]$:

$$\begin{aligned}\mathfrak{OT}_+(\mathfrak{H},c) &:= \{\mathbf{U}_{\theta,c}[s,\mathbf{n},J]\in\mathfrak{OT}(\mathfrak{H},c) \mid s=1,\ |\theta|\le 1\} = \\ &= \{\mathbf{U}_{\theta,c}[1,\mathbf{n},J] \mid \theta\in[-1,1]\setminus\{0\},\ \mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ J\in\mathfrak{U}(\mathfrak{H}_1)\} = \\ &= \left\{\mathbf{U}_{\theta,c}\left[\mathsf{sgn}_+(\theta+1),\mathbf{n},J\right] \mid \theta\in\mathbb{R}\setminus\{0\},\ \mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ J\in\mathfrak{U}(\mathfrak{H}_1)\right\}, \end{aligned}\tag{II.109}$$

where $\mathsf{sgn}_+(\xi)=\begin{cases}\operatorname{sign}(\xi), & \xi\ne 0\\ 1, & \xi=0\end{cases}$ $(\xi\in\mathbb{R})$.

As a contraposition to the class $\mathfrak{OT}_+(\mathfrak{H},c)$, we may introduse the following class:

$$\begin{aligned}\mathfrak{OT}_-(\mathfrak{H},c) &:= \{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]\in\mathfrak{OT}(\mathfrak{H},c) \mid s=-1\} = \\ &= \{\mathbf{W}_{\lambda,c}[-1,\mathbf{n},J] \mid \lambda\in[0,\infty]\setminus\{c\},\ \mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ J\in\mathfrak{U}(\mathfrak{H}_1)\}. \end{aligned}\tag{II.110}$$

Using (II.83), (II.84) and (II.87), we can obtain the following representation of the class $\mathfrak{OT}_-(\mathfrak{H},c)$ in the terms of operators of kind $\mathbf{U}_{\theta,c}[s,\mathbf{n},J]$:

$$\begin{aligned}\mathfrak{OT}_-(\mathfrak{H},c) &:= \{\mathbf{U}_{\theta,c}[s,\mathbf{n},J]\in\mathfrak{OT}(\mathfrak{H},c) \mid s=-1,\ |\theta|\le 1\} = \\ &= \{\mathbf{U}_{\theta,c}[-1,\mathbf{n},J] \mid \theta\in[-1,1]\setminus\{0\},\ \mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ J\in\mathfrak{U}(\mathfrak{H}_1)\} = \\ &= \left\{\mathbf{U}_{\theta,c}\left[-\mathsf{sgn}_+(\theta+1),\mathbf{n},J\right] \mid \theta\in\mathbb{R}\setminus\{0\},\ \mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ J\in\mathfrak{U}(\mathfrak{H}_1)\right\}. \end{aligned}\tag{II.111}$$

**Assertion II.17.8.** *The following equality is true:*

$$\mathfrak{OT}_+(\mathfrak{H},c)\cap\mathfrak{OT}_-(\mathfrak{H},c)=\mathfrak{OT}_\infty(\mathfrak{H},c).$$

*Proof.* Let $L\in\mathfrak{OT}_+(\mathfrak{H},c)\cap\mathfrak{OT}_-(\mathfrak{H},c)$. Then, according to (II.109) and (II.111), operator $L$ may be represented in the form $L=\mathbf{U}_{\theta,c}[1,\mathbf{n},J]=\mathbf{U}_{\theta_1,c}[-1,\mathbf{n}_1,J_1]$, where $\theta,\theta_1\in[-1,1]\setminus\{0\}$, $\mathbf{n},\mathbf{n}_1\in\mathbf{B}_1(\mathfrak{H}_1)$, $J,J_1\in\mathfrak{U}(\mathfrak{H}_1)$. Therefore, according to (II.86), we have:

$$L\mathbf{e}_0=\left(\varphi_0(\theta)\,\mathcal{T}(\mathbf{e}_0)-\varphi_1(\theta)\,\frac{\langle\mathbf{n},\mathbf{e}_0\rangle}{c}\right)\mathbf{e}_0$$

$$
\begin{aligned}
&+ J\left(c\varphi_1(\theta)\mathcal{T}(\mathbf{e}_0)\mathbf{n} - \varphi_0(\theta)\mathbf{X}_1[\mathbf{n}]\mathbf{e}_0 + \mathbf{X}_1^{\perp}[\mathbf{n}]\mathbf{e}_0\right) = \\
&= \varphi_0(\theta)\mathbf{e}_0 + c\varphi_1(\theta)J(\mathbf{n});\\
L\mathbf{e}_0 = &\left(-\varphi_0(\theta_1)\mathcal{T}(\mathbf{e}_0) - \varphi_1(\theta_1)\frac{\langle\mathbf{n}_1,\mathbf{e}_0\rangle}{c}\right)\mathbf{e}_0 + \\
&+ J\left(c\varphi_1(\theta_1)\mathcal{T}(\mathbf{e}_0)\mathbf{n}_1 + \varphi_0(\theta_1)\mathbf{X}_1[\mathbf{n}_1]\mathbf{e}_0 + \mathbf{X}_1^{\perp}[\mathbf{n}_1]\mathbf{e}_0\right) = \\
&= -\varphi_0(\theta_1)\mathbf{e}_0 + c\varphi_1(\theta_1)J(\mathbf{n}_1).
\end{aligned}
$$

From the last two equalities it follows, that:

$$\varphi_0(\theta) = \mathcal{T}(L\mathbf{e}_0) = -\varphi_0(\theta_1).$$

And, since $\varphi_0(\vartheta) \geq 0$ $(\forall\,\vartheta \in [-1,1]\setminus\{0\})$, we obtain $\varphi_0(\theta) = \varphi_0(\theta_1) = 0$. The last equality is possible only if $\theta = -1$. So, according to (II.96), we have:

$$L = \mathbf{U}_{\theta,c}[1,\mathbf{n},J] = \mathbf{U}_{-1,c}[1,\mathbf{n},J] \in \mathfrak{OT}_{\infty}(\mathfrak{H},c).$$

Thus, $\mathfrak{OT}_{+}(\mathfrak{H},c) \cap \mathfrak{OT}_{-}(\mathfrak{H},c) \subseteq \mathfrak{OT}_{\infty}(\mathfrak{H},c)$.
From the other hand, if $L \in \mathfrak{OT}_{\infty}(\mathfrak{H},c)$, then, according to (II.96) and (II.89), $L$ can be represented in the form $L = \mathbf{U}_{-1,c}[s,\mathbf{n},J]$, where $s \in \{-1,1\}$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$, $J \in \mathfrak{U}(\mathfrak{H}_1)$. And, according to (II.86), $L = \mathbf{U}_{-1,c}[s,\mathbf{n},J] = \mathbf{U}_{-1,c}[1,\mathbf{n},J] = \mathbf{U}_{-1,c}[-1,\mathbf{n},J]$. Thus, in accordance with (II.109) and (II.111), we have, $L \in \mathfrak{OT}_{+}(\mathfrak{H},c) \cap \mathfrak{OT}_{-}(\mathfrak{H},c)$. Hence, we obtain the inverse inclusion $\mathfrak{OT}_{\infty}(\mathfrak{H},c) \subseteq \mathfrak{OT}_{+}(\mathfrak{H},c) \cap \mathfrak{OT}_{-}(\mathfrak{H},c)$. □

Main results of this Section were published in [6, 8, 13].

## 18 Algebraic Properties of Tachyon Lorentz Transforms

The aim of this section is to investigate some algebraic properties of introduced in previous section classes of generalized Lorentz transforms $\mathfrak{OT}_{+}(\mathfrak{H},c)$ and $\mathfrak{OT}(\mathfrak{H},c)$ over real Hilbert space $\mathfrak{H}$. Namely, we investigate the group properties of these classes.

Let us introduce the denotation:

$$\mathbf{E}_{\theta,c}[s,\mathbf{n}] := \mathbf{U}_{\theta,c}[s,\mathbf{n},\mathbb{I}_1] \qquad (\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\ \theta \in \mathbb{R}\setminus\{0\},\quad s \in \{-1,1\}), \tag{II.112}$$

where $\mathbb{I}_1 := \mathbb{I}_{\mathfrak{H}_1}$ is the identity operator on the space $\mathfrak{H}_1$. The operators of kind $\mathbf{E}_{\theta,c}[s,\mathbf{n}]$ will be named by ***elementary*** generalized Lorentz transforms.

Let $\mathfrak{H}$ be a real Hilbert space, and let $\mathfrak{H}_1$ be introduced in (II.13) subspace of the Minkowski space $\mathcal{M}(\mathfrak{H})$, isomorphic to $\mathfrak{H}$. Recall, that in (II.35) we had introduced the unitary operator $\widetilde{J} \in \mathfrak{U}(\mathcal{M}(\mathfrak{H}))$ for any unitary on subspace $\mathfrak{H}_1$ operator $J \in \mathfrak{U}(\mathfrak{H}_1)$:

$$\widetilde{J}\mathrm{w} = \widehat{\mathbf{T}}\mathrm{w} + J\mathbf{X}\mathrm{w} = \mathcal{T}(\mathrm{w})\mathbf{e}_0 + J\mathbf{X}\mathrm{w}, \quad \mathrm{w} \in \mathcal{M}(\mathfrak{H}) \tag{II.35: dubl}$$

It is easy to see, that for any operators $J, J_1 \in \mathfrak{U}(\mathfrak{H}_1)$ the following equalities are performed:

$$\widetilde{JJ_1} = \widetilde{J}\,\widetilde{J}_1; \qquad \widetilde{J}^{-1} = \widetilde{(J^{-1})}. \tag{II.113}$$

Recall that, according to (II.89), any generalized Lorentz transform $L \in \mathfrak{OT}(\mathfrak{H},c)$ can be represented in the form $L = \mathbf{U}_{\theta,c}[s,\mathbf{n},J]$, where $s \in \{-1,1\}$, $\theta \in \mathbb{R}\setminus\{0\}$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$, $J \in \mathfrak{U}(\mathfrak{H}_1)$.

**Lemma II.18.1.** *For arbitrary generalized Lorentz transform $\mathbf{U}_{\theta,c}[s,\mathbf{n},J] \in \mathfrak{OT}(\mathfrak{H},c)$ ($s \in \{-1,1\}$, $\theta \in \mathbb{R}\setminus\{0\}$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$, $J \in \mathfrak{U}(\mathfrak{H}_1)$) the following equalities are true:*

$$\widetilde{J}\mathbf{E}_{\theta,c}[s,\mathbf{n}] = \mathbf{U}_{\theta,c}[s,\mathbf{n},J]; \quad \mathbf{E}_{\theta,c}[s,\mathbf{n}]\widetilde{J} = \mathbf{U}_{\theta,c}\left[s,J^{-1}\mathbf{n},J\right]. \tag{II.114}$$

*Proof.* The first equality (II.114) follows from (II.86), (II.112) and (II.35). Hence, we are going to prove the second one. For any, $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ we put:

$$\mathrm{w}' := \widetilde{J}\mathrm{w} = \mathcal{T}(\mathrm{w})\,\mathbf{e}_0 + J\mathbf{X}\mathrm{w}.$$

Applying (II.86) and (II.112) we obtain:

$$\begin{aligned}
\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\widetilde{J}\mathrm{w} = \mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathrm{w}' &= \left(s\varphi_0(\theta)\,\mathcal{T}(\mathrm{w}') - \varphi_1(\theta)\,\frac{\langle \mathbf{n}, \mathrm{w}'\rangle}{c}\right)\mathbf{e}_0 + \\
&+ c\varphi_1(\theta)\,\mathcal{T}(\mathrm{w}')\,\mathbf{n} - s\varphi_0(\theta)\,\mathbf{X}_1[\mathbf{n}]\,\mathrm{w}' + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}' = \\
&= \left(s\varphi_0(\theta)\,\mathcal{T}(\mathcal{T}(\mathrm{w})\,\mathbf{e}_0 + J\mathbf{X}\mathrm{w}) - \varphi_1(\theta)\,\frac{\langle \mathbf{n}, \mathcal{T}(\mathrm{w})\,\mathbf{e}_0 + J\mathbf{X}\mathrm{w}\rangle}{c}\right)\mathbf{e}_0 + \\
&+ c\varphi_1(\theta)\,\mathcal{T}(\mathcal{T}(\mathrm{w})\,\mathbf{e}_0 + J\mathbf{X}\mathrm{w})\,\mathbf{n} - s\varphi_0(\theta)\,\mathbf{X}_1[\mathbf{n}]\,(\mathcal{T}(\mathrm{w})\,\mathbf{e}_0 + J\mathbf{X}\mathrm{w}) + \\
&+ \mathbf{X}_1^{\perp}[\mathbf{n}]\,(\mathcal{T}(\mathrm{w})\,\mathbf{e}_0 + J\mathbf{X}\mathrm{w}) = \\
&= \left(s\varphi_0(\theta)\,\mathcal{T}(\mathrm{w}) - \varphi_1(\theta)\,\frac{\langle \mathbf{n}, J\mathbf{X}\mathrm{w}\rangle}{c}\right)\mathbf{e}_0 + \\
&+ c\varphi_1(\theta)\,\mathcal{T}(\mathrm{w})\,\mathbf{n} - s\varphi_0(\theta)\,\mathbf{X}_1[\mathbf{n}]\,J\mathbf{X}\mathrm{w} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,J\mathbf{X}\mathrm{w}.
\end{aligned} \tag{II.115}$$

Since $J$ is unitary operator, mapping $\mathfrak{H}_1$ into $\mathfrak{H}_1$, we get:

$$\langle \mathbf{n}, J\mathbf{X}\mathrm{w}\rangle = \left\langle J^{-1}\mathbf{n}, \mathbf{X}\mathrm{w}\right\rangle = \left\langle \mathbf{X}J^{-1}\mathbf{n}, \mathrm{w}\right\rangle = \left\langle J^{-1}\mathbf{n}, \mathrm{w}\right\rangle. \tag{II.116}$$

Further, using (II.17),(II.18), (II.116), we deliver:

$$\begin{aligned}
\mathbf{X}_1[\mathbf{n}]\,J\mathbf{X}\mathrm{w} &= \langle \mathbf{n}, J\mathbf{X}\mathrm{w}\rangle\,\mathbf{n}, = \left\langle J^{-1}\mathbf{n}, \mathrm{w}\right\rangle\mathbf{n} = \\
&= J\left\langle J^{-1}\mathbf{n}, \mathrm{w}\right\rangle J^{-1}\mathbf{n} = J\mathbf{X}_1\left[J^{-1}\mathbf{n}\right]\mathrm{w};
\end{aligned} \tag{II.117}$$

$$\begin{aligned}
\mathbf{X}_1^{\perp}[\mathbf{n}]\,J\mathbf{X}\mathrm{w} &= (\mathbf{X} - \mathbf{X}_1[\mathbf{n}])\,J\mathbf{X}\mathrm{w} = \mathbf{X}J\mathbf{X}\mathrm{w} - J\mathbf{X}_1\left[J^{-1}\mathbf{n}\right]\mathrm{w} = \\
&= J\left(\mathbf{X}\mathrm{w} - \mathbf{X}_1\left[J^{-1}\mathbf{n}\right]\mathrm{w}\right) = J\mathbf{X}_1^{\perp}\left[J^{-1}\mathbf{n}\right]\mathrm{w}.
\end{aligned} \tag{II.118}$$

Substituting the right-hand sides of the equalities (II.116), (II.117), (II.118) into the equality (II.115) instead of the expressions $\langle \mathbf{n}, J\mathbf{X}\mathrm{w}\rangle$, $\mathbf{X}_1[\mathbf{n}]\,J\mathbf{X}\mathrm{w}$, $\mathbf{X}_1^{\perp}[\mathbf{n}]\,J\mathbf{X}\mathrm{w}$ and applying the equality (II.86), we deduce:

$$\begin{aligned}
\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\widetilde{J}\mathrm{w} &= \left(s\varphi_0(\theta)\,\mathcal{T}(\mathrm{w}) - \varphi_1(\theta)\,\frac{\langle J^{-1}\mathbf{n}, \mathrm{w}\rangle}{c}\right)\mathbf{e}_0 + \\
&+ c\varphi_1(\theta)\,\mathcal{T}(\mathrm{w})\,\mathbf{n} - s\varphi_0(\theta)\,J\mathbf{X}_1\left[J^{-1}\mathbf{n}\right]\mathrm{w} + J\mathbf{X}_1^{\perp}\left[J^{-1}\mathbf{n}\right]\mathrm{w} = \\
&= \left(s\varphi_0(\theta)\,\mathcal{T}(\mathrm{w}) - \varphi_1(\theta)\,\frac{\langle J^{-1}\mathbf{n}, \mathrm{w}\rangle}{c}\right)\mathbf{e}_0 + \\
&+ J\left(c\varphi_1(\theta)\,\mathcal{T}(\mathrm{w})\,J^{-1}\mathbf{n} - s\varphi_0(\theta)\,\mathbf{X}_1\left[J^{-1}\mathbf{n}\right]\mathrm{w} + \mathbf{X}_1^{\perp}\left[J^{-1}\mathbf{n}\right]\mathrm{w}\right) = \\
&= \mathbf{U}_{\theta,c}\left[s, J^{-1}\mathbf{n}, J\right]\mathrm{w} \quad (\forall \mathrm{w} \in \mathcal{M}(\mathfrak{H})).
\end{aligned}$$

□

**Corollary II.18.1.** *Let* $\mathbf{U}_{\theta,c}[s,\mathbf{n},J] \in \mathfrak{OT}(\mathfrak{H},c)$ *(*$s \in \{-1,1\}$*,* $\theta \in \mathbb{R}\setminus\{0\}$*,* $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$*,* $J \in \mathfrak{U}(\mathfrak{H}_1)$*) and* $J_1 \in \mathfrak{U}(\mathfrak{H}_1)$*.*

*Then* $\widetilde{J_1}\mathbf{U}_{\theta,c}[s,\mathbf{n},J], \mathbf{U}_{\theta,c}[s,\mathbf{n},J]\,\widetilde{J_1} \in \mathfrak{OT}(\mathfrak{H},c)$*, and besides:*

$$\widetilde{J_1}\mathbf{U}_{\theta,c}[s,\mathbf{n},J] = \mathbf{U}_{\theta,c}[s,\mathbf{n},J_1J]; \tag{II.119}$$

$$\mathbf{U}_{\theta,c}[s,\mathbf{n},J]\,\widetilde{J_1} = \mathbf{U}_{\theta,c}\left[s, J_1^{-1}\mathbf{n}, JJ_1\right]. \tag{II.120}$$

*Proof.* The equality (II.119) follows from (II.113) and Lemma II.18.1. So, we are to prove the equality (II.120). Applying Lemma II.18.1 and equality (II.119) we obtain:

$$\mathbf{U}_{\theta,c}\left[s,\mathbf{n},J\right]\widetilde{J_1}=\widetilde{J}\mathbf{E}_{\theta,c}\left[s,\mathbf{n}\right]\widetilde{J_1}=\widetilde{J}\mathbf{U}_{\theta,c}\left[s,J_1^{-1}\mathbf{n},J_1\right]=\mathbf{U}_{\theta,c}\left[s,J_1^{-1}\mathbf{n},JJ_1\right].\qquad\square$$

From Lemma II.18.1 and Corollary II.18.1 we can conclude, that the question about belonging of product (composition) of arbitrary generalized Lorentz transforms into the initial class $\mathfrak{OT}(\mathfrak{H},c)$ can be can be reduced to the question about belonging into the initial class $\mathfrak{OT}(\mathfrak{H},c)$ of product of elementary generalized Lorentz transforms. In the next sections we are going to study just the last question.

## 18.1 Composition of Generalized Lorentz Transforms with Parallel Directions of Motion

At first, we aim to investigate composition of elementary generalized Lorentz transforms with the same directing vectors.

Let us introduce the following denotations:

$$\begin{aligned}\mathfrak{S}(\xi,\eta) &:= \frac{1}{2}(\operatorname{sign}\xi+1)(\operatorname{sign}\eta+1)-1\\ &= \begin{cases}1, & \xi,\eta>0\\ -1, & \xi<0 \text{ or } \eta<0\end{cases}, \qquad \xi,\eta\in\mathbb{R}\setminus\{0\};\end{aligned}\qquad\text{(II.121)}$$

$$\mathbb{I}_{\sigma,\mu}[\mathbf{n}]\,x := \sigma\mathbf{X}_1[\mathbf{n}]\,x+\mu\mathbf{X}_1^{\perp}[\mathbf{n}]\,x=\sigma\langle\mathbf{n},x\rangle\,\mathbf{n}+\mu\mathbf{X}_1^{\perp}[\mathbf{n}]\,x,\quad x\in\mathfrak{H}_1$$
$$(\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ \sigma,\mu\in\{-1,1\}).\qquad\text{(II.122)}$$

It is apparently, that $\mathbb{I}_{\sigma,\mu}[\mathbf{n}]\in\mathfrak{U}(\mathfrak{H}_1)$ (for arbitrary $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$ and $\sigma,\mu\in\{-1,1\}$).

**Lemma II.18.2.** *Let* $\mathbf{E}_{\theta,c}[s,\mathbf{n}],\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}]$ *(*$\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$, $\theta,\theta_1\in\mathbb{R}\setminus\{0\}$, $s,s_1\in\{-1,1\}$*) be any elementary generalized Lorentz transforms with the same directing vector* $\mathbf{n}$. *Then:*

$$\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}]=\mathbf{U}_{\theta\theta_1^{-ss_1},c}\left[s',-ss'\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]\right],\ \textit{where } s'=\mathfrak{S}(ss_1,\theta_1).$$

*Proof.* Consider any fixed vector $\mathrm{w}\in\mathcal{M}(\mathfrak{H})$. Denote:

$$\mathrm{w}':=\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}]\,\mathrm{w}.$$

Applying formulas (II.86),(II.112) and equalities (II.18) we obtain:

$$\begin{aligned}\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}]\,\mathrm{w}&=\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathrm{w}'=\\ &=\left(s\varphi_0(\theta)\,\mathcal{T}(\mathrm{w}')-\varphi_1(\theta)\frac{\langle\mathbf{n},\mathrm{w}'\rangle}{c}\right)\mathbf{e}_0+\\ &\quad+c\varphi_1(\theta)\,\mathcal{T}(\mathrm{w}')\,\mathbf{n}-s\varphi_0(\theta)\,\mathbf{X}_1[\mathbf{n}]\,\mathrm{w}'+\mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}';\end{aligned}\qquad\text{(II.123)}$$

$$\begin{aligned}\mathrm{w}'=&\left(s_1\varphi_0(\theta_1)\,\mathcal{T}(\mathrm{w})-\varphi_1(\theta_1)\frac{\langle\mathbf{n},\mathrm{w}\rangle}{c}\right)\mathbf{e}_0+\\ &+c\varphi_1(\theta_1)\,\mathcal{T}(\mathrm{w})\,\mathbf{n}-s_1\varphi_0(\theta_1)\,\mathbf{X}_1[\mathbf{n}]\,\mathrm{w}+\mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w};\end{aligned}$$

$$\mathcal{T}(\mathrm{w}')=s_1\varphi_0(\theta_1)\,\mathcal{T}(\mathrm{w})-\varphi_1(\theta_1)\frac{\langle\mathbf{n},\mathrm{w}\rangle}{c};\qquad\text{(II.124)}$$

$$\mathbf{X}_1[\mathbf{n}]\,\mathrm{w}'=c\varphi_1(\theta_1)\,\mathcal{T}(\mathrm{w})\,\mathbf{n}-s_1\varphi_0(\theta_1)\,\mathbf{X}_1[\mathbf{n}]\,\mathrm{w};\qquad\text{(II.125)}$$

$$\begin{aligned}\langle \mathbf{n}, \mathrm{w}'\rangle &= \langle \mathbf{X}_1[\mathbf{n}]\,\mathbf{n}, \mathrm{w}'\rangle = \langle \mathbf{n}, \mathbf{X}_1[\mathbf{n}]\,\mathrm{w}'\rangle = \\ &= \langle \mathbf{n}, c\varphi_1(\theta_1)\,\mathcal{T}(\mathrm{w})\,\mathbf{n} - s_1\varphi_0(\theta_1)\,\mathbf{X}_1[\mathbf{n}]\,\mathrm{w}\rangle = \\ &= c\varphi_1(\theta_1)\,\mathcal{T}(\mathrm{w})\,\langle \mathbf{n}, \mathbf{n}\rangle - s_1\varphi_0(\theta_1)\,\langle \mathbf{X}_1[\mathbf{n}]\,\mathbf{n}, \mathrm{w}\rangle = \\ &= c\varphi_1(\theta_1)\,\mathcal{T}(\mathrm{w}) - s_1\varphi_0(\theta_1)\,\langle \mathbf{n}, \mathrm{w}\rangle\,; \end{aligned} \tag{II.126}$$

$$\mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}' = \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}. \tag{II.127}$$

Substitution the values $\mathcal{T}(\mathrm{w}')$, $\mathbf{X}_1[\mathbf{n}]\,\mathrm{w}'$, $\langle \mathbf{n}, \mathrm{w}'\rangle$, $\mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}'$ from (II.124),(II.125),(II.126) and (II.127) into (II.123) gives:

$$\begin{aligned}\mathbf{E}_{\theta,c}[s,\mathbf{n}]\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}]\,\mathrm{w} &= \left(s\varphi_0(\theta)\left(s_1\varphi_0(\theta_1)\,\mathcal{T}(\mathrm{w}) - \varphi_1(\theta_1)\,\frac{\langle \mathbf{n}, \mathrm{w}\rangle}{c}\right) - \right.\\ &\quad \left. - \varphi_1(\theta)\,\frac{c\varphi_1(\theta_1)\,\mathcal{T}(\mathrm{w}) - s_1\varphi_0(\theta_1)\,\langle \mathbf{n}, \mathrm{w}\rangle}{c}\right)\mathbf{e}_0 + \\ &\quad + c\varphi_1(\theta)\left(s_1\varphi_0(\theta_1)\,\mathcal{T}(\mathrm{w}) - \varphi_1(\theta_1)\,\frac{\langle \mathbf{n}, \mathrm{w}\rangle}{c}\right)\mathbf{n} - \\ &\quad - s\varphi_0(\theta)\left(c\varphi_1(\theta_1)\,\mathcal{T}(\mathrm{w})\,\mathbf{n} - s_1\varphi_0(\theta_1)\,\mathbf{X}_1[\mathbf{n}]\,\mathrm{w}\right) + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w} = \\ &= \Bigg(\left(ss_1\varphi_0(\theta)\,\varphi_0(\theta_1) - \varphi_1(\theta)\,\varphi_1(\theta_1)\right)\mathcal{T}(\mathrm{w}) + \\ &\quad + s\left(ss_1\varphi_1(\theta)\,\varphi_0(\theta_1) - \varphi_0(\theta)\,\varphi_1(\theta_1)\right)\frac{\langle \mathbf{n}, \mathrm{w}\rangle}{c}\Bigg)\mathbf{e}_0 + \\ &\quad + cs\left(ss_1\varphi_1(\theta)\,\varphi_0(\theta_1) - \varphi_0(\theta)\,\varphi_1(\theta_1)\right)\mathcal{T}(\mathrm{w})\,\mathbf{n} + \\ &\quad + \left(ss_1\varphi_0(\theta)\,\varphi_0(\theta_1) - \varphi_1(\theta)\,\varphi_1(\theta_1)\right)\mathbf{X}_1[\mathbf{n}]\,\mathrm{w} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}. \end{aligned} \tag{II.128}$$

Using the definitions of the functions $\varphi_0(\cdot)$, $\varphi_1(\cdot)$ (see formula (II.81)) we get:

$$\begin{aligned} & ss_1\varphi_0(\theta)\,\varphi_0(\theta_1) - \varphi_1(\theta)\,\varphi_1(\theta_1) = \\ &= ss_1\frac{1}{2}\left(\frac{1}{|\theta|} + \theta\right)\frac{1}{2}\left(\frac{1}{|\theta_1|} + \theta_1\right) - \frac{1}{2}\left(\frac{1}{|\theta|} - \theta\right)\frac{1}{2}\left(\frac{1}{|\theta_1|} - \theta_1\right) = \\ &= \frac{1}{4}\left(ss_1\left(\frac{1}{|\theta|}\frac{1}{|\theta_1|} + \theta\frac{1}{|\theta_1|} + \frac{1}{|\theta|}\theta_1 + \theta\theta_1\right) - \right.\\ &\qquad \left. - \left(\frac{1}{|\theta|}\frac{1}{|\theta_1|} - \theta\frac{1}{|\theta_1|} - \frac{1}{|\theta|}\theta_1 + \theta\theta_1\right)\right) = \\ &= \begin{cases} \frac{1}{2}\left(\frac{\theta}{|\theta_1|} + \frac{\theta_1}{|\theta|}\right), & ss_1 = 1 \\ -\frac{1}{2}\left(\frac{1}{|\theta\theta_1|} + \theta\theta_1\right), & ss_1 = -1 \end{cases} = \begin{cases} \varphi_0\left(\frac{\theta}{\theta_1}\right), & ss_1 = 1,\ \theta_1 > 0 \\ -\varphi_0\left(\frac{\theta}{\theta_1}\right), & ss_1 = 1,\ \theta_1 < 0 \\ -\varphi_0(\theta\theta_1), & ss_1 = -1 \end{cases} = \\ &= \mathfrak{S}(ss_1, \theta_1)\,\varphi_0\left(\theta\theta_1^{-ss_1}\right) = s'\varphi_0(\theta'), \end{aligned} \tag{II.129}$$

where $s' = \mathfrak{S}(ss_1, \theta_1)$; $\theta' = \theta\theta_1^{-ss_1}$. Similarly we obtain:

$$ss_1\varphi_1(\theta)\,\varphi_0(\theta_1) - \varphi_0(\theta)\,\varphi_1(\theta_1) = s'\varphi_1(\theta'). \tag{II.130}$$

Substituting the right-hand sides of the equalities (II.129),(II.130) instead of the corresponding expressions in the formula (II.128), we conclude:

$$\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[\sigma s_1,\mathbf{n}]\,\mathrm{w} = \left(s'\varphi_0(\theta')\,\mathcal{T}(\mathrm{w}) + ss'\varphi_1(\theta')\,\frac{\langle \mathbf{n}, \mathrm{w}\rangle}{c}\right)\mathbf{e}_0 +$$

$$+ css'\varphi_1(\theta')\mathcal{T}(\mathrm{w})\,\mathbf{n} + s'\varphi_0(\theta')\mathbf{X}_1[\mathbf{n}]\,\mathrm{w} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}.$$

Taking into account formula (II.18), we can rewrite the last equality in the form:

$$\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[\sigma s_1,\mathbf{n}]\,\mathrm{w} = \left(s'\varphi_0(\theta')\mathcal{T}(\mathrm{w}) + \varphi_1(\theta')\frac{\langle ss'\mathbf{n},\mathrm{w}\rangle}{c}\right)\mathbf{e}_0 + \\ + c\varphi_1(\theta')\mathcal{T}(\mathrm{w})(ss'\mathbf{n}) + s'\varphi_0(\theta')\mathbf{X}_1[ss'\mathbf{n}]\,\mathrm{w} + \mathbf{X}_1^{\perp}[ss'\mathbf{n}]\,\mathrm{w} = \\ = \mathbf{U}_{\theta\theta_1^{-ss_1},c}[s',-ss'\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]\,\mathrm{w}.$$

□

Now we consider the composition of elementary generalized Lorentz transforms $\mathbf{E}_{\theta,c}[s,\mathbf{n}]$ and $\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]$ with the parallel direction vectors $\mathbf{n}\parallel\mathbf{n}_1$ (that is under the condition $\mathbf{n}=\sigma\mathbf{n}_1$, where $\sigma\in\{-1,1\}$).

**Lemma II.18.3.** *Let* $\mathbf{E}_{\theta,c}[s,\mathbf{n}], \mathbf{E}_{\theta_1,c}[s_1,\sigma\mathbf{n}]$ $(\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ \theta,\theta_1\in\mathbb{R}\setminus\{0\},\ \sigma,s,s_1\in\{-1,1\})$ *be elementary generalized Lorentz transforms with parallel directions of motion. Then the following equality holds:*

$$\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\sigma\mathbf{n}] = \mathbf{U}_{\theta\theta_1^{-\sigma ss_1},c}[\sigma s',-\sigma ss'\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]],$$

*where* $s'=\mathfrak{S}(\sigma ss_1,\theta_1)$.

*Proof.* Consider any elementary generalized Lorentz transforms with parallel directions of motion, $\mathbf{E}_{\theta,c}[s,\mathbf{n}], \mathbf{E}_{\theta_1,c}[s_1,\sigma\mathbf{n}]\in\mathfrak{OT}(\mathfrak{H},c)$. Applying the formulas (II.86),(II.112) and equalities (II.18) we obtain:

$$\mathbf{E}_{\theta_1,c}[s_1,\sigma\mathbf{n}]\,\mathrm{w} = \left(s_1\varphi_0(\theta_1)\mathcal{T}(\mathrm{w}) - \varphi_1(\theta_1)\frac{\langle\sigma\mathbf{n},\mathrm{w}\rangle}{c}\right)\mathbf{e}_0 + \\ + c\varphi_1(\theta_1)\mathcal{T}(\mathrm{w})\,\sigma\mathbf{n} - s_1\varphi_0(\theta_1)\mathbf{X}_1[\sigma\mathbf{n}]\,\mathrm{w} + \mathbf{X}_1^{\perp}[\sigma\mathbf{n}]\,\mathrm{w} = \\ = \left(s_1\varphi_0(\theta_1)\mathcal{T}(\mathrm{w}) - \varphi_1(\theta_1)\frac{\langle\sigma\mathbf{n},\mathrm{w}\rangle}{c}\right)\mathbf{e}_0 + \\ + c\varphi_1(\theta_1)\mathcal{T}(\mathrm{w})\,\sigma\mathbf{n} - s_1\varphi_0(\theta_1)\mathbf{X}_1[\mathbf{n}]\,\mathrm{w} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w} = \\ = \sigma\left(\sigma s_1\varphi_0(\theta_1)\mathcal{T}(\mathrm{w}) - \varphi_1(\theta_1)\frac{\langle\mathbf{n},\mathrm{w}\rangle}{c}\right)\mathbf{e}_0 + \\ + \sigma\left(c\varphi_1(\theta_1)\mathcal{T}(\mathrm{w})\,\mathbf{n} - \sigma s_1\varphi_0(\theta_1)\mathbf{X}_1[\mathbf{n}]\,\mathrm{w} + \sigma\mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right) = \\ = \sigma\mathbf{U}_{\theta_1,c}[\sigma s_1,\mathbf{n},\mathbb{I}_{1,\sigma}[\mathbf{n}]]\,\mathrm{w} \qquad (\mathrm{w}\in\mathcal{M}(\mathfrak{H})).$$

Taking into account, that $(\mathbb{I}_{1,\sigma}[\mathbf{n}])^{-1}\mathbf{n}=\mathbf{n}$, and using Lemma II.18.1, we get:

$$\mathbf{E}_{\theta_1,c}[s_1,\sigma\mathbf{n}] = \sigma\mathbf{U}_{\theta_1,c}[\sigma s_1,\mathbf{n},\mathbb{I}_{1,\sigma}[\mathbf{n}]] = \sigma\mathbf{E}_{\theta_1,c}[\sigma s_1,\mathbf{n}]\,\widetilde{\mathbb{I}_{1,\sigma}}[\mathbf{n}] \\ (\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ \theta_1\in\mathbb{R}\setminus\{0\},\ \sigma,s_1\in\{-1,1\}). \tag{II.131}$$

Therefore, applying Lemma II.18.2 and Lemma II.18.1, we deduce:

$$\mathbf{E}_{\theta,c}[s,\mathbf{n}]\mathbf{E}_{\theta_1,c}[s_1,\sigma\mathbf{n}] = \sigma\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[\sigma s_1,\mathbf{n}]\,\widetilde{\mathbb{I}_{1,\sigma}}[\mathbf{n}] = \\ = \sigma\mathbf{U}_{\theta\theta_1^{-\sigma ss_1},c}[s',-ss'\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]\,\widetilde{\mathbb{I}_{1,\sigma}}[\mathbf{n}] = \\ = \widetilde{\mathbb{I}_{-1,1}}[\mathbf{n}]\left(\sigma\mathbf{E}_{\theta\theta_1^{-\sigma ss_1},c}[s',-ss'\mathbf{n}]\,\widetilde{\mathbb{I}_{1,\sigma}}[\mathbf{n}]\right), \qquad \text{where } s'=\mathfrak{S}(\sigma ss_1,\theta_1). \tag{II.132}$$

According to the equalities (II.122) and (II.18), $\mathbb{I}_{1,\sigma}[\mathbf{n}]=\mathbb{I}_{1,\sigma}[-ss'\mathbf{n}]$, hence, the equality (II.132) may be rewritten in the form:

$$\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\sigma\mathbf{n}] = \widetilde{\mathbb{I}_{-1,1}}[\mathbf{n}]\left(\sigma\mathbf{E}_{\theta\theta_1^{-\sigma ss_1},c}[\sigma(\sigma s'),-ss'\mathbf{n}]\,\widetilde{\mathbb{I}_{1,\sigma}}[-ss'\mathbf{n}]\right).$$

And, using the equality (II.131), we obtain:

$$\mathbf{E}_{\theta,c}\left[s,\mathbf{n}\right]\mathbf{E}_{\theta_1,c}\left[s_1,\sigma\mathbf{n}\right]=\widetilde{\mathbb{I}_{-1,1}}\left[\mathbf{n}\right]\left(\mathbf{E}_{\theta\theta_1^{-\sigma ss_1},c}\left[\sigma s',-\sigma ss'\mathbf{n}\right]\right)= \\ =\mathbf{U}_{\theta\theta_1^{-\sigma ss_1},c}\left[\sigma s',-\sigma ss'\mathbf{n},\mathbb{I}_{-1,1}\left[\mathbf{n}\right]\right].$$

□

The next assertion shows, that composition of any generalized Lorentz transforms with parallel directions of motion always is generalized Lorentz transform.

**Assertion II.18.1.** *Suppose, that,* $\mathbf{U}_{\theta,c}\left[s,\mathbf{n},J\right],\mathbf{U}_{\theta_1,c}\left[s_1,\sigma J\mathbf{n},J_1\right]\in\mathfrak{OT}\left(\mathfrak{H},c\right)$.
*Then* $\mathbf{U}_{\theta_1,c}\left[s_1,\sigma J\mathbf{n},J_1\right]\mathbf{U}_{\theta,c}\left[s,\mathbf{n},J\right]\in\mathfrak{OT}\left(\mathfrak{H},c\right)$, *and besides:*

$$\mathbf{U}_{\theta_1,c}\left[s_1,\sigma J\mathbf{n},J_1\right]\mathbf{U}_{\theta,c}\left[s,\mathbf{n},J\right]=\mathbf{U}_{\theta_1\theta^{-\sigma s_1 s},c}\left[\sigma s',-s_1s'\mathbf{n},J_1J\mathbb{I}_{-1,1}\left[\mathbf{n}\right]\right],$$

*where* $s'=\mathfrak{S}\left(\sigma s_1 s,\theta\right)$.

*Proof.* In accordance with Lemma II.18.1:

$$\mathbf{U}_{\theta,c}\left[s,\mathbf{n},J\right]=\mathbf{E}_{\theta,c}\left[s,J\mathbf{n}\right]\widetilde{J} \\ \mathbf{U}_{\theta_1,c}\left[s_1,\sigma J\mathbf{n},J_1\right]=\widetilde{J}_1\mathbf{E}_{\theta_1,c}\left[s_1,\sigma J\mathbf{n}\right].$$

Hence, using Lemma II.18.3, we get:

$$\mathbf{U}_{\theta_1,c}\left[s_1,\sigma J\mathbf{n},J_1\right]\mathbf{U}_{\theta,c}\left[s,\mathbf{n},J\right]=\widetilde{J}_1\mathbf{E}_{\theta_1,c}\left[s_1,\sigma J\mathbf{n}\right]\mathbf{E}_{\theta,c}\left[s,J\mathbf{n}\right]\widetilde{J}= \\ =\widetilde{J}_1\mathbf{U}_{\theta_1\theta^{-\sigma s_1 s},c}\left[\sigma s',-\sigma s_1 s'(\sigma J\mathbf{n}),\mathbb{I}_{-1,1}\left[\sigma J\mathbf{n}\right]\right]\widetilde{J}= \\ =\widetilde{J}_1\mathbf{U}_{\theta_1\theta^{-\sigma s_1 s},c}\left[\sigma s',-s_1s'J\mathbf{n},\mathbb{I}_{-1,1}\left[J\mathbf{n}\right]\right]\widetilde{J},$$

where $s'=\mathfrak{S}\left(\sigma s_1 s,\theta\right)$. Applying Corollary II.18.1 to the right-hand side of last formula, we obtain:

$$\mathbf{U}_{\theta_1,c}\left[s_1,\sigma J\mathbf{n},J_1\right]\mathbf{U}_{\theta,c}\left[s,\mathbf{n},J\right]=\widetilde{J}_1\mathbf{U}_{\theta_1\theta^{-\sigma s_1 s},c}\left[\sigma s',-s_1s'\mathbf{n},\mathbb{I}_{-1,1}\left[J\mathbf{n}\right]J\right]= \\ =\mathbf{U}_{\theta_1\theta^{-\sigma s_1 s},c}\left[\sigma s',-s_1s'\mathbf{n},J_1\mathbb{I}_{-1,1}\left[J\mathbf{n}\right]J\right]. \qquad \text{(II.133)}$$

Using equality (II.122) and unitarity of the operator $J$, for all $x\in\mathfrak{H}_1$ we get:

$$\mathbb{I}_{-1,1}\left[J\mathbf{n}\right]Jx=-\mathbf{X}_1\left[J\mathbf{n}\right]Jx+\mathbf{X}_1^{\perp}\left[J\mathbf{n}\right]Jx= \\ =-\mathbf{X}_1\left[J\mathbf{n}\right]Jx+\left(\mathbf{X}-\mathbf{X}_1\left[J\mathbf{n}\right]\right)Jx= \\ =-\left\langle J\mathbf{n},Jx\right\rangle J\mathbf{n}+\mathbf{X}Jx-\left\langle J\mathbf{n},Jx\right\rangle J\mathbf{n}= \\ =-\left\langle \mathbf{n},x\right\rangle J\mathbf{n}+Jx-\left\langle \mathbf{n},x\right\rangle J\mathbf{n}= \\ =J\left(-\left\langle \mathbf{n},x\right\rangle \mathbf{n}+\mathbf{X}x-\left\langle \mathbf{n},x\right\rangle \mathbf{n}\right)= \\ =J\left(-\mathbf{X}_1\left[\mathbf{n}\right]x+\left(\mathbf{X}-\mathbf{X}_1\left[\mathbf{n}\right]\right)x\right)=J\mathbb{I}_{-1,1}\left[\mathbf{n}\right]x.$$

Consequently, according to (II.133), we have:

$$\mathbf{U}_{\theta_1,c}\left[s_1,\sigma J\mathbf{n},J_1\right]\mathbf{U}_{\theta,c}\left[s,\mathbf{n},J\right]=\mathbf{U}_{\theta_1\theta^{-\sigma s_1 s},c}\left[\sigma s',-s_1s'\mathbf{n},J_1J\mathbb{I}_{-1,1}\left[\mathbf{n}\right]\right].$$

□

*Remark* II.18.1. Coordinate transform operators $\mathbf{U}_{\theta,c}\left[s,\mathbf{n},J\right]$ and $\mathbf{U}_{\theta_1,c}\left[s_1,\sigma J\mathbf{n},J_1\right]$ in Assertion II.18.1 indeed have parallel directions of motion. To explain the last statement, let us consider, for example, the case $\theta\neq-1$. Then, by Theorem II.17.3, the coordinate transform $\mathbf{U}_{\theta,c}\left[s,\mathbf{n},J\right]$ is v-determined, and besides $\mathcal{V}\left(\mathbf{U}_{\theta,c}\left[s,\mathbf{n},J\right]\right)=cs\frac{1-\theta|\theta|}{1+\theta|\theta|}\mathbf{n}$. Hence, according to formula (II.66) coordinate transform operator $\left(\mathbf{U}_{\theta,c}\left[s,\mathbf{n},J\right]\right)^{-1}$ also is v-determined with $\mathcal{V}\left(\left(\mathbf{U}_{\theta,c}\left[s,\mathbf{n},J\right]\right)^{-1}\right)=J\mathcal{V}\left(\mathbf{U}_{\theta,c}\left[s,\mathbf{n},J\right]\right)=cs\frac{1-\theta|\theta|}{1+\theta|\theta|}J\mathbf{n}$. Suppose, that (v-determined) coordinate transform operator

$\mathbf{U}_{\theta,c}[s,\mathbf{n},J]$ maps coordinates of any point in fixed reference frame[15] $\boldsymbol{l}$ into coordinates of this point in other reference frame $\boldsymbol{l}'$, moving relatively the frame $\boldsymbol{l}$ with a constant velocity $\mathcal{V}(\mathbf{U}_{\theta,c}[s,\mathbf{n},J]) = cs\frac{1-\theta|\theta|}{1+\theta|\theta|}\mathbf{n}$. Then the frame $\boldsymbol{l}$ moves relatively the frame $\boldsymbol{l}'$ with velocity $\mathcal{V}\left((\mathbf{U}_{\theta,c}[s,\mathbf{n},J])^{-1}\right) = cs\frac{1-\theta|\theta|}{1+\theta|\theta|}J\mathbf{n}$. Hence, the directing vector of motion of the reference frame $\boldsymbol{l}$ relatively the frame $\boldsymbol{l}'$ is parallel to the vector $J\mathbf{n}$. Thus, the reference frame $\boldsymbol{l}''$, connected with the coordinate transform $\mathbf{U}_{\theta_1,c}[s_1,\sigma J\mathbf{n},J_1]$ has directing vector of motion $\sigma J\mathbf{n}$, which is parallel to the vector $J\mathbf{n}$.

**Corollary II.18.2.** *Let $\mathfrak{H}$ be a real Hilbert space such, that* $\dim(\mathfrak{H}) = 1$. *Then for any operators $L, L_1 \in \mathfrak{OT}(\mathfrak{H},c)$ we have $L_1 L \in \mathfrak{OT}(\mathfrak{H},c)$.*

*Proof.* Suppose, that $L, L_1 \in \mathfrak{OT}(\mathfrak{H},c)$, where $\mathfrak{H}$ is a real Hilbert space with $\dim(\mathfrak{H}) = 1$. Then, according to (II.89), operators $L$,$L_1$ may be represened in the form:

$$L = \mathbf{U}_{\theta,c}[s,\mathbf{n},J], \qquad L_1 = \mathbf{U}_{\theta_1,c}[s_1,\mathbf{n}_1,J_1],$$

where $s, s_1 \in \{-1,1\}$, $\theta,\theta_1 \in \mathbb{R}\setminus\{0\}$, $\mathbf{n},\mathbf{n}_1 \in \mathbf{B}_1(\mathfrak{H}_1)$, $J, J_1 \in \mathfrak{U}(\mathfrak{H}_1)$. Since $\dim(\mathfrak{H}_1) = \dim(\mathfrak{H}) = 1$, there exist number $\sigma \in \{-1,1\}$ such, that $\mathbf{n}_1 = \sigma\mathbf{n}$. Since $J$ is unitary operator in one-dimensional space $\mathfrak{H}_1$, there must exist number $\sigma' \in \{-1,1\}$ such, that $J\mathbf{n} = \sigma'\mathbf{n}$. Hence:

$$\mathbf{n}_1 = \sigma\mathbf{n} = \sigma\sigma' J\mathbf{n} = \widetilde{\sigma} J\mathbf{n},$$

where $\widetilde{\sigma} = \sigma\sigma' \in \{-1,1\}$. Hence, $\mathbf{U}_{\theta_1,c}[s_1,\mathbf{n}_1,J_1] = \mathbf{U}_{\theta_1,c}[s_1,\widetilde{\sigma}J\mathbf{n},J_1]$. And, according to Assertion II.18.1, $L_1 L = \mathbf{U}_{\theta_1,c}[s_1,\widetilde{\sigma}J\mathbf{n},J_1]\,\mathbf{U}_{\theta,c}[s,\mathbf{n},J] \in \mathfrak{OT}(\mathfrak{H},c)$. □

The next corollary proves, that the operation of taking inverse operator does not lead outside the class of generalized Lorentz transforms $\mathfrak{OT}(\mathfrak{H},c)$.

**Corollary II.18.3.** *Let,* $\mathbf{U}_{\theta,c}[s,\mathbf{n},J] \in \mathfrak{OT}(\mathfrak{H},c)$. *Then* $\mathbf{U}_{\theta,c}[s,\mathbf{n},J]^{-1} \in \mathfrak{OT}(\mathfrak{H},c)$ *with:*

$$\mathbf{U}_{\theta,c}[s,\mathbf{n},J]^{-1} = \mathbf{U}_{\theta^s,c}\left[s_\theta, s_\theta J\mathbf{n}, J^{-1}\right], \tag{II.134}$$

*where* $s_\theta = \mathfrak{S}(s,\theta)$.

*Proof.* Chose any $\mathbf{U}_{\theta,c}[s,\mathbf{n},J] \in \mathfrak{OT}(\mathfrak{H},c)$. Denote:

$$\theta_1 := \theta^s, \quad s_1 := \sigma := s_\theta = \mathfrak{S}(s,\theta),$$
$$J_1 := J^{-1}.$$

According to Assertion II.18.1:

$$\begin{aligned}\mathbf{U}_{\theta_1,c}[s_1,\sigma J\mathbf{n},J_1]\mathbf{U}_{\theta,c}[s,\mathbf{n},J] &= \mathbf{U}_{\theta_1\theta^{-\sigma s_1 s},c}[\sigma s', -s_1 s'\mathbf{n}, J_1 J\mathbb{I}_{-1,1}[\mathbf{n}]] = \\ &= \mathbf{U}_{\theta^s\theta^{-s_\theta\cdot s_\theta\cdot s},c}\left[s_\theta s', -s_\theta s'\mathbf{n}, J^{-1}J\mathbb{I}_{-1,1}[\mathbf{n}]\right] = \mathbf{U}_{1,c}[s_\theta s', -s_\theta s'\mathbf{n}, \mathbb{I}_{-1,1}[\mathbf{n}]],\end{aligned}$$

where $s' = \mathfrak{S}(\sigma s_1 s,\theta) = \mathfrak{S}(s_\theta s_\theta s,\theta) = \mathfrak{S}(s,\theta) = s_\theta$. Hence:

$$\begin{aligned}\mathbf{U}_{\theta_1,c}[s_1,\sigma J\mathbf{n},J_1]\mathbf{U}_{\theta,c}[s,\mathbf{n},J] &= \mathbf{U}_{1,c}[s_\theta s_\theta,\ -s_\theta s_\theta\mathbf{n}, \mathbb{I}_{-1,1}[\mathbf{n}]] = \\ &= \mathbf{U}_{1,c}[1,-\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]].\end{aligned}$$

Using (II.86), (II.122) and (II.18), it is not hard to verify, that for arbitrary $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ it holds the equality $\mathbf{U}_{1,c}[1,-\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]]\,\mathrm{w} = \mathrm{w}$. Therefore, $\mathbf{U}_{1,c}[1,-\mathbf{n},\mathbb{I}_{-1,1}[\mathbf{n}]] = \mathbb{I}$. Thus, $\mathbf{U}_{\theta_1,c}[s_1,\sigma J\mathbf{n},J_1]\,\mathbf{U}_{\theta,c}[s,\mathbf{n},J] = \mathbb{I}$. Consequently:

$$\mathbf{U}_{\theta,c}[s,\mathbf{n},J]^{-1} = \mathbf{U}_{\theta_1,c}[s_1,\sigma J\mathbf{n},J_1] = \mathbf{U}_{\theta^s,c}[s_\theta, s_\theta J\mathbf{n}, J^{-1}].$$ □

[15] In this remark we understand the reference frames $\boldsymbol{l}$ and $\boldsymbol{l}'$ in a usual physical sense.

*Remark* II.18.2. By means of application (II.86), (II.121), (II.18) and (II.82, two bottom equalities), the equality (II.134) may be rewritten in the form:

$$\mathbf{U}_{\theta,c}[s,\mathbf{n},J]^{-1}=\mathbf{U}_{\theta,c}\left[\widetilde{s}_\theta, s\widetilde{s}_\theta J\mathbf{n}, J^{-1}\right], \quad \text{where } \widetilde{s}_\theta = s\operatorname{sign}\theta. \tag{II.135}$$

Indeed, let $s\in\{-1,1\}$, $\theta\in\mathbb{R}\setminus\{0\}$, $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$ and $J\in\mathfrak{U}(\mathfrak{H}_1)$. Denote, $s_\theta:=\mathfrak{S}(s,\theta)$, $\widetilde{s}_\theta:=s\operatorname{sign}\theta$.

1) In the case $s=1$ we have, $s_\theta=\mathfrak{S}(1,\theta)=\operatorname{sign}\theta$, $\widetilde{s}_\theta:=\operatorname{sign}\theta$. So, according to (II.134), in this case we obtain:

$$\begin{aligned}\mathbf{U}_{\theta,c}[s,\mathbf{n},J]^{-1} &= \mathbf{U}_{\theta^s,c}\left[s_\theta, s_\theta J\mathbf{n}, J^{-1}\right]=\mathbf{U}_{\theta,c}\left[\operatorname{sign}\theta,\operatorname{sign}\theta J\mathbf{n},J^{-1}\right]=\\ &=\mathbf{U}_{\theta,c}\left[\widetilde{s}_\theta, s\widetilde{s}_\theta J\mathbf{n}, J^{-1}\right].\end{aligned}$$

2) In the case $s=-1$ we have, $s_\theta=\mathfrak{S}(-1,\theta)=-1$, $\widetilde{s}_\theta:=-\operatorname{sign}\theta$. Hence, applying (II.134) and (II.87), we deduce:

$$\begin{aligned}\mathbf{U}_{\theta,c}[s,\mathbf{n},J]^{-1} &= \mathbf{U}_{\theta^{-1},c}\left[-1,-J\mathbf{n},J^{-1}\right]=\\ &=\mathbf{U}_{(\theta^{-1})^{-1},c}\left[(-1)\operatorname{sign}\left(\theta^{-1}\right),-\operatorname{sign}\left(\theta^{-1}\right)(-J\mathbf{n}),J^{-1}\right]=\\ &=\mathbf{U}_{\theta,c}\left[-\operatorname{sign}(\theta),\operatorname{sign}\theta J\mathbf{n},J^{-1}\right]=\mathbf{U}_{\theta,c}\left[\widetilde{s}_\theta, s\widetilde{s}_\theta J\mathbf{n}, J^{-1}\right].\end{aligned}$$

Corollary II.18.3 shows, that class of operators $\mathfrak{OT}(\mathfrak{H},c)$ is invariant with respect to the operation of taking inverse. Classes of operators $\mathfrak{O}(\mathfrak{H},c)$ and $\mathfrak{O}_+(\mathfrak{H},c)$ also are invariant with respect to this operation (by assertions II.17.1 and II.17.6 respectively). But, it turns out, that the class $\mathfrak{OT}_+(\mathfrak{H},c)$ is not invariant with respect to the operation of taking inverse.

**Corollary II.18.4.** *If $L\in\mathfrak{OT}_+(\mathfrak{H},c)\setminus(\mathfrak{O}_+(\mathfrak{H},c)\cup\mathfrak{OT}_\infty(\mathfrak{H},c))$ then $L^{-1}\notin\mathfrak{OT}_+(\mathfrak{H},c)$.*

*Proof.* Let $L\in\mathfrak{OT}_+(\mathfrak{H},c)\setminus(\mathfrak{O}_+(\mathfrak{H},c)\cup\mathfrak{OT}_\infty(\mathfrak{H},c))$. Then $L\in\mathfrak{OT}_+(\mathfrak{H},c)$ and $L\notin\mathfrak{O}_+(\mathfrak{H},c)\cup\mathfrak{OT}_\infty(\mathfrak{H},c)$. Since $L\in\mathfrak{OT}_+(\mathfrak{H},c)$, then, by (II.109), operator $L$ can be represented in the form:

$$L=\mathbf{U}_{\theta,c}[1,\mathbf{n},J],$$

where $\theta\in[-1,1]\setminus\{0\}$, $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$, $J\in\mathfrak{U}(\mathfrak{H}_1)$. Since $L\notin\mathfrak{O}_+(\mathfrak{H},c)\cup\mathfrak{OT}_\infty(\mathfrak{H},c)$, then, according to (II.107) and (II.96), $\theta<0$ and $\theta\neq-1$. Hence, by equality (II.135), we get:

$$L^{-1}=\left(\mathbf{U}_{\theta,c}[1,\mathbf{n},J]\right)^{-1}=\mathbf{U}_{\theta,c}\left[-1,-J\mathbf{n},J^{-1}\right].$$

Thus, by (II.111), $L^{-1}\in\mathfrak{OT}_-(\mathfrak{H},c)$. And, since $\theta\neq-1$, then, by (II.95), $L^{-1}=\mathbf{U}_{\theta,c}[-1,-J\mathbf{n},J^{-1}]\in\mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H},c)$. So, by (II.94), we have $L^{-1}\notin\mathfrak{OT}_\infty(\mathfrak{H},c)$. Thus, $L^{-1}\in\mathfrak{OT}_-(\mathfrak{H},c)$ and $L^{-1}\notin\mathfrak{OT}_\infty(\mathfrak{H},c)$. Hence, by Assertion II.17.8, $L^{-1}\notin\mathfrak{OT}_+(\mathfrak{H},c)$. □

Any operator of kind $L=\mathbf{W}_{\lambda,c}[1,\mathbf{n},J]$, where $c<\lambda<\infty$, $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$ and $J\in\mathfrak{U}(\mathfrak{H}_1)$, satisfies condition $L\in\mathfrak{OT}_+(\mathfrak{H},c)\setminus(\mathfrak{O}_+(\mathfrak{H},c)\cup\mathfrak{OT}_\infty(\mathfrak{H},c))$. Indeed, according to (II.108), $L\in\mathfrak{OT}_+(\mathfrak{H},c)$. According to (II.95), $L\in\mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H},c)$, and so, by (II.94), we get $L\notin\mathfrak{OT}_\infty(\mathfrak{H},c)$. Since $c<\lambda<\infty$ then, by Assertion II.17.4, we have $\|\mathcal{V}(L)\|=|\lambda|>c$. So, by Assertion II.17.2, we obtain $L\notin\mathfrak{O}(\mathfrak{H},c)$. And, by (II.98), we get, $L\notin\mathfrak{O}_+(\mathfrak{H},c)$. Thus, we have $L\in\mathfrak{OT}_+(\mathfrak{H},c)$, $L\notin\mathfrak{OT}_\infty(\mathfrak{H},c)$ and $L\notin\mathfrak{O}_+(\mathfrak{H},c)$. Hence, the class of operators $\mathfrak{OT}_+(\mathfrak{H},c)\setminus(\mathfrak{O}_+(\mathfrak{H},c)\cup\mathfrak{OT}_\infty(\mathfrak{H},c))$ is not empty. Therefore Corollary II.18.4 leads to the next corollary.

**Corollary II.18.5.** *Class of operators $\mathfrak{OT}_+(\mathfrak{H},c)$ does not form a group of operators over the space $\mathcal{M}(\mathfrak{H})$.*

The next corollary immediatelly follows from Corollary II.18.2 and Corollary II.18.3.

**Corollary II.18.6.** *Let $\mathfrak{H}$ be a real Hilbert space such, that $\dim(\mathfrak{H})=1$. Then class of operators $\mathfrak{OT}(\mathfrak{H},c)$ is a group of operators over the space $\mathcal{M}(\mathfrak{H})$.*

## 18.2 Composition of Generalized Lorentz Transforms with Orthogonal Directions of Motion

**Lemma II.18.4.** *Let,* $\mathbf{E}_{\theta,c}[s,\mathbf{n}], \mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1] \in \mathfrak{OT}(\mathfrak{H},c)$ *be elementary generalized Lorentz transforms with orthogonal directing vectors, that is* $\langle \mathbf{n},\mathbf{n}_1\rangle = 0$. *Then for any vector* $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ *the following equality is performed:*

$$\begin{aligned}\mathbf{E}_{\theta,c}[s,\mathbf{n}]\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\mathrm{w} = \Bigg(& ss_1\varphi_0(\theta)\varphi_0(\theta_1)\mathcal{T}(\mathrm{w}) - \\ & - \left(s\varphi_0(\theta)\varphi_1(\theta_1)\frac{\langle \mathbf{n}_1,\mathrm{w}\rangle}{c} + \varphi_1(\theta)\frac{\langle \mathbf{n},\mathrm{w}\rangle}{c}\right)\Bigg)\mathbf{e}_0 + \\ & + c\varphi_1(\theta)\left(s_1\varphi_0(\theta_1)\mathcal{T}(\mathrm{w}) - \varphi_1(\theta_1)\frac{\langle \mathbf{n}_1,\mathrm{w}\rangle}{c}\right)\mathbf{n} - \\ & - (s\varphi_0(\theta)+1)\mathbf{X}_1[\mathbf{n}]\mathrm{w} + \\ & + c\varphi_1(\theta_1)\mathcal{T}(\mathrm{w})\mathbf{n}_1 - s_1\varphi_0(\theta_1)\mathbf{X}_1[\mathbf{n}_1]\mathrm{w} + \mathbf{X}_1^{\perp}[\mathbf{n}_1]\mathrm{w}. \end{aligned} \qquad \text{(II.136)}$$

*Proof.* Chose any fixed vector $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$. Denote:

$$\mathrm{w}' := \mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\mathrm{w}.$$

Then, using the formulas (II.86),(II.112), (II.18) and taking into account the fact, that $\langle \mathbf{n},\mathbf{n}_1\rangle = 0$, we get:

$$\begin{aligned}\mathbf{E}_{\theta,c}[s,\mathbf{n}]\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\mathrm{w} &= \mathbf{E}_{\theta,c}[s,\mathbf{n}]\mathrm{w}' = \\ &= \left(s\varphi_0(\theta)\mathcal{T}(\mathrm{w}') - \varphi_1(\theta)\frac{\langle \mathbf{n},\mathrm{w}'\rangle}{c}\right)\mathbf{e}_0 + \\ &\quad + c\varphi_1(\theta)\mathcal{T}(\mathrm{w}')\mathbf{n} - s\varphi_0(\theta)\mathbf{X}_1[\mathbf{n}]\mathrm{w}' + \mathbf{X}_1^{\perp}[\mathbf{n}]\mathrm{w}'; \end{aligned} \qquad \text{(II.137)}$$

$$\begin{aligned}\mathrm{w}' = &\left(s_1\varphi_0(\theta_1)\mathcal{T}(\mathrm{w}) - \varphi_1(\theta_1)\frac{\langle \mathbf{n}_1,\mathrm{w}\rangle}{c}\right)\mathbf{e}_0 + \\ &+ c\varphi_1(\theta_1)\mathcal{T}(\mathrm{w})\mathbf{n}_1 - s_1\varphi_0(\theta_1)\mathbf{X}_1[\mathbf{n}_1]\mathrm{w} + \mathbf{X}_1^{\perp}[\mathbf{n}_1]\mathrm{w}; \end{aligned}$$

$$\mathcal{T}(\mathrm{w}') = s_1\varphi_0(\theta_1)\mathcal{T}(\mathrm{w}) - \varphi_1(\theta_1)\frac{\langle \mathbf{n}_1,\mathrm{w}\rangle}{c}; \qquad \text{(II.138)}$$

$$\begin{aligned}\mathbf{X}_1[\mathbf{n}]\mathrm{w}' &= \mathbf{X}_1[\mathbf{n}]\left(\widehat{\mathbf{T}}\mathrm{w}' + \mathbf{X}\mathrm{w}'\right) = \mathbf{X}_1[\mathbf{n}]\mathbf{X}\mathrm{w}' = \\ &= \mathbf{X}_1[\mathbf{n}]\left(c\varphi_1(\theta_1)\mathcal{T}(\mathrm{w})\mathbf{n}_1 - s_1\varphi_0(\theta_1)\mathbf{X}_1[\mathbf{n}_1]\mathrm{w} + \mathbf{X}_1^{\perp}[\mathbf{n}_1]\mathrm{w}\right) = \\ &= c\varphi_1(\theta_1)\mathcal{T}(\mathrm{w})\mathbf{X}_1[\mathbf{n}]\mathbf{n}_1 - s_1\varphi_0(\theta_1)\mathbf{X}_1[\mathbf{n}]\mathbf{X}_1[\mathbf{n}_1]\mathrm{w} + \\ &\quad + \mathbf{X}_1[\mathbf{n}]\mathbf{X}_1^{\perp}[\mathbf{n}_1]\mathrm{w}. \end{aligned} \qquad \text{(II.139)}$$

Since $\langle \mathbf{n},\mathbf{n}_1\rangle = 0$, we have:

$$\begin{aligned}&\mathbf{X}_1[\mathbf{n}]\mathbf{n}_1 = \langle \mathbf{n},\mathbf{n}_1\rangle\mathbf{n} = \mathbf{0}; \qquad \mathbf{X}_1[\mathbf{n}]\mathbf{X}_1[\mathbf{n}_1] = \mathbb{O}; \\ &\mathbf{X}_1[\mathbf{n}]\mathbf{X}_1^{\perp}[\mathbf{n}_1] = \mathbf{X}_1[\mathbf{n}](\mathbf{X} - \mathbf{X}_1[\mathbf{n}_1]) = \mathbf{X}_1[\mathbf{n}]\mathbf{X} = \mathbf{X}_1[\mathbf{n}]. \end{aligned}$$

Hence, according to (II.139), we obtain:

$$\mathbf{X}_1[\mathbf{n}]\mathrm{w}' = \mathbf{X}_1[\mathbf{n}]\mathrm{w}; \qquad \text{(II.140)}$$

$$\langle \mathbf{n},\mathrm{w}'\rangle = \langle \mathbf{X}_1[\mathbf{n}]\mathbf{n},\mathrm{w}'\rangle = \langle \mathbf{n},\mathbf{X}_1[\mathbf{n}]\mathrm{w}'\rangle = \langle \mathbf{n},\mathbf{X}_1[\mathbf{n}]\mathrm{w}\rangle =$$

$$= \langle \mathbf{X}_1[\mathbf{n}]\mathbf{n}, \mathrm{w}\rangle = \langle \mathbf{n}, \mathrm{w}\rangle. \tag{II.141}$$

Further, applying (II.140), we deliver:

$$\begin{aligned}\mathbf{X}_1^{\perp}[\mathbf{n}]\mathrm{w}' &= (\mathbf{X} - \mathbf{X}_1[\mathbf{n}])\mathrm{w}' = \mathbf{X}\mathrm{w}' - \mathbf{X}_1[\mathbf{n}]\mathrm{w} = \\ &= c\varphi_1(\theta_1)\mathcal{T}(\mathrm{w})\mathbf{n}_1 - s_1\varphi_0(\theta_1)\mathbf{X}_1[\mathbf{n}_1]\mathrm{w} + \\ &\quad + \mathbf{X}_1^{\perp}[\mathbf{n}_1]\mathrm{w} - \mathbf{X}_1[\mathbf{n}]\mathrm{w}.\end{aligned} \tag{II.142}$$

Substitution of the values $\mathcal{T}(\mathrm{w}')$, $\mathbf{X}_1[\mathbf{n}]\mathrm{w}'$, $\langle \mathbf{n}, \mathrm{w}'\rangle$, $\mathbf{X}_1^{\perp}[\mathbf{n}]\mathrm{w}'$ from the formulas (II.138),(II.140),(II.141),(II.142) into (II.137), provides:

$$\begin{aligned}&\mathbf{E}_{\theta,c}[s,\mathbf{n}]\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\mathrm{w} = \mathbf{E}_{\theta,c}[s,\mathbf{n}]\mathrm{w}' = \\ &= \left(s\varphi_0(\theta)\left(s_1\varphi_0(\theta_1)\mathcal{T}(\mathrm{w}) - \varphi_1(\theta_1)\frac{\langle \mathbf{n}_1, \mathrm{w}\rangle}{c}\right) - \varphi_1(\theta)\frac{\langle \mathbf{n}, \mathrm{w}\rangle}{c}\right)\mathbf{e}_0 + \\ &\quad + c\varphi_1(\theta)\left(s_1\varphi_0(\theta_1)\mathcal{T}(\mathrm{w}) - \varphi_1(\theta_1)\frac{\langle \mathbf{n}_1, \mathrm{w}\rangle}{c}\right)\mathbf{n} - \\ &\quad - s\varphi_0(\theta)\mathbf{X}_1[\mathbf{n}]\mathrm{w} + c\varphi_1(\theta_1)\mathcal{T}(\mathrm{w})\mathbf{n}_1 - s_1\varphi_0(\theta_1)\mathbf{X}_1[\mathbf{n}_1]\mathrm{w} + \mathbf{X}_1^{\perp}[\mathbf{n}_1]\mathrm{w} - \mathbf{X}_1[\mathbf{n}]\mathrm{w} = \\ &= \left(ss_1\varphi_0(\theta)\varphi_0(\theta_1)\mathcal{T}(\mathrm{w}) - \left(s\varphi_0(\theta)\varphi_1(\theta_1)\frac{\langle \mathbf{n}_1, \mathrm{w}\rangle}{c} + \varphi_1(\theta)\frac{\langle \mathbf{n}, \mathrm{w}\rangle}{c}\right)\right)\mathbf{e}_0 + \\ &\quad + c\varphi_1(\theta)\left(s_1\varphi_0(\theta_1)\mathcal{T}(\mathrm{w}) - \varphi_1(\theta_1)\frac{\langle \mathbf{n}_1, \mathrm{w}\rangle}{c}\right)\mathbf{n} - \\ &\quad - s\varphi_0(\theta)\mathbf{X}_1[\mathbf{n}]\mathrm{w} + c\varphi_1(\theta_1)\mathcal{T}(\mathrm{w})\mathbf{n}_1 - \\ &\quad - s_1\varphi_0(\theta_1)\mathbf{X}_1[\mathbf{n}_1]\mathrm{w} + \mathbf{X}_1^{\perp}[\mathbf{n}_1]\mathrm{w} - \mathbf{X}_1[\mathbf{n}]\mathrm{w},\end{aligned}$$

that was necessary to be proved. □

**Lemma II.18.5.** *Let,* $\mathbf{E}_{\theta,c}[s,\mathbf{n}], \mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1] \in \mathfrak{OT}(\mathfrak{H}, c)$ *be elementary generalized Lorentz transforms with orthogonal directing vectors,* ($\langle \mathbf{n}, \mathbf{n}_1\rangle = 0$)*. Then:*

*1. The coordinate transform* $\mathbf{E}_{\theta,c}[s,\mathbf{n}]\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]$ *is v-determined if and only if* $\theta, \theta_1 \neq -1$*, moreover, in the case* $\theta, \theta_1 \neq -1$ *it is performed the equality:*

$$\|\mathcal{V}(\mathbf{E}_{\theta,c}[s,\mathbf{n}]\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1])\| = c\sqrt{1 + \frac{1 - \operatorname{sign}\theta_1}{\varphi_0^2(\theta_1)} - \frac{\operatorname{sign}\theta}{\varphi_0^2(\theta)\varphi_0^2(\theta_1)}}. \tag{II.143}$$

*2. For* $\theta, \theta_1 \neq -1$ *the inequality* $\|\mathcal{V}(\mathbf{E}_{\theta,c}[s,\mathbf{n}]\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1])\| < c$ *holds if and only if* $\theta, \theta_1 > 0$.

*Proof.* **1.** Using the Corollary II.18.3, Lemma II.18.1 and equality (II.122), we obtain:

$$\begin{aligned}&\mathbf{E}_{\theta,c}[s,\mathbf{n}]^{-1} = \mathbf{U}_{\theta,c}[s,\mathbf{n},\mathbb{I}_1]^{-1} = \mathbf{U}_{\theta^s,c}\left[s_\theta, s_\theta\mathbf{n}, \mathbb{I}_1^{-1}\right] = \mathbf{U}_{\theta^s,c}[s_\theta, s_\theta\mathbf{n}, \mathbb{I}_1] = \mathbf{E}_{\theta^s,c}[s_\theta, s_\theta\mathbf{n}]; \\ &\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]^{-1} = \mathbf{E}_{\theta_1^{s_1},c}\left[(s_1)_{\theta_1}, (s_1)_{\theta_1}\mathbf{n}_1\right], \\ &\qquad \text{where} \quad s_\theta = \mathfrak{S}(s,\theta),\ (s_1)_{\theta_1} = \mathfrak{S}(s_1,\theta_1).\end{aligned}$$

Hence:

$$\begin{aligned}(\mathbf{E}_{\theta,c}[s,\mathbf{n}]\ \mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1])^{-1} &= \mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]^{-1}\mathbf{E}_{\theta,c}[s,\mathbf{n}]^{-1} = \\ &= \mathbf{E}_{\theta_1^{s_1},c}\left[(s_1)_{\theta_1}, (s_1)_{\theta_1}\mathbf{n}_1\right]\mathbf{E}_{\theta^s,c}[s_\theta, s_\theta\mathbf{n}].\end{aligned} \tag{II.144}$$

Now we substitute the vector $\mathrm{w} = \mathbf{e}_0$, into (II.144) and apply the equality (II.136):

$$(\mathbf{E}_{\theta,c}[s,\mathbf{n}]\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1])^{-1}\mathbf{e}_0 = \mathbf{E}_{\theta_1^{s_1},c}\left[(s_1)_{\theta_1}, (s_1)_{\theta_1}\mathbf{n}_1\right]\mathbf{E}_{\theta^s,c}[s_\theta, s_\theta\mathbf{n}]\mathbf{e}_0 =$$

$$= (s_1)_{\theta_1} s_\theta \varphi_0\left(\theta_1^{s_1}\right) \varphi_0\left(\theta^s\right) \mathcal{T}\left(\mathbf{e}_0\right) \mathbf{e}_0 + $$
$$+ c\varphi_1\left(\theta_1^{s_1}\right) s_\theta \varphi_0\left(\theta^s\right) \mathcal{T}\left(\mathbf{e}_0\right) (s_1)_{\theta_1} \mathbf{n}_1 + c\varphi_1\left(\theta^s\right) \mathcal{T}\left(\mathbf{e}_0\right) (s_\theta \mathbf{n}) =$$
$$= (s_1)_{\theta_1} s_\theta \varphi_0\left(\theta_1^{s_1}\right) \varphi_0\left(\theta^s\right) \mathbf{e}_0$$
$$+ c s_\theta (s_1)_{\theta_1} \varphi_1\left(\theta_1^{s_1}\right) \varphi_0\left(\theta^s\right) \mathbf{n}_1 + c s_\theta \varphi_1\left(\theta^s\right) \mathbf{n}. \qquad \text{(II.145)}$$

From the equality (II.145) it follows, that $\mathcal{T}\left(\left(\mathbf{E}_{\theta,c}[s,\mathbf{n}]\, \mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\right)^{-1} \mathbf{e}_0\right) = (s_1)_{\theta_1} s_\theta \varphi_0\left(\theta_1^{s_1}\right) \varphi_0\left(\theta^s\right)$ (where $(s_1)_{\theta_1}, s_\theta \in \{-1,1\}$). Therefore, the inequality $\mathcal{T}\left(\mathbf{E}_{\theta,c}[s,\mathbf{n}]\, \mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\, \mathbf{e}_0\right) \neq 0$ is true if and only if $\varphi_0\left(\theta_1^{s_1}\right) \varphi_0\left(\theta^s\right) \neq 0$, i.e. if and only if $\theta, \theta_1 \neq -1$. Consequently, by Definition II.17.2, the coordinate transform $\mathbf{E}_{\theta,c}[s,\mathbf{n}]\, \mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]$ is v-determined if and only if $\theta, \theta_1 \neq -1$.

Now we consider the case $\theta, \theta_1 \neq -1$. By Definition II.17.2, applying the equality (II.145), we obtain:

$$\mathcal{V}\left(\mathbf{E}_{\theta,c}[s,\mathbf{n}]\, \mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\right) = \frac{\mathbf{X}\left(\mathbf{E}_{\theta,c}[s,\mathbf{n}]\, \mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\right)^{-1} \mathbf{e}_0}{\mathcal{T}\left(\left(\mathbf{E}_{\theta,c}[s,\mathbf{n}]\, \mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\right)^{-1} \mathbf{e}_0\right)} =$$
$$= \frac{c s_\theta (s_1)_{\theta_1} \varphi_1\left(\theta_1^{s_1}\right) \varphi_0\left(\theta^s\right) \mathbf{n}_1 + c s_\theta \varphi_1\left(\theta^s\right) \mathbf{n}}{(s_1)_{\theta_1} s_\theta \varphi_0\left(\theta_1^{s_1}\right) \varphi_0\left(\theta^s\right)} =$$
$$= c\frac{\varphi_1\left(\theta_1^{s_1}\right) \varphi_0\left(\theta^s\right) \mathbf{n}_1 + \mathfrak{S}(s_1,\theta_1)\, \varphi_1\left(\theta^s\right) \mathbf{n}}{\varphi_0\left(\theta_1^{s_1}\right) \varphi_0\left(\theta^s\right)}.$$

Since $\langle \mathbf{n}, \mathbf{n}_1 \rangle = 0$, we have:

$$\left\|\mathcal{V}\left(\mathbf{E}_{\theta,c}[s,\mathbf{n}]\, \mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\right)\right\| = c\sqrt{\frac{\left(\varphi_1\left(\theta_1^{s_1}\right) \varphi_0\left(\theta^s\right)\right)^2 + \left(\varphi_1\left(\theta^s\right)\right)^2}{\varphi_0^2\left(\theta_1^{s_1}\right) \varphi_0^2\left(\theta^s\right)}}.$$

According to two lower equalities from (II.82) for $s \in \{-1,1\}$ it holds $\left|\varphi_0\left(\theta^s\right)\right| = \left|\varphi_0(\theta)\right|$, $\left|\varphi_1\left(\theta^s\right)\right| = \left|\varphi_1(\theta)\right|$. Hence:

$$\left\|\mathcal{V}\left(\mathbf{E}_{\theta,c}[s,\mathbf{n}]\, \mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\right)\right\| = c\sqrt{\frac{\varphi_1^2(\theta_1)\, \varphi_0^2(\theta) + \varphi_1^2(\theta)}{\varphi_0^2(\theta_1)\, \varphi_0^2(\theta)}}.$$

From here, using equalities (II.82), we deduce:

$$\left\|\mathcal{V}\left(\mathbf{E}_{\theta,c}[s,\mathbf{n}]\, \mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\right)\right\| = c\sqrt{\frac{\varphi_0^2(\theta_1) - \operatorname{sign}\theta_1}{\varphi_0^2(\theta_1)} + \frac{\varphi_0^2(\theta) - \operatorname{sign}\theta}{\varphi_0^2(\theta)\, \varphi_0^2(\theta_1)}} =$$
$$= c\sqrt{1 + \frac{1 - \operatorname{sign}\theta_1}{\varphi_0^2(\theta_1)} - \frac{\operatorname{sign}\theta}{\varphi_0^2(\theta)\, \varphi_0^2(\theta_1)}}.$$

**2.** Let $\theta, \theta_1 \neq -1$.

a) In the case $\theta, \theta_1 > 0$ ($\operatorname{sign}\theta = \operatorname{sign}\theta_1 = 1$), according to (II.143), we deliver:

$$\left\|\mathcal{V}\left(\mathbf{E}_{\theta,c}[s,\mathbf{n}]\, \mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\right)\right\| = c\sqrt{1 - \frac{1}{\varphi_0^2(\theta)\, \varphi_0^2(\theta_1)}} < c.$$

b) Similarly, in the cases $\theta < 0$, $\theta_1 > 0$ ($\operatorname{sign}\theta = -1$, $\operatorname{sign}\theta_1 = 1$) and $\theta_1 < 0$ ($\operatorname{sign}\theta_1 = -1$), using the equalities (II.82) we, correspondingly, obtain:

$$\left\|\mathcal{V}\left(\mathbf{E}_{\theta,c}[s,\mathbf{n}]\, \mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\right)\right\| = c\sqrt{1 + \frac{1}{\varphi_0^2(\theta)\, \varphi_0^2(\theta_1)}} > c;$$

$$\|\mathcal{V}(\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1])\| = c\sqrt{1+\frac{2}{\varphi_0^2(\theta_1)}-\frac{\operatorname{sign}\theta}{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)}} =$$
$$= c\sqrt{1+\frac{\varphi_0^2(\theta)+(\varphi_0^2(\theta)-\operatorname{sign}\theta)}{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)}} = c\sqrt{1+\frac{\varphi_0^2(\theta)+\varphi_1^2(\theta)}{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)}} > c.$$
□

**Lemma II.18.6.** *Suppose, that for elementary generalized Lorentz transforms* $\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,,\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\in\mathfrak{OT}(\mathfrak{H},c)$ $\langle\mathbf{n},\mathbf{n}_1\rangle=0$ *it is performed the equality:*

$$\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1] = \mathbf{U}_{\theta',c}[s',\mathbf{n}',J']\,,$$

*where* $s'\in\{-1,1\}$*,* $\theta'\in[-1,1]\setminus\{0\}$*,* $\mathbf{n}'\in\mathbf{B}_1(\mathfrak{H}_1)$*,* $J'\in\mathfrak{U}(\mathfrak{H}_1)$*. Then the following statements are true:*

1. $\operatorname{sign}\theta' = \mathfrak{S}(\theta,\theta_1)\,;$
2. $\varphi_0(\theta') = |\varphi_0(\theta)\,\varphi_0(\theta_1)|\,;$
3. *if* $\theta,\theta_1\neq-1$*, then* $s' = ss_1\operatorname{sign}(\varphi_0(\theta)\,\varphi_0(\theta_1))\,;$
4. $\varphi_1(\theta') = \sqrt{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)-\mathfrak{S}(\theta,\theta_1)};$
   *If, in addition,* $\theta,\theta_1\neq 1$*, then:*
5. $\mathbf{n}' = \dfrac{s\varphi_0(\theta)\,\varphi_1(\theta_1)\,\mathbf{n}_1+\varphi_1(\theta)\,\mathbf{n}}{\varphi_1(\theta')} = \dfrac{s\varphi_0(\theta)\,\varphi_1(\theta_1)\,\mathbf{n}_1+\varphi_1(\theta)\,\mathbf{n}}{\sqrt{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)-\mathfrak{S}(\theta,\theta_1)}};$
6. $J'\mathbf{n}' = \dfrac{s_1\varphi_1(\theta)\,\varphi_0(\theta_1)\,\mathbf{n}+\varphi_1(\theta_1)\,\mathbf{n}_1}{\sqrt{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)-\mathfrak{S}(\theta,\theta_1)}}.$

*Proof.* **1.** Suppose, that $\theta,\theta_1\neq-1$. Then, by Lemma II.18.5, the coordinate transform $\mathbf{U}_{\theta',c}[s',\mathbf{n}',J'] = \mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]$ is v-determined, moreover the inequality $\|\mathcal{V}(\mathbf{U}_{\theta',c}[s',\mathbf{n}',J'])\|<c$ is true if and only if $\theta,\theta_1>0$. According to Theorem II.17.3 and Corollary II.17.1, we have $\|\mathcal{V}(\mathbf{U}_{\theta',c}[s',\mathbf{n}',J'])\| = c\left|\frac{1-\theta'|\theta'|}{1+\theta'|\theta'|}\right|\|\mathbf{n}'\| = c\left|\frac{1-\theta'|\theta'|}{1+\theta'|\theta'|}\right|$. From this we can see, that the inequality $\theta'>0$ is true if and only if $\|\mathcal{V}(\mathbf{U}_{\theta',c}[s',\mathbf{n}',J'])\|<c$, that is if and only if $\theta,\theta_1>0$. In the case $\theta=-1$ or $\theta_1=-1$, according to Lemma II.18.5, the coordinate transform $\mathbf{U}_{\theta',c}[s',\mathbf{n}',J']$ is not v-determined. But, by Theorem II.17.3, this is possible only if $\theta'=-1$. Thus, in the case $\theta=-1$ or $\theta_1=-1$ the equality $\operatorname{sign}\theta'=\mathfrak{S}(\theta,\theta_1)$ also remains to be true.

**2,3.** According to the conditions of Lemma and Theorem II.17.3, for any $\mathrm{w}\in\mathcal{M}(\mathfrak{H})$ it is performed the equality:

$$\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\,\mathrm{w} = \mathbf{U}_{\theta',c}[s',\mathbf{n}',J']\,\mathrm{w} = \left(s'\varphi_0(\theta')\,\mathcal{T}(\mathrm{w})-\varphi_1(\theta')\frac{\langle\mathbf{n}',\mathrm{w}\rangle}{c}\right)\mathbf{e}_0+$$
$$+J'\left(c\varphi_1(\theta')\,\mathcal{T}(\mathrm{w})\,\mathbf{n}'-s'\varphi_0(\theta')\,\mathbf{X}_1[\mathbf{n}']\,\mathrm{w}+\mathbf{X}_1^{\perp}[\mathbf{n}']\,\mathrm{w}\right). \quad \text{(II.146)}$$

Matching the coefficients near the vector $\mathbf{e}_0$ in right-hand sides of the equalities (II.136) and (II.146) we deduce the equality:

$$ss_1\varphi_0(\theta)\,\varphi_0(\theta_1)\,\mathcal{T}(\mathrm{w})-\left(s\varphi_0(\theta)\,\varphi_1(\theta_1)\frac{\langle\mathbf{n}_1,\mathrm{w}\rangle}{c}+\varphi_1(\theta)\frac{\langle\mathbf{n},\mathrm{w}\rangle}{c}\right) =$$
$$= s'\varphi_0(\theta')\,\mathcal{T}(\mathrm{w})-\varphi_1(\theta')\frac{\langle\mathbf{n}',\mathrm{w}\rangle}{c}, \qquad \mathrm{w}\in\mathcal{M}(\mathfrak{H}). \quad \text{(II.147)}$$

Hence, if we substitute the vector $\mathrm{w}_0=\mathbf{e}_0$ to the last equality, we obtain:

$$ss_1\varphi_0(\theta)\,\varphi_0(\theta_1) = s'\varphi_0(\theta')\,;$$

Therefore, the equality (II.147) leads to:

$$s\varphi_0(\theta)\,\varphi_1(\theta_1)\,\frac{\langle \mathbf{n}_1, \mathrm{w}\rangle}{c} + \varphi_1(\theta)\,\frac{\langle \mathbf{n}, \mathrm{w}\rangle}{c} = \varphi_1(\theta')\,\frac{\langle \mathbf{n}', \mathrm{w}\rangle}{c}, \qquad \mathrm{w} \in \mathcal{M}(\mathfrak{H}),$$

that is:

$$\langle s\varphi_0(\theta)\,\varphi_1(\theta_1)\,\mathbf{n}_1 + \varphi_1(\theta)\,\mathbf{n}, \mathrm{w}\rangle = \langle \varphi_1(\theta')\,\mathbf{n}', \mathrm{w}\rangle, \qquad \mathrm{w} \in \mathcal{M}(\mathfrak{H}).$$

That is why:

$$s\varphi_0(\theta)\,\varphi_1(\theta_1)\,\mathbf{n}_1 + \varphi_1(\theta)\,\mathbf{n} = \varphi_1(\theta')\,\mathbf{n}'.$$

Thus, we have proved the equalities:

$$\begin{aligned} s'ss_1\varphi_0(\theta)\,\varphi_0(\theta_1) &= \varphi_0(\theta') \\ s\varphi_0(\theta)\,\varphi_1(\theta_1)\,\mathbf{n}_1 + \varphi_1(\theta)\,\mathbf{n} &= \varphi_1(\theta')\,\mathbf{n}' \end{aligned} \tag{II.148}$$

By conditions of Lemma, $\theta' \in [-1, 1] \setminus \{0\}$, hence $\varphi_0(\theta') = \frac{1+\theta'|\theta'|}{2|\theta'|} \geq 0$. Consequently, the first equality (II.148) stipulates the equality:

$$\varphi_0(\theta') = |\varphi_0(\theta')| = |\varphi_0(\theta)\,\varphi_0(\theta_1)|.$$

And, taking into account the condition $\theta, \theta_1 \neq -1$ (that is $\varphi_0(\theta)\,\varphi_0(\theta_1) \neq 0$), we get the equality:

$$s'ss_1 = \operatorname{sign}\left(\varphi_0(\theta)\,\varphi_0(\theta_1)\right).$$

**4.** Using the equalities (II.82), as well as first and second items of this Lemma, we obtain, $\varphi_1^2(\theta') = \varphi_0^2(\theta') - \operatorname{sign}\theta' = \varphi_0^2(\theta)\,\varphi_0^2(\theta_1) - \mathfrak{S}(\theta, \theta_1)$. Since, by conditions of Lemma, $\theta' \in [-1, 1] \setminus \{0\}$, then $\varphi_1(\theta') = \frac{1-\theta'|\theta'|}{2|\theta'|} \geq 0$. Hence, $\varphi_1(\theta') = \sqrt{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1) - \mathfrak{S}(\theta, \theta_1)}$.

**5.** Let $\theta \neq 1$ and $\theta_1 \neq 1$. It is easy to verify, that in this case $\sqrt{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1) - \mathfrak{S}(\theta, \theta_1)} > 0$. So, the five statement of this Lemma follows from its four statement together with the equality (II.148).

**6.** Substituting the vector $\mathrm{w} = \mathbf{e}_0$ into the equalities (II.136), (II.146) and taking into account the equalities $\langle \mathbf{n}, \mathbf{e}_0\rangle = \langle \mathbf{n}_1, \mathbf{e}_0\rangle = 0$ and $\mathbf{X}_1[\mathbf{n}]\,\mathbf{e}_0 = \mathbf{X}_1[\mathbf{n}]\,\mathbf{e}_0 = \mathbf{0}$, we receive:

$$\mathbf{E}_{\theta,c}[s, \mathbf{n}]\mathbf{E}_{\theta_1,c}[s_1, \mathbf{n}_1]\,\mathbf{e}_0 = ss_1\varphi_0(\theta)\,\varphi_0(\theta_1)\,\mathbf{e}_0 + cs_1\varphi_1(\theta)\,\varphi_0(\theta_1)\,\mathbf{n} + c\varphi_1(\theta_1)\,\mathbf{n}_1; \tag{II.149}$$

$$\begin{aligned} \mathbf{E}_{\theta,c}[s, \mathbf{n}]\mathbf{E}_{\theta_1,c}[s_1, \mathbf{n}_1]\,\mathbf{e}_0 &= \left(s'\varphi_0(\theta')\,\mathcal{T}(\mathbf{e}_0) - \varphi_1(\theta')\,\frac{\langle \mathbf{n}', \mathbf{e}_0\rangle}{c}\right)\mathbf{e}_0 + \\ &\quad + J'\left(c\varphi_1(\theta')\,\mathcal{T}(\mathbf{e}_0)\,\mathbf{n}' - s'\varphi_0(\theta')\,\mathbf{X}_1[\mathbf{n}']\,\mathbf{e}_0 + \right. \\ &\qquad \left. + \mathbf{X}_1^{\perp}[\mathbf{n}']\,\mathbf{e}_0\right) = s'\varphi_0(\theta')\,\mathbf{e}_0 + c\varphi_1(\theta')\,J'\mathbf{n}' \end{aligned} \tag{II.150}$$

Matching the right-hand sides of the equalities (II.149) we (II.150) deduce:

$$s'\varphi_0(\theta')\,\mathbf{e}_0 + c\varphi_1(\theta')\,J'\mathbf{n}' = ss_1\varphi_0(\theta)\,\varphi_0(\theta_1)\,\mathbf{e}_0 + cs_1\varphi_1(\theta)\,\varphi_0(\theta_1)\,\mathbf{n} + c\varphi_1(\theta_1)\,\mathbf{n}_1.$$

Hence, taking into account the first equality of (II.148) and the statement 4 of this Lemma, we have:

$$c\varphi_1(\theta')\,J'\mathbf{n}' = cs_1\varphi_1(\theta)\,\varphi_0(\theta_1)\,\mathbf{n} + c\varphi_1(\theta_1)\,\mathbf{n}_1;$$

$$J'\mathbf{n}' = \frac{s_1\varphi_1(\theta)\,\varphi_0(\theta_1)\,\mathbf{n} + \varphi_1(\theta_1)\,\mathbf{n}_1}{\varphi_1(\theta')} = \frac{s_1\varphi_1(\theta)\,\varphi_0(\theta_1)\,\mathbf{n} + \varphi_1(\theta_1)\,\mathbf{n}_1}{\sqrt{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1) - \mathfrak{S}(\theta, \theta_1)}}.$$

□

**Theorem II.18.1.** *Let $\mathbf{E}_{\theta,c}[s, \mathbf{n}], \mathbf{E}_{\theta_1,c}[s_1, \mathbf{n}_1] \in \mathfrak{OT}(\mathfrak{H}, c)$ be elementary generalized Lorentz transforms with orthogonal directing vectors ($\langle \mathbf{n}, \mathbf{n}_1\rangle = 0$). The product of the transforms $\mathbf{E}_{\theta,c}[s, \mathbf{n}]$ and $\mathbf{E}_{\theta_1,c}[s_1, \mathbf{n}_1]$ belongs to the class $\mathfrak{OT}(\mathfrak{H}, c)$ if and only if one of the following conditions is satisfied:*

$$1)\ \theta,\theta_1>0,\quad 2)\ \theta=1\ or\ \theta_1=1,\quad 3)\ \theta=\theta_1=-1.$$

*Proof.* The proof of Theorem will be divided into the following cases.

**Case 1**: $\theta,\theta_1>0$. In this case, according to Theorem II.17.3, $\|\mathcal{V}(\mathbf{E}_{\theta,c}[s,\mathbf{n}])\|=c\left|\frac{1-\theta|\theta|}{1+\theta|\theta|}\right|<c$ and $\mathcal{V}(\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1])<c$. Hence, by Lemma II.17.5, $\mathbf{E}_{\theta,c}[s,\mathbf{n}],\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\in\mathfrak{O}(\mathfrak{H},c)$. Since (in accordance with Assertion II.17.1) the set of operators $\mathfrak{O}(\mathfrak{H},c)$ is a group, then $\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\in\mathfrak{O}(\mathfrak{H},c)\subseteq\mathfrak{OT}(\mathfrak{H},c)$.

**Case 2**: $\theta=1$ or $\theta_1=1$. Suppose, that $\theta=1$. Then the coordinate transform $\mathbf{E}_{\theta,c}[s,\mathbf{n}]$ is represented in the form:

$$\begin{aligned}\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathrm{w}=\mathbf{E}_{1,c}[s,\mathbf{n}]\,\mathrm{w}&=s\mathcal{T}(\mathrm{w})\,\mathbf{e}_0-s\mathbf{X}_1[\mathbf{n}]\,\mathrm{w}+\mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}=\\&=s\left(\mathcal{T}(\mathrm{w})\,\mathbf{e}_0-\mathbf{X}_1[\mathbf{n}]\,\mathrm{w}+s\mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right)=\\&=s\left(\mathcal{T}(\mathrm{w})\,\mathbf{e}_0+\mathbb{I}_{-1,s}[\mathbf{n}]\left(\mathbf{X}_1[\mathbf{n}]\,\mathrm{w}+\mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right)\right).\end{aligned}$$

Hence, using the formula (II.35), we get:

$$\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathrm{w}=s\,(\mathbb{I}_{-1,s}[\mathbf{n}])^{\sim}\left(\mathcal{T}(\mathrm{w})\,\mathbf{e}_0+\mathbf{X}_1[\mathbf{n}]\,\mathrm{w}+\mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right)=s\,(\mathbb{I}_{-1,s}[\mathbf{n}])^{\sim}\,\mathrm{w}.$$

Therefore, $\mathbf{E}_{1,c}[s,\mathbf{n}]=s\,(\mathbb{I}_{-1,s}[\mathbf{n}])^{\sim}$. Similarly we can deduce, $\mathbf{E}_{1,c}[s,\mathbf{n}_1]=s\,(\mathbb{I}_{-1,s}[\mathbf{n}_1])^{\sim}$. According to (II.113) and (II.122), we have $((\mathbb{I}_{-1,s}[\mathbf{n}_1])^{\sim})^2=\left((\mathbb{I}_{-1,s}[\mathbf{n}_1])^2\right)^{\sim}=\widetilde{\mathbb{I}_1}=\mathbb{I}$. That is why, using Lemma II.18.1, we obtain:

$$\begin{aligned}\mathbf{E}_{\theta,c}[s,\mathbf{n}]&=s\,(\mathbb{I}_{-1,s}[\mathbf{n}])^{\sim}=s\left((\mathbb{I}_{-1,s}[\mathbf{n}])^{\sim}(\mathbb{I}_{-1,s}[\mathbf{n}_1])^{\sim}\right)(\mathbb{I}_{-1,s}[\mathbf{n}_1])^{\sim}=\\&=(\mathbb{I}_{-1,s}[\mathbf{n}]\,\mathbb{I}_{-1,s}[\mathbf{n}_1])^{\sim}\,\mathbf{E}_{1,c}[s,\mathbf{n}_1]=\mathbf{U}_{1,c}[s,\mathbf{n}_1,\mathbb{I}_{-1,s}[\mathbf{n}]\,\mathbb{I}_{-1,s}[\mathbf{n}_1]].\end{aligned}$$

Thus, in the case $\theta=1$ transforms $\mathbf{E}_{\theta,c}[s,\mathbf{n}]$ and $\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]$ have (in reality) parallel directions of motion. Hence, by Assertion II.18.1 we have:

$$\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]=\mathbf{U}_{1,c}[s,\mathbb{I}_1\mathbf{n}_1,\mathbb{I}_{-1,s}[\mathbf{n}]\,\mathbb{I}_{-1,s}[\mathbf{n}_1]]\ \mathbf{U}_{\theta_1,c}[s_1,\mathbf{n}_1,\mathbb{I}_1]\in\mathfrak{OT}(\mathfrak{H},c).$$

Similarly for $\theta_1=1$ we have $\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]=\mathbf{U}_{1,c}[s_1,\mathbf{n},\mathbb{I}_{-1,s_1}[\mathbf{n}_1]\,\mathbb{I}_{-1,s_1}[\mathbf{n}]]$. Since $\langle\mathbf{n},\mathbf{n}_1\rangle=0$, then $\mathbb{I}_{-1,s_1}[\mathbf{n}_1]\,\mathbb{I}_{-1,s_1}[\mathbf{n}]\,\mathbf{n}=-s_1\mathbf{n}$. Consequently, in accordance with Assertion II.18.1, we get:

$$\begin{aligned}&\mathbf{E}_{\theta,c}[s,\mathbf{n}]\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]=\\&\quad=\mathbf{U}_{\theta,c}[s,-s_1\mathbb{I}_{-1,s_1}[\mathbf{n}_1]\,\mathbb{I}_{-1,s_1}[\mathbf{n}]\,\mathbf{n},\mathbb{I}_1]\ \mathbf{U}_{1,c}[s_1,\mathbf{n},\mathbb{I}_{-1,s_1}[\mathbf{n}_1]\,\mathbb{I}_{-1,s_1}[\mathbf{n}]]\in\mathfrak{OT}(\mathfrak{H},c).\end{aligned}$$

**Case 3**: $\theta=\theta_1=-1$. Since $\varphi_0(-1)=0$, $\varphi_1(-1)=1$, then if this case the operators $\mathbf{E}_{\theta,c}[s,\mathbf{n}]$ and $\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]$ may be represented in the form:

$$\begin{aligned}\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathrm{w}&=-\frac{\langle\mathbf{n},\mathrm{w}\rangle}{c}\mathbf{e}_0+c\mathcal{T}(\mathrm{w})\,\mathbf{n}+\mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w},\\\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\,\mathrm{w}&=-\frac{\langle\mathbf{n}_1,\mathrm{w}\rangle}{c}\mathbf{e}_0+c\mathcal{T}(\mathrm{w})\,\mathbf{n}_1+\mathbf{X}_1^{\perp}[\mathbf{n}_1]\,\mathrm{w}.\end{aligned}$$

Hence, taking into account, that $\langle\mathbf{n},\mathbf{n}_1\rangle=0$, for arbitrary vector $\mathrm{w}\in\mathcal{M}(\mathfrak{H})$ we receive:

$$\begin{aligned}&\mathbf{E}_{\theta,c}[s,\mathbf{n}]\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\,\mathrm{w}=\\&\quad=-\frac{\left\langle\mathbf{n},-\frac{\langle\mathbf{n}_1,\mathrm{w}\rangle}{c}\mathbf{e}_0+c\mathcal{T}(\mathrm{w})\,\mathbf{n}_1+\mathbf{X}_1^{\perp}[\mathbf{n}_1]\,\mathrm{w}\right\rangle}{c}\mathbf{e}_0+\\&\qquad+c\mathcal{T}\left(-\frac{\langle\mathbf{n}_1,\mathrm{w}\rangle}{c}\mathbf{e}_0+c\mathcal{T}(\mathrm{w})\,\mathbf{n}_1+\mathbf{X}_1^{\perp}[\mathbf{n}_1]\,\mathrm{w}\right)\mathbf{n}+\\&\qquad+\mathbf{X}_1^{\perp}[\mathbf{n}]\left(-\frac{\langle\mathbf{n}_1,\mathrm{w}\rangle}{c}\mathbf{e}_0+c\mathcal{T}(\mathrm{w})\,\mathbf{n}_1+\mathbf{X}_1^{\perp}[\mathbf{n}_1]\,\mathrm{w}\right)=\end{aligned}$$

$$= -\frac{\left\langle \mathbf{n}, \mathbf{X}_1^{\perp}[\mathbf{n}_1]\,\mathrm{w}\right\rangle}{c}\mathbf{e}_0 - c\frac{\langle \mathbf{n}_1, \mathrm{w}\rangle}{c}\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\left(c\mathcal{T}(\mathrm{w})\,\mathbf{n}_1 + \mathbf{X}_1^{\perp}[\mathbf{n}_1]\,\mathrm{w}\right) =$$
$$= -\frac{\left\langle \mathbf{X}_1^{\perp}[\mathbf{n}_1]\,\mathbf{n}, \mathrm{w}\right\rangle}{c}\mathbf{e}_0 - \langle \mathbf{n}_1, \mathrm{w}\rangle\,\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\left(c\mathcal{T}(\mathrm{w})\,\mathbf{n}_1 + \mathbf{X}_1^{\perp}[\mathbf{n}_1]\,\mathrm{w}\right). \qquad \text{(II.151)}$$

Thus, using (II.18), we have:

$$\mathbf{X}_1^{\perp}[\mathbf{n}_1]\,\mathbf{n} = \mathbf{X}\mathbf{n} - \mathbf{X}_1[\mathbf{n}_1]\,\mathbf{n} = \mathbf{n} - \langle \mathbf{n}_1, \mathbf{n}\rangle\,\mathbf{n}_1 = \mathbf{n},$$
$$\mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathbf{n}_1 = \mathbf{n}_1.$$

Substituting the last equalities into (II.151), we obtain:

$$\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\,\mathrm{w} = -\frac{\langle \mathbf{n}, \mathrm{w}\rangle}{c}\mathbf{e}_0 + c\mathcal{T}(\mathrm{w})\,\mathbf{n}_1 - \langle \mathbf{n}_1, \mathrm{w}\rangle\,\mathbf{n} +$$
$$+ \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathbf{X}_1^{\perp}[\mathbf{n}_1]\,\mathrm{w} \quad (\mathrm{w} \in \mathcal{M}(\mathfrak{H})). \qquad \text{(II.152)}$$

(emphasize that, since $\langle \mathbf{n}, \mathbf{n}_1\rangle = 0$, then $\mathbf{X}_1[\mathbf{n}]\,\mathbf{X}_1[\mathbf{n}_1] = \mathbb{O}$, and therefore, according to (II.18), operators $\mathbf{X}_1^{\perp}[\mathbf{n}] = \mathbf{X} - \mathbf{X}_1[\mathbf{n}]$ and $\mathbf{X}_1^{\perp}[\mathbf{n}_1] = \mathbf{X} - \mathbf{X}_1[\mathbf{n}_1]$ are commuting).

Denote:

$$\mathcal{J}_{\mathbf{n},\mathbf{n}_1} x = \langle \mathbf{n}, x\rangle\,\mathbf{n}_1 - \langle \mathbf{n}_1, x\rangle\,\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}_1]\,\mathbf{X}_1^{\perp}[\mathbf{n}]\,x, \quad x \in \mathfrak{H}_1.$$

By means of the operator $\mathcal{J}_{\mathbf{n},\mathbf{n}_1}$, using correlations (II.18), we can rewrite the equality (II.152) as follows:

$$\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]\,\mathrm{w} = -\frac{\langle \mathbf{n}, \mathrm{w}\rangle}{c}\mathbf{e}_0 + \mathcal{J}_{\mathbf{n},\mathbf{n}_1}\left(c\mathcal{T}(\mathrm{w})\,\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right) \quad (\mathrm{w} \in \mathcal{M}(\mathfrak{H})). \quad \text{(II.153)}$$

Now, we are going to prove, that $\mathcal{J}_{\mathbf{n},\mathbf{n}_1} \in \mathfrak{U}(\mathfrak{H}_1)$. Since $\langle \mathbf{n}, \mathbf{n}_1\rangle = 0$ and operators $\mathbf{X}_1^{\perp}[\mathbf{n}]$, $\mathbf{X}_1^{\perp}[\mathbf{n}_1]$ are commuting, then for any $x \in \mathfrak{H}_1$ vectors $\mathbf{n}, \mathbf{n}_1$ and $\mathbf{X}_1^{\perp}[\mathbf{n}_1]\,\mathbf{X}_1^{\perp}[\mathbf{n}]\,x$ are pairwise orthogonal. That is why, by definition of operator $\mathcal{J}_{\mathbf{n},\mathbf{n}_1}$, for all $x \in \mathfrak{H}_1$ we have the equality:

$$\|\mathcal{J}_{\mathbf{n},\mathbf{n}_1} x\|^2 = \langle \mathbf{n}, x\rangle^2 + \langle \mathbf{n}_1, x\rangle^2 + \left\|\mathbf{X}_1^{\perp}[\mathbf{n}_1]\,\mathbf{X}_1^{\perp}[\mathbf{n}]\,x\right\|^2.$$

According to (II.18), we have:

$$\mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathbf{X}_1^{\perp}[\mathbf{n}_1] = (\mathbf{X} - \mathbf{X}_1[\mathbf{n}])\,(\mathbf{X} - \mathbf{X}_1[\mathbf{n}_1]) =$$
$$= \mathbf{X} - \mathbf{X}\mathbf{X}_1[\mathbf{n}_1] - \mathbf{X}_1[\mathbf{n}]\,\mathbf{X} + \mathbf{X}_1[\mathbf{n}]\,\mathbf{X}_1[\mathbf{n}_1] = \mathbf{X} - \mathbf{X}_1[\mathbf{n}_1] - \mathbf{X}_1[\mathbf{n}]. \quad \text{(II.154)}$$

Hence, for an arbitrary $\forall x \in \mathfrak{H}_1$ we get:

$$\|\mathcal{J}_{\mathbf{n},\mathbf{n}_1} x\|^2 = \langle \mathbf{n}, x\rangle^2 + \langle \mathbf{n}_1, x\rangle^2 + \left\|\mathbf{X}_1^{\perp}[\mathbf{n}_1]\,\mathbf{X}_1^{\perp}[\mathbf{n}]\,x\right\|^2 =$$
$$= \left\|\langle \mathbf{n}, x\rangle\,\mathbf{n} + \langle \mathbf{n}_1, x\rangle\,\mathbf{n}_1 + \mathbf{X}_1^{\perp}[\mathbf{n}_1]\,\mathbf{X}_1^{\perp}[\mathbf{n}]\,x\right\|^2 =$$
$$= \left\|\mathbf{X}_1[\mathbf{n}]\,x + \mathbf{X}_1[\mathbf{n}_1]\,x + \mathbf{X}_1^{\perp}[\mathbf{n}_1]\,\mathbf{X}_1^{\perp}[\mathbf{n}]\,x\right\|^2 =$$
$$= \|\mathbf{X}_1[\mathbf{n}]\,x + \mathbf{X}_1[\mathbf{n}_1]\,x + (\mathbf{X} - \mathbf{X}_1[\mathbf{n}_1] - \mathbf{X}_1[\mathbf{n}])\,x\|^2 = \|\mathbf{X}x\|^2 = \|x\|^2.$$

Therefore, the operator $\mathcal{J}_{\mathbf{n},\mathbf{n}_1}$ is isometric. Using the definition of the operator $\mathcal{J}_{\mathbf{n},\mathbf{n}_1}$, commutation of the operators $\mathbf{X}_1^{\perp}[\mathbf{n}]$ and $\mathbf{X}_1^{\perp}[\mathbf{n}_1]$ as well as the equality (II.154), it is not hard to verify, that $\mathcal{J}_{\mathbf{n},\mathbf{n}_1}\mathcal{J}_{\mathbf{n}_1,\mathbf{n}} x = \mathcal{J}_{\mathbf{n}_1,\mathbf{n}}\mathcal{J}_{\mathbf{n},\mathbf{n}_1} x = x$ $(x \in \mathfrak{H}_1)$. Consequently the operator $\mathcal{J}_{\mathbf{n},\mathbf{n}_1}$ has the inverse operator $\mathcal{J}_{\mathbf{n}_1,\mathbf{n}}$ on $\mathfrak{H}_1$. Thus, $\mathcal{J}_{\mathbf{n},\mathbf{n}_1} \in \mathfrak{U}(\mathfrak{H}_1)$.

That is why, according to the equality (II.153), we have, $\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1] = \mathbf{U}_{-1,c}[1,\mathbf{n},\mathcal{J}_{\mathbf{n},\mathbf{n}_1}] \in \mathfrak{OT}(\mathfrak{H}, c)$.

**Case 4**: $\theta_1 < 0$, $\theta \notin \{-1, 1\}$. Let us assume, that in this case the coordinate transform $\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]$ belongs to the class $\mathfrak{OT}(\mathfrak{H}, c)$. Then, by Theorem II.17.3, there exist numbers $s' \in \{-1,1\}$, $\theta' \in [-1,1] \setminus \{0\}$, vector $\mathbf{n}' \in \mathbf{B}_1(\mathfrak{H}_1)$ and operator $J' \in \mathfrak{U}(\mathfrak{H}_1)$ such, that $\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1] = \mathbf{U}_{\theta',c}[s',\mathbf{n}',J']$. From here, by Lemma II.18.6, we have $\mathbf{n}' = \frac{s\varphi_0(\theta)\varphi_1(\theta_1)\mathbf{n}_1+\varphi_1(\theta)\mathbf{n}}{\sqrt{\varphi_0^2(\theta)\varphi_0^2(\theta_1)-\mathfrak{S}(\theta,\theta_1)}}$. Consequently, taking into account, that $\langle \mathbf{n},\mathbf{n}_1\rangle = 0$ and applying (II.82), we obtain:

$$\begin{aligned}\|\mathbf{n}'\| &= \sqrt{\frac{\varphi_0^2(\theta)\,\varphi_1^2(\theta_1)+\varphi_1^2(\theta)}{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)-\mathfrak{S}(\theta,\theta_1)}} = \\ &= \sqrt{\frac{\varphi_0^2(\theta)\left(\varphi_0^2(\theta_1)-\operatorname{sign}\theta_1\right)+\left(\varphi_0^2(\theta)-\operatorname{sign}\theta\right)}{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)-\mathfrak{S}(\theta,\theta_1)}} = \\ &= \sqrt{\frac{\varphi_0^2(\theta)\left(\varphi_0^2(\theta_1)+1\right)+\left(\varphi_0^2(\theta)-\operatorname{sign}\theta\right)}{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)+1}} = \sqrt{1+\frac{2\varphi_0^2(\theta)-1-\operatorname{sign}\theta}{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)+1}}.\end{aligned} \tag{II.155}$$

Since $\theta \notin \{-1,1\}$, then $\varphi_0(\theta) = \frac{1+\theta|\theta|}{2|\theta|} \neq 0$, $\varphi_1(\theta) = \frac{1-\theta|\theta|}{2|\theta|} \neq 0$. Hence, in the case $\theta < 0$ from the equality (II.155) we get:

$$\|\mathbf{n}'\| = \sqrt{1+\frac{2\varphi_0^2(\theta)-1-(-1)}{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)+1}} = \sqrt{1+\frac{2\varphi_0^2(\theta)}{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)+1}} > 1,$$

and in the case $\theta > 0$ we receive:

$$\|\mathbf{n}'\| = \sqrt{1+\frac{2\varphi_0^2(\theta)-2}{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)+1}} = \sqrt{1+\frac{2\varphi_1^2(\theta)}{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)+1}} > 1.$$

Thus, in the both cases we have, that $\|\mathbf{n}'\| > 1$, which contradicts to the condition $\mathbf{n}' \in \mathbf{B}_1(\mathfrak{H}_1)$. The last contradiction proves, that the product of the operators $\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]$ can not belong to $\mathfrak{OT}(\mathfrak{H},c)$.

**Case 5**: $\theta < 0$, $\theta_1 \notin \{-1,1\}$. Let us assume, that in this case the coordinate transform $\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1]$ belongs to the class $\mathfrak{OT}(\mathfrak{H},c)$. Then, by Theorem II.17.3, there exist numbers $s' \in \{-1,1\}$, $\theta' \in [-1,1] \setminus \{0\}$, vector $\mathbf{n}' \in \mathbf{B}_1(\mathfrak{H}_1)$ and operator $J' \in \mathfrak{U}(\mathfrak{H}_1)$ such, that $\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1] = \mathbf{U}_{\theta',c}[s',\mathbf{n}',J']$. From here, by Lemma II.18.6, we have, $J'\mathbf{n}' = \frac{s_1\varphi_1(\theta)\varphi_0(\theta_1)\mathbf{n}+\varphi_1(\theta_1)\mathbf{n}_1}{\sqrt{\varphi_0^2(\theta)\varphi_0^2(\theta_1)-\mathfrak{S}(\theta,\theta_1)}}$. Consequently, taking into account, that $\langle \mathbf{n},\mathbf{n}_1\rangle = 0$ and applying (II.82), similarly to the previous case we obtain:

$$\|J'\mathbf{n}'\| = \sqrt{\frac{\varphi_0^2(\theta_1)\,\varphi_1^2(\theta)+\varphi_1^2(\theta_1)}{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)-\mathfrak{S}(\theta,\theta_1)}} = \sqrt{1+\frac{2\varphi_0^2(\theta_1)-1-\operatorname{sign}\theta_1}{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)+1}}. \tag{II.156}$$

Since $\theta_1 \notin \{-1,1\}$, then $\varphi_0(\theta_1) \neq 0$, $\varphi_1(\theta_1) \neq 0$. Hence in the case $\theta_1 < 0$ from the equality (II.156) we get:

$$\|J'\mathbf{n}'\| = \sqrt{1+\frac{2\varphi_0^2(\theta_1)-1-(-1)}{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)+1}} = \sqrt{1+\frac{2\varphi_0^2(\theta_1)}{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)+1}} > 1,$$

and in the case $\theta_1 > 0$ we receive:

$$\|J'\mathbf{n}'\| = \sqrt{1+\frac{2\varphi_0^2(\theta_1)-2}{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)+1}} = \sqrt{1+\frac{2\varphi_1^2(\theta_1)}{\varphi_0^2(\theta)\,\varphi_0^2(\theta_1)+1}} > 1.$$

Thus, in the both cases we have, that $\|J'\mathbf{n}'\| > 1$. But, since $J'$ is unitary operator and $\mathbf{n}' \in \mathbf{B}_1(\mathfrak{H}_1)$, the equality $\|J'\mathbf{n}'\| = 1$ must hold. The last contradiction proves, that in this case we have, that $\mathbf{E}_{\theta,c}[s,\mathbf{n}]\,\mathbf{E}_{\theta_1,c}[s_1,\mathbf{n}_1] \notin \mathfrak{OT}(\mathfrak{H},c)$ also. □

The next corollary immediately follows from Theorem II.18.1.

**Corollary II.18.7.** *Let $\mathfrak{H}$ be a real Hilbert space such, that* $\dim(\mathfrak{H}) > 1$*. Then the class of operators $\mathfrak{OT}(\mathfrak{H}, c)$ does not form a group of operators over the space $\mathcal{M}(\mathfrak{H})$.*

*Proof.* Indeed, in the case $\dim(\mathfrak{H}) > 1$ there exist vectors $\mathbf{n}, \mathbf{n}_1 \in \mathbf{B}_1(\mathfrak{H}_1)$ such, that $\langle \mathbf{n}, \mathbf{n}_1 \rangle = 0$. $\square$

Main results of this Section were published in [7].

# 19 Kinematic Sets, Generated by Special Relativity and its Tachyon Extensions.

Let $(\mathfrak{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a Hilbert space over the Real field. Space $\mathfrak{H}$ generates the coordinate space $\widehat{\mathfrak{H}} = (\mathfrak{H}, \mathcal{T}_{\mathfrak{H}}, \mathbb{L}_{\mathfrak{H}}, \rho_{\mathfrak{H}}, \|\cdot\|, \langle\cdot,\cdot\rangle)$, where $\rho_{\mathfrak{H}}$ and $\mathcal{T}_{\mathfrak{H}}$ are metrics and topology, generated by the norm $\|\cdot\|$ on the space $\mathfrak{H}$, as well as $\mathbb{L}_{\mathfrak{H}}$ is the natural linear structure of the space $\mathfrak{H}$.

Recall that in Subsection 17.1 (page 99) we have denoted by $\mathcal{L}(\mathfrak{H})$ the space of (homogeneous) linear continuous operators over the space $\mathfrak{H}$. Denote by $\mathcal{L}^{\times}(\mathfrak{H})$ the space of all operators of affine transformations over the space $\mathfrak{H}$, that is $\mathcal{L}^{\times}(\mathfrak{H}) = \left\{\mathbf{A}_{[\mathbf{a}]} \mid \mathbf{A} \in \mathcal{L}(\mathfrak{H}), \mathbf{a} \in \mathfrak{H}\right\}$, where $\mathbf{A}_{[\mathbf{a}]}x = \mathbf{A}x + \mathbf{a}$, $x \in \mathfrak{H}$.

Denote via $\mathbf{Pk}(\mathfrak{H})$ the set of all operators $\mathbf{S} \in \mathcal{L}^{\times}(\mathcal{M}(\mathfrak{H}))$, which has the continuous inverse operator $\mathbf{S}^{-1} \in \mathcal{L}^{\times}(\mathcal{M}(\mathfrak{H}))$. Operators $\mathbf{S} \in \mathbf{Pk}(\mathfrak{H})$ will be named as ***(affine) coordinate transform operators***.

Let, $\mathcal{B}$ be any base changeable set such, that $\mathfrak{Bs}(\mathcal{B}) \subseteq \mathfrak{H} = \mathbf{Zk}\left(\widehat{\mathfrak{H}}\right)$ and $\mathbb{T}\mathbf{m}(\mathcal{B}) = (\mathbb{R}, \leq)$, where $\leq$ is the standard order in the field of real numbers $\mathbb{R}$ {[16]}. Then $\mathbb{B}\mathfrak{s}(\mathcal{B}) \subseteq \mathbb{R} \times \mathfrak{H} = \mathcal{M}(\mathfrak{H})$. Any set $\mathbb{S} \subseteq \mathbf{Pk}(\mathfrak{H})$ is the transforming set of bijections relatively the $\mathcal{B}$ on $\mathfrak{H} = \mathbf{Zk}\left(\widehat{\mathfrak{H}}\right)$ (in the sense of Example I.11.2). Therefore, we can put:

$$\mathfrak{Kim}(\mathbb{S}, \mathcal{B}; \mathfrak{H}) := \mathfrak{Kim}\left(\mathbb{S}, \mathcal{B}, \widehat{\mathfrak{H}}\right),$$

where the kinematic set $\mathfrak{Kim}\left(\mathbb{S}, \mathcal{B}, \widehat{\mathfrak{H}}\right)$ is defined in (II.11). Now, we deduce the following corollary from Theorem II.16.2.

**Corollary II.19.1.** *The kinematic set $\mathfrak{Kim}(\mathbb{S}, \mathcal{B}; \mathfrak{H})$ allows universal coordinate transform.*

In Section 17 we have defined the operator of kind $\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J]$ for any fixed values $c \in (0,\infty)$, $\lambda \in [0,\infty] \setminus \{c\}$, $s \in \{-1,1\}$, $J \in \mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ (see formula (II.76)). Now we extend definition of this operator to the case $c = \infty$. Namely, in this case we put:

$$\mathbf{W}_{\lambda,\infty}[s, \mathbf{n}, J]\,\mathrm{w} := s\mathcal{T}(\mathrm{w})\mathbf{e}_0 + J\left((\lambda\mathcal{T}(\mathrm{w}) - s\langle\mathbf{n}, \mathrm{w}\rangle)\,\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right) \quad (\forall \mathrm{w} \in \mathcal{M}(\mathfrak{H})),$$

where $\lambda \in [0,\infty)$. Thus, for any fixed values $c \in (0,\infty]$, $\lambda \in [0,\infty] \setminus \{c\}$, $s \in \{-1,1\}$, $J \in \mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ and $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ we have:

$$\mathbf{W}_{\lambda,c}[s, \mathbf{n}, J]\,\mathrm{w} =$$

$$= \begin{cases} \dfrac{\left(s\mathcal{T}(\mathrm{w}) - \frac{\lambda}{c^2}\langle\mathbf{n},\mathrm{w}\rangle\right)}{\sqrt{\left|1-\frac{\lambda^2}{c^2}\right|}}\mathbf{e}_0 + J\left(\dfrac{\lambda\mathcal{T}(\mathrm{w}) - s\langle\mathbf{n},\mathrm{w}\rangle}{\sqrt{\left|1-\frac{\lambda^2}{c^2}\right|}}\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right), & \lambda < \infty,\ c < \infty; \\ -\frac{\langle\mathbf{n},\mathrm{w}\rangle}{c}\mathbf{e}_0 + J\left(c\mathcal{T}(\mathrm{w})\,\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right), & \lambda = \infty,\ c < \infty; \\ s\mathcal{T}(\mathrm{w})\mathbf{e}_0 + J\left((\lambda\mathcal{T}(\mathrm{w}) - s\langle\mathbf{n},\mathrm{w}\rangle)\,\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right), & \lambda < \infty,\ c = \infty. \end{cases} \quad \text{(II.157)}$$

[16] Such base changeable set $\mathcal{B}$ exists, because, for example, we may put $\mathcal{B} := \mathcal{A}t(\mathbb{R}, \mathcal{R})$, where $\mathcal{R}$ is a system of abstract trajectories from $\mathbb{R}$ to a set $\mathbf{M} \subseteq \mathfrak{H}$.

In the case $\mathfrak{H} = \mathbb{R}^3$ operators of kind $\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J]$ $(\lambda < \infty)$ become Galilean transforms, "started" from the origin at zero time point. It is not difficult prove, that $\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J] = \lim_{c\to\infty}\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]$, where the convergence is understood in the sense of uniform operator topology.

**Assertion II.19.1.** *For any fixed* $\lambda \in [0,\infty)$, $s \in \{-1,1\}$, $J \in \mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ *operator* $\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J]$ *is a linear coordinate transform operator (that is* $\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J]$ *has the inverse operator* $\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J]^{-1} \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$*). Moreover:*

$$\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J]^{-1} = \mathbf{W}_{\lambda,\infty}\left[s,J\mathbf{n},J^{-1}\right]$$

*Proof.* It is easy to verify, that $\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J] \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$. Now, we are going to prove the equality:

$$\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J]\,\mathbf{W}_{\lambda,\infty}\left[s,J\mathbf{n},J^{-1}\right] = \mathbb{I}, \tag{II.158}$$

where $\mathbb{I} = \mathbb{I}_{\mathcal{M}(\mathfrak{H})}$ is the identity operator on the space $\mathcal{M}(\mathfrak{H})$. Chose any $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$. According to (II.157) we have:

$$\begin{aligned}\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J]\mathbf{W}_{\lambda,\infty}\left[s,J\mathbf{n},J^{-1}\right]\mathrm{w} &= \mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J]\,\widetilde{\mathrm{w}}, \quad \text{where} \\ \widetilde{\mathrm{w}} = \mathbf{W}_{\lambda,\infty}\left[s,J\mathbf{n},J^{-1}\right]\mathrm{w} &= s\mathcal{T}(\mathrm{w})\mathbf{e}_0 + \\ &+ J^{-1}\left((\lambda\mathcal{T}(\mathrm{w}) - s\langle J\mathbf{n},\mathrm{w}\rangle)J\mathbf{n} + \mathbf{X}_1^{\perp}[J\mathbf{n}]\mathrm{w}\right).\end{aligned} \tag{II.159}$$

Next, applying (II.14), (II.17) and taking into account that $J$ is the unitary operator on the subspace $\mathfrak{H}_1 \subseteq \mathcal{M}(\mathfrak{H})$, we obtain:

$$\begin{aligned}\mathcal{T}(\widetilde{\mathrm{w}}) &= s\mathcal{T}(\mathrm{w}); \\ \langle\mathbf{n},\widetilde{\mathrm{w}}\rangle &= (\lambda\mathcal{T}(\mathrm{w}) - s\langle J\mathbf{n},\mathrm{w}\rangle)\langle\mathbf{n},\mathbf{n}\rangle + \left\langle\mathbf{n},J^{-1}\mathbf{X}_1^{\perp}[J\mathbf{n}]\mathrm{w}\right\rangle = \\ &= (\lambda\mathcal{T}(\mathrm{w}) - s\langle J\mathbf{n},\mathrm{w}\rangle) + \left\langle J\mathbf{n},\mathbf{X}_1^{\perp}[J\mathbf{n}]\mathrm{w}\right\rangle = (\lambda\mathcal{T}(\mathrm{w}) - s\langle J\mathbf{n},\mathrm{w}\rangle); \\ \mathbf{X}_1^{\perp}[\mathbf{n}]\,\widetilde{\mathrm{w}} &= (\mathbf{X} - \mathbf{X}_1[\mathbf{n}])\,\widetilde{\mathrm{w}} = \\ &= J^{-1}\left((\lambda\mathcal{T}(\mathrm{w}) - s\langle J\mathbf{n},\mathrm{w}\rangle)J\mathbf{n} + \mathbf{X}_1^{\perp}[J\mathbf{n}]\mathrm{w}\right) - \langle\mathbf{n},\widetilde{\mathrm{w}}\rangle\mathbf{n} = \\ &= J^{-1}\left(\langle\mathbf{n},\widetilde{\mathrm{w}}\rangle J\mathbf{n} + \mathbf{X}_1^{\perp}[J\mathbf{n}]\mathrm{w}\right) - \langle\mathbf{n},\widetilde{\mathrm{w}}\rangle\mathbf{n} = J^{-1}\mathbf{X}_1^{\perp}[J\mathbf{n}]\mathrm{w}.\end{aligned}$$

Herefrom, using (II.159), (II.157) and (II.18) we deduce:

$$\begin{aligned}&\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J]\,\mathbf{W}_{\lambda,\infty}\left[s,J\mathbf{n},J^{-1}\right]\mathrm{w} = \\ &\quad = s\mathcal{T}(\widetilde{\mathrm{w}})\mathbf{e}_0 + J\left((\lambda\mathcal{T}(\widetilde{\mathrm{w}}) - s\langle\mathbf{n},\widetilde{\mathrm{w}}\rangle)\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\widetilde{\mathrm{w}}\right) = s(s\mathcal{T}(\mathrm{w}))\mathbf{e}_0 + \\ &\qquad + J\left((\lambda(s\mathcal{T}(\mathrm{w})) - s(\lambda\mathcal{T}(\mathrm{w}) - s\langle J\mathbf{n},\mathrm{w}\rangle))\mathbf{n} + J^{-1}\mathbf{X}_1^{\perp}[J\mathbf{n}]\mathrm{w}\right) = \\ &\quad = \mathcal{T}(\mathrm{w})\mathbf{e}_0 + \langle J\mathbf{n},\mathrm{w}\rangle J\mathbf{n} + \mathbf{X}_1^{\perp}[J\mathbf{n}]\mathrm{w} = \widehat{\mathbf{T}}\mathrm{w} + \mathbf{X}_1[J\mathbf{n}]\mathrm{w} + \mathbf{X}_1^{\perp}[J\mathbf{n}]\mathrm{w} = \mathrm{w}.\end{aligned}$$

Equality (II.158) is proved. Applying the equality (II.158) to the operator $\mathbf{W}_{\lambda,\infty}[s,J\mathbf{n},J^{-1}]$, we obtain the equality: $\mathbf{W}_{\lambda,\infty}[s,J\mathbf{n},J^{-1}]\,\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J] = \mathbb{I}$. Thus, $\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J]^{-1} = \mathbf{W}_{\lambda,\infty}[s,J\mathbf{n},J^{-1}] \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$. □

Let $J \in \mathfrak{U}(\mathfrak{H}_1)$, $s \in \{-1,1\}$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$. Denote:

$$J_{(s,\mathbf{n})} := J\,\mathbb{I}_{-s,1}[\mathbf{n}], \tag{II.160}$$

where operator $\mathbb{I}_{-s,1}[\mathbf{n}]$ is defined by (II.122). Using operator (II.160) we rewrite representation (II.157) of operator $\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J]$ in more convenient (for some considerations) form. Applying the equality $(\mathbb{I}_{-s,1}[\mathbf{n}])^2 = \mathbb{I}$ as well as the equalities (II.157), (II.122) and (II.17) for any $\lambda \in [0,\infty)$, $s \in \{-1,1\}$, $J \in \mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ and $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ we obtain:

$$\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J]\mathrm{w} = s\mathcal{T}(\mathrm{w})\mathbf{e}_0 + J_{(s,\mathbf{n})}\mathbb{I}_{-s,1}[\mathbf{n}]\left((\lambda\mathcal{T}(\mathrm{w}) - s\langle\mathbf{n},\mathrm{w}\rangle)\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\mathrm{w}\right) =$$

$$= s\mathcal{T}(\mathrm{w})\mathbf{e}_0 + J_{(s,\mathbf{n})}\left(\left(\langle \mathbf{n}, \mathrm{w}\rangle - \lambda s\mathcal{T}(\mathrm{w})\right)\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\right) =$$
$$= s\mathcal{T}(\mathrm{w})\mathbf{e}_0 + J_{(s,\mathbf{n})}\left(\left(\langle \mathbf{n}, \mathrm{w}\rangle - \lambda s\mathcal{T}(\mathrm{w})\right)\mathbf{n} + \mathbf{X}\mathrm{w} - \langle \mathbf{n}, \mathrm{w}\rangle\,\mathbf{n}\right) =$$
$$= s\mathcal{T}(\mathrm{w})\mathbf{e}_0 + J_{(s,\mathbf{n})}\left(\mathbf{X}\mathrm{w} - \lambda s\mathcal{T}(\mathrm{w})\mathbf{n}\right).$$

Hence, for any $\lambda \in [0,\infty)$, $s \in \{-1,1\}$, $J \in \mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ and $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ we have:

$$\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J]\,\mathrm{w} = s\mathcal{T}(\mathrm{w})\mathbf{e}_0 + J_{(s,\mathbf{n})}\left(\mathbf{X}\mathrm{w} - \lambda s\mathcal{T}(\mathrm{w})\mathbf{n}\right); \tag{II.161}$$
$$\mathcal{T}\left(\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J]\,\mathrm{w}\right) = s\mathcal{T}(\mathrm{w}); \tag{II.162}$$
$$\mathbf{X}\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J]\,\mathrm{w} = J_{(s,\mathbf{n})}\left(\mathbf{X}\mathrm{w} - \lambda s\mathcal{T}(\mathrm{w})\mathbf{n}\right). \tag{II.163}$$

Denote by $\mathfrak{O}(\mathfrak{H},\infty)$ the following class of operators:

$$\mathfrak{O}(\mathfrak{H},\infty) := \left\{\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J] \mid \lambda \in [0,\infty),\ s \in \{-1,1\},\ J \in \mathfrak{U}(\mathfrak{H}_1),\ \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)\right\}.$$

Using (II.161), (II.162), (II.163) for any operators $L = \mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J] \in \mathfrak{O}(\mathfrak{H},\infty)$, $L_1 = \mathbf{W}_{\lambda_1,\infty}[s_1,\mathbf{n}_1,J_1] \in \mathfrak{O}(\mathfrak{H},\infty)$ (where $\lambda,\lambda_1 \in [0,\infty)$, $s,s_1 \in \{-1,1\}$, $J,J_1 \in \mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n},\mathbf{n}_1 \in \mathbf{B}_1(\mathfrak{H}_1)$) and arbitrary $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ we get:

$$L_1L\mathrm{w} = \mathbf{W}_{\lambda_1,\infty}[s_1,\mathbf{n}_1,J_1]\,\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J]\,\mathrm{w} =$$
$$= s_1\left(s\mathcal{T}(\mathrm{w})\right)\mathbf{e}_0 + (J_1)_{(s_1,\mathbf{n}_1)}\left(J_{(s,\mathbf{n})}\left(\mathbf{X}\mathrm{w} - \lambda s\mathcal{T}(\mathrm{w})\mathbf{n}\right) - \lambda_1 s_1\left(s\mathcal{T}(\mathrm{w})\right)\mathbf{n}_1\right) =$$
$$= ss_1\mathcal{T}(\mathrm{w})\,\mathbf{e}_0 + (J_1)_{(s_1,\mathbf{n}_1)}\,J_{(s,\mathbf{n})}\left(\mathbf{X}\mathrm{w} - \lambda s\mathcal{T}(\mathrm{w})\mathbf{n} - \lambda_1 s_1 s\mathcal{T}(\mathrm{w})\,J_{(s,\mathbf{n})}^{-1}\mathbf{n}_1\right) =$$
$$= ss_1\mathcal{T}(\mathrm{w})\,\mathbf{e}_0 + (J_1)_{(s_1,\mathbf{n}_1)}\,J_{(s,\mathbf{n})}\left(\mathbf{X}\mathrm{w} - ss_1\mathcal{T}(\mathrm{w})\left(\lambda s_1\mathbf{n} - \lambda_1 J_{(s,\mathbf{n})}^{-1}\mathbf{n}_1\right)\right) =$$
$$= ss_1\mathcal{T}(\mathrm{w})\,\mathbf{e}_0 + (J_2)_{(ss_1,\mathbf{n}_2)}\left(\mathbf{X}\mathrm{w} - \lambda_2 ss_1\mathcal{T}(\mathrm{w})\mathbf{n}_2\right) = \mathbf{W}_{\lambda_2,\infty}[ss_1,\mathbf{n}_2,J_2]\,\mathrm{w},$$

where

$$\lambda_2 = \left\|\lambda s_1\mathbf{n} - \lambda_1 J_{(s,\mathbf{n})}^{-1}\mathbf{n}_1\right\|; \quad \mathbf{n}_2 = \begin{cases} \frac{\lambda s_1\mathbf{n} - \lambda_1 J_{(s,\mathbf{n})}^{-1}\mathbf{n}_1}{\lambda_2}, & \lambda_2 \neq 0 \\ \mathbf{n}, & \lambda_2 = 0 \end{cases}; \quad J_2 = (J_1)_{(s_1,\mathbf{n}_1)}\,J_{(s,\mathbf{n})}\mathbb{I}_{-ss_1,1}[\mathbf{n}_2].$$

It is easy to see, that $\lambda_2 \in [0,\infty)$, $\mathbf{n}_2 \in \mathbf{B}_1(\mathfrak{H}_1)$ and $J_2 \in \mathfrak{U}(\mathfrak{H}_1)$. That is why the product of operators $\mathbf{W}_{\lambda_1,\infty}[s_1,\mathbf{n}_1,J_1]\,\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J]$ may be represented in the form:

$$\mathbf{W}_{\lambda_1,\infty}[s_1,\mathbf{n}_1,J_1]\,\mathbf{W}_{\lambda,\infty}[s,\mathbf{n},J] = \mathbf{W}_{\lambda_2,\infty}[ss_1,\mathbf{n}_2,J_2], \tag{II.164}$$

where $\lambda_2 \in [0,\infty)$, $ss_1 \in \{-1,1\}$, $\mathbf{n}_2 \in \mathbf{B}_1(\mathfrak{H}_1)$, $J_2 \in \mathfrak{U}(\mathfrak{H}_1)$.

Thus we have seen, that for any operators $L_1, L \in \mathfrak{O}(\mathfrak{H},\infty)$ the operator $L_1L$ belongs to $\mathfrak{O}(\mathfrak{H},\infty)$. The last result togrther with Assertion II.19.1 leads to the following conclusion:

**Corollary II.19.2.** *The set of operators $\mathfrak{O}(\mathfrak{H},\infty)$ is a group of operators over the Minkowski space $\mathcal{M}(\mathfrak{H})$ over the Hilbert space $\mathfrak{H}$.*

By analogy with (II.99) we may introduce the class of operators $\mathfrak{O}_+(\mathfrak{H},\infty)$ (in the case, where the velocity of light is equal to infinity):

$$\mathfrak{O}_+(\mathfrak{H},\infty) = \left\{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J] \in \mathfrak{O}(\mathfrak{H},\infty) \mid s = 1\right\}.$$

Applying Assertion II.19.1 and formula (II.164) it is easy to obtain the following corollary.

**Corollary II.19.3.** *The set of operators $\mathfrak{O}_+(\mathfrak{H},\infty)$ is a group of operators over $\mathcal{M}(\mathfrak{H})$.*

Chose any fixed values $c \in (0,\infty]$, $\lambda \in [0,\infty] \setminus \{c\}$, $s \in \{-1,1\}$, $J \in \mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ and $\mathbf{a} \in \mathcal{M}(\mathfrak{H})$. Introduce the following operator:

$$\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}]\mathrm{w} := \mathbf{W}_{\lambda,c}[s,\mathbf{n},J]\,(\mathrm{w}+\mathbf{a}). \tag{II.165}$$

**Corollary II.19.4.** *Let* $c \in (0,\infty]$, $\lambda \in [0,\infty] \setminus \{c\}$, $s \in \{-1,1\}$, $J \in \mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$, *and* $\mathbf{a} \in \mathcal{M}(\mathfrak{H})$. *Then:*

$$\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}] \in \mathbf{Pk}(\mathfrak{H}).$$

*Proof.* It is sufficient to prove, that for $c \in (0,\infty]$, $\lambda \in [0,\infty] \setminus \{c\}$, $s \in \{-1,1\}$, $J \in \mathfrak{U}(\mathfrak{H}_1)$ and $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ ***the operator*** $\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]$ ***has the continuous inverse*** $\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]^{-1} \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$, because the existence of inverse operator $\mathbf{W}_{\lambda,c}[s,\mathbf{n},J]^{-1} \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ leads to the existence of operator $\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}]^{-1} \in \mathcal{L}^{\times}(\mathcal{M}(\mathfrak{H}))$ (for any $\mathbf{a} \in \mathcal{M}(\mathfrak{H})$) in accordance with the formula:

$$\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}]^{-1}\mathrm{w} = \mathbf{W}_{\lambda,c}[s,\mathbf{n},J]^{-1}\mathrm{w} - \mathbf{a} \qquad (\forall \mathrm{w} \in \mathcal{M}(\mathfrak{H})).$$

For the case $c < \infty$ the highlighted statement had been proved in Section 18 (see Corollary II.18.3). While in the case $c = \infty$ this statement was proved in Assertion II.19.1. □

For $0 < c \leq \infty$ we introduce the following classes of (affine) coordinate transform operators:

$$\begin{aligned}
\mathfrak{PT}(\mathfrak{H},c) &:= \{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}] \mid s \in \{-1,1\},\ \lambda \in [0,\infty] \setminus \{c\},\\
&\qquad\qquad \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\ J \in \mathfrak{U}(\mathfrak{H}_1),\ \mathbf{a} \in \mathcal{M}(\mathfrak{H})\};\\
\mathfrak{PT}_+(\mathfrak{H},c) &:= \{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}] \in \mathfrak{PT}(\mathfrak{H},c) \mid s = 1\};\\
\mathfrak{P}(\mathfrak{H},c) &:= \{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}] \in \mathfrak{PT}(\mathfrak{H},c) \mid \lambda < c\};\\
\mathfrak{P}_+(\mathfrak{H},c) &:= \{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}] \in \mathfrak{P}(\mathfrak{H},c) \mid s = 1\}.
\end{aligned}$$

(It is apparently, that $\mathfrak{PT}(\mathfrak{H},\infty) = \mathfrak{P}(\mathfrak{H},\infty)$, $\mathfrak{PT}_+(\mathfrak{H},\infty) = \mathfrak{P}_+(\mathfrak{H},\infty)$). It is not hard to see, that:

$$\left.\begin{aligned}
\mathfrak{PT}(\mathfrak{H},c) &= \{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}] \mid \mathbf{W}_{\lambda,c}[s,\mathbf{n},J] \in \mathfrak{OT}(\mathfrak{H},c),\ \mathbf{a} \in \mathcal{M}(\mathfrak{H})\};\\
\mathfrak{PT}_+(\mathfrak{H},c) &= \{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}] \mid \mathbf{W}_{\lambda,c}[s,\mathbf{n},J] \in \mathfrak{OT}_+(\mathfrak{H},c),\ \mathbf{a} \in \mathcal{M}(\mathfrak{H})\};\\
\mathfrak{P}(\mathfrak{H},c) &= \{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}] \mid \mathbf{W}_{\lambda,c}[s,\mathbf{n},J] \in \mathfrak{O}(\mathfrak{H},c),\ \mathbf{a} \in \mathcal{M}(\mathfrak{H})\};\\
\mathfrak{P}_+(\mathfrak{H},c) &= \{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}] \mid \mathbf{W}_{\lambda,c}[s,\mathbf{n},J] \in \mathfrak{O}_+(\mathfrak{H},c),\ \mathbf{a} \in \mathcal{M}(\mathfrak{H})\}.
\end{aligned}\right\} \qquad \text{(II.166)}$$

Applying representations (II.166) for classes of operators $\mathfrak{P}(\mathfrak{H},c)$ and $\mathfrak{P}_+(\mathfrak{H},c)$ as well as Assertion II.17.1, Assertion II.17.6, Corollary II.19.2, Corollary II.19.3 and formula (II.165), we obtain the following conclusion:

**Corollary II.19.5.** *For arbitrary* $c \in (0,\infty]$ *classes of operators* $\mathfrak{P}(\mathfrak{H},c)$ *and* $\mathfrak{P}_+(\mathfrak{H},c)$ *are groups of operators in the space* $\mathcal{M}(\mathfrak{H})$.
(Note, that $\mathfrak{P}(\mathfrak{H},c), \mathfrak{P}_+(\mathfrak{H},c) \subseteq \mathcal{L}^{\times}(\mathcal{M}(\mathfrak{H}))$.)

*Remark* II.19.1. In the case $\mathfrak{H} = \mathbb{R}^3$, $c < \infty$ the group of operators $\mathfrak{P}_+(\mathfrak{H},c)$ coincides with the famous Poincare group (for definition of Poincare group see, for example [58, 59]). In the case $\mathfrak{H} = \mathbb{R}^3$, $c = \infty$ the group of operators $\mathfrak{P}_+(\mathfrak{H},\infty)$ coincides with the Galilean group (for definition of Galilean group see, for example [58–60]).

Also applying representations (II.166) for classes of operators $\mathfrak{PT}(\mathfrak{H},c)$ and $\mathfrak{PT}_+(\mathfrak{H},c)$ as well as Corollary II.18.5, Corollary II.18.7 and formula (II.165), we deduce the following conclusion:

**Corollary II.19.6.** *For arbitrary* $c \in (0,\infty)$ *the following assertions are true:*

1. *Class of operators* $\mathfrak{PT}_+(\mathfrak{H},c)$ *is not group of operators in the space* $\mathcal{M}(\mathfrak{H})$*;*
2. *Class* $\mathfrak{PT}(\mathfrak{H},c)$ *is not group of operators in the space* $\mathcal{M}(\mathfrak{H})$ *in the case* $\dim(\mathfrak{H}) > 1$.

*Proof.* Indeed, if we assume, that $\mathfrak{PT}_+(\mathfrak{H},c)$ is group of operators over $\mathcal{M}(\mathfrak{H})$, then the set of operators:

$$\mathfrak{OT}_+(\mathfrak{H},c)=\{\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}]\in\mathfrak{PT}_+(\mathfrak{H},c)\mid\mathbf{a}=\mathbf{0}\}$$

will be subgroup of it, which is impossible, according to Corollary II.18.5. Thus, the first item of Corollary has been proved. The proof of the second item is similar. □

Using the introduced above classes of operators, we may define the following kinematic sets:

$$\begin{aligned}\mathfrak{KPT}_0(\mathfrak{H},\mathcal{B},c)&:=\mathfrak{Kim}(\mathfrak{PT}(\mathfrak{H},c),\mathcal{B};\mathfrak{H});\\ \mathfrak{KPT}(\mathfrak{H},\mathcal{B},c)&:=\mathfrak{Kim}(\mathfrak{PT}_+(\mathfrak{H},c),\mathcal{B};\mathfrak{H});\\ \mathfrak{KP}_0(\mathfrak{H},\mathcal{B},c)&:=\mathfrak{Kim}(\mathfrak{P}(\mathfrak{H},c),\mathcal{B};\mathfrak{H});\\ \mathfrak{KP}(\mathfrak{H},\mathcal{B},c)&:=\mathfrak{Kim}(\mathfrak{P}_+(\mathfrak{H},c),\mathcal{B};\mathfrak{H}).\end{aligned}$$

In the case $\dim(\mathfrak{H})=3$, $c<\infty$ the kinematic set $\mathfrak{KP}(\mathfrak{H},\mathcal{B},c)$ represents the simplest mathematically strict model of the kinematics of special relativity theory in inertial frames of reference. Kinematic set $\mathfrak{KP}_0(\mathfrak{H},\mathcal{B},c)$ is constructed on the basis of general Lorentz-Poincare group, and it includes apart from usual reference frames (with positive direction of time), which have understandable physical interpretation, also reference frames with negative direction of time. Kinematic sets $\mathfrak{KPT}(\mathfrak{H},\mathcal{B},c)$ and $\mathfrak{KPT}_0(\mathfrak{H},\mathcal{B},c)$ include apart from standard ("tardyon") reference frames also "tachyon" reference frames, which are moving relatively the "tardyon" reference frames with velocity, greater than the velocity of light $c$. Kinematic set $\mathfrak{KP}(\mathfrak{H},\mathcal{B},\infty)=\mathfrak{KPT}(\mathfrak{H},\mathcal{B},\infty)$ in the case $\dim(\mathfrak{H})=3$, $c=\infty$ represents the mathematically strict model of the Galilean kinematics in the inertial frames of reference. The next corollary follows from Corollary II.19.1.

**Corollary II.19.7.** *Kinematic sets $\mathfrak{KPT}_0(\mathfrak{H},\mathcal{B},c)$, $\mathfrak{KPT}(\mathfrak{H},\mathcal{B},c)$, $\mathfrak{KP}_0(\mathfrak{H},\mathcal{B},c)$, $\mathfrak{KP}(\mathfrak{H},\mathcal{B},c)$ allow universal coordinate transform.*

*Remark* II.19.2. From Corollary II.19.5 it follows, that the sets of operators $\mathfrak{P}(\mathfrak{H},c)$ and $\mathfrak{P}_+(\mathfrak{H},c)$ form the groups of operators over the space $\mathcal{M}(\mathfrak{H})$. At the same time, in Corollary II.19.6 it is proved, that the classes of operators $\mathfrak{PT}_+(\mathfrak{H},c)$ and $\mathfrak{PT}(\mathfrak{H},c)$ (for $\dim(\mathfrak{H})>1$) do not form a group over $\mathcal{M}(\mathfrak{H})$. This means, that the kinematics $\mathfrak{KPT}(\mathfrak{H},\mathcal{B},c)$ and $\mathfrak{KPT}_0(\mathfrak{H},\mathcal{B},c)$, constructed on the basis of these classes, do not satisfy the relativity principle, because, according to Theorem II.16.2, the subset of universal coordinate transforms (II.12), providing transition from one reference frame to all other, is different for different frames. But, in kinematics $\mathfrak{KPT}_0(\mathfrak{H},\mathcal{B},c)$ and $\mathfrak{KPT}(\mathfrak{H},\mathcal{B},c)$ the relativity principle is violated only in the superluminal diapason, because the kinematics sets $\mathfrak{KPT}_0(\mathfrak{H},\mathcal{B},c)$ and $\mathfrak{KPT}(\mathfrak{H},\mathcal{B},c)$ are formed by the "addition" of new, superlight reference frames to the kinematics sets $\mathfrak{KP}_0(\mathfrak{H},\mathcal{B},c)$ and $\mathfrak{KP}(\mathfrak{H},\mathcal{B},c)$, which satisfy the principle of relativity. It should be noted that the principle of relativity is only one of the experimentally established facts. Therefore, it is possible that this principle is not satisfied when we exit out of the light barrier. Possibility of revision of the relativity principle is now discussed in the physical literature (see for example, [47, 61–66]).

Main results of this Section were anonced in [11] and published in [12].

## 20 Kinematic Sets, which do not Allow Universal Coordinate Transform.

In this section, it is constructed one interesting class of kinematic sets, in which every particle at each time moment can can have its own "velocity of light". On a physical level, the similar models (with particle-dependent velocity of light) were considered in the papers [67–71].

Let a set $\mathfrak{V}_\mathfrak{f} \subseteq (0,\infty]$ be such, that $\mathfrak{V}_\mathfrak{f} \neq \emptyset$ and $(0,\infty] \setminus \mathfrak{V}_\mathfrak{f} \neq \emptyset$. Denote:

$$\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}} := \mathfrak{H} \times \mathfrak{V}_\mathfrak{f} = \{(x,c) \mid x \in \mathfrak{H},\ c \in \mathfrak{V}_\mathfrak{f}\}; \quad \mathcal{M}\left(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}}\right) := \mathbb{R} \times \mathfrak{H}_{\mathfrak{V}_\mathfrak{f}}.$$

The set $\mathcal{M}\left(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}}\right)$ will be named by the Minkowski space ***with the set of forbidden velocities*** $\mathfrak{V}_\mathfrak{f}$ over $\mathfrak{H}$. The set $\widetilde{\mathfrak{V}_\mathfrak{f}} := [0,\infty] \setminus \mathfrak{V}_\mathfrak{f}$ will be named as the ***set of allowed velocities*** for the space $\mathcal{M}\left(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}}\right)$.

For an arbitrary $\omega = (t,(x,c)) \in \mathcal{M}\left(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}}\right)$ we put $\omega^* := (t,x) \in \mathcal{M}(\mathfrak{H})$. Also for $\lambda \in \widetilde{\mathfrak{V}_\mathfrak{f}}$, $s \in \{-1,1\}$, $J \in \mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$, $\mathbf{a} \in \mathcal{M}(\mathfrak{H})$ and $\omega = (t,(x,c)) \in \mathcal{M}\left(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}}\right)$ we introduce the denotation:

$$\mathbf{W}_{\lambda;\mathfrak{V}_\mathfrak{f}}[s,\mathbf{n},J;\mathbf{a}]\,\omega := \left(\mathsf{tm}\left(\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}]\,\omega^*\right),\ \left(\mathsf{bs}\left(\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}]\,\omega^*\right),c\right)\right). \tag{II.167}$$

Therefore, for any $\omega = (t,(x,c)) \in \mathcal{M}\left(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}}\right)$ we have the equality:

$$\left(\mathbf{W}_{\lambda;\mathfrak{V}_\mathfrak{f}}[s,\mathbf{n},J;\mathbf{a}]\,\omega\right)^* = \mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}]\,\omega^*. \tag{II.168}$$

**Assertion II.20.1.** *For arbitrary $\lambda \in \widetilde{\mathfrak{V}_\mathfrak{f}}$, $s \in \{-1,1\}$, $J \in \mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$, $\mathbf{a} \in \mathcal{M}(\mathfrak{H})$ the mapping $\mathbf{W}_{\lambda;\mathfrak{V}_\mathfrak{f}}[s,\mathbf{n},J;\mathbf{a}]$ is bijection on $\mathcal{M}\left(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}}\right)$.*

*Proof.* Suppose, that $\mathbf{W}_{\lambda;\mathfrak{V}_\mathfrak{f}}[s,\mathbf{n},J;\mathbf{a}]\,\omega_1 = \mathbf{W}_{\lambda;\mathfrak{V}_\mathfrak{f}}[s,\mathbf{n},J;\mathbf{a}]\,\omega_2$, where $\omega_1 = (t_1,(x_1,c_1)) \in \mathcal{M}\left(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}}\right)$, $\omega_2 = (t_2,(x_2,c_2)) \in \mathcal{M}\left(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}}\right)$. Then,

$$\begin{aligned}&\left(\mathsf{tm}\left(\mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J;\mathbf{a}]\,\omega_1^*\right),\ \left(\mathsf{bs}\left(\mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J;\mathbf{a}]\,\omega_1^*\right),c_1\right)\right) = \\ &\quad = \left(\mathsf{tm}\left(\mathbf{W}_{\lambda,c_2}[s,\mathbf{n},J;\mathbf{a}]\,\omega_2^*\right),\ \left(\mathsf{bs}\left(\mathbf{W}_{\lambda,c_2}[s,\mathbf{n},J;\mathbf{a}]\,\omega_2^*\right),c_2\right)\right).\end{aligned}$$

Consequently, $c_1 = c_2$. Hence, we have proved the equalities:

$$\begin{aligned}\mathsf{tm}\left(\mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J;\mathbf{a}]\,\omega_1^*\right) &= \mathsf{tm}\left(\mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J;\mathbf{a}]\,\omega_2^*\right) \\ \mathsf{bs}\left(\mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J;\mathbf{a}]\,\omega_1^*\right) &= \mathsf{bs}\left(\mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J;\mathbf{a}]\,\omega_2^*\right).\end{aligned}$$

Therefore, $\mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J;\mathbf{a}]\,\omega_1^* = \mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J;\mathbf{a}]\,\omega_2^*$. And, taking into account the fact, that the mapping $\mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J;\mathbf{a}]$ is bijection on $\mathcal{M}(\mathfrak{H})$, we conclude, that, $\omega_1^* = \omega_2^*$, ie $t_1 = t_2$, $x_1 = x_2$. Hence, $\omega_1 = (t_1,(x_1,c_1)) = (t_2,(x_2,c_2)) = \omega_2$. Thus, the mapping $\mathbf{W}_{\lambda;\mathfrak{V}_\mathfrak{f}}[s,\mathbf{n},J;\mathbf{a}]$ is one-to-one correspondence.

Now it remains to prove, that $\mathbf{W}_{\lambda;\mathfrak{V}_\mathfrak{f}}[s,\mathbf{n},J;\mathbf{a}]$ reflects the set $\mathcal{M}\left(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}}\right)$ on $\mathcal{M}\left(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}}\right)$. Consider any $\omega = (t,(x,c)) \in \mathcal{M}\left(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}}\right)$. Denote:

$$\widetilde{\omega} := \left(\mathsf{tm}\left(\left(\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}]\right)^{[-1]}\omega^*\right),\ \left(\mathsf{bs}\left(\left(\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}]\right)^{[-1]}\omega^*\right),c\right)\right).$$

Then,

$$\begin{aligned}\widetilde{\omega}^* &= \left(\mathsf{tm}\left(\left(\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}]\right)^{[-1]}\omega^*\right),\ \mathsf{bs}\left(\left(\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}]\right)^{[-1]}\omega^*\right)\right) = \\ &= \left(\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}]\right)^{[-1]}\omega^*.\end{aligned}$$

Consequently, $\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}]\,\widetilde{\omega}^* = \omega^*$. Hence,

$$\begin{aligned}\mathbf{W}_{\lambda;\mathfrak{V}_\mathfrak{f}}[s,\mathbf{n},J;\mathbf{a}]\,\widetilde{\omega} &= \left(\mathsf{tm}\left(\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}]\,\widetilde{\omega}^*\right),\ \left(\mathsf{bs}\left(\mathbf{W}_{\lambda,c}[s,\mathbf{n},J;\mathbf{a}]\,\widetilde{\omega}^*\right),c\right)\right) = \\ &= \left(\mathsf{tm}(\omega^*),\ (\mathsf{bs}(\omega^*),c)\right) = (t,(x,c)) = \omega.\end{aligned}$$

Thus, $\mathbf{W}_{\lambda;\mathfrak{V}_\mathfrak{f}}[s,\mathbf{n},J;\mathbf{a}]$ is bijection from $\mathcal{M}\left(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}}\right)$ onto $\mathcal{M}\left(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}}\right)$. □

Denote:

$$\begin{aligned}\mathfrak{PT}(\mathfrak{H};\mathfrak{V}_{\mathfrak{f}}) &:= \Big\{\mathbf{W}_{\lambda;\mathfrak{V}_{\mathfrak{f}}}[s,\mathbf{n},J;\mathbf{a}] \mid \lambda\in\widetilde{\mathfrak{V}_{\mathfrak{f}}},\ s\in\{-1,1\},\\ &\qquad J\in\mathfrak{U}(\mathfrak{H}_1),\ \mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ \mathbf{a}\in\mathcal{M}(\mathfrak{H})\Big\};\\ \mathfrak{PT}_+(\mathfrak{H};\mathfrak{V}_{\mathfrak{f}}) &:= \big\{\mathbf{W}_{\lambda;\mathfrak{V}_{\mathfrak{f}}}[s,\mathbf{n},J;\mathbf{a}]\in\mathfrak{PT}(\mathfrak{H};\mathfrak{V}_{\mathfrak{f}}) \mid s=1\big\}.\end{aligned}$$

Let, $\mathcal{B}$ be a base changeable set such, that $\mathfrak{Bs}(\mathcal{B})\subseteq\mathfrak{H}_{\mathfrak{V}_{\mathfrak{f}}}$, $\mathbb{Tm}(\mathcal{B})=(\mathbb{R},\leq)$. Then we have, $\mathbb{Bs}(\mathcal{B})\subseteq\mathbb{R}\times\mathfrak{H}_{\mathfrak{V}_{\mathfrak{f}}}=\mathcal{M}\left(\mathfrak{H}_{\mathfrak{V}_{\mathfrak{f}}}\right)$. Hence, we deliver the following kinematic multi-projectors:

$$\begin{aligned}\mathfrak{PT}(\mathfrak{H};\mathfrak{V}_{\mathfrak{f}})^{\wedge} &= \left(\left((\mathbb{R},\leq),\mathfrak{H}_{\mathfrak{V}_{\mathfrak{f}}},\mathbf{S},\widehat{\mathfrak{H}},\mathbf{q}\right) \mid \mathbf{S}\in\mathfrak{PT}(\mathfrak{H};\mathfrak{V}_{\mathfrak{f}})\right);\\ \mathfrak{PT}_+(\mathfrak{H};\mathfrak{V}_{\mathfrak{f}})^{\wedge} &= \left(\left((\mathbb{R},\leq),\mathfrak{H}_{\mathfrak{V}_{\mathfrak{f}}},\mathbf{S},\widehat{\mathfrak{H}},\mathbf{q}\right) \mid \mathbf{S}\in\mathfrak{PT}_+(\mathfrak{H};\mathfrak{V}_{\mathfrak{f}})\right),\quad\text{where}\\ \mathbf{q}(\widetilde{x}) &= x\quad\left(\forall\,\widetilde{x}=(x,c)\in\mathfrak{H}_{\mathfrak{V}_{\mathfrak{f}}}\right)\end{aligned}\tag{II.169}$$

for $\mathcal{B}$. In accordance with Theorem II.16.1 and Definition II.16.2, we can denote:

$$\begin{aligned}\mathfrak{KPT}_0(\mathfrak{H},\mathcal{B};\mathfrak{V}_{\mathfrak{f}}) &:= \mathfrak{Kim}\left[\mathfrak{PT}(\mathfrak{H};\mathfrak{V}_{\mathfrak{f}})^{\wedge},\mathcal{B}\right];\\ \mathfrak{KPT}(\mathfrak{H},\mathcal{B};\mathfrak{V}_{\mathfrak{f}}) &:= \mathfrak{Kim}\left[\mathfrak{PT}_+(\mathfrak{H};\mathfrak{V}_{\mathfrak{f}})^{\wedge},\mathcal{B}\right].\end{aligned}$$

It turns out, that the kinematic sets $\mathfrak{KPT}_0(\mathfrak{H},\mathcal{B};\mathfrak{V}_{\mathfrak{f}})$ and $\mathfrak{KPT}(\mathfrak{H},\mathcal{B};\mathfrak{V}_{\mathfrak{f}})$, in the general case, do not allow universal coordinate transform. More precisely, they allow universal coordinate transform if and only if only one value of forbidden velocity $c\in(0,\infty]$ is actually realized. In the last case, kinematics in $\mathfrak{KPT}_0(\mathfrak{H},\mathcal{B};\mathfrak{V}_{\mathfrak{f}})$ or $\mathfrak{KPT}(\mathfrak{H},\mathcal{B};\mathfrak{V}_{\mathfrak{f}})$ can be reduced to kinematics of type $\mathfrak{KPT}_0(\mathfrak{H},\mathcal{B},c)$ or $\mathfrak{KPT}(\mathfrak{H},\mathcal{B},c)$ (for $c<\infty$), and to Galilean kinematics (for $c=\infty$).

**Theorem II.20.1.** *Let the set of forbidden velocities $\mathfrak{V}_{\mathfrak{f}}$ be separated from zero (ie there exists a number $\eta>0$ such, that $\mathfrak{V}_{\mathfrak{f}}\subseteq[\eta,\infty]$).*

*Kinematic set $\mathfrak{KPT}(\mathfrak{H},\mathcal{B};\mathfrak{V}_{\mathfrak{f}})$ allows universal coordinate transform if and only if there don't exist elementary states $\widetilde{x}_1=(x_1,c_1)$, $\widetilde{x}_2=(x_2,c_2)\in\mathfrak{Bs}(\mathcal{B})$ such, that $c_1\neq c_2$.*

To prove Theorem II.20.1 we need the following two lemmas.

**Lemma II.20.1.** *Chose any fixed $c_1,c_2\in(0,\infty]$, $c_1\neq c_2$, $s\in\{-1,1\}$ and $J\in\mathfrak{U}(\mathfrak{H}_1)$.*

*Then, for any fixed number $\varepsilon\in(0,\min(c_1,c_2))$ and arbitrary fixed vectors $\mathrm{w}_1,\mathrm{w}_2\in\mathcal{M}(\mathfrak{H})$ such, that $\mathrm{w}_1\neq\mathrm{w}_2$, there exist $\lambda\in(0,\varepsilon)$, $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$ and $\mathbf{a}\in\mathcal{M}(\mathfrak{H})$, for which the following equality holds:*

$$\mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J;\mathbf{a}]\,\mathrm{w}_1=\mathbf{W}_{\lambda,c_2}[s,\mathbf{n},J;\mathbf{a}]\,\mathrm{w}_2.$$

*Proof.* Further, for convenience, we assume, that $c_1<c_2$. Obviously, this assumption does not restrict the the generality of our conclusions.

**1.** At first, we are going to prove Lemma in the special case $\mathrm{w}_1=\mathbf{0}$, $\mathrm{w}_2=\mathrm{w}\neq\mathbf{0}$. Consider any, $\varepsilon\in(0,\min(c_1,c_2))$. According to the specifics of this case, we should find $\lambda\in(0,\varepsilon)$, $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$ and $\mathbf{a}\in\mathcal{M}(\mathfrak{H})$, such, that:

$$\mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J;\mathbf{a}]\,\mathbf{0}=\mathbf{W}_{\lambda,c_2}[s,\mathbf{n},J;\mathbf{a}]\,\mathrm{w}.\tag{II.170}$$

Taking into account (II.165), we can rewrite the last condition in the form:

$$\mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J]\,\mathbf{a}=\mathbf{W}_{\lambda,c_2}[s,\mathbf{n},J]\,(\mathrm{w}+\mathbf{a}).\tag{II.171}$$

Denote:

$$t:=\mathcal{T}(\mathrm{w}),\qquad x:=\mathbf{X}\mathrm{w}.\tag{II.172}$$

Then we can write, $\mathrm{w}=t\mathbf{e}_0+x$.

Consider any fixed vector $\mathbf{n}_0 \in \mathbf{B}_1(\mathfrak{H}_1)$. Denote:

$$\mathbf{n} := \begin{cases} \frac{x}{\|x\|}, & x \neq \mathbf{0} \\ \mathbf{n}_0, & x = \mathbf{0}. \end{cases} \tag{II.173}$$

Then, we have:

$$\begin{aligned} x &= \|x\| \, \mathbf{n}, \\ \langle \mathbf{n}, \mathrm{w} \rangle &= \langle \mathbf{n}, x \rangle = \|x\| \, , \\ \mathbf{X}_1^{\perp}[\mathbf{n}] \, \mathrm{w} &= \mathbf{X}\mathrm{w} - \langle \mathbf{n}, \mathrm{w} \rangle \, \mathbf{n} = x - \|x\| \, \mathbf{n} = x - x = \mathbf{0}. \end{aligned} \tag{II.174}$$

Vector $\mathbf{a}$ we seek in the form:

$$\mathbf{a} = \tau \mathbf{e}_0 + \mu \mathbf{n}, \qquad \text{where} \quad \tau, \mu \in \mathbb{R}. \tag{II.175}$$

**1.a)** At first we consider the case $c_1, c_2 < \infty$.

Substituting the value of the vector $\mathbf{a}$ from (II.175) into the condition (II.171) and applying (II.172), (II.174), (II.157), we obtain the following condition:

$$\begin{aligned} &\left(s\tau - \frac{\lambda}{c_1^2}\mu\right)\gamma\left(\frac{\lambda}{c_1}\right)\mathbf{e}_0 + (\lambda\tau - s\mu)\,\gamma\left(\frac{\lambda}{c_1}\right) J\mathbf{n} = \\ &= \left(s\,(t+\tau) - \frac{\lambda}{c_2^2}\left(\|x\| + \mu\right)\right)\gamma\left(\frac{\lambda}{c_2}\right)\mathbf{e}_0 + \\ &\quad + \left(\lambda\,(t+\tau) - s\left(\|x\| + \mu\right)\right)\gamma\left(\frac{\lambda}{c_2}\right) J\mathbf{n}, \\ &\text{where} \quad \gamma(\xi) = \frac{1}{\sqrt{|1-\xi^2|}}, \qquad \xi \geq 0,\ \xi \neq 1. \end{aligned} \tag{II.176}$$

Taking into account orthogonality of the vector $\mathbf{e}_0$ to the subspace $\mathfrak{H}_1$ and unitarity of the operator $J$ on the subspace $\mathfrak{H}_1$, we get the following system of equations:

$$\begin{cases} \left(s\tau - \frac{\lambda}{c_1^2}\mu\right)\gamma\left(\frac{\lambda}{c_1}\right) = \left(s\,(t+\tau) - \frac{\lambda}{c_2^2}\left(\|x\| + \mu\right)\right)\gamma\left(\frac{\lambda}{c_2}\right) \\ (\lambda\tau - s\mu)\,\gamma\left(\frac{\lambda}{c_1}\right) = \left(\lambda\,(t+\tau) - s\left(\|x\| + \mu\right)\right)\gamma\left(\frac{\lambda}{c_2}\right) \end{cases} \tag{II.177}$$

By means of simple transformations, the system (II.177) can be reduced to the form:

$$\begin{cases} \tau\left(\gamma\left(\frac{\lambda}{c_2}\right) - \gamma\left(\frac{\lambda}{c_1}\right)\right) = \lambda s\left(\left(\frac{\|x\|+\mu}{c_2^2}\right)\gamma\left(\frac{\lambda}{c_2}\right) - \frac{\mu}{c_1^2}\gamma\left(\frac{\lambda}{c_1}\right)\right) - t\gamma\left(\frac{\lambda}{c_2}\right) \\ \lambda\tau\left(\gamma\left(\frac{\lambda}{c_2}\right) - \gamma\left(\frac{\lambda}{c_1}\right)\right) = s\left(\left(\|x\| + \mu\right)\gamma\left(\frac{\lambda}{c_2}\right) - \mu\gamma\left(\frac{\lambda}{c_1}\right)\right) - \lambda t\gamma\left(\frac{\lambda}{c_2}\right) \end{cases} \tag{II.178}$$

Replacing the expression $\tau\left(\gamma\left(\frac{\lambda}{c_2}\right) - \gamma\left(\frac{\lambda}{c_1}\right)\right)$ in the second equation of the system (II.178) by the right-hand side of the first equation of this system, we deliver the equation:

$$\lambda^2\left(\frac{\|x\|+\mu}{c_2^2}\gamma\left(\frac{\lambda}{c_2}\right) - \frac{\mu}{c_1^2}\gamma\left(\frac{\lambda}{c_1}\right)\right) = \left(\|x\| + \mu\right)\gamma\left(\frac{\lambda}{c_2}\right) - \mu\gamma\left(\frac{\lambda}{c_1}\right).$$

After simple transformations the last equation takes a form:

$$\frac{\left(\|x\| + \mu\right)\left(1 - \frac{\lambda^2}{c_2^2}\right)}{\sqrt{\left|1 - \frac{\lambda^2}{c_2^2}\right|}} - \frac{\mu\left(1 - \frac{\lambda^2}{c_1^2}\right)}{\sqrt{\left|1 - \frac{\lambda^2}{c_1^2}\right|}} = 0. \tag{II.179}$$

Now, we introduce the denotations:

$$\Phi_1(y) := \operatorname{sign}(y)\sqrt{|y|}; \quad \Phi_2(y) = y\,|y| \qquad (y \in \mathbb{R}). \tag{II.180}$$

In the case $y \neq 0$ the function $\Phi_1(y)$ may be represented in the form, $\Phi_1(y) = \frac{y}{\sqrt{|y|}}$.

In view of denotation (II.180) the equation (II.179) becomes:

$$\Phi_1\left((\|x\|+\mu)\,|\|x\|+\mu|\left(1-\frac{\lambda^2}{c_2^2}\right)\right) = \Phi_1\left(\mu\,|\mu|\left(1-\frac{\lambda^2}{c_1^2}\right)\right).$$

Taking into account, that the function $\Phi_1$ is strictly monotone on $\mathbb{R}$, we get the equation:

$$(\|x\|+\mu)\,|\|x\|+\mu|\left(1-\frac{\lambda^2}{c_2^2}\right) = \mu\,|\mu|\left(1-\frac{\lambda^2}{c_1^2}\right),$$

which after simple transformations is reduced to the form:

$$\lambda^2\left(\Phi_2\left(\frac{\|x\|+\mu}{c_2}\right)-\Phi_2\left(\frac{\mu}{c_1}\right)\right) = \Phi_2\left(\|x\|+\mu\right)-\Phi_2\left(\mu\right). \tag{II.181}$$

Since $c_1 < c_2$, then for $\mu < -\|x\|$ we have $\frac{\|x\|+\mu}{c_2} > \frac{\mu}{c_1}$. Therefore, taking into account, that the function $\Phi_2$ is strictly increasing on $\mathbb{R}$, we may define the function:

$$\Phi_{3;x}(\mu) = \sqrt{\frac{\Phi_2\left(\|x\|+\mu\right)-\Phi_2\left(\mu\right)}{\Phi_2\left(\frac{\|x\|+\mu}{c_2}\right)-\Phi_2\left(\frac{\mu}{c_1}\right)}} = \sqrt{\frac{\mu^2-\left(\|x\|+\mu\right)^2}{\left(\frac{\mu}{c_1}\right)^2-\left(\frac{\|x\|+\mu}{c_2}\right)^2}}, \qquad \mu < -\|x\|.$$

It is easy to verify, that $\Phi_{3;x}(\mu) \to 0$, $\mu \to -\infty$. Hence, there exists the number $\mu_0 < -\|x\|$ such, that $\Phi_{3;x}(\mu_0) \in [0,\varepsilon)$.

In the case $x \neq \mathbf{0}$ we have $\Phi_{3;x}(\mu) > 0$ for all $\mu$ such, that $\mu < -\|x\|$. In the case $x = \mathbf{0}$, the equation (II.181) becomes the true equality for $\mu = 0$ and arbitrary $\lambda \in \mathbb{R}$. That is why, if we put:

$$\mu := \begin{cases} \mu_0, & x \neq \mathbf{0} \\ 0, & x = \mathbf{0} \end{cases}; \qquad \lambda := \begin{cases} \Phi_{3;x}(\mu_0), & x \neq \mathbf{0} \\ \frac{\varepsilon}{2}, & x = \mathbf{0}, \end{cases} \tag{II.182}$$

we will obtain the values $\mu \in \mathbb{R}$ and $\lambda \in (0,\varepsilon)$, for which the equality (II.181) holds.

Since $0 < \lambda < \varepsilon < \min(c_1, c_2)$, then for values $\lambda, \mu$, determined by the formula (II.182), the second equation from the system (II.178) takes the form:

$$\lambda\tau\left(\frac{1}{\sqrt{1-\frac{\lambda^2}{c_2^2}}}-\frac{1}{\sqrt{1-\frac{\lambda^2}{c_1^2}}}\right) = s\left(\frac{\|x\|+\mu}{\sqrt{1-\frac{\lambda^2}{c_2^2}}}-\frac{\mu}{\sqrt{1-\frac{\lambda^2}{c_1^2}}}\right)-\frac{\lambda t}{\sqrt{1-\frac{\lambda^2}{c_2^2}}}, \tag{II.183}$$

where, considering that $\lambda > 0$ and $c_1 < c_2$, we have $\lambda\left(\frac{1}{\sqrt{1-\frac{\lambda^2}{c_2^2}}}-\frac{1}{\sqrt{1-\frac{\lambda^2}{c_1^2}}}\right) \neq 0$. Hence, the number $\tau$ is uniquely determined by the equality (II.183). Then, the vector $\mathbf{a}$ we calculate by the formula (II.175). And, substituting the delivered values of the parameters $\lambda \in (0,\varepsilon)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ and $\mathbf{a} \in \mathcal{M}(\mathfrak{H})$ into (II.171), we guarantee the valid equality. In the case $c_2, c_2 < \infty$ and $\mathrm{w}_1 = \mathbf{0}$, Lemma is proved.

**1.b)** Thus, it remains to consider only the case $c_2 = \infty$, $c_1 < \infty$ ($\mathrm{w}_1 = \mathbf{0}$, $\mathrm{w}_2 = \mathrm{w} \neq \mathbf{0}$). Note, that the case $c_1 = \infty$ is impossible, because $c_1 < c_2$.

Substituting the value of the vector $\mathbf{a}$ from (II.175) into the condition (II.171) and applying (II.172), (II.174), (II.157), we obtain the following condition:

$$\left(s\tau - \frac{\lambda}{c_1^2}\mu\right)\gamma\left(\frac{\lambda}{c_1}\right)\mathbf{e}_0 + (\lambda\tau - s\mu)\,\gamma\left(\frac{\lambda}{c_1}\right)\, J\mathbf{n} = \\ = s\,(t+\tau)\,\mathbf{e}_0 + \left(\lambda\,(t+\tau) - s\,(\|x\| + \mu)\right)\, J\mathbf{n}.$$

Hence, taking into account orthogonality of the vector $\mathbf{e}_0$ to the subspace $\mathfrak{H}_1$ and unitarity of the operator $J$ on the subspace $\mathfrak{H}_1$, we get the following system of equations:

$$\begin{cases} \left(s\tau - \frac{\lambda}{c_1^2}\mu\right)\gamma\left(\frac{\lambda}{c_1}\right) = s\,(t+\tau) \\ (\lambda\tau - s\mu)\,\gamma\left(\frac{\lambda}{c_1}\right) = \lambda\,(t+\tau) - s\,(\|x\| + \mu) \end{cases} \tag{II.184}$$

After simple transformations, the system (II.184) may be reduced to the form:

$$\begin{cases} \tau\left(1 - \gamma\left(\frac{\lambda}{c_1}\right)\right) = -\lambda s\,\frac{\mu}{c_1^2}\gamma\left(\frac{\lambda}{c_1}\right) - t \\ \lambda\tau\left(1 - \gamma\left(\frac{\lambda}{c_1}\right)\right) = s\left(\|x\| + \mu - \mu\gamma\left(\frac{\lambda}{c_1}\right)\right) - \lambda t \end{cases} \tag{II.185}$$

Replacing the expression $\tau\left(1 - \gamma\left(\frac{\lambda}{c_1}\right)\right)$ in the second equation of the system (II.185) by the right-hand side of the first equation of this system, we obtain the equation:

$$-\lambda^2\frac{\mu}{c_1^2}\gamma\left(\frac{\lambda}{c_1}\right) = \|x\| + \mu - \mu\gamma\left(\frac{\lambda}{c_1}\right),$$

which, by means of simple transformations takes a form:

$$\Phi_1\left((\|x\| + \mu)\,|\|x\| + \mu|\right) = \Phi_1\left(\mu\,|\mu|\left(1 - \frac{\lambda^2}{c_1^2}\right)\right), \tag{II.186}$$

where the function $\Phi_1$ is determined by the formula (II.180). Taking into account, that the function $\Phi_1$ is strictly monotone on $\mathbb{R}$, we get the equation:

$$(\|x\| + \mu)\,|\|x\| + \mu| = \mu\,|\mu|\left(1 - \frac{\lambda^2}{c_1^2}\right),$$

which after simple transformations is reduced to the form:

$$-\lambda^2\Phi_2\left(\frac{\mu}{c_1}\right) = \Phi_2\left(\|x\| + \mu\right) - \Phi_2\left(\mu\right). \tag{II.187}$$

Therefore, taking into account, that the function $\Phi_2(y) = y\,|y|$ is strictly increasing on $\mathbb{R}$, we may define the function:

$$\Phi_{3;x}^{\infty}(\mu) = \sqrt{\frac{\Phi_2\left(\|x\| + \mu\right) - \Phi_2\left(\mu\right)}{-\Phi_2\left(\frac{\mu}{c_1}\right)}} = \sqrt{\frac{\mu^2 - \left(\|x\| + \mu\right)^2}{\left(\frac{\mu}{c_1}\right)^2}}, \qquad \mu < -\|x\|.$$

It is easy to verify, that $\Phi_{3;x}^{\infty}(\mu) \to 0$, $\mu \to -\infty$. Hence, there exists the number $\mu_0 < -\|x\|$ such, that $\Phi_{3;x}^{\infty}(\mu_0) \in [0, \varepsilon)$.

In the case $x \neq \mathbf{0}$ we have $\Phi_{3;x}^{\infty}(\mu) > 0$ for all $\mu$ such, that $\mu < -\|x\|$. In the case $x = \mathbf{0}$, the equation (II.187) becomes the true equality for $\mu = 0$ and arbitrary $\lambda \in \mathbb{R}$. That is why, if we put:

$$\mu := \begin{cases} \mu_0, & x \neq \mathbf{0} \\ 0, & x = \mathbf{0} \end{cases} \qquad \lambda := \begin{cases} \Phi_{3;x}^{\infty}(\mu_0), & x \neq \mathbf{0} \\ \frac{\varepsilon}{2}, & x = \mathbf{0}, \end{cases} \tag{II.188}$$

we will obtain the values $\mu \in \mathbb{R}$ and $\lambda \in (0,\varepsilon)$, for which the equality (II.187) is true.

Since $0 < \lambda < \varepsilon < \min(c_1, c_2) = c_1$, then for values $\lambda, \mu$, determined by the formula (II.188), the second equation from the system (II.185) may be rewritten in the form:

$$\lambda\tau \left(1 - \frac{1}{\sqrt{1-\frac{\lambda^2}{c_1^2}}}\right) = s\left(\|x\| + \mu - \frac{\mu}{\sqrt{1-\frac{\lambda^2}{c_1^2}}}\right) - \lambda t, \tag{II.189}$$

where, considering that $\lambda, c_1 > 0$, we have, $\lambda\left(1 - \frac{1}{\sqrt{1-\frac{\lambda^2}{c_1^2}}}\right) \neq 0$. Hence, the number $\tau$ is uniquely determined by the equality (II.189). Then, the vector $\mathbf{a}$ we calculate by the formula (II.175). And, substituting the delivered values of the parameters $\lambda \in (0,\varepsilon)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ and $\mathbf{a} \in \mathcal{M}(\mathfrak{H})$ into (II.171), we obtain the valid equality. Hence, in the case $c_1 < \infty$, $c_2 = \infty$ and $\mathrm{w}_1 = \mathbf{0}$, Lemma is proved.

**2.** We now turn to the general case, where $\mathrm{w}_1, \mathrm{w}_2$ are arbitrary vectors of the space $\mathcal{M}(\mathfrak{H})$ such, that $\mathrm{w}_1 \neq \mathrm{w}_2$. According to the result, proved in the first item of Lemma, parameters $\lambda \in (0,\varepsilon)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ and $\widetilde{\mathbf{a}} \in \mathcal{M}(\mathfrak{H})$, exist such, that $\mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J]\,\widetilde{\mathbf{a}} = \mathbf{W}_{\lambda,c_2}[s,\mathbf{n},J](\mathrm{w}_2 - \mathrm{w}_1 + \widetilde{\mathbf{a}})$. Denote, $\mathbf{a} := \widetilde{\mathbf{a}} - \mathrm{w}_1$. Then, taking into account, (II.165), we receive the desired equality $\mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J;\mathbf{a}]\,\mathrm{w}_1 = \mathbf{W}_{\lambda,c_2}[s,\mathbf{n},J;\mathbf{a}]\,\mathrm{w}_2$. □

For $y_1, y_2 \in (0,\infty]$, such, that $y_1 \neq \infty$ or $y_2 \neq \infty$ we put:

$$\sigma(y_1,y_2) = \begin{cases} \left(\frac{y_1^{-2}+y_2^{-2}}{2}\right)^{-\frac{1}{2}}, & y_1, y_2 < \infty \\ \sqrt{2}\,y_1, & y_1 < \infty,\ y_2 = \infty \\ \sqrt{2}\,y_2, & y_1 = \infty,\ y_2 < \infty \end{cases}. \tag{II.190}$$

**Lemma II.20.2.** *Suppose, that for some vector* $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ *it holds the equality*

$$\mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J]\,\mathrm{w} = \mathbf{W}_{\lambda,c_2}[s,\mathbf{n},J]\,\mathrm{w},$$

*where* $c_1, c_2 \in (0,\infty]$, $\lambda \in (0,\infty] \setminus \{c_1, c_2, \sigma(c_1,c_2)\}$, $s \in \{-1,1\}$, $J \in \mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ *with* $c_1 \neq c_2$. *Then,* $\mathcal{T}(\mathrm{w}) = \langle \mathbf{n}, \mathrm{w}\rangle = 0$.

*Proof of Lemma* we divide into a few cases.

**Case 1**: $c_1, c_2 < \infty$, $\lambda < \infty$. In this case, by the formula (II.157), we get

$$\begin{aligned}\mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J]\,\mathrm{w} - \mathbf{W}_{\lambda,c_2}[s,\mathbf{n},J]\,\mathrm{w} = \\ = \left(\left(\gamma\left(\frac{\lambda}{c_1}\right) - \gamma\left(\frac{\lambda}{c_2}\right)\right) s\mathcal{T}(\mathrm{w}) - \left(\frac{\lambda}{c_1^2}\gamma\left(\frac{\lambda}{c_1}\right) - \frac{\lambda}{c_2^2}\gamma\left(\frac{\lambda}{c_2}\right)\right)\langle \mathbf{n}, \mathrm{w}\rangle\right)\mathbf{e}_0 + \\ + \left(\gamma\left(\frac{\lambda}{c_1}\right) - \gamma\left(\frac{\lambda}{c_2}\right)\right)\left(\lambda\mathcal{T}(\mathrm{w}) - s\langle \mathbf{n}, \mathrm{w}\rangle\right) J\mathbf{n},\end{aligned} \tag{II.191}$$

where the function $\gamma : [0,\infty) \mapsto \mathbb{R}$ is determined by the formula (II.176). By conditions of Lemma, $\mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J]\,\mathrm{w} - \mathbf{W}_{\lambda,c_2}[s,\mathbf{n},J]\,\mathrm{w} = \mathbf{0}$, where $\mathbf{0}$ is zero vector of the space $\mathcal{M}(\mathfrak{H})$. Hence, the right-hand side of the equality (II.191) is equal to zero vector. Therefore, taking into account orthogonality of the vector $\mathbf{e}_0$ to the subspace $\mathfrak{H}_1$ and unitarity of the operator $J$

on the subspace $\mathfrak{H}_1$, we get the following equalities:

$$\begin{aligned} s\left(\gamma\left(\tfrac{\lambda}{c_1}\right)-\gamma\left(\tfrac{\lambda}{c_2}\right)\right)\mathcal{T}(\mathrm{w})-\left(\tfrac{\lambda}{c_1^2}\gamma\left(\tfrac{\lambda}{c_1}\right)-\tfrac{\lambda}{c_2^2}\gamma\left(\tfrac{\lambda}{c_2}\right)\right)\langle\mathbf{n},\mathrm{w}\rangle &= 0;\\ \left(\gamma\left(\tfrac{\lambda}{c_1}\right)-\gamma\left(\tfrac{\lambda}{c_2}\right)\right)\ \left(\lambda\mathcal{T}(\mathrm{w})-s\langle\mathbf{n},\mathrm{w}\rangle\right) &= 0. \end{aligned} \tag{II.192}$$

According to the conditions of Lemma, $\lambda>0$ and $\lambda\neq\sigma(c_1,c_2)=\sqrt{\frac{2}{\frac{1}{c_1^2}+\frac{1}{c_2^2}}}$. Consequently, $\gamma\left(\frac{\lambda}{c_1}\right)-\gamma\left(\frac{\lambda}{c_2}\right)\neq 0$. Thus, the equalities (II.192) may be rewritten in the form:

$$\left\{\begin{aligned} s\left(\gamma\left(\tfrac{\lambda}{c_1}\right)-\gamma\left(\tfrac{\lambda}{c_2}\right)\right)\mathcal{T}(\mathrm{w})-\left(\tfrac{\lambda}{c_1^2}\gamma\left(\tfrac{\lambda}{c_1}\right)-\tfrac{\lambda}{c_2^2}\gamma\left(\tfrac{\lambda}{c_2}\right)\right)\langle\mathbf{n},\mathrm{w}\rangle &= 0;\\ \lambda\mathcal{T}(\mathrm{w})-s\langle\mathbf{n},\mathrm{w}\rangle &= 0. \end{aligned}\right. \tag{II.193}$$

The system (II.193) is a system of linear homogeneous equations relatively the variables $\mathcal{T}(\mathrm{w})$ and $\langle\mathbf{n},\mathrm{w}\rangle$. Determinant of this system is:

$$\begin{aligned} \Delta &= -\left[\left(\gamma\left(\frac{\lambda}{c_1}\right)-\gamma\left(\frac{\lambda}{c_2}\right)\right)-\left(\frac{\lambda^2}{c_1^2}\gamma\left(\frac{\lambda}{c_1}\right)-\frac{\lambda^2}{c_2^2}\gamma\left(\frac{\lambda}{c_2}\right)\right)\right]=\\ &= -\left(\boldsymbol{g}\left(\frac{\lambda}{c_1}\right)-\boldsymbol{g}\left(\frac{\lambda}{c_2}\right)\right), \qquad \text{where}\\ \boldsymbol{g}(\xi) &= \left(1-\xi^2\right)\gamma(\xi)=\operatorname{sign}\,(1-\xi)\sqrt{|1-\xi^2|} \qquad (\xi\geq 0,\ \xi\neq 1). \end{aligned}$$

Since the function $\boldsymbol{g}(\xi)=\operatorname{sign}\,(1-\xi)\sqrt{|1-\xi^2|}$ is strictly decreasing on $[0,\infty)$, determinant $\Delta$ of the system (II.193) is nonzero. Hence, $\mathcal{T}(\mathrm{w})=\langle\mathbf{n},\mathrm{w}\rangle=0$, that was necessary to prove.

**Case 2**: $c_1,c_2<\infty$, $\lambda=\infty$.

In this case, by the formula (II.157), we receive:

$$\begin{aligned} \mathbf{0} &= \mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J]\,\mathrm{w}-\mathbf{W}_{\lambda,c_2}[s,\mathbf{n},J]\,\mathrm{w}=\\ &= -\frac{\langle\mathbf{n},\mathrm{w}\rangle}{c_1}\mathbf{e}_0+c_1\mathcal{T}(\mathrm{w})\,J\mathbf{n}-\left(-\frac{\langle\mathbf{n},\mathrm{w}\rangle}{c_2}\mathbf{e}_0+c_2\mathcal{T}(\mathrm{w})\,J\mathbf{n}\right)=\\ &= -\left(\frac{1}{c_1}-\frac{1}{c_2}\right)\langle\mathbf{n},\mathrm{w}\rangle\,\mathbf{e}_0+(c_1-c_2)\,\mathcal{T}(\mathrm{w})\,J\mathbf{n}. \end{aligned}$$

And since $c_1\neq c_2$, taking into account orthogonality of the vector $\mathbf{e}_0$ to the subspace $\mathfrak{H}_1$ and unitarity of the operator $J$ on the subspace $\mathfrak{H}_1$, we get the equality $\mathcal{T}(\mathrm{w})=\langle\mathbf{n},\mathrm{w}\rangle=0$.

**Case 3**: $c_1<\infty$, $c_2=\infty$.

By the conditions of Lemma $\lambda\neq c_2$. Hence, in this case we have $\lambda<\infty$. And, according to (II.157), we obtain:

$$\begin{aligned} \mathbf{0} &= \mathbf{W}_{\lambda,c_1}[s,\mathbf{n},J]\,\mathrm{w}-\mathbf{W}_{\lambda,c_2}[s,\mathbf{n},J]\,\mathrm{w}=\\ &= \left(\left(\gamma\left(\frac{\lambda}{c_1}\right)-1\right)s\mathcal{T}(\mathrm{w})-\frac{\lambda}{c_1^2}\gamma\left(\frac{\lambda}{c_1}\right)\langle\mathbf{n},\mathrm{w}\rangle\right)\mathbf{e}_0+\\ &\quad+\left(\gamma\left(\frac{\lambda}{c_1}\right)-1\right)\left(\lambda\mathcal{T}(\mathrm{w})-s\langle\mathbf{n},\mathrm{w}\rangle\right)J\mathbf{n}. \end{aligned} \tag{II.194}$$

By the conditions of Lemma, $\lambda>0$ and $\lambda\neq\sigma(c_1,c_2)=\sqrt{2}\,c_1$. Thus, $\gamma\left(\frac{\lambda}{c_1}\right)-1\neq 0$. Hence, taking into account orthogonality of the vector $\mathbf{e}_0$ to the subspace $\mathfrak{H}_1$ and unitarity of the operator $J$ on the subspace $\mathfrak{H}_1$, from the equality (II.194) we receive the system of equations:

$$\left\{\begin{aligned} \left(\gamma\left(\tfrac{\lambda}{c_1}\right)-1\right)s\mathcal{T}(\mathrm{w})-\tfrac{\lambda}{c_1^2}\gamma\left(\tfrac{\lambda}{c_1}\right)\langle\mathbf{n},\mathrm{w}\rangle &= 0\\ \lambda\mathcal{T}(\mathrm{w})-s\langle\mathbf{n},\mathrm{w}\rangle &= 0. \end{aligned}\right. \tag{II.195}$$

The system (II.195) is a system of linear homogeneous equations relatively the variables $\mathcal{T}(\mathrm{w})$ and $\langle \mathbf{n}, \mathrm{w}\rangle$. Determinant of this system is:

$$\Delta_1 = -\left(\left(\gamma\left(\frac{\lambda}{c_1}\right) - 1\right) - \frac{\lambda^2}{c_1^2}\gamma\left(\frac{\lambda}{c_1}\right)\right) = -\left(\boldsymbol{g}\left(\frac{\lambda}{c_1}\right) - \boldsymbol{g}(0)\right).$$

Since, by the conditions of Lemma, $\lambda > 0$ and $c_1 < \infty$, then $\frac{\lambda}{c_1} \neq 0$. That is why, $\Delta_1 \neq 0$. Thus, $\mathcal{T}(\mathrm{w}) = \langle \mathbf{n}, \mathrm{w}\rangle = 0$.

**Case 4**: $c_1 = \infty$, $c_2 < \infty$ is considered similarly to the case 3.

**Case** $c_1, c_2 = \infty$ is impossible, because, by the conditions of Lemma, $c_1 \neq c_2$. □

**Corollary II.20.1.** *Let,* $c_1, c_2 \in (0, \infty]$, $c_1 \neq c_2$, $s \in \{-1, 1\}$, $J \in \mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$.

*Then for any* $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ *and* $\varepsilon \in (0, \min(c_1, c_2))$ *there exist* $\lambda \in (0, \varepsilon)$ *and* $\mathbf{a} \in \mathcal{M}(\mathfrak{H})$, *such, that*

$$\mathbf{W}_{\lambda, c_1}[s, \mathbf{n}, J; \mathbf{a}]\,\mathrm{w} \neq \mathbf{W}_{\lambda, c_2}[s, \mathbf{n}, J; \mathbf{a}]\,\mathrm{w},$$

*Proof.* Let us chose any $\mathbf{a} \in \mathcal{M}(\mathfrak{H})$ such, that:

$$\mathcal{T}(\mathrm{w} + \mathbf{a}) \neq 0. \tag{II.196}$$

Also we chose any number $\lambda \in (0, \varepsilon) \setminus \{\sigma(c_1, c_2)\}$. If we assume, that $\mathbf{W}_{\lambda, c_1}[s, \mathbf{n}, J; \mathbf{a}]\,\mathrm{w} = \mathbf{W}_{\lambda, c_2}[s, \mathbf{n}, J; \mathbf{a}]\,\mathrm{w}$, then, according to (II.165), we will obtain:

$$\mathbf{W}_{\lambda, c_1}[s, \mathbf{n}, J](\mathrm{w} + \mathbf{a}) = \mathbf{W}_{\lambda, c_2}[s, \mathbf{n}, J](\mathrm{w} + \mathbf{a}).$$

Hence, by Lemma II.20.2, $\mathcal{T}(\mathrm{w} + \mathbf{a}) = 0$, contrary to the correlation (II.196). Thus, $\mathbf{W}_{\lambda, c_1}[s, \mathbf{n}, J; \mathbf{a}]\,\mathrm{w} \neq \mathbf{W}_{\lambda, c_2}[s, \mathbf{n}, J; \mathbf{a}]\,\mathrm{w}$. □

*Proof of Theorem II.20.1.* **1**. For any fixed vector $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ we are going to prove the equality:

$$\mathbf{W}_{0;\mathfrak{V}_\mathfrak{f}}[1, \mathbf{n}, \mathbb{I}_{-1,1}[\mathbf{n}], \mathbf{0}] = \mathbb{I}_{\mathcal{M}(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}})}, \tag{II.197}$$

where $\mathbb{I}_{\mathcal{M}(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}})}$ is the the identity operator on $\mathcal{M}(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}})$, and operator $\mathbb{I}_{-1,1}[\mathbf{n}]$ is defined by (II.122). Indeed, according to (II.167), (II.165) and (II.157) for an arbitrary element $\omega = (t, (x, c)) \in \mathcal{M}(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}})$, we have:

$$\begin{aligned}
&\mathbf{W}_{0;\mathfrak{V}_\mathfrak{f}}[1, \mathbf{n}, \mathbb{I}_{-1,1}[\mathbf{n}], \mathbf{0}]\,\omega = \\
&\quad = \left(\mathsf{tm}\left(\mathbf{W}_{0,c}[1, \mathbf{n}, \mathbb{I}_{-1,1}[\mathbf{n}]]\,\omega^*\right),\ \left(\mathsf{bs}\left(\mathbf{W}_{0,c}[1, \mathbf{n}, \mathbb{I}_{-1,1}[\mathbf{n}]]\,\omega^*\right), c\right)\right) = \\
&\quad = \left(\mathsf{tm}(\omega^*),\ (\mathsf{bs}(\omega^*), c)\right) = (t,\ (x, c)) = \omega,
\end{aligned}$$

that was necessary to prove. From the equality (II.197) it follows, that $\mathbb{I}_{\mathcal{M}(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}})} \in \mathfrak{PT}_+(\mathfrak{H}; \mathfrak{V}_\mathfrak{f})$. Besides this, in accordance with Remark I.11.3, $\mathbb{I}_{\mathcal{M}(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}})}[\mathcal{B}] = \mathcal{B}$. Hence, by Property II.16.1(1), we can define the reference frame:

$$\mathfrak{l}_0 = \left(\mathbb{I}_{\mathcal{M}(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}})}, \mathbb{I}_{\mathcal{M}(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}})}[\mathcal{B}]\right) = \left(\mathbb{I}_{\mathcal{M}(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}})}, \mathcal{B}\right) \in \mathcal{L}k\left(\mathfrak{KPT}(\mathfrak{H}, \mathcal{B}; \mathfrak{V}_\mathfrak{f})\right).$$

Now, we fix any reference frame $\mathfrak{l} = (U, U[\mathcal{B}]) \in \mathcal{L}k\left(\mathfrak{KPT}(\mathfrak{H}, \mathcal{B}; \mathfrak{V}_\mathfrak{f})\right)$, where $U = \mathbf{W}_{\lambda,\mathfrak{V}_\mathfrak{f}}[1, \mathbf{n}, J; \mathbf{a}] \in \mathfrak{PT}_+(\mathfrak{H}; \mathfrak{V}_\mathfrak{f})$.

According to Properties II.16.1(3, 5), we obtain:

$$\mathbb{M}k(\mathfrak{l}) = \mathbb{R} \times \mathbf{Zk}\left(\widehat{\mathfrak{H}}\right) = \mathbb{R} \times \mathfrak{H} = \mathcal{M}(\mathfrak{H}); \tag{II.198}$$

$$\langle !\, \mathfrak{l} \leftarrow \mathfrak{l}_0 \rangle\, \omega = U\left(\mathbb{I}^{[-1]}_{\mathcal{M}(\mathfrak{H}_{\mathfrak{V}_\mathfrak{f}})}\omega\right) = U\omega = \mathbf{W}_{\lambda,\mathfrak{V}_\mathfrak{f}}[1, \mathbf{n}, J; \mathbf{a}]\,\omega \tag{II.199}$$

$$(\forall \omega \in \mathbb{Bs}(\mathfrak{l}_0) = \mathbb{Bs}(\mathcal{B}) \subseteq \mathcal{M}(\mathfrak{H}_{\mathfrak{V}_\mathsf{f}}).$$

Using Property II.16.1(3) as well as equality (II.169), for an elementary-time state $\omega = (t,(x,c)) \in \mathbb{Bs}(\mathfrak{l})$ we get:

$$\mathbf{Q}^{\langle \mathfrak{l} \rangle}(\omega) = (\mathsf{tm}(\omega), \mathbf{q}(\mathsf{bs}(\omega))) = (t, \mathbf{q}((x,c))) = (t,x) = \omega^*. \tag{II.200}$$

Hence, using Definition II.15.1 (item 1) and equality (II.168), we deduce:

$$\mathbf{Q}^{\langle \mathfrak{l} \leftarrow \mathfrak{l}_0 \rangle}(\omega) = \mathbf{Q}^{\langle \mathfrak{l} \rangle}(\langle ! \, \mathfrak{l} \leftarrow \mathfrak{l}_0 \rangle \omega) = \left(\mathbf{W}_{\lambda,\mathfrak{V}_\mathsf{f}}[1,\mathbf{n},J;\mathbf{a}]\,\omega\right)^* = \\ = \mathbf{W}_{\lambda,c}[1,\mathbf{n},J;\mathbf{a}]\,\omega^* \quad \left(\forall \omega \in \mathbb{Bs}(\mathfrak{l}_0) = \mathbb{Bs}(\mathcal{B}) \subseteq \mathcal{M}(\mathfrak{H}_{\mathfrak{V}_\mathsf{f}})\right). \tag{II.201}$$

**2**. By conditions of Theorem a number $\eta > 0$ exists such, that $\mathfrak{V}_\mathsf{f} \subseteq [\eta, \infty)$.

**2.1**. Suppose, that there exist elementary states $\widetilde{x}_1 = (x_1, c_1)$, $\widetilde{x}_2 = (x_2, c_2) \in \mathfrak{Bs}(\mathcal{B})$ such, that $c_1 \neq c_2$. Since, by Property I.6.1(9), $\mathfrak{Bs}(\mathcal{B}) = \{\mathsf{bs}(\omega) \mid \omega \in \mathbb{Bs}(\mathcal{B})\}$, then there exist elementary-time states of kind $\omega_1 = (t_1, \widetilde{x}_1) = (t_1,(x_1,c_1)) \in \mathbb{Bs}(\mathcal{B})$, $\omega_2 = (t_2, \widetilde{x}_2) = (t_2,(x_2,c_2)) \in \mathbb{Bs}(\mathcal{B})$. Now, we consider two cases.

**Case 2.1.1**: $\omega_1^* \neq \omega_2^*$. Consider any fixed operator $J_1 \in \mathfrak{U}(\mathfrak{H}_1)$. By Lemma II.20.1, there exist $\lambda_1 \in (0,\eta)$, $\mathbf{n}_1 \in \mathbf{B}_1(\mathfrak{H}_1)$ and $\mathbf{a}_1 \in \mathcal{M}(\mathfrak{H})$, such, that

$$\mathbf{W}_{\lambda_1,c_1}[1,\mathbf{n}_1,J_1;\mathbf{a}_1]\,\omega_1^* = \mathbf{W}_{\lambda_1,c_2}[1,\mathbf{n}_1,J_1;\mathbf{a}_1]\,\omega_2^*. \tag{II.202}$$

Let us introduce the reference frame:

$$\mathfrak{l}_1 := (U_1, U_1[\mathcal{B}]) \in \mathcal{L}k(\mathfrak{KPT}(\mathfrak{H},\mathcal{B};\mathfrak{V}_\mathsf{f})), \quad \text{where} \\ U_1 := \mathbf{W}_{\lambda_1;\mathfrak{V}_\mathsf{f}}[1,\mathbf{n}_1,J_1,\mathbf{a}_1] \in \mathfrak{PT}_+(\mathfrak{H};\mathfrak{V}_\mathsf{f}).$$

According to (II.201), and (II.202), we receive:

$$\mathbf{Q}^{\langle \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle}(\omega_1) = \mathbf{W}_{\lambda_1,c_1}[1,\mathbf{n}_1,J_1;\mathbf{a}_1]\,\omega_1^* = \mathbf{W}_{\lambda_1,c_2}[1,\mathbf{n}_1,J_1;\mathbf{a}_1]\,\omega_2^* = \mathbf{Q}^{\langle \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle}(\omega_2).$$

From the other hand, by the formula (II.200), we obtain $\mathbf{Q}^{\langle \mathfrak{l}_0 \rangle}(\omega_1) = \omega_1^* \neq \omega_2^* = \mathbf{Q}^{\langle \mathfrak{l}_0 \rangle}(\omega_2)$. Thus, for the elementary-time states $\omega_1$,$\omega_2$ we have $\mathbf{Q}^{\langle \mathfrak{l} \leftarrow \mathfrak{l}_0 \rangle}(\omega_1) = \mathbf{Q}^{\langle \mathfrak{l} \leftarrow \mathfrak{l}_0 \rangle}(\omega_2)$, while $\mathbf{Q}^{\langle \mathfrak{l}_0 \rangle}(\omega_1) \neq \mathbf{Q}^{\langle \mathfrak{l}_0 \rangle}(\omega_2)$. Hence, by Theorem II.15.1, the reference frames $\mathfrak{l}_0$ and $\mathfrak{l}$ do not allow universal coordinate transform. Therefore, in accordance with Assertion II.15.2, item 2, the kinematic set $\mathfrak{KPT}(\mathfrak{H},\mathcal{B};\mathfrak{V}_\mathsf{f})$ do not allow universal coordinate transform in this case.

**Case 2.1.2**: $\omega_1^* = \omega_2^*$. Consider any fixed operator $J_2 \in \mathfrak{U}(\mathfrak{H}_1)$ and vector $\mathbf{n}_2 \in \mathbf{B}_1(\mathfrak{H}_1)$. According to Corollary II.20.1, there exist $\lambda_2 \in (0,\eta)$ and $\mathbf{a}_2 \in \mathcal{M}(\mathfrak{H})$, such, that

$$\mathbf{W}_{\lambda_2,c_1}[1,\mathbf{n}_2,J_2;\mathbf{a}_2]\,\omega_1^* \neq \mathbf{W}_{\lambda_2,c_2}[1,\mathbf{n}_2,J_2;\mathbf{a}_2]\,\omega_2^*. \tag{II.203}$$

Let us consider the reference frame:

$$\mathfrak{l}_2 := (U_2, U_2[\mathcal{B}]) \in \mathcal{L}k(\mathfrak{KPT}(\mathfrak{H},\mathcal{B};\mathfrak{V}_\mathsf{f})), \quad \text{where} \\ U_2 = \mathbf{W}_{\lambda_2;\mathfrak{V}_\mathsf{f}}[1,\mathbf{n}_2,J_2,\mathbf{a}_2] \in \mathfrak{PT}_+(\mathfrak{H};\mathfrak{V}_\mathsf{f}).$$

According to (II.201), and (II.203), we receive:

$$\mathbf{Q}^{\langle \mathfrak{l}_2 \leftarrow \mathfrak{l}_0 \rangle}(\omega_1) = \mathbf{W}_{\lambda_2,c_1}[1,\mathbf{n}_2,J_2;\mathbf{a}_2]\,\omega_1^* \neq \mathbf{W}_{\lambda_2,c_2}[1,\mathbf{n}_2,J_2;\mathbf{a}_2]\,\omega_2^* = \mathbf{Q}^{\langle \mathfrak{l}_2 \leftarrow \mathfrak{l}_0 \rangle}(\omega_2).$$

From the other hand, by the formula (II.200), we obtain: $\mathbf{Q}^{\langle \mathfrak{l}_0 \rangle}(\omega_1) = \omega_1^* = \omega_2^* = \mathbf{Q}^{\langle \mathfrak{l}_0 \rangle}(\omega_2)$.

Thus, for the elementary-time states $\omega_1$,$\omega_2$ we have $\mathbf{Q}^{\langle \mathfrak{l}_2 \leftarrow \mathfrak{l}_0 \rangle}(\omega_1) \neq \mathbf{Q}^{\langle \mathfrak{l}_2 \leftarrow \mathfrak{l}_0 \rangle}(\omega_2)$, while $\mathbf{Q}^{\langle \mathfrak{l}_0 \rangle}(\omega_1) = \mathbf{Q}^{\langle \mathfrak{l}_0 \rangle}(\omega_2)$. Hence, by Theorem II.15.1, the reference frames $\mathfrak{l}_0$ and $\mathfrak{l}_2$ do not allow universal coordinate transform. Therefore, in accordance with Assertion II.15.2, item 2, the kinematic set $\mathfrak{KPT}(\mathfrak{H},\mathcal{B};\mathfrak{V}_\mathsf{f})$ does not allow universal coordinate transform.

***Thus***, if the kinematic set $\mathfrak{KPT}(\mathfrak{H},\mathcal{B};\mathfrak{V}_\mathsf{f})$ allows universal coordinate transform, then there not exist elementary states $\widetilde{x}_1 = (x_1,c_1)$, $\widetilde{x}_2 = (x_2,c_2) \in \mathfrak{Bs}(\mathcal{B})$ such, that $c_1 \neq c_2$.

**2.2**. Now we suppose, that in base changeable set $\mathcal{B}$ there not exist elementary states $\widetilde{x}_1 = (x_1, c_1)$, $\widetilde{x}_2 = (x_2, c_2) \in \mathfrak{Bs}(\mathcal{B})$ such, that $c_1 \neq c_2$. Under this assumption a number $c_0 \in \mathfrak{V}_{\mathfrak{f}}$ must exist such, that arbitrary elementary state $\widetilde{x} \in \mathfrak{Bs}(\mathcal{B})$ can be represented in the form: $\widetilde{x} = (x, c_0)$, where $x \in \mathfrak{H}$. Chose any reference frame:

$$\mathfrak{l} := (U, U[\mathcal{B}]) \in \mathcal{L}k\left(\mathfrak{KPT}\left(\mathfrak{H}, \mathcal{B}; \mathfrak{V}_{\mathfrak{f}}\right)\right), \quad \text{where}$$
$$U = \mathbf{W}_{\lambda;\mathfrak{V}_{\mathfrak{f}}}\left[1, \mathbf{n}, J, \mathbf{a}\right] \in \mathfrak{PT}_{+}\left(\mathfrak{H}; \mathfrak{V}_{\mathfrak{f}}\right).$$

According to (II.201), (II.200), for arbitrary elementary-time state $\omega = (t, (x, c_0)) \in \mathbb{B}\mathfrak{s}\left(\mathfrak{l}_0\right) = \mathbb{B}\mathfrak{s}(\mathcal{B})$ we obtain:

$$\mathbf{Q}^{\langle \mathfrak{l} \leftarrow \mathfrak{l}_0 \rangle}(\omega) = \mathbf{W}_{\lambda, c_0}\left[1, \mathbf{n}, J; \mathbf{a}\right]\omega^* = \mathbf{W}_{\lambda, c_0}\left[1, \mathbf{n}, J; \mathbf{a}\right]\left(\mathbf{Q}^{\langle \mathfrak{l}_0 \rangle}(\omega)\right),$$

where $\mathbf{W}_{\lambda, c_0}\left[1, \mathbf{n}, J; \mathbf{a}\right]$ is a bijection from $\mathcal{M}\left(\mathfrak{H}\right)$ onto $\mathcal{M}\left(\mathfrak{H}\right)$ (and, by (II.198), $\mathbf{W}_{\lambda, c_0}\left[1, \mathbf{n}, J; \mathbf{a}\right]$ is a bijection from $\mathbb{M}k\left(\mathfrak{l}_0\right)$ onto $\mathbb{M}k(\mathfrak{l})$). Hence, in accordance with Definition II.15.1, the mapping $\mathbf{W}_{\lambda, c_0}\left[1, \mathbf{n}, J; \mathbf{a}\right]$ is universal coordinate transform from $\mathfrak{l}_0$ to $\mathfrak{l}$. Consequently, the reference frames $\mathfrak{l}_0$ and $\mathfrak{l}$ allow universal coordinate transform, ie $\mathfrak{l}_0 \rightleftarrows \mathfrak{l}$ (for any reference frame $\mathfrak{l} \in \mathcal{L}k\left(\mathfrak{KPT}\left(\mathfrak{H}, \mathcal{B}; \mathfrak{V}_{\mathfrak{f}}\right)\right)$). Thus, by Assertion II.15.2, kinematic set $\mathfrak{KPT}\left(\mathfrak{H}, \mathcal{B}; \mathfrak{V}_{\mathfrak{f}}\right)$ allows universal coordinate transform. □

Similarly to Theorem II.20.1 it can be proved the following theorem.

**Theorem II.20.2.** *Let the set of forbidden velocities $\mathfrak{V}_{\mathfrak{f}} \subseteq (0, \infty]$ be separated from zero (ie there exists a number $\eta > 0$ such, that $\mathfrak{V}_{\mathfrak{f}} \subseteq [\eta, \infty]$).*

*Kinematic set $\mathfrak{KPT}_0\left(\mathfrak{H}, \mathcal{B}; \mathfrak{V}_{\mathfrak{f}}\right)$ allows universal coordinate transform if and only if there don't exist elementary states $\widetilde{x}_1 = (x_1, c_1)$, $\widetilde{x}_2 = (x_2, c_2) \in \mathfrak{Bs}(\mathcal{B})$ such, that $c_1 \neq c_2$.*

The main results of this Section were announced in the paper [11] and published in [12].

# Part III
# Kinematic Changeable Sets with Given Universal Coordinate Transforms

In this Part we are going to explain the theory of universal kinematics (that is kinematic changeable sets with given universal coordinate transforms). Results, to be explained in this Part, are published in the papers [13–15].

## Sorry!

Now this Part of the paper is under development and it will be available in the next versions of the preprint as soon as possible.

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