"*An Introduction to Functions of a Quaternion Hypercomplex Variable - PART 1/6.*"

by Stephen C. Pearson MRACI
[T/A S. C. Pearson Technical Services],
Dorset. ENGLAND.

Email Address:- *scpearson1952@talktalk.net* .

I. PRELIMINARY REMARKS.

(1) The author accordingly wishes to thank

(a) the following mathematicians, namely – (i) Mr. J. R. Holmes (MA, BSc, MEd, MACE); (ii) Mr. P. G. Brown (MA); (iii) Mr. L. Lobb (BSc, MSc, FIMA) & (iv) Mr. S. J. Crothers (BA, Adv.Cert.Comp.Tech., Affil.IEAust., Affil.AC S), who refereed his original paper on 18th February 1985; 15th April 1985; 10th March 1986 & 8th June 1995 respectively;

(b) the Mathematical Association of New South Wales (M.A.N.S.W) for publishing three (3) expository articles thus pertaining to his original paper in *"Reflections" (Journal of M.A.N.S.W), Volume 11 – Nos. 2; 3 & 4 of May 1986; August 1986 & November 1986* respectively;

(c) the Mathematical Association of Tasmania (M.A.T) for publishing one (1) expository article thus pertaining to his original paper in *"Delta" (Journal of M.A.T), Volume 26 – No. 3 of September 1986*;

(d) the American Mathematical Society (A.M.S) for publishing an abstract thus pertaining to his original paper in the A.M.S journal, *"ABSTRACTS of Papers presented to the American Mathematical Society", Volume 22 – No.2 of 2001 (Issue 124)*.

(2) Pages [2] – [5] of this submission contain copies of references from those mathematicians mentioned in Statement (1)(a) above.

(3) Pages [6] – [16] of this submission contain copies of announcements pertaining to those expository articles described in Statements (1)(b) & (1)(c) above.

(4) Pages [17] – [18] of this submission contain a copy of the abstract described in Statement (1)(d) above.

II. COPY OF AUTHOR'S ORIGINAL PAPER – PART 1/6.

For further details, the reader should accordingly refer to the remainder of this submission (as per Statement (4) of Section I) from Page [19] onwards.

*   *   *   *   *   *   *   *

****
****
****
****
****
****
****
****

****

J.R. Holmes MA, BSc, MEd, MACE

3 Lucia Avenue,
Baulkham Hills. N.S.W. 2153

18th February, 1985

TO WHOM IT MAY CONCERN

I have known Stephen Christopher Pearson only for some three months after responding to an advertisement placed by him in the Newsletter of the Mathematical Association of New South Wales for some person to read the draft of a paper written by him.

The paper was entitled "An Introduction to Functions of a Quaternion Hypercomplex Variable".

I have read the paper and found that he has applied himself to an advanced area of Mathematics in a most enthusiastic and conscientious manner.

Using very few published references he has produced a carefully structured paper. He has, in this paper, demonstrated a high degree of ability in analytical thought and in logical argument.

From my reading, Mr Pearson appears to be a most conscientious and able student of Mathematics, of high potential.

J.R. Holmes

J.R. Holmes

****

****

26 CARRINGTON STREET
SUMMER HILL 2130
15/4/85

TO WHOM IT MAY CONCERN

I came into contact with Mr. Stephen Christopher Pearson through the organisation known as MANSWA (Mathematical Association of New South Wales). He kindly invited me to read and assess his paper "An introduction to Functions of a Quaternion Hypercomplex Variable" completed on 26th May 1984. Accordingly, I have made several comments and suggestions which he may find helpful.

Mr. Pearson's understanding of the mathematics involved and his rigourous analysis of the topic have been very impressive. He has shown great care and insight investigating this previously ignored field of Mathematics and has proved many interesting theorems and helped to clarify the difficulties involved in studying this non-commutative ring from an analytic rather than algebraic viewpoint. He deserves high commendation for his work thus far and every encouragement to continue to tread this untrodden ground.

Yours sincerely

P.G. BROWN M.A.

P. G. Brown M.A.

(Teacher of Mathematics at Bankstown High School)

****

****

MITCHELL
COLLEGE OF ADVANCED EDUCATION

BATHURST NSW 2795.
AUSTRALIA. Ph. (063) 311022

SCHOOL OF MATHEMATICS
APPLIED SCIENCE AND PLANNING

10 March 1986

TO WHOM IT MAY CONCERN

I have known Stephen C. Pearson through correspondence and telephone communication for over a year. I first became aware of his work through an advertisement placed by him in the Newsletter of the Mathematical Association of New South Wales. This concerned a paper written by him called "An Introduction to Functions of a Quaternion Hypercomplex Variable".

I referred the paper to one of my mathematics external students who was interested in quaternions. That student certainly gained a great deal from the clear exposition available from Mr. Pearson's paper.

My own reading of the paper has brought a realisation of the great amount of work Mr. Pearson has devoted to drawing together, in a rigorous manner, the concepts of this somewhat neglected area. The author's precise statements and proofs of the properties of quaternion hypercomplex functions are very scholarly. He exhibits much careful logical thought and analysis. Furthermore, in keeping with the independent and even "romantic" background of Hamilton's first "break-throughs" on quaternions, the paper exhibits a pleasing degree of enthusiasm, as well as skill!

From my reading, Mr. Pearson deserves commendation for his diligence and skill in gathering and clarifying the interesting "analogous" and "diverging" results that become evident from such an attempt to extend real and complex analysis to hypercomplex analysis. In his conclusion he has posed a number of acute questions. I trust that he (and others) will be encouraged to delve further into this field.

Lawson Lobb

LAWSON LOBB B.Sc.,M.S.,FIMA.

Senior Lecturer.

****

****

476 Lyons Road West
Five Dock. 2046
Sydney
Australia

8th June, 1995

TO WHOM IT MAY CONCERN

I have known Mr. Stephen C. Pearson since 1984 when, as a student of mathematics at Mitchell College of Advanced Education (now Charles Sturt University), I responded to an advertisement placed by him in the newsletter of the Mathematical Association of New South Wales which referenced his research on analytic functions of a hypercomplex quaternion variable. I was provided with a copy of the aforesaid advertisement by one of my lecturers after an inquiry.

My continued association with Mr. Pearson has given me an opportunity to read his unpublished research papers.

I believe Mr. Pearson has carried out a cogent and concise development of this aspect of the calculus. He has demonstrated a sound and informed mathematical method along with an admirable ability to overcome certain conceptual difficulties associated with such ideas as noncommutativity, multivalued functions, and fields in the context peculiar to hypercomplex quaternion analysis.

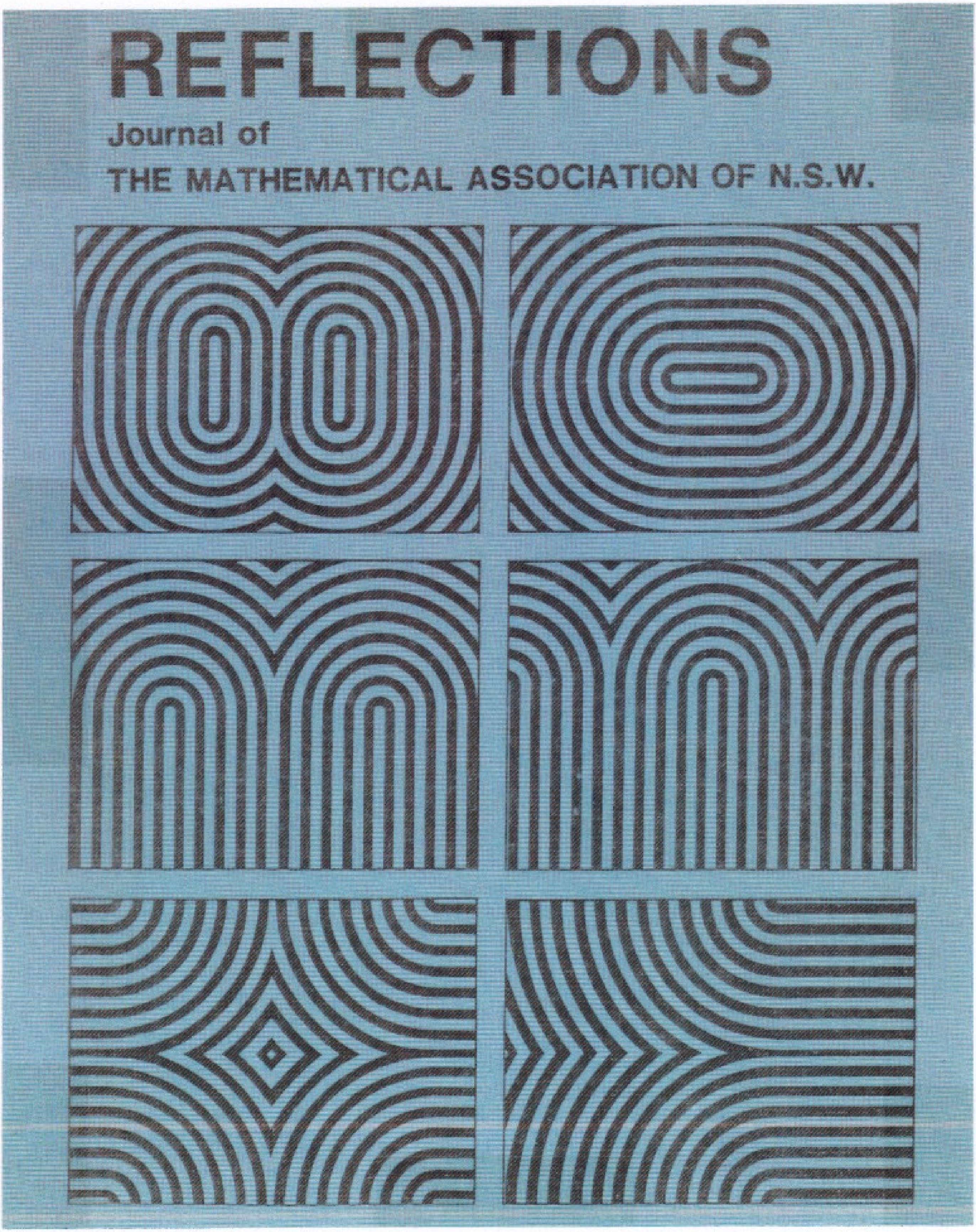
Mr. Pearson's research has motivated me to conduct my own investigations into the quaternion analogues of particular aspects of the analytic theory of complex variables.

Mr. Pearson's work is, as far as I am aware, an original and significant contribution to the theory of mathematical analysis.

Stephen J. Crothers

Stephen J. Crothers B A., Adv.Cert.Comp.Tech., Affil.IEAust, Affil.ACS.

****


REFLECTIONS
Journal of
THE MATHEMATICAL ASSOCIATION OF N.S.W.

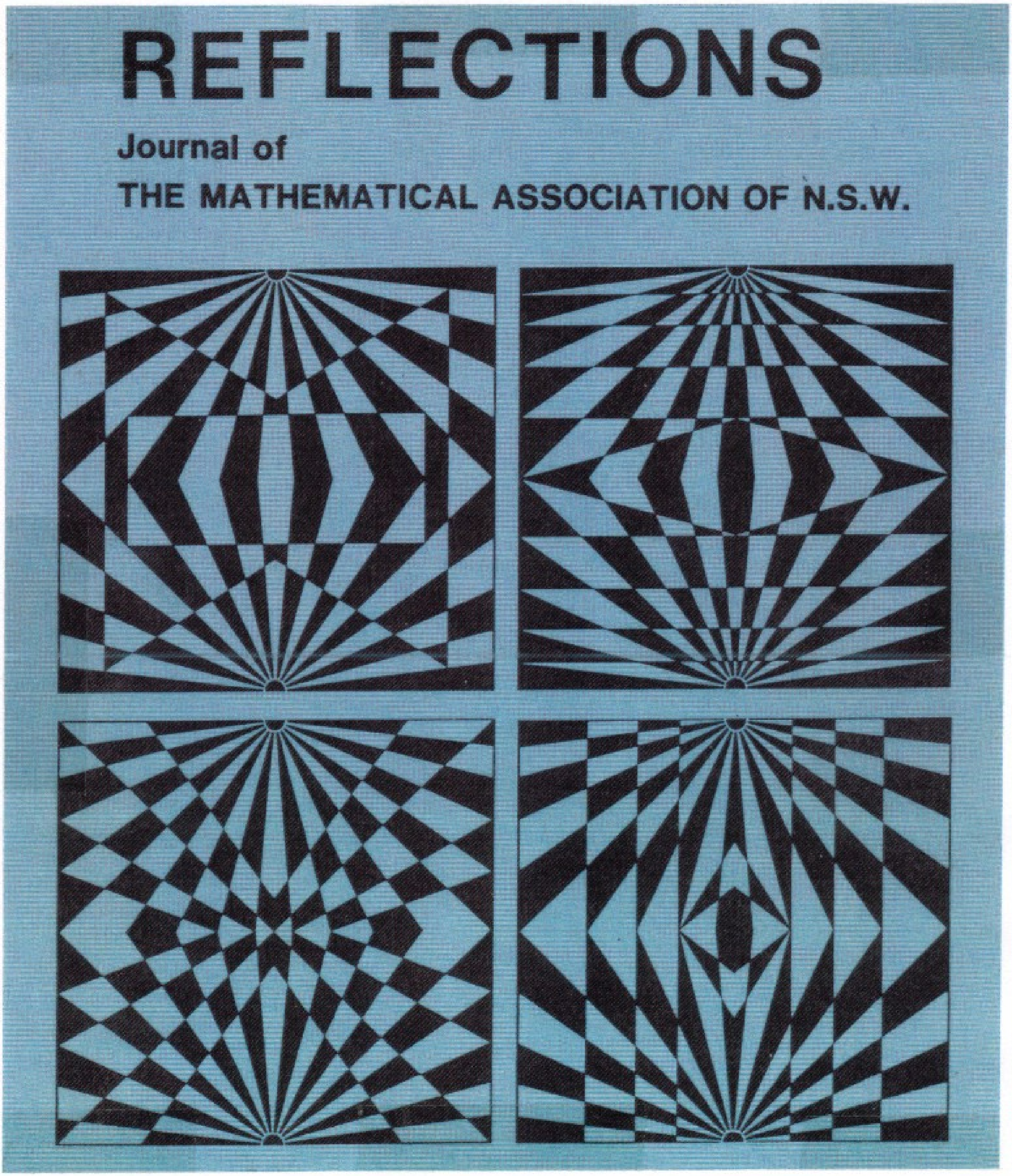
****

VOL 11
Nº 2
MAY 1986

Registered for posting as a publication—Category B
Publication Nº NBH 2995

CONTENTS

****


REFLECTIONS
Journal of
THE MATHEMATICAL ASSOCIATION OF N.S.W.

VOL. 11

Nº 3

AUGUST 1986

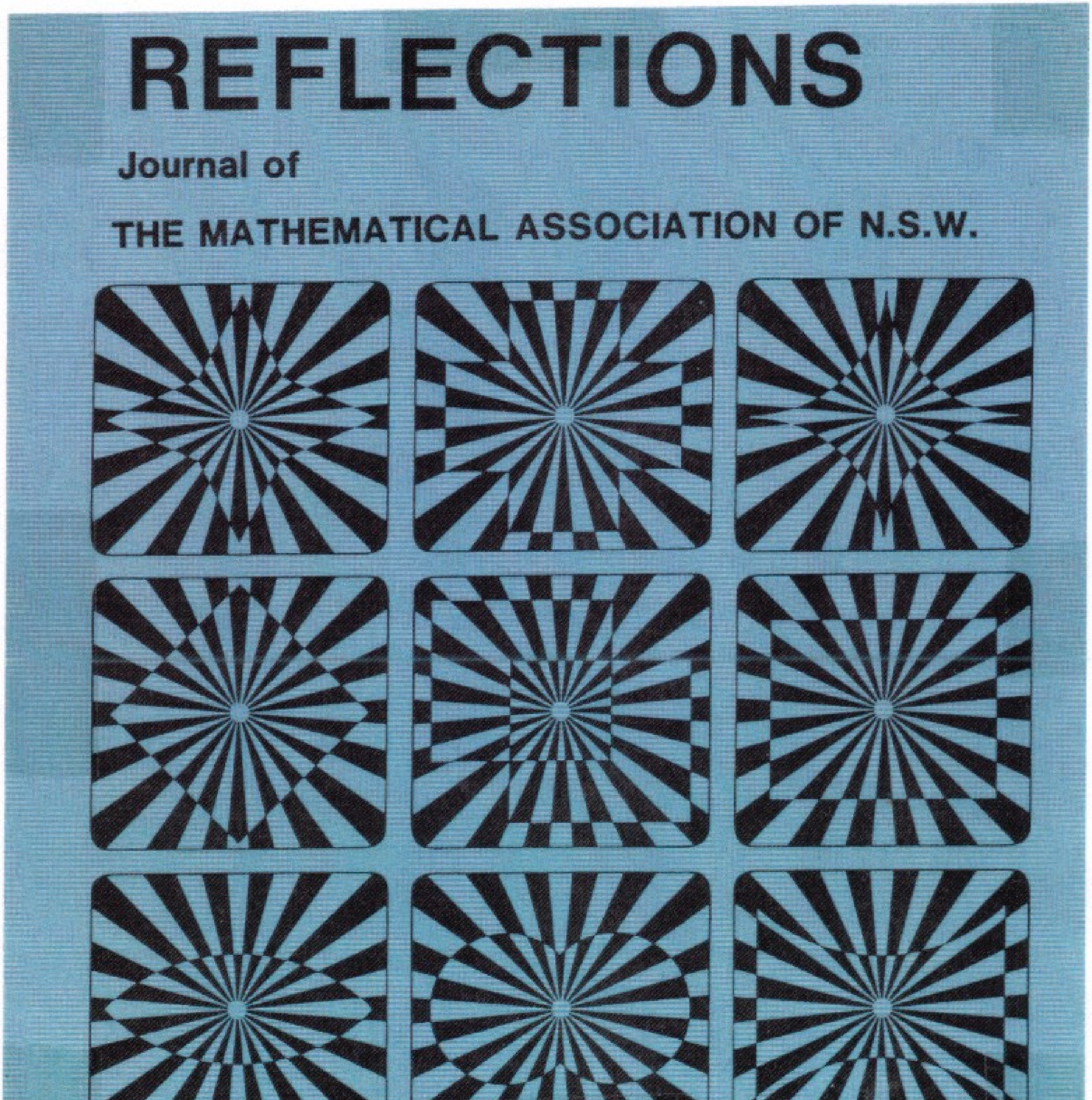
Registered for posting as a publication–Category B

Publication No. NBH 2995

CONTENTS

****

****


REFLECTIONS
Journal of
THE MATHEMATICAL ASSOCIATION OF N.S.W.

****

VOL. 11

Nº 4 NOVEMBER 1986

Registered for posting as a publication–Category B

Publication No. NBH 2995

CONTENTS

****

****

****

DELTA

VOL: 26 SEPTEMBER '86

Nº : 3

"Journal of the Mathematical Association of Tasmania."

Registered by Australia Post Publication Number TBH0295

MAT EXECUTIVE 1986

ASSOCIATION ADDRESS:
PO Box 313
Sandy Bay, Tasmania 7005

CONTENTS

****

ABSTRACTS

of Papers Presented to the American Mathematical Society

Volume 22, Number 2 Issue 124

Abstracts for

ISSN 0192-5857 Pages 271-434

AMS

****

****

BY-TITLE ABSTRACTS

The abstracts printed in this section were accepted by the American Mathematical Society for written presentation. An individual may present two abstracts per issue, including abstracts co-authored.

00 ► General

01T-00-25 **Saad H. Mohamed**, Ain Shams University, Cairo, Egypt, and **Bruno J. Mueller***, McMaster University, Hamilton, Ontario, Canada. *Direct sums of lifting modules.*

A module is said to be lifting, or (D_1), if it is supplemented and every supplement is a direct summand. Let $M = A \oplus B$ be modules. Recall that B is A-objective if for any complement C of B in M, M decomposes as $M = C \oplus A' \oplus B'$ with $A' \subseteqq A$ and $B' \subseteqq B$. Dually we define A to be B-cojective if $M = S \oplus A' \oplus B'$ holds for every supplement S of B in M.

Theorem 1: If A and B are lifting and mutually cojective, then $M = S \oplus A' \oplus B'$ holds for any supplement S in M whatsoever.

We call a module semidiscrete if it is lifting and enjoys the finite internal exchange property.

Theorem 2: If A and B are semidiscrete and mutually cojective, then M is semidiscrete.

[For terminology cf. Mohamed-Mueller, Continuous and Discrete Modules, Cambridge Univ. Press (1990), and AMS Abstracts 21, 00T-16-21.] (Received December 31, 2000)

01T-00-28 **Stephen C. Pearson***, S.C. Pearson Scientific (SCI) Software, S.C. Pearson Engineering Services, 83 Pimelea Drive, Woodford, N.S.W. 2778, Australia. *An introduction to functions of a quaternion hypercomplex variable.* Preliminary report.

This paper was completed on 31st March 1984 and comprises a total of 161 foolscap pages. Its purpose is to enunciate various definitions and theorems which pertain to the following topics, i.e. (a) the algebra of quaternion hypercomplex numbers; (b) functions of a single quaternion hypercomplex varible; (c) the concepts of limit and continuity applied to such functions; (d) the elementary principles of differentiation and integration applied to quaternion hypercomplex functions. Many of the concepts presented therein are analogous to well established notions from real and complex variable analysis with any divergent results being due to the non-commutativity of quaternion products. (Received December 31, 2000)

13 ► Commutative Rings and Algebras

01T-13-21 **Jim Coykendall*** (Jim_Coykendall@ndsu.nodak.edu), Department of Mathematics, North Dakota State University, Fargo, ND 58105-5075, and **David E Dobbs** (dobbs@math.utk.edu), Department of Mathematics, Ayres Hall, The University of Tennessee, Knoxville, TN 37996. *Lying over and survival pairs of commutative rings.*

Abstract: In the paper "Lying-over Pairs of Commutative Rings," (*Canad. J. Math.* **33** (1981), no. 2, 454–475),the notions of lying-over-pair (LO-pair) and survival pair of commutative rings were introduced by the second author. It is evident that any LO-pair is a survival pair, and the cited paper established the converse in many cases. In the present work, we establish the converse in general. Thus, in combination with some previous results of the cited paper, we conclude that if $R \subseteq T$ are commutative rings (with the same 1), then: (R,T) is a survival-pair iff (R,T) is an LO-pair iff (R,T) is a GU-pair. (Received December 08, 2000)

30 ► Functions of a Complex Variable

01T-30-23 **Mark Burgin***, Department of Mathematics, UCLA, Los Angeles, CA 90095. *Classes of Extrafunctions.* Preliminary report.

As it is demonstrated in [M. Burgin, Topological Characterization of Hypernumbers, VIII Conf. on General Topology and Appl., New York, 1992] theory of hypernumbers and extrafunctions is based on topological technique like non-standard analysis stems from set theoretical constructions. However, topology in functional spaces may be introduced in different ways. Taking the pointwise convergence topology, we obtain extrafunctions studied in [M. Burgin, Differential Calculus For Extrafunctions, Doklady Nat. Acad. Sci. of Ukraine, 1993, No. 11]. We call them pointwise extrafunctions. To define compactwise extrafunctions, we take the space $FR(FC)$

****

***

"An Introduction to Functions of a Quaternion Hypercomplex Variable"

by Stephen. C. Pearson,
Student (No. 7952007),
School of Mathematical Sciences,
N. S. W Institute of Technology.

31st March 1984.

FINAL DRAFT pending further assessment.

***

PREFACE

As its title suggests, the overall aim of this dissertation is to provide a theoretical introduction to the properties of functions of a quaternion hypercomplex variable. Hence, it is only natural that we should commence any discourse on this subject by firstly defining the basic algebraic properties of quaternion hypercomplex numbers (cf. Sections I and II).

***

Once these particular properties are established, it then becomes a relatively easy task to formulate a suitable definition for the function of a single quaternion hypercomplex variable, whereupon further algebraic properties thereof may be determined as well as the appropriate conditions governing both the existence of limits and the continuity of these functions (cf. Section III). Finally, we conclude our formal analysis of this topic with various applications of the elementary principles of differential and integral calculus in relation to quaternion hypercomplex functions (cf. Section IV).

As a means of developing a totally rigorous approach to the hitherto described material, the author has subsequently adopted Churchill et al. (cf. Reference B2, Section VII) as a suitable guideline for that very purpose, in as much as the latter named authors have enunciated many notions, thus pertaining to complex variable analysis, which fortunately can be treated in an analogous manner with respect to quaternion hypercomplex numbers and their corresponding functions. However, it should also be noted that quaternion hypercomplex numbers, unlike real and complex numbers, are generally not commutative – indeed, it is the virtual absence of this very fundamental property which is largely responsible for any significant divergence in the behavior of such numbers by direct contrast with their real and complex counterparts.

In summary, the author has made a number of pertinent comments in respect of both the above mentioned factors and other related matters – essentially, this was conducted in the spirit of an appraisal of all such results having been elucidated in the ensuing Sections I – IV

****

of this dissertation (cf. Section V).

Stephen. C. Pearson,
Student (No. 7952007),
School of Mathematical Sciences,
N.S.W Institute of Technology.

31st March 1984.

****

TABLE OF CONTENTS

****

***

***

****

****

I. Introduction

Before defining the existence and subsequent properties of a quaternion hypercomplex number, we will firstly assume that the properties of both real and complex numbers are clearly understood. For the purposes of our discussion, we shall accordingly denote the set of all real numbers by the letter, $\mathbb{R}$, and similarly the set of all complex numbers by the letter, $\mathbb{C}$.

Since it has already been established that

(i) real and complex numbers obey the common properties of associativity and commutativity

AND

(ii) the set of all real numbers is a subset of the set of all complex numbers, ie.

$$\mathbb{R} \subset \mathbb{C} \ ,$$

the question naturally arises as to whether or not one may logically postulate the existence of an additional extension to $\mathbb{C}$. Indeed, the Irish mathematician, Hamilton (1805 – 1865), initially realised that, if one were to discard the law of commutativity of multiplication, then we may extend $\mathbb{C}$ to a <u>set, $\mathbb{H}$, of quaternions</u>, which is defined as a <u>two-dimensional vector space</u> over $\mathbb{C}$ and hence a <u>four-dimensional vector space</u> over $\mathbb{R}$. Such an algebraic entity, in which all the field axioms except that of the commutativity of multiplication are satisfied, is called a <u>skew field</u>.

We now construct the set, $\mathbb{H}$, by expressing it as a vector space over $\mathbb{R}$ with a set of <u>basis vectors, $\{1, i, j, k\}$</u>. In order to define an operation for multiplication which satisfies the distributive law on any vector space, V, over $\mathbb{R}$ with basis vectors, $\{e_1, \ldots\ldots, e_n\}$, it is sufficient that we define all the products, $e_r e_s$, such that for any given

$v, v' \in V$, we accordingly obtain

$$\left.\begin{aligned} v &= \sum v_r e_r \\ v' &= \sum v'_r e_r \end{aligned}\right\} \ , \ \forall\, v_r, v'_r \in \mathbb{R} \qquad (1-1),$$

and hence, by the law of distribution,

$$vv' = \sum_{r,s} v_r v'_s (e_r e_s) \qquad (1-2).$$

Since each $e_r e_s \in V$, we further deduce that

$$e_r e_s = \sum_t \alpha_{rst} e_t, \quad (\alpha_{rst} \in \mathbb{R}) \qquad (1-3),$$

where the numbers, α_{rst}, are called structure constants. In this manner, every product, vv', may be evaluated, once all the numbers (n^3 in total), α_{rst}, have been similarly determined.

Any vector space, V, which obeys Eq. (1-3), is called a linear algebra over $\mathbb{R}$, or more simply, a real algebra. In particular, whenever $(V, e_1, \ldots\ldots, e_n)$ is written as $(\mathbb{H}, 1, i, j, k)$, we henceforth define both the structure of a quaternion and the products of its constituent basis elements in accordance with the following definition:–

Definition DI–1

A quaternion number,

$$q = x + iy + j\hat{x} + k\hat{y}, \text{ where } x, y, \hat{x}, \hat{y} \in \mathbb{R},$$

is defined by the following rules, with respect to its constituent basis elements, $1, i, j, k$:–

****

$1 \cdot i = i \cdot 1 = i$, $1 \cdot j = j \cdot 1 = j$, $1 \cdot k = k \cdot 1 = k$;

$1^2 = 1$, $i^2 = j^2 = k^2 = -1$;

$ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$,

where '1' denotes the identity element of multiplication. Furthermore, we define

$0 \cdot 1 = 1 \cdot 0 = 0 \cdot i = i \cdot 0 = 0 \cdot j = j \cdot 0 = 0 \cdot k = k \cdot 0 = 0$,

the identity element of addition.

As regards any variation on notation, some authors may prefer to denote a given quaternion, q, by the equation —

$$q = 1 \cdot x + i \cdot y + j \cdot \hat{x} + k \cdot \hat{y} \qquad (1-4),$$

but we will instead take the liberty of suppressing the element '1' from such notation.†

Finally, we remark that finite-dimensional vector spaces over $\mathbb{R}$, admitting an associative multiplication as indicated by Eq. (1-2), are often referred to as systems of hypercomplex numbers. Clearly, the set, $\mathbb{H}$, of quaternions falls into this category and, from this point onwards, we shall otherwise refer to quaternions as quaternion hypercomplex numbers, insofar as these synonymous terms will be liberally interchanged throughout the remainder of this text.

****

****

† Further details regarding choice of notation are provided in Part 1 of Section III.

****

II. The Algebra of Quaternion Hypercomplex Numbers

Having previously encountered complex variables, the reader will no doubt be familiar with the following algebraic operations and concepts pertaining thereto :-

(a) addition and multiplication,
(b) conjugate,
(c) inverse,
(d) quotient,
(e) real and imaginary parts,
(f) modulus,
(g) triangle inequality.

Because of their relative simplicity, the above mentioned notions can therefore be readily extended to the set of all quaternion hypercomplex numbers, $\mathbb{H}$ — indeed, we shall reinforce this fact by explicitly stating these same notions in both their complex and quaternion hypercomplex forms, wherever practicable.

Once again, we emphasise and will in due course verify our original assertion that the multiplication of quaternions is generally not commut-

****

****

ative. This is by no means a trivial assertion, since it will soon become apparent that this overall lack of commutativity with respect to the multiplication of quaternions is ultimately the determining factor as regards any significant deviation in the behaviour of such variables compared with the concomitant properties of their real and complex counterparts.

1. Basic Operations of Addition, Subtraction and Multiplication.

From Definition DI-1, we have already established that the

****

quaternion,

$q = x + iy + j\hat{x} + k\hat{y}$,

whence it naturally follows that any two quaternions, q_1 and q_2, may be written as

$q_1 = x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1$ (2-1),

$q_2 = x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2$ (2-2).

Consequently, the addition of Eqs. (2-1) and (2-2) yields -

****

$$q_1 + q_2 = x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1 + x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2$$

$$= x_1 + x_2 + i(y_1 + y_2) + j(\hat{x}_1 + \hat{x}_2) + k(\hat{y}_1 + \hat{y}_2)$$

$$= x_2 + x_1 + i(y_2 + y_1) + j(\hat{x}_2 + \hat{x}_1) + k(\hat{y}_2 + \hat{y}_1)$$

$$= q_2 + q_1 \qquad (2-3).$$

Similarly, the subtraction of Eq. (2-2) from Eq. (2-1) yields –

$$q_1 - q_2 = x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1 - (x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2)$$

$$= x_1 - x_2 + i(y_1 - y_2) + j(\hat{x}_1 - \hat{x}_2) + k(\hat{y}_1 - \hat{y}_2) \qquad (2-4).$$

The above stated results may henceforth be summarised by way of the following definition :–

Definition DII – 1

For any two quaternions,

$$q_1 = x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1 ,$$

$$q_2 = x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2 ,$$

the sum and difference thereof, $q_1 + q_2$ and $q_1 - q_2$, are respectively defined as –

(i) $q_1 + q_2 = q_2 + q_1 = x_1 + x_2 + i(y_1 + y_2) + j(\hat{x}_1 + \hat{x}_2) + k(\hat{y}_1 + \hat{y}_2)$,

(ii) $q_1 - q_2 = x_1 - x_2 + i(y_1 - y_2) + j(\hat{x}_1 - \hat{x}_2) + k(\hat{y}_1 - \hat{y}_2)$.

Our next task is to investigate the associativity of the addition of quaternions, for which we shall accordingly enunciate the following theorem :–

Theorem TII – 1

The addition of any three quaternions, q_1, q_2, q_3, is always associative, that is ~~to~~ say

$$(q_1 + q_2) + q_3 = q_1 + (q_2 + q_3).$$

* * *

PROOF :–

From Definitions DI–1 and DII–1, we initially obtain

$$q_1 = x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1,$$

$$q_2 = x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2,$$

$$q_3 = x_3 + iy_3 + j\hat{x}_3 + k\hat{y}_3,$$

****

$$q_1 + q_2 = x_1 + x_2 + i(y_1 + y_2) + j(\hat{x}_1 + \hat{x}_2) + k(\hat{y}_1 + \hat{y}_2),$$

$$q_2 + q_3 = x_2 + x_3 + i(y_2 + y_3) + j(\hat{x}_2 + \hat{x}_3) + k(\hat{y}_2 + \hat{y}_3).$$

Hence, we further deduce that

$$\begin{aligned}
(q_1 + q_2) + q_3 &= [x_1 + x_2 + i(y_1 + y_2) + j(\hat{x}_1 + \hat{x}_2) + k(\hat{y}_1 + \hat{y}_2)] + \\
&\quad x_3 + iy_3 + j\hat{x}_3 + k\hat{y}_3 \\
&= x_1 + x_2 + i(y_1 + y_2) + j(\hat{x}_1 + \hat{x}_2) + k(\hat{y}_1 + \hat{y}_2) + \\
&\quad x_3 + iy_3 + j\hat{x}_3 + k\hat{y}_3 \\
&= x_1 + x_2 + x_3 + i((y_1 + y_2) + y_3) + j((\hat{x}_1 + \hat{x}_2) + \hat{x}_3) + \\
&\quad k((\hat{y}_1 + \hat{y}_2) + \hat{y}_3) \\
&= x_1 + x_2 + x_3 + i(y_1 + y_2 + y_3) + j(\hat{x}_1 + \hat{x}_2 + \hat{x}_3) + \\
&\quad k(\hat{y}_1 + \hat{y}_2 + \hat{y}_3)
\end{aligned}$$

and similarly

$$\begin{aligned}
q_1 + (q_2 + q_3) &= x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1 + \\
&\quad [x_2 + x_3 + i(y_2 + y_3) + j(\hat{x}_2 + \hat{x}_3) + k(\hat{y}_2 + \hat{y}_3)] \\
&= x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1 + \\
&\quad x_2 + x_3 + i(y_2 + y_3) + j(\hat{x}_2 + \hat{x}_3) + k(\hat{y}_2 + \hat{y}_3)
\end{aligned}$$

****

$$= x_1 + x_2 + x_3 + i(y_1 + (y_2 + y_3)) + j(\hat{x}_1 + (\hat{x}_2 + \hat{x}_3)) + k(\hat{y}_1 + (\hat{y}_2 + \hat{y}_3))$$

$$= x_1 + x_2 + x_3 + i(y_1 + y_2 + y_3) + j(\hat{x}_1 + \hat{x}_2 + \hat{x}_3) + k(\hat{y}_1 + \hat{y}_2 + \hat{y}_3)$$

$= (q_1 + q_2) + q_3$, as required. **Q.E.D.**

-8-

Having thus determined the fundamental properties of addition in relation to quaternion hypercomplex numbers, we will proceed further by evaluating the properties of their corresponding products. As a matter of fact, we may effectively ascertain these properties by deriving various products of the hitherto mentioned quaternions, q_1, q_2 and q_3.

Bearing in mind the provisions of Definition DI-1, we likewise perceive that the quaternion product,

$$q_1 q_2 = (x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1)(x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2)$$

$$= x_1(x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2) + iy_1(x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2) + j\hat{x}_1(x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2) + k\hat{y}_1(x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2)$$

$$= x_1x_2 + ix_1y_2 + jx_1\hat{x}_2 + kx_1\hat{y}_2 + ix_2y_1 + i^2y_1y_2 + ij\hat{x}_2y_1 + j\hat{x}_1x_2 + ji\hat{x}_1y_2 + j^2\hat{x}_1\hat{x}_2 + jk\hat{x}_1\hat{y}_2 + kx_2\hat{y}_1 + ki\hat{y}_1y_2 + kj\hat{x}_2\hat{y}_1 + k^2\hat{y}_1\hat{y}_2 + ik\,y_1\hat{y}_2$$

$$= x_1x_2 + ix_1y_2 + jx_1\hat{x}_2 + kx_1y_2 + ix_2y_1 - y_1y_2 + k\hat{x}_2y_1 - j y_1\hat{y}_2 + j\hat{x}_1x_2 - k\hat{x}_1y_2 - \hat{x}_1\hat{x}_2 + i\hat{x}_1\hat{y}_2 + kx_2\hat{y}_1 + j\hat{y}_1y_2 - i\hat{x}_2\hat{y}_1 - \hat{y}_1\hat{y}_2$$

$$= x_1x_2 - y_1y_2 - \hat{x}_1\hat{x}_2 - \hat{y}_1\hat{y}_2 + i(x_1y_2 + x_2y_1 + \hat{x}_1\hat{y}_2 - \hat{x}_2\hat{y}_1) + j(x_1\hat{x}_2 - y_1\hat{y}_2 + \hat{x}_1x_2 + \hat{y}_1y_2) + k(x_1\hat{y}_2 + \hat{x}_2y_1 - \hat{x}_1y_2 + x_2\hat{y}_1) \tag{2-5}$$

and similarly the quaternion product,

$$q_2q_1 = (x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2)(x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1)$$

$$= x_2(x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1) + iy_2(x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1) + j\hat{x}_2(x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1) + k\hat{y}_2(x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1)$$

$$= x_1x_2 + ix_2y_1 + j\hat{x}_1x_2 + kx_2\hat{y}_1 + ix_1y_2 + i^2y_1y_2 + ij\hat{x}_1y_2 + ik\hat{y}_1y_2 + jx_1\hat{x}_2 + ji\hat{x}_2y_1 + j^2\hat{x}_1\hat{x}_2 + jk\hat{x}_2\hat{y}_1 + kx_1\hat{y}_2 + ki\,y_1\hat{y}_2 + kj\hat{x}_1\hat{y}_2 + k^2\hat{y}_1\hat{y}_2$$

$$= x_1x_2 + ix_2y_1 + j\hat{x}_1x_2 + kx_2\hat{y}_1 + ix_1y_2 - y_1y_2 + k\hat{x}_1y_2 - j\hat{y}_1y_2 + jx_1\hat{x}_2 - k\hat{x}_2y_1 - \hat{x}_1\hat{x}_2 + i\hat{x}_2\hat{y}_1 + kx_1\hat{y}_2 + jy_1\hat{y}_2 - i\hat{x}_1\hat{y}_2 - \hat{y}_1\hat{y}_2$$

$$= x_1x_2 - y_1y_2 - \hat{x}_1\hat{x}_2 - \hat{y}_1\hat{y}_2 + i(x_2y_1 + x_1y_2 + \hat{x}_2\hat{y}_1 - \hat{x}_1\hat{y}_2) + j(\hat{x}_1x_2 - \hat{y}_1y_2 + x_1\hat{x}_2 + y_1\hat{y}_2) + k(x_2\hat{y}_1 + \hat{x}_1y_2 - \hat{x}_2y_1 + x_1\hat{y}_2)$$

(2–6).

Accordingly, we shall summarise the essence of Eqs. (2–5) and (2–6) by means of the following definition :–

Definition DII – 2

For any two quaternions,

$q_1 = x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1,$

$q_2 = x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2,$

the quaternion products pertaining thereto, q_1q_2 and q_2q_1, are respectively defined as –

(i) $q_1 q_2 = x_1x_2 - y_1y_2 - \hat{x}_1\hat{x}_2 - \hat{y}_1\hat{y}_2 + i(x_1y_2 + x_2y_1 + \hat{x}_1\hat{y}_2 - \hat{x}_2\hat{y}_1) + j(x_1\hat{x}_2 - y_1\hat{y}_2 + \hat{x}_1x_2 + \hat{y}_1y_2) + k(x_1\hat{y}_2 + \hat{x}_2y_1 - \hat{x}_1y_2 + x_2\hat{y}_1),$

(ii) $q_2 q_1 = x_1x_2 - y_1y_2 - \hat{x}_1\hat{x}_2 - \hat{y}_1\hat{y}_2 + i(x_2y_1 + x_1y_2 + \hat{x}_2\hat{y}_1 - \hat{x}_1\hat{y}_2) + j(\hat{x}_1x_2 - \hat{y}_1y_2 + x_1\hat{x}_2 + y_1\hat{y}_2) + k(x_2\hat{y}_1 + \hat{x}_1y_2 - \hat{x}_2y_1 + x_1\hat{y}_2).$

From the above definition, we observe that

$q_1q_2 \neq q_2q_1, \ \forall x_1, y_1, \ldots\ldots\ldots, \hat{x}_2, \hat{y}_2 \in \mathbb{R},$

in other words the multiplication of quaternions is generally not commutative – indeed, the preceding Eqs. (2–5) and (2–6) have played a vital role in formally justifying our original assertion to that effect.

As a direct consequence of Definition DII – 2, we may now state and likewise verify two more theorems which respectively confirm both the associativity of multiplication and also the existence of distributive laws with respect to quaternion hypercomplex numbers;–

Theorem TII – 2

The multiplication of any three quaternions, q_1, q_2 and q_3 is always associative, that is to say

$$q_1(q_2q_3) = (q_1q_2)q_3.$$

*　　　　*　　　　*

PROOF:–

As previously indicated, we once again set

$$q_1 = x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1,$$

$$q_2 = x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2,$$

$$q_3 = x_3 + iy_3 + j\hat{x}_3 + k\hat{y}_3.$$

Furthermore, from Definition DII – 2, we deduce that

(a) $q_1q_2 = (x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1)(x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2)$

$$= x_1x_2 - y_1y_2 - \hat{x}_1\hat{x}_2 - \hat{y}_1\hat{y}_2 + i(x_1y_2 + x_2y_1 + \hat{x}_1\hat{y}_2 - \hat{x}_2\hat{y}_1) + j(x_1\hat{x}_2 - y_1\hat{y}_2 + \hat{x}_1x_2 + \hat{y}_1y_2) + k(x_1\hat{y}_2 + \hat{x}_2y_1 - \hat{x}_1y_2 + x_2\hat{y}_1)$$

$$= X_1 + iY_1 + j\hat{X}_1 + k\hat{Y}_1,$$

thus implying that

$$X_1 = x_1x_2 - y_1y_2 - \hat{x}_1\hat{x}_2 - \hat{y}_1\hat{y}_2,$$

$$Y_1 = x_1y_2 + x_2y_1 + \hat{x}_1\hat{y}_2 - \hat{x}_2\hat{y}_1,$$

$$\hat{X}_1 = x_1\hat{x}_2 - y_1\hat{y}_2 + \hat{x}_1x_2 + \hat{y}_1y_2,$$

$$\hat{Y}_1 = x_1\hat{y}_2 + \hat{x}_2y_1 - \hat{x}_1y_2 + x_2\hat{y}_1,$$

AND

(b) $$q_2q_3 = (x_2 + iy_2 + j\hat{x}_2 + k\hat{y}_2)(x_3 + iy_3 + j\hat{x}_3 + k\hat{y}_3)$$

$$= x_2x_3 - y_2y_3 - \hat{x}_2\hat{x}_3 - \hat{y}_2\hat{y}_3 + i(x_2y_3 + x_3y_2 + \hat{x}_2\hat{y}_3 - \hat{x}_3\hat{y}_2) + j(x_2\hat{x}_3 - y_2\hat{y}_3 + \hat{x}_2x_3 + \hat{y}_2y_3) + k(x_2\hat{y}_3 + \hat{x}_3y_2 - \hat{x}_2y_3 + x_3\hat{y}_2)$$

$$= X_2 + iY_2 + j\hat{X}_2 + k\hat{Y}_2,$$

thereby implying that

$$X_2 = x_2x_3 - y_2y_3 - \hat{x}_2\hat{x}_3 - \hat{y}_2\hat{y}_3,$$

****

$$Y_2 = x_2 y_3 + x_3 y_2 + \hat{x}_2 \hat{y}_3 - \hat{x}_3 \hat{y}_2 ,$$

$$\hat{X}_2 = x_2 \hat{x}_3 - y_2 \hat{y}_3 + \hat{x}_2 x_3 + \hat{y}_2 y_3 ,$$

$$\hat{Y}_2 = x_2 \hat{y}_3 + \hat{x}_3 y_2 - \hat{x}_2 y_3 + x_3 \hat{y}_2 .$$

****
****
****
****
****

Consequently, in view of Definition DII-2, it also follows that the quaternion product,

$$
\begin{aligned}
q_1(q_2 q_3) &= q_1(X_2 + iY_2 + j\hat{X}_2 + k\hat{Y}_2) \\
&= (x_1 + iy_1 + j\hat{x}_1 + k\hat{y}_1)(X_2 + iY_2 + j\hat{X}_2 + k\hat{Y}_2) \\
&= x_1 X_2 - y_1 Y_2 - \hat{x}_1 \hat{X}_2 - \hat{y}_1 \hat{Y}_2 + i(x_1 Y_2 + X_2 y_1 + \hat{x}_1 \hat{Y}_2 - \hat{X}_2 \hat{y}_1) + \\
&\quad j(x_1 \hat{X}_2 - y_1 \hat{Y}_2 + \hat{x}_1 X_2 + \hat{y}_1 Y_2) + k(x_1 \hat{Y}_2 + \hat{X}_2 y_1 - \hat{x}_1 Y_2 + X_2 \hat{y}_1) \\
&= \begin{bmatrix} x_1(x_2 x_3 - y_2 y_3 - \hat{x}_2 \hat{x}_3 - \hat{y}_2 \hat{y}_3) - y_1(x_2 y_3 + x_3 y_2 + \hat{x}_2 \hat{y}_3 - \hat{x}_3 \hat{y}_2) - \\ \hat{x}_1(x_2 \hat{x}_3 - y_2 \hat{y}_3 + \hat{x}_2 x_3 + \hat{y}_2 y_3) - \hat{y}_1(x_2 \hat{y}_3 + \hat{x}_3 y_2 - \hat{x}_2 y_3 + x_3 \hat{y}_2) \end{bmatrix} \\
&\quad + i \begin{bmatrix} x_1(x_2 y_3 + x_3 y_2 + \hat{x}_2 \hat{y}_3 - \hat{x}_3 \hat{y}_2) + \\ y_1(x_2 x_3 - y_2 y_3 - \hat{x}_2 \hat{x}_3 - \hat{y}_2 \hat{y}_3) + \\ \hat{x}_1(x_2 \hat{y}_3 + \hat{x}_3 y_2 - \hat{x}_2 y_3 + x_3 \hat{y}_2) - \\ \hat{y}_1(x_2 \hat{x}_3 - y_2 \hat{y}_3 + \hat{x}_2 x_3 + \hat{y}_2 y_3) \end{bmatrix}
\end{aligned}
$$

****

$$+j\begin{bmatrix} x_1(x_2\hat{x}_3 - y_2\hat{y}_3 + \hat{x}_2x_3 + \hat{y}_2y_3) - \\ y_1(x_2\hat{y}_3 + \hat{x}_3y_2 - \hat{x}_2y_3 + x_3\hat{y}_2) + \\ \hat{x}_1(x_2x_3 - y_2y_3 - \hat{x}_2\hat{x}_3 - \hat{y}_2\hat{y}_3) + \\ \hat{y}_1(x_2y_3 + x_3y_2 + \hat{x}_2\hat{y}_3 - \hat{x}_3\hat{y}_2) \end{bmatrix}$$

$$+k\begin{bmatrix} x_1(x_2\hat{y}_3 + \hat{x}_3y_2 - \hat{x}_2y_3 + x_3\hat{y}_2) + \\ y_1(x_2\hat{x}_3 - y_2\hat{y}_3 + \hat{x}_2x_3 + \hat{y}_2y_3) - \\ \hat{x}_1(x_2y_3 + x_3y_2 + \hat{x}_2\hat{y}_3 - \hat{x}_3\hat{y}_2) + \\ \hat{y}_1(x_2x_3 - y_2y_3 - \hat{x}_2\hat{x}_3 - \hat{y}_2\hat{y}_3) \end{bmatrix}$$

$$= \begin{bmatrix} x_1x_2x_3 - x_1y_2y_3 - x_1\hat{x}_2\hat{x}_3 - x_1\hat{y}_2\hat{y}_3 - y_1y_3x_2 - y_1x_3y_2 - \\ y_1\hat{x}_2\hat{y}_3 + y_1\hat{x}_3y_2 - \hat{x}_1x_2\hat{x}_3 + \hat{x}_1y_2\hat{y}_3 - \hat{x}_1\hat{x}_2x_3 - \hat{x}_1\hat{y}_2y_3 - \\ \hat{y}_1x_2\hat{y}_3 - \hat{y}_1\hat{x}_3y_2 + \hat{y}_1\hat{x}_2y_3 - y_1x_3\hat{y}_2 \end{bmatrix}$$

—13—

$$+i\begin{bmatrix} x_1x_2y_3 + x_1x_3y_2 + x_1\hat{x}_2\hat{y}_3 - x_1x_3\hat{y}_2 + \\ y_1x_2x_3 - y_1y_2y_3 - y_1\hat{x}_2\hat{x}_3 - y_1\hat{y}_2\hat{y}_3 + \\ \hat{x}_1\hat{y}_3x_2 + \hat{x}_1\hat{x}_3y_2 - \hat{x}_1\hat{x}_2y_3 + \hat{x}_1x_3\hat{y}_2 - \\ \hat{y}_1x_2\hat{x}_3 + \hat{y}_1y_2\hat{y}_3 - \hat{y}_1\hat{x}_2x_3 - \hat{y}_1\hat{y}_2y_3 \end{bmatrix}$$

$$+j\begin{bmatrix} x_1x_2\hat{x}_3 - x_1y_2\hat{y}_3 + x_1\hat{x}_2x_3 + x_1\hat{y}_2y_3 - \\ y_1x_2\hat{y}_3 - y_1\hat{x}_3y_2 + y_1\hat{x}_2y_3 - y_1x_3\hat{y}_2 + \\ \hat{x}_1x_2x_3 - \hat{x}_1y_2y_3 - \hat{x}_1\hat{x}_2\hat{x}_3 - x_1\hat{y}_2\hat{y}_3 + \\ \hat{y}_1x_2y_3 + \hat{y}_1x_3y_2 + \hat{y}_1\hat{x}_2\hat{y}_3 - \hat{y}_1\hat{x}_3\hat{y}_2 \end{bmatrix}$$

$$
+k\begin{bmatrix} x_1x_2\hat{y}_3 + x_1\hat{x}_3y_2 - x_1\hat{x}_2y_3 + x_1x_3\hat{y}_2 + \\ y_1x_2\hat{x}_3 - y_1y_2\hat{y}_3 + y_1\hat{x}_2x_3 + y_1\hat{y}_2y_3 - \\ \hat{x}_1x_2y_3 - \hat{x}_1x_3y_2 - \hat{x}_1\hat{x}_2\hat{y}_3 + \hat{x}_1\hat{x}_3\hat{y}_2 + \\ \hat{y}_1x_2x_3 - \hat{y}_1y_2y_3 - \hat{y}_1\hat{x}_2\hat{x}_3 - \hat{y}_1\hat{y}_2\hat{y}_3 \end{bmatrix}
$$

and similarly the quaternion product,

$$
\begin{aligned}
(q_1q_2)q_3 &= (X_1 + iY_1 + j\hat{X}_1 + k\hat{Y}_1)\, q_3 \\
&= (X_1 + iY_1 + j\hat{X}_1 + k\hat{Y}_1)(x_3 + iy_3 + j\hat{x}_3 + k\hat{y}_3) \\
&= X_1x_1 - Y_1y_3 - \hat{X}_1\hat{x}_3 - \hat{Y}_1\hat{y}_3 + i(X_1y_3 + x_3Y_1 + \hat{X}_1\hat{y}_3 - \hat{x}_3\hat{Y}_1) + \\
&\quad j(X_1\hat{x}_3 - Y_1\hat{y}_3 + \hat{X}_1x_3 + \hat{Y}_1y_3) + k(X_1\hat{y}_3 + \hat{x}_3Y_1 - \hat{X}_1y_3 + x_3\hat{Y}_1) \\
&= \begin{bmatrix} x_3(x_1x_2 - y_1y_2 - \hat{x}_1\hat{x}_2 - \hat{y}_1\hat{y}_2) - y_3(x_1y_2 + x_2y_1 + \hat{x}_1\hat{y}_2 - \hat{x}_2\hat{y}_1) - \\ \hat{x}_3(x_1\hat{x}_2 - y_1\hat{y}_2 + \hat{x}_1x_2 + \hat{y}_1y_2) - \hat{y}_3(x_1\hat{y}_2 + \hat{x}_2y_1 - \hat{x}_1y_2 + x_2\hat{y}_1) \end{bmatrix} \\
&\quad + i\begin{bmatrix} y_3(x_1x_2 - y_1y_2 - \hat{x}_1\hat{x}_2 - \hat{y}_1\hat{y}_2) + \\ x_3(x_1y_2 + x_2y_1 + \hat{x}_1\hat{y}_2 - \hat{x}_2\hat{y}_1) + \\ \hat{y}_3(x_1\hat{x}_2 - y_1\hat{y}_2 + \hat{x}_1x_2 + \hat{y}_1y_2) - \\ \hat{x}_3(x_1\hat{y}_2 + \hat{x}_2y_1 - \hat{x}_1y_2 + x_2\hat{y}_1) \end{bmatrix}
\end{aligned}
$$

To be continued via the author's next submission, namely -

"An Introduction to Functions of a Quaternion Hypercomplex Variable - PART 2/6."

* * * * * * * *

* * * * * * * *

[THIS PAGE HAS BEEN INTENTIONALLY LEFT BLANK.]