

The Smarandache Bryant Schneider Group Of A Smarandache Loop ^{*†}

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Abstract

The concept of Smarandache Bryant Schneider Group of a Smarandache loop is introduced. Relationship(s) between the Bryant Schneider Group and the Smarandache Bryant Schneider Group of an S-loop are discovered and the later is found to be useful in finding Smarandache isotopy-isomorphy condition(s) in S-loops just like the formal is useful in finding isotopy-isomorphy condition(s) in loops. Some properties of the Bryant Schneider Group of a loop are shown to be true for the Smarandache Bryant Schneider Group of a Smarandache loop. Some interesting and useful cardinality formulas are also established for a type of finite Smarandache loop

1 Introduction

The study of Smarandache loops was initiated by W. B. Vasantha Kandasamy in 2002. In her book [16], she defined a Smarandache loop (S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. For more on loops and their properties, readers should check [14], [3], [5], [7], [8] and [16]. In her book, she introduced over 75 Smarandache concepts in loops but the concept Smarandache Bryant Schneider Group which is to be studied here for the first time is not among. In her first paper [17], she introduced some types of Smarandache loops. The present author has contributed to the study of S-quasigroups and S-loops in [9], [10] and [11] while Muktibodh [13] did a study on the first.

Robinson [15] introduced the idea of Bryant-Schneider group of a loop because its importance and motivation stem from the work of Bryant and Schneider [4]. Since the advent of the Bryant-Schneider group, some studies by Adeniran [1], [2] and Chiboka [6] have been

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done on it relative to CC-loops, C-loops and extra loops after Robinson [15] studied the Bryant-Schneider group of a Bol loop. The judicious use of it was earlier predicted by Robinson [15]. As mentioned in [Section 5, Robinson [15]], the Bryant-Schneider group of a loop is extremely useful in investigating isotopy-isomorphy condition(s) in loops.

In this study, the concept of Smarandache Bryant Schneider Group of a Smarandache loop is introduced. Relationship(s) between the Bryant Schneider Group and the Smarandache Bryant Schneider Group of an S-loop are discovered and the later is found to be useful in finding Smarandache isotopy-isomorphy condition(s) in S-loops just like the formal is useful in finding isotopy-isomorphy condition(s) in loops. Some properties of the Bryant Schneider Group of a loop are shown to be true for the Smarandache Bryant Schneider Group of a Smarandache loop. Some interesting and useful cardinality formulas are also established for a type of finite Smarandache loop. But first, we state some important definitions.

2 Definitions and Notations

Definition 2.1 *Let L be a non-empty set. Define a binary operation (\cdot) on L : If $x \cdot y \in L \forall x, y \in L$, (L, \cdot) is called a groupoid. If the system of equations ; $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for x and y respectively, then (L, \cdot) is called a quasigroup. Furthermore, if there exists a unique element $e \in L$ called the identity element such that $\forall x \in L$, $x \cdot e = e \cdot x = x$, (L, \cdot) is called a loop.*

Furthermore, if there exist at least a non-empty subset M of L such that (M, \cdot) is a non-trivial subgroup of (L, \cdot) , then L is called a Smarandache loop(S-loop) with Smarandache subgroup(S-subgroup) M .

The set $SYM(L, \cdot) = SYM(L)$ of all bijections in a loop (L, \cdot) forms a group called the permutation(symmetric) group of the loop (L, \cdot) . The triple (U, V, W) such that $U, V, W \in SYM(L, \cdot)$ is called an autotopism of L if and only if $xU \cdot yV = (x \cdot y)W \forall x, y \in L$. The group of autotopisms(under componentwise multiplication([14]) of L is denoted by $AUT(L, \cdot)$. If $U = V = W$, then the group $AUM(L, \cdot) = AUM(L)$ formed by such U 's is called the automorphism group of (L, \cdot) . If L is an S-loop with an arbitrary S-subgroup H , then the group $SSYM(L, \cdot) = SSYM(L)$ formed by all $\theta \in SYM(L)$ such that $h\theta \in H \forall h \in H$ is called the Smarandache permutation(symmetric) group of L . Hence, the group $SA(L, \cdot) = SA(L)$ formed by all $\theta \in SSYM(L) \cap AUM(L)$ is called the Smarandache automorphism group of L .

Let (G, \cdot) be a loop. The bijection $L_x : G \longrightarrow G$ defined as $yL_x = x \cdot y \forall x, y \in G$ is called a left translation(multiplication) of G while the bijection $R_x : G \longrightarrow G$ defined as $yR_x = y \cdot x \forall x, y \in G$ is called a right translation(multiplication) of G .

Definition 2.2 *(Robinson [15])*

Let (G, \cdot) be a loop. A mapping $\theta \in SYM(G, \cdot)$ is a special map for G means that there exist $f, g \in G$ so that $(\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot)$.

Definition 2.3 Let (G, \cdot) be a Smarandache loop with S -subgroup (H, \cdot) . A mapping $\theta \in SSYM(G, \cdot)$ is a Smarandache special map (S -special map) for G if and only if there exist $f, g \in H$ such that $(\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot)$.

Definition 2.4 (Robinson [15])

Let the set

$$BS(G, \cdot) = \{\theta \in SYM(G, \cdot) : \exists f, g \in G \ni (\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot)\}$$

i.e the set of all special maps in a loop, then $BS(G, \cdot) \leq SYM(G, \cdot)$ is called the Bryant-Schneider group of the loop (G, \cdot) .

Definition 2.5 Let the set

$$SBS(G, \cdot) = \{\theta \in SSYM(G, \cdot) : \text{there exist } f, g \in H \ni (\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot)\}$$

i.e the set of all S -special maps in a S -loop, then $SBS(G, \cdot)$ is called the Smarandache Bryant-Schneider group (SBS group) of the S -loop (G, \cdot) with S -subgroup H if $SBS(G, \cdot) \leq SYM(G, \cdot)$.

Definition 2.6 The triple $\phi = (R_g, L_f, I)$ is called an f, g -principal isotopism of a loop (G, \cdot) onto a loop (G, \circ) if and only if

$$x \cdot y = x R_g \circ y L_f \quad \forall x, y \in G \quad \text{or} \quad x \circ y = x R_g^{-1} \cdot y L_f^{-1} \quad \forall x, y \in G.$$

f and g are called translation elements of G or at times written in the pair form (g, f) , while (G, \circ) is called an f, g -principal isotope of (G, \cdot) .

On the other hand, (G, \otimes) is called a Smarandache f, g -principal isotope of (G, \oplus) if for some $f, g \in S$,

$$x R_g \otimes y L_f = (x \oplus y) \quad \forall x, y \in G$$

where (S, \oplus) is a S -subgroup of (G, \oplus) . In these cases, f and g are called Smarandache elements (S -elements).

Let (L, \cdot) and (G, \circ) be S -loops with S -subgroups L' and G' respectively such that $x A \in G' \quad \forall x \in L'$, where $A : (L, \cdot) \longrightarrow (G, \circ)$. Then the mapping A is called a Smarandache isomorphism if $(L, \cdot) \cong (G, \circ)$, hence we write $(L, \cdot) \simeq (G, \circ)$. An S -loop (L, \cdot) is called a G -Smarandache loop (GS -loop) if and only if $(L, \cdot) \simeq (G, \circ)$ for all S -loop isotopes (G, \circ) of (L, \cdot) .

Definition 2.7 Let (G, \cdot) be a Smarandache loop with an S -subgroup H .

$$\Omega(G, \cdot) = \left\{ (\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot) \text{ for some } f, g \in H : h\theta \in H \quad \forall h \in H \right\}$$

3 Main Results

The Smarandache Bryant Schneider Group

Theorem 3.1 *Let (G, \cdot) be a Smarandache loop. $SBS(G, \cdot) \leq BS(G, \cdot)$.*

Proof

Let (G, \cdot) be an S-loop with S-subgroup H . Comparing Definition 2.4 and Definition 2.5, it can easily be observed that $SBS(G, \cdot) \subset BS(G, \cdot)$. The case $SBS(G, \cdot) \subseteq BS(G, \cdot)$ is possible when $G = H$ where H is the S-subgroup of G but this will be a contradiction since G is an S-loop.

Identity If I is the identity mapping on G , then $hI = h \in H \forall h \in H$ and there exists $e \in H$ where e is the identity element in G such that $(IR_e^{-1}, IL_e^{-1}, I) = (I, I, I) \in AUT(G, \cdot)$. So, $I \in SBS(G, \cdot)$. Thus $SBS(G, \cdot)$ is non-empty.

Closure and Inverse Let $\alpha, \beta \in SBS(G, \cdot)$. Then there exist $f_1, g_1, f_2, g_2 \in H$ such that

$$A = (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha), B = (\beta R_{g_2}^{-1}, \beta L_{f_2}^{-1}, \beta) \in AUT(G, \cdot).$$

$$\begin{aligned} AB^{-1} &= (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha)(R_{g_2} \beta^{-1}, L_{f_2} \beta^{-1}, \beta^{-1}) \\ &= (\alpha R_{g_1}^{-1} R_{g_2} \beta^{-1}, \alpha L_{f_1}^{-1} L_{f_2} \beta^{-1}, \alpha \beta^{-1}) \in AUT(G, \cdot). \end{aligned}$$

Let $\delta = \beta R_{g_1}^{-1} R_{g_2} \beta^{-1}$ and $\gamma = \beta L_{f_1}^{-1} L_{f_2} \beta^{-1}$. Then,

$$(\alpha \beta^{-1} \delta, \alpha \beta^{-1} \gamma, \alpha \beta^{-1}) \in AUT(G, \cdot) \Leftrightarrow (x \alpha \beta^{-1} \delta) \cdot (y \alpha \beta^{-1} \gamma) = (x \cdot y) \alpha \beta^{-1} \forall x, y \in G.$$

Putting $y = e$ and replacing x by $x \beta \alpha^{-1}$, we have $(x \delta) \cdot (e \alpha \beta^{-1} \gamma) = x$ for all $x \in G$. Similarly, putting $x = e$ and replacing y by $y \beta \alpha^{-1}$, we have $(e \alpha \beta^{-1} \delta) \cdot (y \gamma) = y$ for all $y \in G$. Thence, $x \delta R_{(e \alpha \beta^{-1} \gamma)} = x$ and $y \gamma L_{(e \alpha \beta^{-1} \delta)} = y$ which implies that

$$\delta = R_{(e \alpha \beta^{-1} \gamma)}^{-1} \text{ and } \gamma = L_{(e \alpha \beta^{-1} \delta)}^{-1}.$$

Thus, since $g = e \alpha \beta^{-1} \gamma, f = e \alpha \beta^{-1} \delta \in H$ then

$$AB^{-1} = (\alpha \beta^{-1} R_g^{-1}, \alpha \beta^{-1} L_f^{-1}, \alpha \beta^{-1}) \in AUT(G, \cdot) \Leftrightarrow \alpha \beta^{-1} \in SBS(G, \cdot).$$

$\therefore SBS(G, \cdot) \leq BS(G, \cdot)$.

Corollary 3.1 *Let (G, \cdot) be a Smarandache loop. Then, $SBS(G, \cdot) \leq SSYM(G, \cdot) \leq SYM(G, \cdot)$. Hence, $SBS(G, \cdot)$ is the Smarandache Bryant-Schneider group(SBS group) of the S-loop (G, \cdot) .*

Proof

Although the fact that $SBS(G, \cdot) \leq SYM(G, \cdot)$ follows from Theorem 3.1 and the fact in [Theorem 1, [15]] that $BS(G, \cdot) \leq SYM(G, \cdot)$. Nevertheless, it can also be traced from the facts that $SBS(G, \cdot) \leq SSYM(G, \cdot)$ and $SSYM(G, \cdot) \leq SYM(G, \cdot)$.

It is easy to see that $SSYM(G, \cdot) \subset SYM(G, \cdot)$ and that $SBS(G, \cdot) \subset SSYM(G, \cdot)$ while the trivial cases $SSYM(G, \cdot) \subseteq SYM(G, \cdot)$ and $SBS(G, \cdot) \subseteq SSYM(G, \cdot)$ will contradict the fact that G is an S-loop because these two are possible if the S-subgroup H is G . Reasoning through the axioms of a group, it is easy to show that $SSYM(G, \cdot) \leq SYM(G, \cdot)$. By using the same steps in Theorem 3.1, it will be seen that $SBS(G, \cdot) \leq SSYM(G, \cdot)$.

The SBS Group of a Smarandache f, g -principal isotope

Theorem 3.2 *Let (G, \cdot) be a S-loop with a Smarandache f, g -principal isotope (G, \circ) . Then, (G, \circ) is an S-loop.*

Proof

Let (G, \cdot) be an S-loop, then there exist an S-subgroup (H, \cdot) of G . If (G, \circ) is a Smarandache f, g -principal isotope of (G, \cdot) , then

$$x \cdot y = xR_g \circ yL_f \quad \forall x, y \in G \text{ which implies } x \circ y = xR_g^{-1} \cdot yL_f^{-1} \quad \forall x, y \in G$$

where $f, g \in H$. So

$$h_1 \circ h_2 = h_1R_g^{-1} \cdot h_2L_f^{-1} \quad \forall h_1, h_2 \in H \text{ for some } f, g \in H.$$

Let us now consider the set H under the operation "o". That is the pair (H, \circ) .

Groupoid Since $f, g \in H$, then by the definition $h_1 \circ h_2 = h_1R_g^{-1} \cdot h_2L_f^{-1}$, $h_1 \circ h_2 \in H \quad \forall h_1, h_2 \in H$ since (H, \cdot) is a groupoid. Thus, (H, \circ) is a groupoid.

Quasigroup With the definition $h_1 \circ h_2 = h_1R_g^{-1} \cdot h_2L_f^{-1} \quad \forall h_1, h_2 \in H$, it is clear that (H, \circ) is a quasigroup since (H, \cdot) is a quasigroup.

Loop It can easily be seen that $f \cdot g$ is an identity element in (H, \circ) . So, (H, \circ) is a loop.

Group Since (H, \cdot) is a associative, it is easy to show that (H, \circ) is associative.

Hence, (H, \circ) is an S-subgroup in (G, \circ) since the latter is a loop(a quasigroup with identity element $f \cdot g$). Therefore, (G, \circ) is an S-loop.

Theorem 3.3 *Let (G, \cdot) be a Smarandache loop with an S-subgroup (H, \cdot) . A mapping $\theta \in SYM(G, \cdot)$ is a S-special map if and only if θ is an S-isomorphism of (G, \cdot) onto some Smarandache f, g -principal isotopes (G, \circ) where $f, g \in H$.*

Proof

By Definition 2.3, a mapping $\theta \in SSYM(G)$ is a S-special map implies there exist $f, g \in H$ such that $(\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot)$. It can be observed that

$$(\theta R_g^{-1}, \theta L_f^{-1}, \theta) = (\theta, \theta, \theta)(R_g^{-1}, L_f^{-1}, I) \in AUT(G, \cdot).$$

But since $(R_g^{-1}, L_f^{-1}, I) : (G, \circ) \longrightarrow (G, \cdot)$ then for $(\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot)$ we must have $(\theta, \theta, \theta) : (G, \cdot) \longrightarrow (G, \circ)$ which means $(G, \cdot) \cong^\theta (G, \circ)$, hence $(G, \cdot) \succsim^\theta (G, \circ)$ because $(H, \cdot)\theta = (H, \circ)$. $(R_g, L_f, I) : (G, \cdot) \longrightarrow (G, \circ)$ is an f, g -principal isotopism so (G, \circ) is a Smarandache f, g -principal isotope of (G, \cdot) by Theorem 3.2.

Conversely, if θ is an S-isomorphism of (G, \cdot) onto some Smarandache f, g -principal isotopes (G, \circ) where $f, g \in H$ such that (H, \cdot) is a S-subgroup of (G, \cdot) means $(\theta, \theta, \theta) : (G, \cdot) \longrightarrow (G, \circ)$, $(R_g, L_f, I) : (G, \cdot) \longrightarrow (G, \circ)$ which implies $(R_g^{-1}, L_f^{-1}, I) : (G, \circ) \longrightarrow (G, \cdot)$ and $(H, \cdot)\theta = (H, \circ)$. Thus, $(\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G, \cdot)$. Therefore, θ is a S-special map because $f, g \in H$.

Corollary 3.2 *Let (G, \cdot) be a Smarandache loop with a an S-subgroup (H, \cdot) . A mapping $\theta \in SBS(G, \cdot)$ if and only if θ is an S-isomorphism of (G, \cdot) onto some Smarandache f, g -principal isotopes (G, \circ) such that $f, g \in H$ where (H, \cdot) is an S-subgroup of (G, \cdot) .*

Proof

This follows from Definition 2.5 and Theorem 3.3.

Theorem 3.4 *Let (G, \cdot) and (G, \circ) be S-loops. (G, \circ) is a Smarandache f, g -principal isotope of (G, \cdot) if and only if (G, \cdot) is a Smarandache g, f -principal isotope of (G, \circ) .*

Proof

Let (G, \cdot) and (G, \circ) be S-loops such that if (H, \cdot) is an S-subgroup in (G, \cdot) , then (H, \circ) is an S-subgroup of (G, \circ) . The left and right translation maps relative to an element x in (G, \circ) shall be denoted by \mathcal{L}_x and \mathcal{R}_x respectively.

If (G, \circ) is a Smarandache f, g -principal isotope of (G, \cdot) then, $x \cdot y = xR_g \circ yL_f \forall x, y \in G$ for some $f, g \in H$. Thus, $xR_y = xR_g \mathcal{R}_y L_f$ and $yL_x = yL_f \mathcal{L}_x R_g$ $x, y \in G$ and we have $R_y = R_g \mathcal{R}_y L_f$ and $L_x = L_f \mathcal{L}_x R_g$ $x, y \in G$. So, $\mathcal{R}_y = R_g^{-1} R_y L_f^{-1}$ and $\mathcal{L}_x = L_f^{-1} L_x R_g^{-1} = x, y \in G$. Putting $y = f$ and $x = g$ respectively, we now get $\mathcal{R}_f = R_g^{-1} R_f L_f^{-1} = R_g^{-1}$ and $\mathcal{L}_g = L_f^{-1} L_g R_g^{-1} = L_f^{-1}$. That is, $\mathcal{R}_f = R_g^{-1}$ and $\mathcal{L}_g = L_f^{-1}$ for some $f, g \in H$.

Recall that

$$x \cdot y = xR_g \circ yL_f \forall x, y \in G \Leftrightarrow x \circ y = xR_g^{-1} \cdot yL_f^{-1} \forall x, y \in G.$$

So using the last two translation equations,

$$x \circ y = x\mathcal{R}_f \cdot y\mathcal{L}_g \forall x, y \in G \Leftrightarrow \text{the triple } (\mathcal{R}_f, \mathcal{L}_g, I) : (G, \circ) \longrightarrow (G, \cdot)$$

is a Smarandache g, f -principal isotopism. Therefore, (G, \cdot) is a Smarandache g, f -principal isotope of (G, \circ) .

The proof of the converse is achieved by doing the reverse of the procedure described above.

Theorem 3.5 *If (G, \cdot) is an S-loop with a Smarandache f, g -principal isotope (G, \circ) , then $SBS(G, \cdot) = SBS(G, \circ)$.*

Proof

Let (G, \circ) be the Smarandache f, g -principal isotope of the S-loop (G, \cdot) with S-subgroup (H, \cdot) . By Theorem 3.2, (G, \circ) is an S-loop with S-subgroup (H, \circ) . The left and right translation maps relative to an element x in (G, \circ) shall be denoted by \mathcal{L}_x and \mathcal{R}_x respectively.

Let $\alpha \in SBS(G, \cdot)$, then there exist $f_1, g_1 \in H$ so that $(\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha) \in AUT(G, \cdot)$. Recall that the triple $(R_{g_1}, L_{f_1}, I) : (G, \cdot) \longrightarrow (G, \circ)$ is a Smarandache f, g -principal isotopism, so $x \cdot y = x R_g \circ y L_f \forall x, y \in G$ and this implies

$$R_x = R_g \mathcal{R}_{x L_f} \text{ and } L_x = L_f \mathcal{L}_{x R_g} \forall x \in G \text{ which also implies that}$$

$$\mathcal{R}_{x L_f} = R_g^{-1} R_x \text{ and } \mathcal{L}_{x R_g} = L_f^{-1} L_x \forall x \in G \text{ which finally gives}$$

$$\mathcal{R}_x = R_g^{-1} R_{x L_f^{-1}} \text{ and } \mathcal{L}_x = L_f^{-1} L_{x R_g^{-1}} \forall x \in G.$$

Set $f_2 = f \alpha R_{g_1}^{-1} R_g$ and $g_2 = g \alpha L_{f_1}^{-1} L_f$. Then

$$\mathcal{R}_{g_2} = R_g^{-1} R_{g \alpha L_{f_1}^{-1} L_f L_f^{-1}} = R_g^{-1} R_{g \alpha L_{f_1}^{-1}} \quad (1)$$

$$\text{and } \mathcal{L}_{f_2} = L_f^{-1} L_{f \alpha R_{g_1}^{-1} R_g R_g^{-1}} = L_f^{-1} L_{f \alpha R_{g_1}^{-1}} \forall x \in G. \quad (2)$$

Since, $(\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha) \in AUT(G, \cdot)$, then

$$(x \alpha R_{g_1}^{-1}) \cdot (y \alpha L_{f_1}^{-1}) = (x \cdot y) \alpha \forall x, y \in G. \quad (3)$$

Putting $y = g$ and $x = f$ separately in the last equation,

$$x \alpha R_{g_1}^{-1} R_{(g \alpha L_{f_1}^{-1})} = x R_g \alpha \text{ and } y \alpha L_{f_1}^{-1} L_{(f \alpha R_{g_1}^{-1})} = y L_f \alpha \forall x, y \in G.$$

Thus by applying (1) and (2), we now have

$$\alpha R_{g_1}^{-1} = R_g \alpha R_{(g \alpha L_{f_1}^{-1})}^{-1} = R_g \alpha \mathcal{R}_{g_2}^{-1} R_g^{-1} \text{ and } \alpha L_{f_1}^{-1} = L_f \alpha L_{(f \alpha R_{g_1}^{-1})}^{-1} = L_f \alpha \mathcal{L}_{f_2}^{-1} L_f^{-1}. \quad (4)$$

We shall now compute $(x \circ y) \alpha$ by (3) and (4) and then see the outcome.

$$(x \circ y) \alpha = (x R_g^{-1} \cdot y L_f^{-1}) \alpha = x R_g^{-1} \alpha R_{g_1}^{-1} \cdot y L_f^{-1} \alpha L_{f_1}^{-1} = x R_g^{-1} R_g \alpha \mathcal{R}_{g_2}^{-1} R_g^{-1} \cdot y L_f^{-1} L_f \alpha \mathcal{L}_{f_2}^{-1} L_f^{-1} = x \alpha \mathcal{R}_{g_2}^{-1} R_g^{-1} \cdot y \alpha \mathcal{L}_{f_2}^{-1} L_f^{-1} = x \alpha \mathcal{R}_{g_2}^{-1} \circ y \alpha \mathcal{L}_{f_2}^{-1} \forall x, y \in G.$$

Thus,

$$(x \circ y) \alpha = x \alpha \mathcal{R}_{g_2}^{-1} \circ y \alpha \mathcal{L}_{f_2}^{-1} \forall x, y \in G \Leftrightarrow (\alpha \mathcal{R}_{g_2}^{-1}, \alpha \mathcal{L}_{f_2}^{-1}, \alpha) \in AUT(G, \circ) \Leftrightarrow \alpha \in SBS(G, \circ).$$

Whence, $SBS(G, \cdot) \subseteq SBS(G, \circ)$.

Since (G, \circ) is the Smarandache f, g -principal isotope of the S-loop (G, \cdot) , then by Theorem 3.4, (G, \cdot) is the Smarandache g, f -principal isotope of (G, \circ) . So following the steps above, it can similarly be shown that $SBS(G, \circ) \subseteq SBS(G, \cdot)$. Therefore, the conclusion that $SBS(G, \cdot) = SBS(G, \circ)$ follows.

Cardinality Formulas

Theorem 3.6 Let (G, \cdot) be a finite Smarandache loop with n distinct S -subgroups. If the SBS group of (G, \cdot) relative to an S -subgroup (H_i, \cdot) is denoted by $SBS_i(G, \cdot)$, then

$$|BS(G, \cdot)| = \frac{1}{n} \sum_{i=1}^n |SBS_i(G, \cdot)| [BS(G, \cdot) : SBS_i(G, \cdot)].$$

Proof

Let the n distinct S -subgroups of G be denoted by H_i , $i = 1, 2, \dots, n$. Note here that $H_i \neq H_j \forall i, j = 1, 2, \dots, n$. By Theorem 3.1, $SBS_i(G, \cdot) \leq BS(G, \cdot) \forall i = 1, 2, \dots, n$. Hence, by the Lagrange's theorem of classical group theory,

$$|BS(G, \cdot)| = |SBS_i(G, \cdot)| [BS(G, \cdot) : SBS_i(G, \cdot)] \forall i = 1, 2, \dots, n.$$

Thus, adding the equation above for all $i = 1, 2, \dots, n$, we get

$$n|BS(G, \cdot)| = \sum_{i=1}^n |SBS_i(G, \cdot)| [BS(G, \cdot) : SBS_i(G, \cdot)] \forall i = 1, 2, \dots, n, \text{ thence,}$$

$$|BS(G, \cdot)| = \frac{1}{n} \sum_{i=1}^n |SBS_i(G, \cdot)| [BS(G, \cdot) : SBS_i(G, \cdot)].$$

Theorem 3.7 Let (G, \cdot) be a Smarandache loop. Then, $\Omega(G, \cdot) \leq AUT(G, \cdot)$.

Proof

Let (G, \cdot) be an S -loop with S -subgroup H . By Definition 2.7, it can easily be observed that $\Omega(G, \cdot) \subseteq AUT(G, \cdot)$.

Identity If I is the identity mapping on G , then $hI = h \in H \forall h \in H$ and there exists $e \in H$ where e is the identity element in G such that $(IR_e^{-1}, IL_e^{-1}, I) = (I, I, I) \in AUT(G, \cdot)$. So, $(I, I, I) \in \Omega(G, \cdot)$. Thus $\Omega(G, \cdot)$ is non-empty.

Closure and Inverse Let $A, B \in \Omega(G, \cdot)$. Then there exist $\alpha, \beta \in SSYM(G, \cdot)$ and some $f_1, g_1, f_2, g_2 \in H$ such that

$$A = (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha), B = (\beta R_{g_2}^{-1}, \beta L_{f_2}^{-1}, \beta) \in AUT(G, \cdot).$$

$$\begin{aligned} AB^{-1} &= (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha)(R_{g_2} \beta^{-1}, L_{f_2} \beta^{-1}, \beta^{-1}) \\ &= (\alpha R_{g_1}^{-1} R_{g_2} \beta^{-1}, \alpha L_{f_1}^{-1} L_{f_2} \beta^{-1}, \alpha \beta^{-1}) \in AUT(G, \cdot). \end{aligned}$$

Using the same techniques for the proof of closure and inverse in Theorem 3.1 here and by letting $\delta = \beta R_{g_2}^{-1} R_{g_2} \beta^{-1}$ and $\gamma = \beta L_{f_2}^{-1} L_{f_2} \beta^{-1}$, it can be shown that,

$$AB^{-1} = (\alpha \beta^{-1} R_g^{-1}, \alpha \beta^{-1} L_f^{-1}, \alpha \beta^{-1}) \in AUT(G, \cdot) \text{ where } g = e \alpha \beta^{-1} \gamma, f = e \alpha \beta^{-1} \delta \in H$$

$$\text{such that } \alpha \beta^{-1} \in SSYM(G, \cdot) \Leftrightarrow AB^{-1} \in \Omega(G, \cdot).$$

$\therefore \Omega(G, \cdot) \leq AUT(G, \cdot)$.

Theorem 3.8 *Let (G, \cdot) be a Smarandache loop with an S -subgroup H such that $f, g \in H$ and $\alpha \in SBS(G, \cdot)$. If the mapping*

$$\Phi : \Omega(G, \cdot) \longrightarrow SBS(G, \cdot) \text{ is defined as } \Phi : (\alpha R_g^{-1}, \alpha L_f^{-1}, \alpha) \mapsto \alpha,$$

then Φ is an homomorphism.

Proof

Let $A, B \in \Omega(G, \cdot)$. Then there exist $\alpha, \beta \in SSYM(G, \cdot)$ and some $f_1, g_1, f_2, g_2 \in H$ such that

$$A = (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha), B = (\beta R_{g_2}^{-1}, \beta L_{f_2}^{-1}, \beta) \in AUT(G, \cdot).$$

$\Phi(AB) = \Phi[(\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha)(\beta R_{g_2}^{-1}, \beta L_{f_2}^{-1}, \beta)] = \Phi(\alpha R_{g_1}^{-1} \beta R_{g_2}^{-1}, \alpha L_{f_1}^{-1} \beta L_{f_2}^{-1}, \alpha \beta)$. It will be good if this can be written as; $\Phi(AB) = \Phi(\alpha \beta \delta, \alpha \beta \gamma, \alpha \beta)$ such that $h \alpha \beta \in H \forall h \in H$ and $\delta = R_g^{-1}, \gamma = L_f^{-1}$ for some $g, f \in H$.

This is done as follows: If

$$(\alpha R_{g_1}^{-1} \beta R_{g_2}^{-1}, \alpha L_{f_1}^{-1} \beta L_{f_2}^{-1}, \alpha \beta) = (\alpha \beta \delta, \alpha \beta \gamma, \alpha \beta) \in AUT(G, \cdot) \text{ then,}$$

$$x \alpha \beta \delta \cdot y \alpha \beta \gamma = (x \cdot y) \alpha \beta \forall x, y \in G.$$

Put $y = e$ and replace x by $x \beta^{-1} \alpha^{-1}$ then $x \delta \cdot e \alpha \beta \gamma = x \Leftrightarrow \delta = R_{e \alpha \beta \gamma}^{-1}$.

Similarly, put $x = e$ and replace y by $y \beta^{-1} \alpha^{-1}$. Then, $e \alpha \beta \delta \cdot y \gamma = y \Leftrightarrow \gamma = L_{e \alpha \beta \delta}^{-1}$. So,

$$\Phi(AB) = (\alpha \beta R_{e \alpha \beta \gamma}^{-1}, \alpha \beta L_{e \alpha \beta \delta}^{-1}, \alpha \beta) = \alpha \beta = \Phi(\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha) \Phi(\beta R_{g_2}^{-1}, \beta L_{f_2}^{-1}, \beta) = \Phi(A) \Phi(B).$$

$\therefore \Phi$ is an homomorphism.

Theorem 3.9 *Let (G, \cdot) be a Smarandache loop with an S -subgroup H such that $f, g \in H$ and $\alpha \in SSYM(G, \cdot)$. If the mapping*

$$\Phi : \Omega(G, \cdot) \longrightarrow SBS(G, \cdot) \text{ is defined as } \Phi : (\alpha R_g^{-1}, \alpha L_f^{-1}, \alpha) \mapsto \alpha$$

then,

$$A = (\alpha R_g^{-1}, \alpha L_f^{-1}, \alpha) \in \ker \Phi \text{ if and only if } \alpha$$

is the identity map on G , $g \cdot f$ is the identity element of (G, \cdot) and $g \in N_\mu(G, \cdot)$ the middle nucleus of (G, \cdot) .

Proof

Necessity $\ker \Phi = \{A \in \Omega(G, \cdot) : \Phi(A) = I\}$. So, if $A = (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha) \in \ker \Phi$, then $\Phi(A) = \alpha = I$. Thus, $A = (R_{g_1}^{-1}, L_{f_1}^{-1}, I) \in AUT(G, \cdot) \Leftrightarrow$

$$x \cdot y = xR_g^{-1} \cdot yL_f^{-1} \forall x, y \in G. \quad (5)$$

Replace x by xR_g and y by yL_f in (5) to get

$$x \cdot y = xg \cdot fy \forall x, y \in G. \quad (6)$$

Putting $x = y = e$ in (6), we get $g \cdot f = e$. Replace y by yL_f^{-1} in (6) to get

$$x \cdot yL_f^{-1} = xg \cdot y \forall x, y \in G. \quad (7)$$

Put $x = e$ in (7), then we have $yL_f^{-1} = g \cdot y \forall y \in G$ and so (7) now becomes

$$x \cdot (gy) = xg \cdot y \forall x, y \in G \Leftrightarrow g \in N_\mu(G, \cdot).$$

Sufficieny Let α be the identity map on G , $g \cdot f$ the identity element of (G, \cdot) and $g \in N_\mu(G, \cdot)$. Thus, $fg \cdot f = f \cdot gf = fe = f$. Thus, $f \cdot g = e$. Then also, $y = fg \cdot y = f \cdot gy \forall y \in G$ which results into $yL_f^{-1} = gy \forall y \in G$. Thus, it can be seen that $x\alpha R_g^{-1} \cdot y\alpha L_f^{-1} = xR_g^{-1} \cdot yL_f^{-1} = xR_g^{-1}\alpha \cdot yL_f^{-1}\alpha = xR_g^{-1} \cdot yL_f^{-1} = xR_g^{-1} \cdot gy = (xR_g^{-1} \cdot g)y = xR_g^{-1}R_g \cdot y = x \cdot y = (x \cdot y)\alpha \forall x, y \in G$. Thus, $\Phi(A) = \Phi(\alpha R_g^{-1}, \alpha L_f^{-1}, \alpha) = \Phi(R_g^{-1}, L_f^{-1}, I) = I \Rightarrow A \in \ker \Phi$.

Theorem 3.10 *Let (G, \cdot) be a Smarandache loop with an S -subgroup H such that $f, g \in H$ and $\alpha \in SSYM(G, \cdot)$. If the mapping*

$$\Phi : \Omega(G, \cdot) \longrightarrow SBS(G, \cdot) \text{ is defined as } \Phi : (\alpha R_g^{-1}, \alpha L_f^{-1}, \alpha) \mapsto \alpha$$

then,

$$|N_\mu(G, \cdot)| = |\ker \Phi| \text{ and } |\Omega(G, \cdot)| = |SBS(G, \cdot)||N_\mu(G, \cdot)|.$$

Proof

Let the identity map on G be I . Using Theorem 3.9, if

$$g\theta = (R_g^{-1}, L_{g^{-1}}^{-1}, I) \forall g \in N_\mu(G, \cdot) \text{ then, } \theta : N_\mu(G, \cdot) \longrightarrow \ker \Phi.$$

θ is easily seen to be a bijection, hence $|N_\mu(G, \cdot)| = |\ker \Phi|$.

Since Φ is an homomorphism by Theorem 3.8, then by the first isomorphism theorem in classical group theory, $\Omega(G, \cdot)/\ker \Phi \cong \text{Im}\Phi$. Φ is clearly onto, so $\text{Im}\Phi = SBS(G, \cdot)$, so that $\Omega(G, \cdot)/\ker \Phi \cong SBS(G, \cdot)$. Thus, $|\Omega(G, \cdot)/\ker \Phi| = |SBS(G, \cdot)|$. By Lagrange's theorem, $|\Omega(G, \cdot)| = |\ker \Phi||\Omega(G, \cdot)/\ker \Phi|$, so, $|\Omega(G, \cdot)| = |\ker \Phi||SBS(G, \cdot)|$, $\therefore |\Omega(G, \cdot)| = |N_\mu(G, \cdot)||SBS(G, \cdot)|$.

Theorem 3.11 Let (G, \cdot) be a Smarandache loop with an S -subgroup H . If

$$\Theta(G, \cdot) = \left\{ (f, g) \in H \times H : (G, \circ) \simeq (G, \cdot) \right.$$

$\left. \text{for } (G, \circ) \text{ the Smarandache principal } f, g\text{-isotope of } (G, \cdot) \right\}$ then,

$$|\Omega(G, \cdot)| = |\Theta(G, \cdot)||SA(G, \cdot)|.$$

Proof

Let $A, B \in \Omega(G, \cdot)$. Then there exist $\alpha, \beta \in SSYM(G, \cdot)$ and some $f_1, g_1, f_2, g_2 \in H$ such that

$$A = (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha), B = (\beta R_{g_2}^{-1}, \beta L_{f_2}^{-1}, \beta) \in AUT(G, \cdot).$$

Define a relation \sim on $\Omega(G, \cdot)$ such that

$$A \sim B \iff f_1 = f_2 \text{ and } g_1 = g_2.$$

It is very easy to show that \sim is an equivalence relation on $\Omega(G, \cdot)$. It can easily be seen that the equivalence class $[A]$ of $A \in \Omega(G, \cdot)$ is the inverse image of the mapping

$$\Psi : \Omega(G, \cdot) \longrightarrow \Theta(G, \cdot) \text{ defined as } \Psi : (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha) \mapsto (f, g).$$

If $A, B \in \Omega(G, \cdot)$ then $\Psi(A) = \Psi(B)$ if and only if $(f_1, g_1) = (f_2, g_2)$ so, $f_1 = f_2$ and $g_1 = g_2$. Thus, since $\Omega(G, \cdot) \leq AUT(G, \cdot)$ by Theorem 3.7, then $AB^{-1} = (\alpha R_{g_1}^{-1}, \alpha L_{f_1}^{-1}, \alpha)(\beta R_{g_2}^{-1}, \beta L_{f_2}^{-1}, \beta)^{-1} = (\alpha R_{g_1}^{-1} R_{g_2} \beta^{-1}, \alpha L_{f_1}^{-1} L_{f_2} \beta^{-1}, \alpha \beta^{-1}) = (\alpha \beta^{-1}, \alpha \beta^{-1}, \alpha \beta^{-1}) \in AUT(G, \cdot) \iff \alpha \beta^{-1} \in SA(G, \cdot)$. So,

$$A \sim B \iff \alpha \beta^{-1} \in SA(G, \cdot) \text{ and } (f_1, g_1) = (f_2, g_2).$$

$\therefore |[A]| = |SA(G, \cdot)|$. But each $A = (\alpha R_g^{-1}, \alpha L_f^{-1}, \alpha) \in \Omega(G, \cdot)$ is determined by some $f, g \in H$. So since the set $\{[A] : A \in \Omega(G, \cdot)\}$ of all equivalence classes partitions $\Omega(G, \cdot)$ by the fundamental theorem of equivalence Relation,

$$|\Omega(G, \cdot)| = \sum_{f, g \in H} |[A]| = \sum_{f, g \in H} |SA(G, \cdot)| = |\Theta(G, \cdot)||SA(G, \cdot)|.$$

$$\therefore |\Omega(G, \cdot)| = |\Theta(G, \cdot)||SA(G, \cdot)|.$$

Theorem 3.12 Let (G, \cdot) be a finite Smarandache loop with a finite S -subgroup H . (G, \cdot) is S -isomorphic to all its S -loop S -isotopes if and only if

$$|(H, \cdot)|^2 |SA(G, \cdot)| = |SBS(G, \cdot)||N_\mu(G, \cdot)|.$$

Proof

As shown in [Corollary 5.2, [12]], an S-loop is S-isomorphic to all its S-loop S-isotopes if and only if it is S-isomorphic to all its Smarandache f, g principal isotopes. This will happen if and only if $H \times H = \Theta(G, \cdot)$ where $\Theta(G, \cdot)$ is as defined in Theorem 3.11.

Since $\Theta(G, \cdot) \subseteq H \times H$ then it is easy to see that for a finite Smarandache loop with a finite S-subgroup H , $H \times H = \Theta(G, \cdot)$ if and only if $|H|^2 = |\Theta(G, \cdot)|$. So the proof is complete by Theorem 3.10 and Theorem 3.11.

Corollary 3.3 *Let (G, \cdot) be a finite Smarandache loop with a finite S-subgroup H . (G, \cdot) is a GS-loop if and only if*

$$|(H, \cdot)|^2 |SA(G, \cdot)| = |SBS(G, \cdot)| |N_\mu(G, \cdot)|.$$

Proof

This follows by the definition of a GS-loop and Theorem 3.12.

Lemma 3.1 *Let (G, \cdot) be a finite GS-loop with a finite S-subgroup H and a middle nucleus $N_\mu(G, \cdot)$.*

$$|(H, \cdot)| = |N_\mu(G, \cdot)| \iff |(H, \cdot)| = \frac{|SBS(G, \cdot)|}{|SA(G, \cdot)|}.$$

Proof

From Corollary 3.3,

$$|(H, \cdot)|^2 |SA(G, \cdot)| = |SBS(G, \cdot)| |N_\mu(G, \cdot)|.$$

Necessity If $|(H, \cdot)| = |N_\mu(G, \cdot)|$, then

$$|(H, \cdot)| |SA(G, \cdot)| = |SBS(G, \cdot)| \implies |(H, \cdot)| = \frac{|SBS(G, \cdot)|}{|SA(G, \cdot)|}.$$

Sufficiency If $|(H, \cdot)| = \frac{|SBS(G, \cdot)|}{|SA(G, \cdot)|}$ then, $|(H, \cdot)| |SA(G, \cdot)| = |SBS(G, \cdot)|$. Hence, multiplying both sides by $|(H, \cdot)|$,

$$|(H, \cdot)|^2 |SA(G, \cdot)| = |SBS(G, \cdot)| |(H, \cdot)|.$$

So that

$$|SBS(G, \cdot)| |N_\mu(G, \cdot)| = |SBS(G, \cdot)| |(H, \cdot)| \implies |(H, \cdot)| = |N_\mu(G, \cdot)|.$$

Corollary 3.4 *Let (G, \cdot) be a finite GS-loop with a finite S-subgroup H . If $|N_\mu(G, \cdot)| \geq 1$, then,*

$$|(H, \cdot)| = \frac{|SBS(G, \cdot)|}{|SA(G, \cdot)|}. \text{ Hence, } |(G, \cdot)| = \frac{n|SBS(G, \cdot)|}{|SA(G, \cdot)|} \text{ for some } n \geq 1.$$

Proof

By hypothesis, $\{e\} \neq H \neq G$. In a loop, $N_\mu(G, \cdot)$ is a subgroup, hence if $|N_\mu(G, \cdot)| \geq 1$, then, we can take $(H, \cdot) = N_\mu(G, \cdot)$ so that $|(H, \cdot)| = |N_\mu(G, \cdot)|$. Thus by Lemma 3.1, $|(H, \cdot)| = \frac{|SBS(G, \cdot)|}{|SA(G, \cdot)|}$.

As shown in [Section 1.3, [8]], a loop L obeys the Lagrange's theorem relative to a subloop H if and only if $H(hx) = Hx$ for all $x \in L$ and for all $h \in H$. This condition is obeyed by $N_\mu(G, \cdot)$, hence

$$|(H, \cdot)| \mid |(G, \cdot)| \implies \frac{|SBS(G, \cdot)|}{|SA(G, \cdot)|} \mid |(G, \cdot)| \implies$$

there exists $n \in \mathbb{N}$ such that

$$|(G, \cdot)| = \frac{n|SBS(G, \cdot)|}{|SA(G, \cdot)|}.$$

But if $n = 1$, then $|(G, \cdot)| = |(H, \cdot)| \implies (G, \cdot) = (H, \cdot)$ hence (G, \cdot) is a group which is a contradiction to the fact that (G, \cdot) is an S-loop.

$$\therefore |(G, \cdot)| = \frac{n|SBS(G, \cdot)|}{|SA(G, \cdot)|} \text{ for some natural numbers } n \geq 1.$$

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