A New $3n-1$ Conjecture Akin to Collatz Conjecture

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Abstract

The Collatz conjecture is an open conjecture in mathematics named so after Lothar Collatz who proposed it in 1937. It is also known as $3n+1$ conjecture, the Ulam conjecture (after Stanislaw Ulam), Kakutani's problem (after Shizuo Kakutani) and so on.

In this paper a new conjecture called as the $3n-1$ conjecture which is akin to the Collatz conjecture is proposed. It functions on $3n-1$, for any starting number n, its sequence eventually reaches either 1, 5 or 17. The $3n-1$ conjecture is compared with the Collatz conjecture.

Keywords: Collatz Conjecture, $3n+1$ Conjecture, Kakutani's Conjecture, $3n-1$ Conjecture

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1. Introduction

The Collatz conjecture is long standing open conjecture in number theory. Paul Erdos had commented about the Collatz conjecture that *"Mathematics may not be ready for such problems"*. The Collatz conjecture has been extensively studied by several researchers [1, 2, 3, 4, 5]. A novel theoretical framework was formulated for information discovery using the Collatz conjecture data by

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Idowu [6]. Generalizing the odd part of the Collatz conjecture was studied by [7].

This paper proposes a new conjecture akin to the Collatz conjecture and compares it with the Collatz conjecture. This new conjecture is called as the $3n-1$ conjecture. This paper is organised into five sections. First section is introductory in nature. Section two recalls the Collatz conjecture so that the paper is self contained. Section three proposes the new $3n-1$ conjecture and illustrates it by examples. Comparison of the $3n-1$ conjecture with the Collatz conjecture is carried out in section four. The conclusions and future study are given in the last section.

2. Collatz's Conjecture

The $3n+1$ conjecture or the Collatz conjecture is summarized as follows:

Take any positive integer n. If n is even divide it by 2 to get $n/2$. If n is odd multiply it by 3 and add 1 to obtain $3n+1$. Repeat the process (which has been called "Half Or Triple Plus One" or HOTPO) indefinitely. The conjecture states that no matter what number you start with you will always eventually reach 1.

Consider the following operation on an arbitrary positive integer: If the number is even divide it by two, if the number is odd, triple it and add one. This is illustrated by example of taking numbers from 4 to 10 and the related sequence is obtained:

- n = 4; related sequence is 4, 2, 1.
- n = 5, related sequence is 5, 16, 8, 4, 2, 1.
- n = 6, related sequence is 6, 3, 10, 5, 16, 8, 4, 2, 1.
- n = 7, related sequence is 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.
- n = 8, related sequence is 8, 4, 2, 1.

- n = 9, related sequence is 9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.
- n = 10, related sequence is 10, 5, 16, 8, 4, 2, 1.

In simple modular arithmetic notation the Collatz conjecture can be represented as

$$f(n) = \begin{cases} n/2 & \text{if } n \equiv 0 (\text{mod } 2)\,, \\ 3n+1 & \text{if } n \equiv 1 (\text{mod } 2)\,. \end{cases} \tag{1}$$

Note: Only powers of two converge to one quickly.

The total stopping time i is number of times the function given in equation 1 is called on to reach 1. The longest sequence is created for numbers under 100000 has a total stopping time 350, which is done with the starting number 77031. Different maximum stopping times of the Collatz conjecture are given in Table 2.

3. The $3n-1$ Conjecture

The $3n-1$ conjecture which is akin to the Collatz conjecture is proposed in this section. The $3n-1$ conjecture is as follows:

Take any arbitrary positive integer n. If n is even divide it by two and get $n/2$ if n is odd multiply it by 3 and subtract 1 and obtain $3n-1$, repeat this process indefinitely. We call this process as "Half Or Triple Minus One" or HOTMO. The conjecture states that immaterial of which number you begin with, you will eventually reach 1 or 5 or 17.

3.1. Statement of the Problem/Conjecture

On any arbitrary positive integer, consider the operation

- If the number is even, divide it by two
- Else triple it and subtract one

continue this process recursively. The $3n-1$ conjecture is that this process which will eventually reach either 1 or 5 or 17, regardless of which positive integer is selected at the beginning.

The smallest i such that $a_i = 1$ or 5 or 17 is called as the total stopping time of n. The $3n-1$ conjecture asserts that every n has a well defined total stopping time i. If for some n (any positive integer) such i (total stopping time) doesn't exist, then n has an infinite total stopping time then the conjecture is false. It can happen only because there is some starting number which gives a sequence that does not contain 1, 5 or 17. Such a sequence may have a repeating cycle that does not contain 1, 5 or 17 or it might increase without bounds. Till now such a sequence or number has not been found. Experiments have been carried out till 1 billion.

In simple modular arithmetic notation the $3n-1$ conjecture can be represented as

$$f(n) = \begin{cases} n/2 & \text{if } n \equiv 0 (\text{mod } 2)\,, \\ 3n-1 & \text{if } n \equiv 1 (\text{mod } 2)\,. \end{cases} \tag{2}$$

A sequence is formed by performing this operation repeatedly, it starts with any arbitrary positive integer and takes the result each step as the input for the next.

$$a_i = \begin{cases} n & \text{if i} = 0\,, \\ f(a_{i-1}) & \text{if i} \neq 0\,. \end{cases} \tag{3}$$

$a_i = f^i(n)$ that is a_i is the value of f applied to n recursively i times; n is the starting number and i at the end of the sequence is called the total stopping time.

3.2. Examples

The conjecture states that the sequence will reach 1, 5 or 17. The following repeated sequences / cycles happen for 1, 5 or 17.

1. $n = 1$; the repeated sequence is 4, 2, 1.
2. $n = 5$; the repeated sequence is 14, 7, 20, 10, 5.

3. $n = 17$; the repeated sequence is 50, 25, 74, 37, 110, 55, 164, 82, 41, 122, 61, 182, 91, 272, 136, 68, 34, 17.

We will illustrate this conjecture by some examples using the $3n-1$ formula and taking numbers from 4 to 10. It is tabulated in Table 1

Table 1: Illustration of the $3n-1$ conjecture

n	Sequence	i	Ends in
4	4, 2, 1	3	1
5	5, 14, 7, 20, 10, 5	1	5
6	6, 3, 8, 4, 2, 1	6	1
7	7, 20, 10, 5	4	5
8	8, 4, 2, 1	4	1
9	9, 26, 13, 38, 19, 56, 28, 14, 7, 20, 10, 5	12	5
10	10, 5	2	5

A sequence for n = 87 using the $3n-1$ formula given in equation 2 is as follows: 260, 130, 65, 194, 97, 290, 145, 434, 217, 650, 325, 974, 487, 1460, 730, 365, 1094, 547, 1640, 820, 410, 205, 614, 307, 920, 460, 230, 115, 344, 172, 86, 43, 128, 64, 32, 16, 8, 4, 2, 1. It is represented as a graph in Figure 3.2.

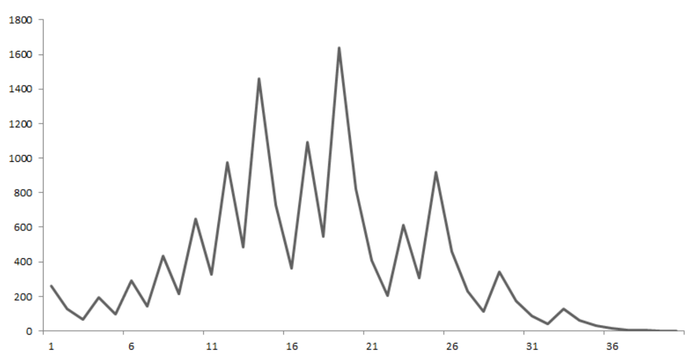

Figure 1: The graph of the sequence for 87

Similar to $3n+1$ conjecture in $3n-1$ conjecture also the powers of 2, converge quickly. Figure 3.2 gives the scatter plot that takes the starting number n from 1 to 1000 along the x-axis and the total stopping number i along the y-axis. Depending on which number the sequence ends, the colour is given. If the sequence ends in 1, then blue colour is given, if it ends in 5 then red colour is given and if it ends in 17 green colour is given.

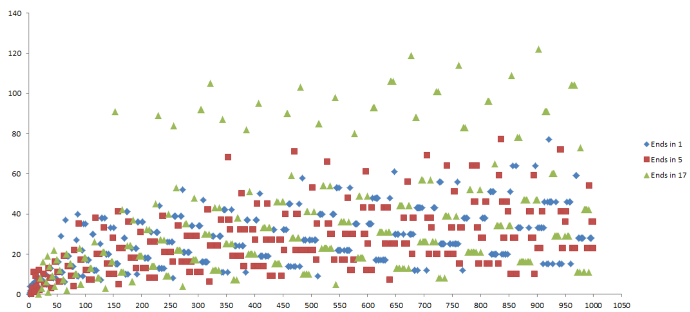

Figure 2: The scatter plot of first 1000 numbers and their stopping times

The histogram of the first 1000000 numbers and their stopping time is given in Figure 3.

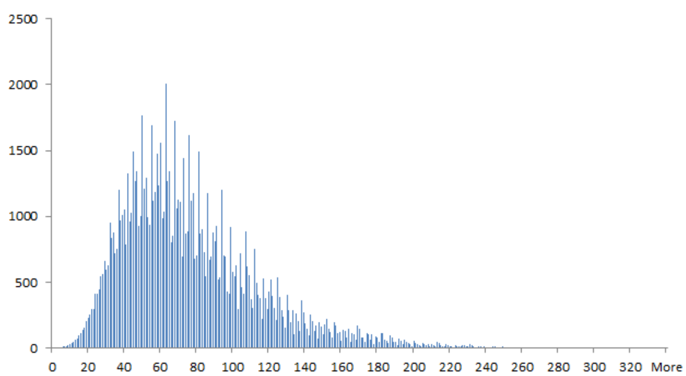

Figure 3: The histogram of first 1 lakh numbers and their stopping times

4. Comparison of $3n + 1$ conjecture with $3n - 1$ conjecture

The $3n-1$ conjecture generally has a smaller stopping time when compared to $3n + 1$ conjecture. The $3n - 1$ conjecture creates a sequence that ends in 3 different numbers with the sequence having a repeated sequence of

1. $n = 1$; the repeated sequence is 4, 2, 1.
2. $n = 5$; the repeated sequence is 14, 7, 20, 10, 5.
3. $n = 17$; the repeated sequence is 50, 25, 74, 37, 110, 55, 164, 82, 41, 122, 61, 182, 91, 272, 136, 68, 34, 17.

The total stopping times of the $3n - 1$ conjecture and $3n + 1$ conjecture are compared in Table 2.

Table 2: Comparison of the $3n + 1$ and the $3n - 1$ conjecture

Numbers	$3n + 1$	Conjecture	$3n - 1$	Conjecture
less than	i	n	i	n
10	19	9	12	9
100	118	97	41	87
1000	178	871	123	903
10000	261	6171	235	9735
100000	350	77031	338	85463
1 million	524	837799	562	858341
10 million	685	8400511	624	5012343
100 million	949	63728127	656	59338533
1 billion	986	670617279	688	702478602

From the comparison table it is clearly seen that the $3n - 1$ conjecture has a smaller total stopping time when compared to the $3n + 1$ conjecture, except for the case of ‘less than 1 million”’ where the $3n - 1$ conjecture has a total stopping time greater than $3n + 1$ conjecture. In case of ‘less than 1 million’ the

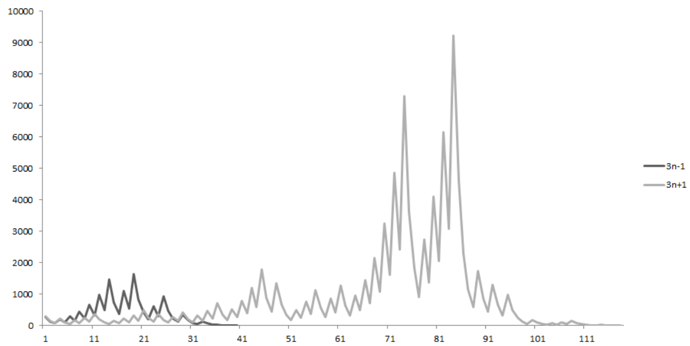

Figure 4: The comparison graph for $n = 87$ for $3n - 1$ and $n = 97$ for $3n + 1$ conjectures respectively

$3n - 1$ conjecture reaches the maximum stopping time of 562 for 858341, when compared to the total stopping time of 524 for 837799 for the $3n + 1$ conjecture.

Figure 4 shows the comparison of graph for the sequence created by taking the starting number n as 87 using the $3n-1$ conjecture and the sequence created by taking the starting number n as 97 using the $3n + 1$ conjecture.

5. Results and further study

The proposed $3n - 1$ conjecture is a new conjecture akin to the $3n + 1$ conjecture or the Collatz conjecture. Given any starting number n, the conjecture states that the sequence will result in either 1, 5 or 17. It has a smaller total stopping time when compared to the Collatz conjecture. Cycles related to the $3n - 1$, resulting hailstone numbers and parity sequence are left open for study.

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