Study on the development of neutrosophic triplet ring and neutrosophic triplet field

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Definition 1. Let $(\mathrm{NTR},*,\#)$ be a set together with two binary operations $*$ and $\#$. Then NTR is called a neutrosophic triplet ring if the following conditions are holds.

1) $(\mathrm{NTR},*)$ is a commutative neutrosophic triplet group with respect to $*$.
2) $(\mathrm{NTR},\#)$ is a semineutrosophic triplet monoid with respect to $\#$.
3) $a\#(b*c)=(a\#b)*(a\#c)$ and $(b*c)\#\mathrm{a}=(b\#a)*(c\#a)$ for all $a,b,c\in NTR$.

Example 1. Consider Z_{10}. Let $NTR=\{0,2,4,6,8\}\subseteq Z_{10}$. Define two operations $*$ and $\#$ on NTR by the following way respectively:

1. $a*b=a\times b(\mathrm{mod}10)$ for all $a,\mathrm{b}\in NTR$.
2. $a\#b=\max(a,b)$ for all $a,\mathrm{b}\in NTR$.

Then clearly $(\mathrm{NT}R,*)$ is a commutative neutrosophic triplet group with respect to multiplication modulo 10, as NTR is well defined and associative. Also $(0,0,0),(2,6,8),(4,6,4),(6,6,6)$ and $(8,6,2)$ are neutrosophic triplets in NTR with respect to multiplication modulo 10. Clearly $a*b=b*a$ for all $a,b\in NTR$.

Now $(R,\#)$ is a semineutrosophic triplet monoid with respect to $\#$. Since $\max(0,0)=0,\max(0,2)=2,\max(2,4)=4,\max(4,6)=6$ and $\max(6,8)=8$, that is for every $a\in NTR$, there exist at least one $neut(a)$ in NTR.

Finally $\#$ is distributive over $*$. For instance

$$2\#(4*6)=(2\#4)*(2\#6)$$

$$2\#(4\times 6)=(2\#4)\times(2\#6)$$

$$2\#4=\max(2,4)\times\max(2,6)$$

$$\max(2,4)=4\times 6$$

$$4=4\text{, and so on.}$$

Thus for all $a, \mathrm{b}, \mathrm{c} \in NTR$, # is distributive over $*$. Therefore $(NTR,*,\#)$ is a neutrosophic triplet ring.

Definition. Let $(\mathrm{NT}R,*,\#)$ be a neutrosophic triplet ring and let $0 \neq a \in NTR$. If there exist a non-zero neutrosophic triplet $b \in NTR$ such that $b\#a = 0$. Then b is called left neutrosophic triplet zero divisors of a. Similarly a neutrosophic triplet $b \in NTR$ **is** called a right neutrosophic triplet zero divisor if $a\#b = 0$.

A neutrosophic triplet zero divisor is one which is left neutrosophic triplet zero divisor as well as right neutrosophic triplet zero divisor.

Theorem. Let NTR be a neutrosophic triplet ring and $a, b \in NTR$. If $b\#a = a\#b = 0$, then

1. $neut(a)\#neut(b) = \mathrm{neut(b)\#neut(a)} = 0$ and
2. $anti(a)\#anti(b) = \mathrm{anti(b)\#anti(a)} = 0$.

Proof 1. Let NTR be a neutrosophic triplet ring and $a, b \in NTR$ such that b is a neutrosophic triplet zero divisor of a. Then $a\#b = b\#a = 0$. Now consider

$$neut(a)\#neut(b) = neut(a\#b)$$

Or

$$= neut(0), \text{ since } a\#b = 0.$$

Or

$$= 0, \text{ as } neut(0) = 0.$$

Also

$$neut(\mathrm{b})\#\mathrm{neut(a)} = \mathrm{neut(b\#a)}$$

$$= \mathrm{neut}(0), \text{ as } \mathrm{b\#a} = 0$$

$$neut(0) = 0.$$

Hence $neut(a)\#neut(b) = \mathrm{neut(b)\#neut(a)} = 0$.

2. The proof is similar to 1.

Definition. Let $(NTR,*,\#)$ be a neutrosophic triplet ring and let S be a subset of NTR. Then S is called a neutrosophic triplet subring of NTR if $(S,*,\#)$ is a neutrosphic triplet ring.

Definition. Let $(NTR,*,\#)$ be a neutrosophic triplet ring and I is a subset of NTR. Then I is called a neutrosophic triplet ideal of NTR if the following conditions are satisfied.

1. $(I,*)$ is a neutrosophic triplet subgroup of $(NTR,\#)$, and
2. For all $x \in I$ and $r \in NTR$, $x\#r \in I$ and $r\#x \in I$.

Theorem. Every neutrosophic triplet ideal is trivially a neutrosophic triplet subring but the converse is not true in general.

Remark. Let $(NTR,*,\#)$ is a neutrosophic triplet ring and let $a \in NTR$. Then the following are true.

1. $neut(a)$ and $anti(a)$ is not unique in NTR with respect to $*$.
2. $neut(a)$ and $anti(a)$ is not unique in NTR with respect to $\#$.

Theorem. Let $(NTR,*,\#)$ is a neutrosophic triplet ring . Then for any element a in a neutrosophic triplet ring NTR, one has $0\#a = a\#0 = 0$.

Definition. Let NTR is a neutrosophic triplet ring and let $a \in NTR$. Then a is called nilpotent neutrosophic triplet if $a^n = 0$, for some positive integer $n > 1$.

Theorem. Let NTR is a neutrosophic triplet ring and let $a \in NTR$. If a is a nilpotent neutrosophic triplet. Then the following are true.

1. $(neut(a))^n = 0$ and
2. $(anti(a))^n = 0$.

Proof: 1:

Suppose that a is a nilpotent neutrosophic triplet in a neutrosophic triplet ring NTR. Then by definition $a^n = 0$ for some positive integer $n > 1$. Now we consider Left hand side of 1:

Since $$(neut(a))^n = (neut(a))\#(neut(a))^{n-1}$$

$$= neut(a\#a^{n-1})$$

$$= neut(a^n)$$

$$= neut(0) \text{, by definition.}$$

$$= 0.$$

This completes the proof.

2: The proof of 2 is similar to 1.

Integral Neutrosophic triplet domain

Definition: Let $(NTR,*,\#)$ be a neutrosophic triplet ring. Then NTR is called a commutative neutrosophic triplet ring if $a\#b=b\#a$ for all $a,b\in NTR$.

Definition: A commutative neutrosophic triplet ring NTR is called integral neutrosophic triplet domain if for all $a,b\in NTR$, $a\#b=0$ implies $a=0$ or $b=0$.

Theorem: Let NTR be an integral neutrosophic triplet domain. Then the following are true.

1. $neut(a)\#neut(b)=0$ implies $neut(a)=0$ or $neut(b)=0$ and
2. $\text{anti}(a)\#\text{anti}(b)=0$ implies $\text{anti}(a)=0$ or $\text{anti}(b)=0$ for all $a,b\in NTR$.

Proof: 1.

Suppose that NTR is an integral neutrosophic triplet domain. Then for all $a,b\in NTR$, $a\#b=0$ implies $a=0$ or $b=0$. Consider $neut(a)\#neut(b)$. Then

$$neut(a)\#neut(b)=neut(a\#b)$$

$$=neut(a\#b)$$

$$=neut(0), \text{ as } a\#b=0.$$

$$neut(a)\#neut(b)=0$$

Which implies that either $neut(a)=0$ or $neut(b)=0$.

2: The proof is similar to 1.

Proposition: A commutative neutrosophic triplet ring NTF is an integral neutrosophic triplet domain if and only if whenever $a,b,c\in NTR$ such that $a\#b=a\#\text{c}$ and $a\neq 0$, then $b=c$.

Proof: Suppose that NTR is an integral neutrosophic triplet domain and let $a,b,c\in NTR$. Since $a\neq 0$ and $a\in NTR$. a is not zero divisor then a is cancellable i.e.,

$$a\#b=a\#\text{c}\Rightarrow a\#b-a\#\text{c}=0\Rightarrow a\#(b-c)=0$$

Since $a\neq 0, b-c=0\Rightarrow b=c$.

$\Leftarrow$ Let $a\in NTR$, such that $a\neq 0$, then by hypothesis a is cancellable. a is not a zero divisor. NTR is an integral neutrosophic triplet domain.

Neutrosophic Triplet Ring Homomorphism.

Definition: Let $(NTR_1,*,\#)$ and $(NTR_2,\oplus,\otimes)$ be two neutrosophic triplet rings. Let $f: NTR_1 \to NTR_2$ be a mapping. Then f is called neutrosophic triplet ring homomorphism if the following conditions are true.

1. $f(a*b) = f(a)\oplus f(b)$.
2. $f(a\#b) = f(a)\otimes f(b)$, for all $a, b \in NTR_1$.
3. $f(neut(a)) = neut(f(a))$.
4. $f(anti(a)) = anti(f(a))$.

Neutrosophic Triplet Field

Definition. Let $(NTR,*,\#)$ be a neutrosophic triplet set together with two binary operations $*$ and #. Then $(NTR,*,\#)$ is called neutrosophic triplet field if the following conditions are holds.

1. $(NTR,*)$ is a commutative neutrosophic triplet group with respect to $*$.
2. $(NTR,\#)$ is a neutrosophic triplet group with respect to $\#$.
3. $a\#(b*c) = (a\#b)*(a\#c)$ and $(b*c)\#a = (b\#a)*(c\#a)$ for all $a,b,c \in \mathrm{NTF}$.

Example. Let X be a set and $P(X)$ be the power set of X. Then $(P(X),\cup,\cap)$ is a neutrosophic triplet field if $neut(\mathrm{A}) = \mathrm{A}$ and $anti(A) = A$ for all $A \in P(X)$.

Proposition. A neutrosophic triplet field NTF has no neutrosophic triplet zero divisors.

Proof. Suppose that a neutrosophic triplet field NTF has neutrosophic triplet zero divisor say $0 \neq a,b$. Then by definition of neutrosophic triplet zero divisor, $a\#b = 0$. This implies either $a = 0$ or $b = 0$ which clearly contradicts our supposition. Hence this shows that a neutrosophic triplet field NTF has no zero divisors.

Proposition. A neutrosophic triplet field NTF has always $anti(a)'s$ for all $a \in NTF$.

Proof. The proof is straightforward.

Theorem. If NTF is a field and $a \in NTF$. Then $a\#0 = 0$.

Proof. Since $(a\#0)*(a\#0) = a\#(0*0)$, by commutative law.

Also $0*0 = 0$. Thus $(a\#0)*(a\#0) = a\#0$ which implies $a\#0 = 0$.

Theorem. Every finite integral neutrosophic triplet domain NTD is a neutrosophic triplet field NTF.

Proof. Let NTD be a finite integral neutrosophic triplet domain. NTD is commutative ring with unity. To show that D is a neutrosophic triplet field NTF, it is enough to show that every non-zero element of NTD is a unit. Let the elements of NTD be labelled as

$r_0 (= 0_{NTD})$ and $r_1 (= 1_{NTD}), r_2, , , , , r_n$.

Let $r_i \in NTD$ such that $r_i \neq 0_{NTD} = r_0$. Thus consider the elements

$r_i \# r_0, r_i \# r_1, , , , , , , , r_i \# r_n \in NTD$ and are distinct $(\because if\ r_i \# r_s = r_i \# r_k \Rightarrow r_s = r_k)$.

Now since $1_{NTD} \in NTD$, therefore there must be some j such that $r_i \# r_j = r_j \# r_i \Rightarrow r_j$ is inverse of r_i i.e. r_i is invertible or r_i is a unit. Thus NTD is a neutrosophic triplet field NTF.

Theorem. Every neurosophic triplet field NTF is an integral neutrosophic triplet domain NTD.

Proof. Let NTF be a neutrosophic triplet field. Then NTF is a commutative neutrosophic triplet ring with unity. To show that NTF is an integral neutrosophic triplet domain NTD, it is enough to show that every non-zero element is not a zero-divisor.

Now suppose that $a, b \in NTF$ such that $a \neq 0\ and\ a \# b = 0$. Consider $a \# b = 0$, since $a \neq 0 \in NTF$. $a \# b = a \# 0\ (\because\ a \# 0 = 0) \Rightarrow a \# b - a \# 0 = 0 \Rightarrow a \# (b - 0) = 0 \Rightarrow a \neq 0, b - 0 = 0 \Rightarrow b = 0$. a is not a zero-divisor.

Theorem. If $f: NTR_1 \to NTR_2$ is a neutrosophic triplet ring homomorphism then

(1) If S is a neutrosophic triplet subring of NTR_1, then $f(S)$ is a neutrosophic triplet subring of NTR_2.

(2) If U is a neutrosophic triplet ring of NTR_2, then $f^{-1}(U)$ is a neutrosophic triplet subring of NTR_1.

(3) If I is a neutrosophic triplet ideal of NTR_2, then $f^{-1}(I)$ is a neutrosophic triplet ideal of NTR_1.

(4) If f is onto, then $f(I)$ is a neutrosophic triplet ideal of NTR_2.
$(I\ is\ neutrosophic\ triplet\ ideal\ of\ NTR_1)$.

Proof. Given that $f: NTR_1 \to NTR_2$ is a neutrosophic triplet ring homomorphism.

(1) If S is a neutrosophic triplet subring of NTR_1, we need to show that $f(S)$ is a neutrosophic triplet subring of NTR_2. To do this, $f(S) \neq \emptyset$, $(\because S\ is\ a\ neutrosophic\ triplet\ subring)$ and

$f(0_{NTR1}) = 0_S$. Also let $a, b \in f(S) \Rightarrow \exists\ \acute{a}, \acute{b} \in S$ such that $f(\acute{a}) = a\ and\ f(\acute{b}) = b$. Since S is neutrosophic triplet subring so for $\acute{a}, \acute{b} \in S \Rightarrow \acute{a} - \acute{b} \in S\ and\ \acute{a}\#\acute{b} \in S$.

Consider $f(\acute{a} - \acute{b}) = f\left(\acute{a} * (-\acute{b})\right) = f(\acute{a}) \oplus f(-\acute{b}) = f(\acute{a}) - f(\acute{b}), \left(\because f(-\acute{b}) = -f(\acute{b})\right),$

i.e., $f(\acute{a} - \acute{b}) = f(\acute{a}) - f(\acute{b}) = a - b \in f(S)$.

Also $f(\acute{a}\#\acute{b}) = f(\acute{a}) \otimes f(\acute{b}) = a \otimes b \in f(S)$.

$f(S)$ is a neutrosophic triplet subring of NTR_2.

(2) If U is a neutrosophic triplet subring of NTR_2, then $f^{-1}(U)$ is a neutrosophic triplet subring of NTR_1, so $f^{-1}(U) = \{r \in U \mid r \in NTR_1\ \&\ f(r) \in U\}$. Clearly $f^{-1}(U) \neq \emptyset, \because U$ is a neutrosophic triplet subring. Let $a, b \in f^{-1}(U) \Rightarrow f(a), f(b) \in U$. Since U is a neutrosophic triplet subring of $NTR_2 \Rightarrow f(a) - f(b) \in U\ and\ f(a)\#f(b) \in U$.

Now $f(a) - f(b) = f(a - b) \in U \Rightarrow a - b \in f^{-1}(U)$.

And $f(a)\#f(b) = f(a \otimes b) \in U \Rightarrow a \otimes b \in f^{-1}(U)$.

$f^{-1}(U)$ is a neutrosophic triplet subring of NTR_1.

(3) If I is an ideal of NTR_2, we need to show that $f^{-1}(I)$ is an ideal of NTR_1. Since $f^{-1}(I) \neq \emptyset$, $\because I$ is an ideal of NTR_2. Also let $a, b \in f^{-1}(I) \Rightarrow f(a), f(b) \in I$. Since I is an ideal of NTR_2. $f(a) - f(b) \in I\ and\ f(a)\#f(b) \in I$.

Now consider,

$$f(a) - f(b) = f(a - b) \in I \Rightarrow a - b \in f^{-1}(I)$$

and

$$f(a)\#f(b) = f(a \otimes b) \in I \Rightarrow a \otimes b \in f^{-1}(I).$$

Let $f(r) \in NTR_2$ and $a \in f^{-1}(I) \Rightarrow f(a) \in I$. $f(r) \in NTR_2$ and since I is an ideal of NTR_2.

$$f(a)\#f(r) \in I\ and\ f(r)\#f(a) \in I,$$

$$f(a \otimes r) \in I\ and\ f(r \otimes a) \in I,$$

$$(a \otimes r) \in f^{-1}(I)\ and\ (r \otimes a) \in f^{-1}(I).$$

Hence $f^{-1}(I)$ is an ideal of NTR_1.

(4) If f is onto, then $f(I)$ is an ideal of NTR_2, where I is an ideal of NTR_1. Since $f(I) \neq \emptyset$. Let $a, b \in f(I) \Rightarrow \exists\ \acute{a}, \acute{b} \in I\ such\ that\ f(\acute{a}) = a\ and\ f(\acute{b}) = b$. Now since I is an ideal of NTR_1, so for $\acute{a}, \acute{b} \in I \Rightarrow \acute{a} - \acute{b} \in I$. Consider,

$$a - b = f(\acute{a}) - f(\acute{b}) = f(\acute{a} - \acute{b}) \in f(I),$$

$$a - b \in f(I).$$

And let $a \in f(I)$ and let $t \in NTR_2 \Rightarrow \exists\ \acute{a} \in I\ such\ that\ f(\acute{a}) = a$, also f is onto $\Rightarrow$ for $t \in NTR_2\ \exists\ r \in NTR_1\ such\ that\ f(r) = t$.

Since I is an ideal of NTR_2, so $\acute{a}\#r\ and\ r\#\acute{a} \in I$. Now

$t\#a = f(r) \otimes f(\acute{a}) = f(r \otimes \acute{a}) \in f(I), t\#a \in f(I)$. Similarly $a\#t \in f(I)$. Hence $f(I)$ is an ideal of NTR_2.