Multiple attribute group decision making method based on some normal neutrosophic number Heronian Mean operator

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Abstract: In this paper, similar to the extension from intuitionistic fuzzy numbers (IFNs) to neutrosophic numbers (NNs), we propose the normal neutrosophic numbers (NNNs) based on the normal intuitionistic fuzzy numbers (NIFNs) to handle the incompleteness, indeterminacy and inconsistency of the evaluation information. In addition, because Heronian mean (HM) operators can consider capture the correlations of the aggregated arguments, we further extend the HM operator to deal with the NNNs, and propose some new HM operators and apply them to solve the multiple attribute group decision making (MAGDM) problems. Firstly, we briefly introduce the definition, the operational laws, the properties, the score function, and the accuracy function of the NNNs. Secondly, some Heronian mean (HM) operators are introduced, such as generalized Heronian mean (GHM) operator, generalized weighted Heronian mean (GWHM) operator, improved generalized weighted Heronian mean (IGWHM) operator, generalized geometric Heronian mean (GGHM) operator, improved generalized geometric Heronian mean (IGGHM) operator, and improved generalized geometric weighted Heronian mean (IGGWHM) operator. Moreover, we propose the normal neutrosophic number improved generalized weighted Heronian mean (NNNIGWHM) operator and normal neutrosophic number improved generalized geometric weighted Heronian mean (NNNIGGWHM) operator, and discuss their properties and some special cases. Furthermore, we propose two multiple attribute group decision making methods respectively based on the NNNIGWHM and NNNIGGWHM operators. Finally, we give an illustrative example to demonstrate the practicality and effectiveness of the two methods.

Keywords: Multiple attribute decision making; Heronian mean; Normal fuzzy number; Normal neutrosophic numbers; Normal neutrosophic number Heronian mean operator.

1. Introduction

Multiple attribute decision making (MADM) or Multiple attribute group decision making (MAGDM) have the wide applications in many fields such as economy, politics, management, and so on. Because the evaluation information usually has the properties of incompleteness, indeterminacy and inconsistency in real decision making, it's not suitable to express the evaluation values by the real numbers in same situations. Compared to the real numbers, fuzzy numbers can be more appropriate to express these evaluation values. Zadeh [1] firstly proposed the fuzzy set (FS) theory which only has a membership function to process the fuzzy information. Base on this, Atanassov [2] further proposed the intuitionistic fuzzy set (IFS) which has the non-membership function (or can be called falsity-membership) and the membership function (or can be called truth-membership). So the most obvious difference between IFS and FS is that whether it has the non-membership function. Obviously, IFS can process the incomplete information in multiple attribute decision making or the multiple attribute group decision making (MAGDM), but it cannot process the indeterminate information and inconsistent information. The reason is that the hesitation degree (or can be called indeterminacy -membership) has been neglected. So, Smarandache [3] added an independent indeterminacy -membership function to IFS and further proposed the neutrosophic set (NS). In other words, the neutrosophic set (NS) is made up of truth-membership, falsity-membership and indeterminacy-membership. We can find IFS is the special case of NS in which the three parts are completely independent. NS has attracted more and more attention. Wang et al. [4, 5] proposed a single valued neutrosophic set (SVNS) and further proposed the interval neutrosophic set (INS) in which the truth-membership, indeterminacy-membership, and false-membership were expressed by interval numbers. In addition, the normal distribution, which widely existed in the natural phenomena or social phenomena, has been widely applied in many fields. To this day, the research about the normal distribution is limited. Yang and Ko [6] firstly proposed the normal fuzzy numbers (NFNs) which was used to describe the normal distribution phenomena. Wang and Li [7] further proposed the normal intuitionistic fuzzy numbers (NIFNs), and their corresponding operations, the score function, the comparative method and so on. But neither NS nor INS considers the normal distribution. So similar to the extension from IFS to NS, it is very necessary to combine the normal distribution with the NS and to define a new concept which can describe this type of information i.e., we can give the definition of the normal neutrosophic set (INS) and the relative theorems in this paper.

The information aggregation operators have attracted more and more attention[8-21], and they have become a hot research topic. Heronian mean (HM) operator is an important aggregation operator which can be used to capture the correlations of the aggregated arguments. Beliakov [8] firstly proved that Heronian mean was an aggregation operator. It is a pity that Beliakov didn't do further researches the HM operator. Based on this, Skora [9, 10] further proposed the generalized Heronian means(GHM) operator, and discussed the generalized arithmetic Heronian mean (GAHM)operator and generalized geometric Heronian mean(GGHM) operator. Yu and Wu [11] further proposed generalized interval-valued intuitionistic fuzzy Heronian mean (GIVIFHM) operator and generalized interval-valued intuitionistic fuzzy weighted Heronian mean (GIVIFWHM) operator, which is an extension from crisp numbers to interval intuitionistic fuzzy numbers. Yu [12] proposed some intuitionistic fuzzy aggregation operators based on HM, including the intuitionistic fuzzy geometric Heronian mean (IFGHM) operator and the intuitionistic fuzzy geometric weighed Heronian mean (IFGWHM) operator. At the same time, Yu [12] further discussed the relative theorems and found that IFGWHM operator has not reducibility and idempotency. Obviously, there is not the research on the HM operators for the normal neutrosophic numbers. Liu et al. [20] proposed some intuitionistic uncertain linguistic Heronian mean operators and applied them to multiple attribute group decision making, and developed the decision-making method based on these operators. Chen and Liu [21] proposed the intuitionistic trapezoidal fuzzy general Heronian OWA operator, and the multi-attribute decision-making approach based on this operator.

Normal neutrosophic number (NNN) is produced by combining the normal fuzzy number and the neutrosophic number, so it is a generalization of FS, IFS, NS, NFN, and so on, and it not only can handle incompleteness, indeterminacy and inconsistency of evaluation information but also can handle the information of normal distribution. Obviously, it can provide an easier way to express the uncertain information. In addition, Heronian mean (HM) operator can capture the correlations of the aggregated arguments. However, the traditional HM operator can only process the crisp number and not NNNs, So, it is very necessary to extend HM operator to process the information with NNNs. In this paper, we will propose some normal neutrosophic number Heronian mean (NNNHM) operators, including the normal neutrosophic number improved generalized weighted Heronian mean (NNNIGWHM) operator and the normal neutrosophic number improved generalized geometric weighted Heronian mean (NNNIGGWHM) operator, and then we will apply them to MAGDM problems.

In order to achieve this goal, the remainder of this paper is shown as follows. In section 2, we briefly introduce the basic concepts, the operation rules and the relevant characteristics of NNNs and some Heronian mean operators and their properties, including generalized Heronian mean(GHM) operator, generalized geometric Heronian mean(GGHM) operator and generalized improved Heronian mean(GIHM) operator. In section 3, we propose the normal neutrosophic numbers improved generalized weighted Heronian mean (NNNIGWHM) operator, the normal neutrosophic numbers improved generalized weighted geometric Heronian mean (NNNIGGWHM) operator. In section 4, the multiple attribute group decision making methods based on NNNIGWHM operator and NNNIGGWHM operators were proposed. In section 5, we give an numerical example to prove the function of the proposed method.

2. Preliminaries

2.1 The normal intuitionistic fuzzy set and the neutrosophic set

Definition 1[22]**.** Let X be a real number set. A denoted as $A=(a,\sigma)$ is an normal fuzzy number (NFN) when its membership function is expressed as :

$$A(x)=e^{-(\frac{x-a}{\sigma})^2} \quad (\sigma>0) \tag{1}$$

The set of all normal fuzzy numbers is denoted as $\tilde{N}$.

Definition 2[23, 24]**.** Let X be an ordinary finite non-empty set and $(a,\sigma)\in\tilde{N}$, $A=\langle(a,\sigma),\mu_A,\nu_A\rangle$ is a normal intuitionistic fuzzy number (NIFN) if its membership function satisfies:

$$\mu_A(x)=\mu_A e^{-(\frac{x-a}{\sigma})^2}, x\in X \tag{2}$$

and its non-membership function satisfies:

$$\nu_A(x)=1-(1-\nu_A)e^{-(\frac{x-a}{\sigma})^2}, x\in X. \tag{3}$$

where $0\le\mu_A(x)\le 1, 0\le\nu_A(x)\le 1, 0\le\mu_A+\nu_A\le 1$. When $\mu_A=1$ and $\nu_A=0$, the NIFN will be a NFN. Compared to NFN, NIFN adds the non-membership function that expresses the degree of alternatives not belonging to (a,σ) . In addition, $\pi_A(x)=1-\mu_A(x)-\nu_A(x)$ expresses the degree of hesitance.

Definition 3 [3]**.** Let X be a universe of discourse, with a generic element in X denoted by x. A neutrosophic set A in X is

$$A=\{\langle x,(T_A(x),I_A(x),F_A(x))\rangle \mid x\in X\} \tag{4}$$

where, $T_A(x)$ is the truth-membership function, $I_A(x)$ is the indeterminacy-membership function, and $F_A(x)$ is the falsity-membership function. $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]0^-,1^+[$.

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \le T_A(x) + I_A(x) + F_A(x) \le 3^+$.

The neutrosophic set was difficult to apply to real life because it was proposed from philosophical point of view. So Wang et al. [4] further narrow it to the single valued neutrosophic set (SVNS) from scientific or engineering point of view. Obviously, SVNS is a generalization of classical set, fuzzy set, intuitionistic fuzzy set and paraconsistent sets etc., and it was defined as follows.

Definition 4 [4]**.** Let X be a universe of discourse, with a generic element in X denoted by x. A single valued neutrosophic set A in X is

$$A = \left\{ \left\langle x, (T_A(x), I_A(x), F_A(x)) \right\rangle \mid x \in X \right\} \tag{5}$$

Where $T_A(x)$ is the truth-membership function, $I_A(x)$ is the indeterminacy-membership function, and $F_A(x)$ is the falsity-membership function. For each point x in X, we have $T_A(x), I_A(x), F_A(x) \in [0,1]$, and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

2.2 The normal neutrosophic set

Because the neutrosophic number can provide the independent indeterminacy-membership function, and it is a generalization of fuzzy set, intuitionistic fuzzy set and paraconsistent sets etc. Motived by the normal intuitionistic fuzzy number which is produced by combining the normal fuzzy number and intuitionistic fuzzy number, we will propose the normal neutrosophic number by combining the normal fuzzy number and neutrosophic number, which will be a generalization of the normal intuitionistic fuzzy number. We can give the definition of the neutrosophic number as follows.

Definition 5. Let X be a universe of discourse, with a generic element in X denoted by x, and $(a, \sigma) \in \tilde{N}$, A normal neutrosophic set A in X is

$$A = \left\{ \left\langle x, (a, \sigma), (T_A(x), I_A(x), F_A(x)) \right\rangle \mid x \in X \right\} \tag{6}$$

where, $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership function, the indeterminacy-membership function, and the falsity-membership function. For each point x in X, we have $T_A(x), I_A(x), F_A(x) \in [0,1]$, and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Further, we can call the $\left\langle (a, \sigma), (T_A(x), I_A(x), F_A(x)) \right\rangle$ as a normal neutrosophic number (NNN).

Definition 6. Let $\tilde{a}_1 = \left\langle (a_1, \sigma_1), (T_1, I_1, F_1) \right\rangle$ and $\tilde{a}_2 = \left\langle (a_2, \sigma_2), (T_2, I_2, F_2) \right\rangle$ be two NNNs, then the operational laws are defined as follows.

(1) $$\tilde{a}_1 \oplus \tilde{a}_2 = \left\langle (a_1 + a_2, \sigma_1 + \sigma_2), (T_1 + T_2 - T_1 T_2, I_1 I_2, F_1 F_2) \right\rangle \tag{7}$$

(2) $$\tilde{a}_1 \otimes \tilde{a}_2 = \left\langle (a_1 a_2, a_1 a_2 \sqrt{\frac{\sigma_1^2}{a_1^2} + \frac{\sigma_2^2}{a_2^2}}), (T_1 T_2, I_1 + I_2 - I_1 I_2, F_1 + F_2 - F_1 F_2) \right\rangle \tag{8}$$

(3) $$\lambda \tilde{a}_1 = \left\langle (\lambda a_1, \lambda \sigma_1), (1 - (1 - T_1)^\lambda, I_1^\lambda, F_1^\lambda) \right\rangle \quad \lambda > 0 \tag{9}$$

(4) $\tilde{a}_1^{\lambda} = \left\langle (a_1^{\lambda}, \lambda^{1/2} a_1^{\lambda-1}\sigma_1), \left(T_1^{\lambda}, 1-(1-I_1)^{\lambda}, 1-(1-F_1)^{\lambda}\right) \right\rangle \ \lambda > 0$ (10)

In the following, we will discuss some properties about the operational laws shown as follows.

Theorem 1. Let $\tilde{a}_1 = \langle (a_1, \sigma_1), (T_1, I_1, F_1) \rangle$ and $\tilde{a}_2 = \langle (a_2, \sigma_2), (T_2, I_2, F_2) \rangle$ be two NNNs, and $\eta, \eta_1, \eta_2 > 0$, then we have

(1) $\tilde{a}_1 \oplus \tilde{a}_2 = \tilde{a}_2 \oplus \tilde{a}_1$ (11)

(2) $\tilde{a}_1 \otimes \tilde{a}_2 = \tilde{a}_2 \otimes \tilde{a}_1$ (12)

(3) $\eta(\tilde{a}_1 \oplus \tilde{a}_2) = \eta\tilde{a}_1 \oplus \eta\tilde{a}_2$ (13)

(4) $\eta_1\tilde{a}_1 \oplus \eta_2\tilde{a}_1 = (\eta_1 + \eta_2)\tilde{a}_1$ (14)

(5) $\tilde{a}_1^{\eta} \otimes \tilde{a}_2^{\eta} = (\tilde{a}_1 \otimes \tilde{a}_2)^{\eta}$ (15)

(6) $\tilde{a}_1^{\eta_1} \otimes \tilde{a}_1^{\eta_2} = \tilde{a}_1^{\eta_1+\eta_2}$ (16)

Proof.

(1) The formula (11) is obviously right according to the operational rule (1) expressed by (7).

(2) The formula (12) is obviously right according to the operational rule (2) expressed by (8).

(3) For the left of formula (13), we have

$$\eta(\tilde{a}_1 \oplus \tilde{a}_2) = \eta\langle (a_1 + a_2, \sigma_1 + \sigma_2), (T_1 + T_2 - T_1T_2, I_1I_2, F_1F_2) \rangle$$

$$= \left\langle (\eta(a_1 + a_2), \eta(\sigma_1 + \sigma_2)), \left(1-(1-(T_1 + T_2 - T_1T_2))^{\eta}, (I_1I_2)^{\eta}, (F_1F_2)^{\eta}\right) \right\rangle$$

$$= \left\langle (\eta(a_1 + a_2), \eta(\sigma_1 + \sigma_2)), \left(1-((1-T_1)(1-T_2))^{\eta}, (I_1I_2)^{n}, (F_1F_2)^{n}\right) \right\rangle$$

and for the right of formula (13), we have

$$= \left\langle (\eta a_1 + \eta a_2, \eta\sigma_1 + \eta\sigma_2), \left((1-(1-T_1)^{\eta}) + (1-(1-T_2)^{\eta}) - (1-(1-T_1)^{\eta})(1-(1-T_2)^{\eta}, I_1^{\eta}I_2^{\eta}, F_1^{\eta}F_2^{\eta}\right) \right\rangle$$

$$= \left\langle (\eta(a_1 + a_2), \eta(\sigma_1 + \sigma_2)), \left(1-((1-T_1)(1-T_2))^{\eta}, (I_1I_2)^{\eta}, (F_1F_2)^{\eta}\right) \right\rangle$$

So, we can get $\eta(\tilde{a}_1 \oplus \tilde{a}_2) = \eta\tilde{a}_1 \oplus \eta\tilde{a}_2$,which completes the proof of formula(13).

(4) For the left of formula (14), we have

$$\eta_1\tilde{a}_1 \oplus \eta_2\tilde{a}_1 = \left\langle (\eta_1 a_1, \eta_1\sigma_1), \left(1-(1-T_1)^{\eta_1}, I_1^{\eta_1}, F_1^{\eta_1}\right) \right\rangle \oplus \left\langle (\eta_2 a_1, \eta_2\sigma_1), \left(1-(1-T_1)^{\eta_2}, I_1^{\eta_2}, F_1^{\eta_2}\right) \right\rangle$$

$$= \left\langle (\eta_1 a_1 + \eta_2 a_1, \eta_1\sigma_1 + \eta_2\sigma_1), \left(1-(1-T_1)^{\eta_1} + 1-(1-T_1)^{\eta_2} - (1-(1-T_1)^{\eta_1})(1-(1-T_1)^{\eta_2}), I_1^{\eta_1}I_1^{\eta_2}, F_1^{\eta_1}F_1^{\eta_2}\right) \right\rangle$$

$$= \left\langle ((\eta_1 + \eta_2)a_1, (\eta_1 + \eta_2)\sigma_1), \left(1-(1-T_1)^{\eta_1+\eta_2}, I_1^{\eta_1+\eta_2}, F_1^{\eta_1+\eta_2}\right) \right\rangle$$

$$= (\eta_1 + \eta_2)\tilde{a}_1$$

So, we can get the formula (14) is right.

(5)For the left of the formula (15), we have

$$\tilde{a}_1^{\eta} \otimes \tilde{a}_2^{\eta} = \left\langle \left(a_1^{\eta}, \eta^{\frac{1}{2}} a_1^{\eta-1}\sigma_1 \right), \left(T_1^{\eta}, 1-(1-I_1)^{\eta}, 1-(1-F_1)^{\eta}\right) \right\rangle$$

$$\otimes\left\langle\left(a_2^{\eta},\eta^{\frac{1}{2}}a_2^{\eta-1}\sigma_2\right),\left(T_2^{\eta},1-(1-I_2)^{\eta},1-(1-F_2)^{\eta}\right)\right\rangle$$

$$=\left\langle\left(a_1^{\eta}a_2^{\eta},a_1^{\eta}a_2^{\eta}\sqrt{\frac{\eta a_1^{2(\eta-1)}\sigma_1^2}{a_1^{2\eta}}+\frac{\eta a_2^{2(\eta-1)}\sigma_2^2}{a_2^{2\eta}}}\right),\right.$$

$$\left(T_1^{\eta}T_2^{\eta},(1-(1-I_1)^{\eta})+(1-(1-I_2)^{\eta})-(1-(1-I_1)^{\eta})(1-(1-I_2)^{\eta}),\right.$$

$$\left.\left.(1-(1-F_1)^{\eta})+(1-(1-F_2)^{\eta})-(1-(1-F_1)^{\eta})(1-(1-F_2)^{\eta})\right)\right\rangle$$

$$=\left\langle\left((a_1a_2)^{\eta},\eta^{\frac{1}{2}}(a_1a_2)^{\eta}\sqrt{\frac{\sigma_1^2}{a_1^2}+\frac{\sigma_2^2}{a_2^2}}\right),\left((T_1T_2)^{\eta},1-((1-I_1)(1-I_2))^{\eta},1-((1-F_1)(1-F_2))^{\eta}\right)\right\rangle$$

and for the right of the formula (15), we have

$$(\tilde{a}_1\otimes\tilde{a}_2)^{\eta}=\left\langle\left(a_1a_2,a_1a_2\sqrt{\frac{\sigma_1^2}{a_1^2}+\frac{\sigma_2^2}{a_2^2}}\right),\left(T_1T_2,I_1+I_2-I_1I_2,F_1+F_2-F_1F_2\right)\right\rangle^{\eta}$$

$$=\left\langle\left((a_1a_2)^{\eta},\eta^{\frac{1}{2}}(a_1a_2)^{\eta}\sqrt{\frac{\sigma_1^2}{a_1^2}+\frac{\sigma_2^2}{a_2^2}}\right),\left((T_1T_2)^{\eta},1-(1-(I_1+I_2-I_1I_2))^{\eta},1-(1-(F_1+F_2-F_1F_2))^{\eta}\right)\right\rangle$$

$$=\left\langle\left((a_1a_2)^{\eta},\eta^{\frac{1}{2}}(a_1a_2)^{\eta}\sqrt{\frac{\sigma_1^2}{a_1^2}+\frac{\sigma_2^2}{a_2^2}}\right),\left((T_1T_2)^{\eta},1-((1-I_1)(1-I_2))^{\eta},1-((1-F_1)(1-F_2))^{\eta}\right)\right\rangle$$

So, we can get the formula (15) is right.

(6) For the formula (16), we have

$$\tilde{a}_1^{\eta_1}\otimes\tilde{a}_1^{\eta_2}=\left\langle\left(a_1^{\eta_1},\eta_1^{\frac{1}{2}}a_1^{\eta_1-1}\sigma_1\right),\left(T_1^{\eta_1},1-(1-I_1)^{\eta_1},1-(1-F_1)^{\eta_1}\right)\right\rangle$$

$$\otimes\left\langle\left(a_1^{\eta_2},\eta_2^{\frac{1}{2}}a_1^{\eta_2-1}\sigma_1\right),\left(T_1^{\eta_2},1-(1-I_1)^{\eta_2},1-(1-F_1)^{\eta_2}\right)\right\rangle$$

$$=\left\langle\left(a_1^{\eta_1}a_1^{\eta_2},a_1^{\eta_1}a_1^{\eta_2}\sqrt{\frac{\eta_1 a_1^{2(\eta_1-1)}\sigma_1^2}{a_1^{2\eta_1}}+\frac{\eta_2 a_1^{2(\eta_2-1)}\sigma_1^2}{a_1^{2\eta_2}}}\right),\right.$$

$$\left.\left(T_1^{\eta_1}T_1^{\eta_2},1-(1-I_1)^{\eta_1}(1-I_1)^{\eta_2},1-(1-F_1)^{\eta_1}(1-F_1)^{\eta_2}\right)\right\rangle$$

$$=\left\langle\left(a_1^{\eta_1+\eta_2},(\eta_1+\eta_1)^{\frac{1}{2}}a_1^{\eta_1+\eta_2-1}\sigma_1\right),\left(T_1^{\eta_1+\eta_2},1-(1-I_1)^{\eta_1+\eta_2},1-(1-F_1)^{\eta_1+\eta_2}\right)\right\rangle$$

$$=\tilde{a}_1^{\eta_1+\eta_2}$$

So, we can get the formula (16) is right.

In the following, we will give the comparative method for two NNNs.

Definition 7. Suppose $\tilde{a}_1=\langle(a_1,\sigma_1),(T_1,I_1,F_1)\rangle$ and $\tilde{a}_2=\langle(a_2,\sigma_2),(T_2,I_2,F_2)\rangle$ are two NNNs. If and only if $a_1\ge a_2$, $\sigma_1\le\sigma_2$, $T_1\ge T_2$, $I_1\le I_2$, $F_1\le F_2$ for all x in X, then $\tilde{a}_1>\tilde{a}_2$.

Generally speaking, for any two NNNs, it is difficult to meet the Definition 7, so we can give a new comparative method by extending the comparative method of INNs to NNNs..

Definition 8. Let $\tilde{a}_k=\langle(a_k,\sigma_k),(T_k,I_k,F_k)\rangle$ be a NNN, and then its score function is

$$s_1(\tilde{a}_k)=a_k(2+T_k-I_k-F_k),\quad s_2(\tilde{a}_k)=\sigma_k(2+T_k-I_k-F_k) \tag{17}$$

and its accuracy function is

$$h_1(\tilde{a}_k)=a_k(2+T_k-I_k+F_k),\quad h_2(\tilde{a}_k)=\sigma_k(2+T_k-I_k+F_k) \tag{18}$$

Definition 9. Let $\tilde{a}_1=\langle(a_1,\sigma_1),(T_1,I_1,F_1)\rangle$ and $\tilde{a}_2=\langle(a_2,\sigma_2),(T_2,I_2,F_2)\rangle$ be two NNNs, the values of score functions of $\tilde{a}_1$ and $\tilde{a}_2$ are $s_1(\tilde{a}_1), s_2(\tilde{a}_1)$ and $s_1(\tilde{a}_2), s_2(\tilde{a}_2)$, and the values of accuracy functions of $\tilde{a}_1$ and $\tilde{a}_2$ are $h_1(\tilde{a}_1), h_2(\tilde{a}_1)$ and $h_1(\tilde{a}_2), h_2(\tilde{a}_2)$, respectively. Then, we have

(1) If $s_1(\tilde{a}_1)>s_1(\tilde{a}_2)$,then, $\tilde{a}_1>\tilde{a}_2$;

(2) If $s_1(\tilde{a}_1)=s_1(\tilde{a}_2)$,then

①If $h_1(\tilde{a}_1)>h_1(\tilde{a}_2)$,then, $\tilde{a}_1>\tilde{a}_2$;

②If $h_1(\tilde{a}_1)=h_1(\tilde{a}_2)$,then

(i) If $s_2(\tilde{a}_1)<s_2(\tilde{a}_2)$,then, $\tilde{a}_1>\tilde{a}_2$;

(ii)If $s_2(\tilde{a}_1)=s_2(\tilde{a}_2)$,then

(a) If $h_2(\tilde{a}_1)<h_2(\tilde{a}_2)$,then, $\tilde{a}_1>\tilde{a}_2$;

(b)If $h_2(\tilde{a}_1)=h_2(\tilde{a}_2)$,then, $\tilde{a}_1>\tilde{a}_2$.

2.3. Heronian mean (HM) operator

Heronian mean (HM) operator can be regard as a useful tool which is used to catch the interrelations of the aggregated arguments [9, 25] and can be defined as follows.

Definition 10[9]**.** A HM operator of dimension n is a mapping HM: $I^n\to I, I=[0,1]$. If

$$HM(x_1,x_2,\cdots,x_n)=\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{x_i x_j} \tag{19}$$

So the HM operator is called the Heronian mean (HM) operator.

Definition11[9, 25]**.** A GHM operator of dimension n is a mapping GHM $:I^n\to I, I=[0,1]$. If

$$GHM^{p,q}(x_1,x_2,\cdots,x_n)=\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}x_i^p x_j^q\right)^{\frac{1}{p+q}} \tag{20}$$

where $p,q\ge 0$. So the $GHM^{p,q}$ operator is called the generalized Heronian mean (GHM) operator.

It is easy to prove that the GHM operator has the following properties [9, 25].

Theorem 2. (Idempotency)

Suppose $x_i=x$ ($i=1,2,\cdots,n$), then

$$GHM^{p,q}(x_1, x_2, \cdots, x_n) = x \tag{21}$$

Theorem 3. (Monotonicity)

Let $x_i\ (i = 1,2,\ldots,n)$ and $y_i\ (i = 1,2,\ldots,n)$ be two sets of the nonnegative numbers satisfying $x_i \le y_i$, for all $i, i = 1,2,\cdots,n$, then

$$GHM^{p,q}(x_1, x_2, \cdots, x_n) \le GHM^{p,q}(y_1, y_2, \cdots, y_n) \tag{22}$$

Theorem 4. (Boundary)

The GHM operator satisfies:

$$\min(x_1, x_2, \ldots, x_n) \le GHM^{p,q}(x_1, x_2, \cdots, x_n) \le \max(x_1, x_2, \cdots, x_n) \tag{23}$$

Because the GHM operator didn't consider the weight of inputs, in the following, we will give the generalized weighted Heronian mean operator as follows.

Definition 12[25]**.** Let $p, q \ge 0$, and $x_i (i = 1,2,\cdots,n)$ be a set of nonnegative numbers. $w = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of $x_i (i = 1,2,\cdots,n)$, which satisfying $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$.If

$$GWHM^{p,q}(x_1, x_2, \cdots, x_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} (w_i x_i)^p (w_j x_j)^q \right)^{\frac{1}{p+q}} \tag{24}$$

then $GWHM^{p,q}$ operator is called the generalized weighted Heronian mean (GWHM) operator.

However, the GWHM operator has not the idempotency, in order to overcome this deficiency, Liu [25] further proposed an improved generalized weighted Heronian mean (IGWHM) operator.

Definition 13[25]**.** Let $p, q \ge 0$, and $x_i (i = 1,2,\cdots,n)$ be a set of nonnegative numbers. $w = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of $x_i (i = 1,2,\cdots,n)$, which satisfying $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$.If

$$IGWHM^{p,q}(x_1, x_2, \cdots, x_n) = \frac{\left(\sum_{i=1}^{n} \sum_{j=i}^{n} w_i w_j x_i^p x_j^q \right)^{\frac{1}{p+q}}}{\left(\sum_{i=1}^{n} \sum_{j=i}^{n} w_i w_j \right)^{\frac{1}{p+q}}} \tag{25}$$

The $IGWHM^{p,q}$ operator is called the improved generalized weighted Heronian mean (IGWHM) operator.

It is easy to prove the $IGWHM^{p,q}$ operator has these properties [25].

Theorem 5 (Idempotency)

Let $x_i = x$ for all $i = 1,2,\cdots,n$, then

$$IGWHM^{p,q}(x_1, x_2, \cdots, x_n) = x \tag{26}$$

Theorem 6 (Monotonicity)

Let $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ be two sets of the nonnegative numbers, if $x_i \le y_i$ for all $i, i = 1,2,\cdots,n$, then

$$IGWHM^{p,q}(x_1, x_2, \cdots, x_n) \le IGWHM^{p,q}(y_1, y_2, \cdots, y_n). \tag{27}$$

Theorem 7 (Boundary)

Let $(x_1, x_2, \cdots, x_n)$ be a set of the nonnegative numbers, if $x_{\min} = \min(x_1, x_2, \ldots, x_n)$ and $x_{\max} = \max(x_1, x_2, \ldots, x_n)$, then

$$x_{\min} \le IGWHM^{p,q}(x_1, x_2, \cdots, x_n) \le x_{\max} \tag{28}$$

In the following, we can analyze some special cases of the IGWHM operator

(1) When $q = 0$, then

$$IGWHM^{p,0}(x_1, x_2, \cdots, x_n) = \frac{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j x_i^p\right)^{\frac{1}{p}}}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j\right)^{\frac{1}{p}}}. \tag{29}$$

Further, when $p = 1$, there is

$$IGWHM^{1,0}(x_1, x_2, \cdots, x_n) = \frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j x_i}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}. \tag{30}$$

(2) When $p = 0$, then

$$IGWHM^{0,q}(x_1, x_2, \cdots, x_n) = \frac{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j x_j^q\right)^{\frac{1}{q}}}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j\right)^{\frac{1}{q}}}. \tag{31}$$

Further, when $q = 1$, there is

$$IGWHM^{0,1}(x_1, x_2, \cdots, x_n) = \frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j x_j}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j} \tag{32}$$

According to the above special cases, we can find that the parameters *p* and *q* don't have the interchangeability.

(3) When $p = q = 1$, then

$$IGWHM^{1,1}(x_1, x_2, \cdots, x_n) = \frac{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j x_i x_j\right)^{\frac{1}{2}}}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j\right)^{\frac{1}{2}}}. \tag{33}$$

2.4. The geometric Heronian mean (GHM) operator

Based on the generalized Heronian mean (GHM) operator, Yu [12] further proposed the generalized geometric Heronian mean (GGHM) operator.

Definition 14[12]. Let $x_i(i=1,2,\cdots,n)$ be a set of nonnegative numbers and $p,q\geq 0$, the value of p and q is not set to 0 at the same time. If

$$GGHM^{p,q}(x_1,x_2,\cdots,x_n)=\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(px_i+qx_j\right)^{\frac{2}{n(n+1)}} \tag{34}$$

then $GGHM^{p,q}$ is called the generalized geometric Heronian mean (GGHM) operator.

It is easy to prove the $GGHM^{p,q}$ operator has these properties [12].

Theorem 8 (Idempotency)

Let $x_i=x$ for all $i, i=1,2,\cdots,n$, then

$$GGHM^{p,q}(x_1,x_2,\cdots,x_n)=x \tag{35}$$

Theorem 9 (Monotonicity)

Let $(x_1,x_2,\cdots,x_n)$ and $(y_1,y_2,\cdots,y_n)$ be two sets of the nonnegative numbers, if $x_i\leq y_i$ for all $i, i=1,2,\cdots,n$, then

$$GGHM^{p,q}(x_1,x_2,\cdots,x_n)\leq GGHM^{p,q}(y_1,y_2,\cdots,y_n). \tag{36}$$

Theorem 10 (Boundary)

Let $(x_1,x_2,\cdots,x_n)$ be a set of the nonnegative numbers, if $x_{\min}=\min(x_1,x_2,\ldots,x_n)$ and $x_{\max}=\max(x_1,x_2,\ldots,x_n)$, then

$$x_{\min}\leq GGHM^{p,q}(x_1,x_2,\cdots,x_n)\leq x_{\max} \tag{37}$$

The GGHM operator brought more attentions on the correlations of the aggregated arguments so that it also neglected the weights, which is similar to GHM operator. So based on this, Yu [12] further proposed the generalized geometric weighted Heronian mean (GGWHM) operator.

Definition 15[12]. Let $x_i(i=1,2,\cdots,n)$ be a collection of nonnegative numbers and $p,q\geq 0$, the value of p and q is not set to 0 at the same time. $w=(w_1,w_2,\cdots,w_n)^T$ is the weight vector of $x_i(i=1,2,\cdots,n)$ satisfying $w_i\in[0,1]$, $\sum_{i=1}^{n}w_i=1$. If

$$GGWHM^{p,q}(x_1,x_2,\cdots,x_n)=\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left((px_i)^{w_i}+(qx_j)^{w_j}\right)^{\frac{2}{n(n+1)}} \tag{38}$$

$GGWHM^{p,q}$ operator is called the generalized geometric weighted Heronian mean (GGWHM) operator.

In order to overcome the counterintuitive of the $GGWHM^{p,q}$ operator, Yu [12] further proposed an improved generalized geometric weighted Heronian mean (IGGWHM) operator.

Definition 16[12]. Let $x_i(i=1,2,\cdots,n)$ be a set of nonnegative numbers and $p,q\geq 0$, the value of p and q is not set to 0 at the same time. $w=(w_1,w_2,\cdots,w_n)^T$ is the weight vector of

$x_i\left(i=1,2,\cdots,n\right)$ satisfying $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$. If

$$IGGWHM^{p,q}\left(x_1,x_2,\cdots,x_n\right)=\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(px_i+qx_j\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}. \tag{39}$$

then $IGGWHM^{p,q}$ is called the improved generalized geometric weighted Heronian mean (IGGWHM) operator.

It is easy to prove the $IGGWHM^{p,q}$ operator has these properties [12].

Theorem 11 (Reducibility).

Let $W=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)^T$, then

$$IGGWHM^{p,q}\left(x_1,x_2,\cdots,x_n\right)=GGHM^{p,q}\left(x_1,x_2,\cdots,x_n\right) \tag{40}$$

Theorem 12 (Idempotency)

Let $x_i = x$ for all $i=1,2,\cdots,n$, then

$$IGGWHM^{p,q}\left(x_1,x_2,\cdots,x_n\right)=x. \tag{41}$$

Theorem 13 (Monotonicity)

Let $\left(x_1,x_2,\cdots,x_n\right)$ and $\left(y_1,y_2,\cdots,y_n\right)$ be two collections of the nonnegative numbers, if $x_i \le y_i$ for all $i=1,2,\cdots,n$, then

$$IGGWHM^{p,q}\left(x_1,x_2,\cdots,x_n\right)\le IGGWHM^{p,q}\left(y_1,y_2,\cdots,y_n\right). \tag{42}$$

Theorem 14 (Boundary)

Let $\left(x_1,x_2,\cdots,x_n\right)$ be a set of the nonnegative numbers, if $x_{\min}=\min(x_1,x_2,\ldots,x_n)$ and $x_{\max}=\max(x_1,x_2,\ldots,x_n)$, then

$$x_{\min}\le IGGWHM^{p,q}\left(x_1,x_2,\cdots,x_n\right)\le x_{\max} \tag{43}$$

In the following, we can analyze some special cases of the $IGGWHM^{p,q}$ operator

(1) When $q=0$, then

$$IGGWHM^{p,0}\left(x_1,x_2,\cdots,x_n\right)=\prod_{i=1}^{n}\prod_{j=i}^{n}\left(x_i\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}. \tag{44}$$

From here we see that $IGGWHM^{p,0}$ does not have any relationship with p.

(2) When $p=0$, then

$$IGGWHM^{0,q}\left(x_1,x_2,\cdots,x_n\right)=\prod_{i=1}^{n}\prod_{j=i}^{n}\left(x_j\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}. \tag{45}$$

Similarly, $IGGWHM^{0,q}$ does not have any relationship with q.

(3) When $p=q=1$, then

$$IGGWHM^{1,1}\left(x_1,x_2,\cdots,x_n\right)=\frac{1}{2}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(x_i+x_j\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}. \tag{46}$$

3. Some Heronian mean operators based on the normal neutrosophic numbers

In this section, we combine the IGWHM and IGGWHM operators with the normal neutrosophic numbers, and propose the normal neutrosophic number improved generalized weighted Heronian mean (NNNIGWHM) operator and the normal neutrosophic number improved generalized geometric weighted Heronian mean (NNNIGGWHM) operator.

3.1 The NNNIGWHM operator

Definition 17. Let $p,q \geq 0$, and $\tilde{a}_i = \langle (a_i, \sigma_i), (T_i, I_i, F_i) \rangle$ $(i = 1,2,\cdots,n)$ be a set of the normal neutrosophic numbers. $w = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of $\tilde{a}_i (i = 1,2,\cdots,n)$ satisfying $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$. If

$$NNNIGWHM^{p,q}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j \tilde{a}_i^{p} \tilde{a}_j^{q}}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j} \right)^{\frac{1}{p+q}} \tag{47}$$

$NNNIGWHM^{p,q}$ is called the normal neutrosophic number improved generalized weighted Heronian mean (NNNIGWHM) operator.

Theorem 15. Let $p,q \geq 0$, and $\tilde{a}_i = \langle (a_i, \sigma_i), (T_i, I_i, F_i) \rangle$ $(i = 1,2,\cdots,n)$ be a set of normal neutrosophic numbers. $w = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of $\tilde{a}_i (i = 1,2,\cdots,n)$,and satisfies $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$. then, the result aggregated from Definition 17 is still a NNN, and

$$NNNIGWHM^{p,q}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j \tilde{a}_i^{p} \tilde{a}_j^{q}}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j} \right)^{\frac{1}{p+q}}$$

$$= \left\langle \left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j a_i^{p} a_j^{q}}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j} \right)^{\frac{1}{p+q}}, (\frac{1}{p+q})^{\frac{1}{2}} \left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j a_i^{p} a_j^{q}}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j} \right)^{\frac{1}{p+q}-1} \left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j a_i^{p} a_j^{q} \sqrt{\frac{p\sigma_i^2}{a_i^2} + \frac{q\sigma_j^2}{a_j^2}}}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j} \right) \right), \right.$$

$$\left\langle \left(1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i^{p}T_j^{q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}, 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_i)^{p}(1-I_j)^{q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}, \right.\right.$$

$$\left.\left. 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-F_i)^{p}(1-F_j)^{q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}} \right)\right\rangle \quad (48)$$

Proof.

Since

$$\tilde{a}_i^{\,p} = \left\langle \left(a_i^{p}, p^{\frac{1}{2}} a_i^{p-1}\sigma_i \right), \left(T_i^{p}, 1-(1-I_i)^{p}, 1-(1-F_i)^{p}\right)\right\rangle$$

$$\tilde{a}_j^{\,q} = \left\langle \left(a_j^{q} . q^{\frac{1}{2}} a_j^{q-1}\sigma_j \right), \left(T_j^{q}, 1-(1-I_j)^{q}, 1-(1-F_j)^{q}\right)\right\rangle$$

and

$$\tilde{a}_i^{p}\tilde{a}_j^{q} = \left\langle \left(a_i^{p}a_j^{q}, a_i^{p}a_j^{q}\sqrt{\frac{p\sigma_i^2}{a_i^2}+\frac{q\sigma_j^2}{a_j^2}} \right), \left(T_i^{p}T_j^{q}, 1-(1-I_i)^{p}(1-I_j)^{q}, 1-(1-F_i)^{p}(1-F_j)^{q}\right)\right\rangle$$

$$w_iw_j\tilde{a}_i^{\,p}\tilde{a}_j^{\,q} = \left\langle \left(w_iw_ja_i^{p}a_j^{q}, w_iw_ja_i^{p}a_j^{q}\sqrt{\frac{p\sigma_i^2}{a_i^2}+\frac{q\sigma_j^2}{a_j^2}} \right), \right.$$

$$\left. \left(1-\left(1-T_i^{p}T_j^{q}\right)^{w_iw_j}, \left(1-(1-I_i)^{p}(1-I_j)^{q}\right)^{w_iw_j}, \left(1-(1-F_i)^{p}(1-F_j)^{q}\right)^{w_iw_j}\right)\right\rangle$$

then

$$\sum_{i=1}^{n}\sum_{j=1}^{n} w_iw_j\tilde{a}_i^{\,p}\tilde{a}_j^{\,q} = \left\langle \left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_iw_ja_i^{p}a_j^{q}, \sum_{i=1}^{n}\sum_{j=i}^{n} w_iw_ja_i^{p}a_j^{q}\sqrt{\frac{p\sigma_i^2}{a_i^2}+\frac{q\sigma_j^2}{a_j^2}} \right), \right.$$

$$\left. \left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i^{p}T_i^{q}\right)^{w_iw_j}, \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_i)^{p}(1-I_j)^{q}\right)^{w_iw_j}, \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-F_i)^{p}(1-F_j)^{q}\right)^{w_iw_j}\right)\right\rangle$$

$$\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_iw_j\tilde{a}_i^{\,p}\tilde{a}_j^{\,q}}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_iw_j} = \left\langle \left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_iw_ja_i^{p}a_j^{q}}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_iw_j}, \frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_iw_ja_i^{p}a_j^{q}\sqrt{\frac{p\sigma_i^2}{a_i^2}+\frac{q\sigma_j^2}{a_j^2}}}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_iw_j} \right), \right.$$

$$\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i^{\,p}T_j^{\,q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}},\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_i)^p(1-I_j)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}},\right.$$

$$\left.\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-F_i)^p(1-F_j)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)\right\rangle$$

and

$$\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j\tilde{a}_i^{\,p}\tilde{a}_j^{\,q}}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{p+q}}=\left\langle\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_ja_i^pa_j^q}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{p+q}},(\frac{1}{p+q})^{\frac{1}{2}}\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_ja_i^pa_j^q}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{p+q}-1}\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_ja_i^pa_j^q\sqrt{\frac{p\sigma_i^2}{a_i^2}+\frac{q\sigma_j^2}{a_j^2}}}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)\right)\right.$$

$$\left(1-\left(\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i^{\,p}T_j^{\,q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}},1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_i)^p(1-I_j)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}},\right.$$

$$\left.\left.1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-F_i)^p(1-F_j)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}\right)\right\rangle$$

So, we can get

$$NNNIGWHM^{p,q}\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j\tilde{a}_i^{\,p}\tilde{a}_j^{\,q}}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{p+q}}$$

$$=\left\langle\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_ja_i^pa_j^q}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{p+q}},(\frac{1}{p+q})^{\frac{1}{2}}\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_ja_i^pa_j^q}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{p+q}-1}\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_ja_i^pa_j^q\sqrt{\frac{p\sigma_i^2}{a_i^2}+\frac{q\sigma_j^2}{a_j^2}}}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)\right)\right.$$

$$\left(1-\left(\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i^{\,p}T_j^{\,q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}},1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_i)^p(1-I_j)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}},\right.$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-F_i)^p(1-F_j)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}\Bigg)\Bigg\rangle$$

Which completes the proof of the theorem 15.

Theorem 16. (Idempotency) .

Let all $\tilde{a}_i=\tilde{a}$ for all i $(i=1,2,\cdots,n)$, then

$$NNNIGWHM^{p,q}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\tilde{a}.$$

Proof

Since all $\tilde{a}_i=\tilde{a}=\langle(a,\sigma),(T,I,F)\rangle$ for all i $(i=1,2,\cdots,n)$, then we have

$$NNNIGWHM^{p,q}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j\tilde{a}^p\tilde{a}^q}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{p+q}}$$

$$=\left\langle\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_ja^{p+q}}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{p+q}},\left(\frac{1}{p+q}\right)^{\frac{1}{2}}\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_ja^{p+q}}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{p+q}-1}\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_ja^{p+q}\sqrt{p+q}\,\frac{\sigma}{a}}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)\right),\right.$$

$$\left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T^{p+q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}},1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I)^{p+q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p-q}},\right.$$

$$\left.\left.1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-F)^{p+q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}\right)\right\rangle=\langle(a,\sigma),(T,I,F)\rangle$$

In the following, we will discuss some special cases of the NNNIGGWHM operator in regard to the parameters p and q .

(1) when $p=0$, then

$$NNNIGWHM^{0,q}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left\langle\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_ja_j^q}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{q}},(\frac{1}{q})^{\frac{1}{2}}\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_ja_j^q}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{q}-1}\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_ja_j^q\sqrt{\frac{q\sigma_j^2}{a_j^2}}}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)\right)\right.$$

$$\left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_j^{q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{q}},1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-I_j\right)^{q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{q}},\right.$$

$$\left.1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-F_j\right)^{q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{q}}\right)\Bigg\rangle \qquad (49)$$

(2) when $q=0$, then

$$NNNIGWHM^{p,0}\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\left\langle\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_ja_i^{p}}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{p}},(\frac{1}{p})^{\frac{1}{2}}\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_ja_i^{p}}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{p}-1}\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_ja_i^{p}\sqrt{\frac{p\sigma_i^2}{a_i^2}}}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)\right),\right.$$

$$\left(1-\left(\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i{}^{p}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p}},1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-I_i\right)^{p}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p}},\right.$$

$$\left.\left.1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-F_i\right)^{p}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p}}\right)\right\rangle \qquad (50)$$

(3) when $p=q=1$, then

$$NNNIGWHM^{1,1}\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\left\langle\left(\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_ja_i\,a_j}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)^{\frac{1}{2}},(\frac{1}{2})^{\frac{1}{2}}\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_ja_i\,a_j}\right)^{\frac{1}{2}}\left(\frac{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_ja_i\,a_j\sqrt{\frac{\sigma_i^2}{a_i^2}+\frac{\sigma_j^2}{a_j^2}}}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\right)\right)\right.$$

$$\left(1-\left(\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i\,T_j\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{2}},1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-I_i\right)\left(1-I_j\right)\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{2}},\right.$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-F_i)(1-F_j)\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_i w_j}}\right)^{\frac{1}{2}}\Bigg)\Bigg\rangle \tag{51}$$

3.2 The NNNIGGWHM operator

Definition 18. Let $p,q \geq 0$, and $\tilde{a}_i = \langle (a_i,\sigma_i),(T_i,I_i,F_i)\rangle$ $(i=1,2,\cdots,n)$ be a collection of single normal neutrosophic numbers. $w=(w_1,w_2,\cdots,w_n)^T$ is the weight vector of $\tilde{a}_i(i=1,2,\cdots,n)$, and satisfies $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$.If

$$NNNIGGWHM^{p,q}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(p\tilde{a}_i+q\tilde{a}_j\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}} \tag{52}$$

then $NNNIGGWHM^{p,q}$ is called the normal neutrosophic number improved generalized geometric weighted Heronian mean (NNNIGGWHM) operator.

Theorem 17. Let $p,q \geq 0$, and $\tilde{a}_i = \langle (a_i,\sigma_i),(T_i,I_i,F_i)\rangle$ $(i=1,2,\cdots,n)$ be a collection of normal neutrosophic numbers. $w=(w_1,w_2,\cdots,w_n)^T$ is the weight vector of $\tilde{a}_i(i=1,2,\cdots,n)$, and satisfies $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$. then the result aggregated from Definition 18 is still a NNN, then the aggregated value by (52) can be expressed as

$$NNNIGGWHM^{p,q}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(p\tilde{a}_i+q\tilde{a}_j\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}$$

$$=\left\langle\left(\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(pa_i+qa_j\right)^H,\frac{1}{p+q}\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(pa_i+qa_j\right)^H\right)\sqrt{\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{H(p\sigma_i+q\sigma_j)^2}{\left(pa_i+qa_j\right)^2}}\right),\right.$$

$$\left.\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-T_i)^p(1-T_j)^q\right)^H\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-I_i^p I_j^q\right)^H\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-F_i^p F_j^q\right)^H\right)^{\frac{1}{p+q}}\right)\right\rangle \tag{53}$$

Where

$$H=\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k} \tag{54}$$

Proof.

According to the operational rules of NNNs, we have

$$p\tilde{a}_i = \left\langle (pa_i,p\sigma_i),(1-(1-T_i)^p, I_i^p, F_i^p)\right\rangle$$

$$q\tilde{a}_j=\left\langle\left(qa_j,q\sigma_j\right),(1-(1-T_j)^q,I_j^q,F_j^q)\right\rangle$$

and

$$p\tilde{a}_i+q\tilde{a}_j=\left\langle\left(pa_i+qa_j,p\sigma_i+q\sigma_j\right),(1-(1-T_i)^p(1-T_j)^q,I_i^pI_j^q,F_i^pF_j^q)\right\rangle$$

Due to $H=\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}$

$$\left(p\tilde{a}_i+q\tilde{a}_j\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}=\left(p\tilde{a}_i+q\tilde{a}_j\right)^H=\left\langle\left(\left(pa_i+qa_j\right)^H,H^{\frac{1}{2}}\left(pa_i+qa_j\right)^{H-1}\left(p\sigma_i+q\sigma_j\right)\right),\right.$$

$$\left.\left(\left(1-(1-T_i)^p(1-T_j)^q\right)^H,1-\left(1-I_i^pI_j^q\right)^H,1-\left(1-F_i^pF_j^q\right)^H\right)\right\rangle$$

and

$$\prod_{i=1}^{n}\prod_{j=i}^{n}\left(p\tilde{a}_i+q\tilde{a}_j\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}=\left\langle\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(pa_i+qa_j\right)^H,\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(pa_i+qa_j\right)^H\right)\sqrt{\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{H(p\sigma_i+q\sigma_j)^2}{(pa_i+qa_j)^2}}\right),\right.$$

$$\left.\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-T_i)^p(1-T_j)^q\right)^H,1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-I_i^pI_j^q\right)^H,1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-F_i^pF_j^q\right)^H\right)\right\rangle$$

So

$$\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(p\tilde{a}_i+q\tilde{a}_j\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}=\left\langle\left(\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(pa_i+qa_j\right)^H,\right.\right.$$

$$\left.\frac{1}{p+q}\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(pa_i+qa_j\right)^H\right)\sqrt{\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{H(p\sigma_i+q\sigma_j)^2}{(pa_i+qa_j)^2}}\right),$$

$$\left.\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-T_i)^p(1-T_j)^q\right)^H\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-I_i^pI_j^q\right)^H\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-F_i^pF_j^q\right)^H\right)^{\frac{1}{p+q}}\right)\right\rangle$$

So, we complete the proof that

$$NNNIGGWHM^{p,q}\left(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n\right)=\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(p\tilde{a}_i+q\tilde{a}_j\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}$$

$$=\left\langle\left(\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(pa_i+qa_j\right)^H,\frac{1}{p+q}\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(pa_i+qa_j\right)^H\right)\sqrt{\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{H(p\sigma_i+q\sigma_j)^2}{\left(pa_i+qa_j\right)^2}}\right),\right.$$

$$\left.\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-T_i\right)^p\left(1-T_j\right)^q\right)^H\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-I_i^pI_j^q\right)^H\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-F_i^pF_j^q\right)^H\right)^{\frac{1}{p+q}}\right)\right\rangle$$

Moreover, the NNNIGGWHM operator has the following properties.

Theorem 18. (Reducibility) .

Suppose $w=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)^{T}$, then

$$NNNIGGWHM^{p,q}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=NNNGGHM^{p,q}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$$

Proof. Since $w=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)^{T}$,then according to (53), we have

$$H=\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}=\frac{2}{n(n+1)}$$

$$NNNIGGWHM^{p,q}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left\langle\left(\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(pa_i+qa_j\right)^{H},\frac{1}{p+q}\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(pa_i+qa_j\right)^{H}\right)\sqrt{\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{H(p\sigma_i+q\sigma_j)^2}{\left(pa_i+qa_j\right)^2}}\right),\right.$$

$$\left.\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-T_i)^p(1-T_j)^q\right)^{H}\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-I_i^pI_j^q\right)^{H}\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-F_i^pF_j^q\right)^{H}\right)^{\frac{1}{p+q}}\right)\right\rangle$$

$$=\left\langle\left(\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(pa_i+qa_j\right)^{\frac{2}{n(n+1)}},\frac{1}{p+q}\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(pa_i+qa_j\right)^{\frac{2}{n(n+1)}}\right)\sqrt{\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{\frac{2}{n(n+1)}(p\sigma_i+q\sigma_j)^2}{\left(pa_i+qa_j\right)^2}}\right),\right.$$

$$\left.\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-T_i)^p(1-T_j)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-I_i^pI_j^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-F_i^pF_j^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\right)\right\rangle$$

$$=NNNGGHM^{pq}(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n)$$

Theorem 19. (Idempotency) .

Let all $\tilde{a}_i=a=\langle(a,\sigma),(T,I,F)\rangle$ for all i $(i=1,2,\cdots,n)$, then

$$NNNIGGWHM^{p,q}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\tilde{a}.$$

Proof. since $\tilde{a}=\langle(a,\sigma),(T,I,F)\rangle$ $(i=1,2,\cdots,n)$, then, we have

$$NNNIGGWHM^{p,q}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left(p\tilde{a}+q\tilde{a}\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}$$

$$=\left\langle\left(\frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}(pa+qa)^{H},\frac{1}{p+q}\left(\prod_{i=1}^{n}\prod_{j=i}^{n}(pa+qa)^{H}\right)\sqrt{\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{H(p\sigma+q\sigma)^{2}}{(pa+qa)^{2}}}\right),\right.$$

$$\left.\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-T)^{p+q}\right)^{H}\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-I^{p+q}\right)^{H}\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-F^{p+q}\right)^{H}\right)^{\frac{1}{p+q}}\right)\right\rangle$$

$$=\left\langle\left(\frac{1}{p+q}(pa+qa)^{\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=1}^{n}w_k}},\frac{1}{p+q}(pa+qa)^{\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=1}^{n}w_k}}\sqrt{\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=1}^{n}w_k}\sigma^{2}}{a^{2}}}\right),\right.$$

$$\left.\left(1-\left(1-\left(1-(1-T)^{p+q}\right)^{\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=1}^{n}w_k}}\right)^{\frac{1}{p+q}},\left(1-\left(1-I^{p+q}\right)^{\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=1}^{n}w_k}}\right)^{\frac{1}{p+q}},\left(1-\left(1-F^{p+q}\right)^{\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=1}^{n}w_k}}\right)^{\frac{1}{p+q}}\right)\right\rangle$$

$$=\left\langle\left(\frac{1}{p+q}(pa+qa)^{\sum_{i=1}^{n}\frac{2(n+1-i)}{n(n+1)}},\frac{1}{p+q}\frac{\sigma}{a}(pa+qa)^{\sum_{i=1}^{n}\frac{2(n+1-i)}{n(n+1)}}\sqrt{\sum_{i=1}^{n}\frac{2(n+1-i)}{n(n+1)}}\right),\right.$$

$$\left.\left(1-\left(1-\left(1-(1-T)^{p+q}\right)^{\sum_{i=1}^{n}\frac{2(n+1-i)}{n(n+1)}}\right)^{\frac{1}{p+q}},\left(1-\left(1-I^{p+q}\right)^{\sum_{i=1}^{n}\frac{2(n+1-i)}{n(n+1)}}\right)^{\frac{1}{p+q}},\left(1-\left(1-F^{p+q}\right)^{\sum_{i=1}^{n}\frac{2(n+1-i)}{n(n+1)}}\right)^{\frac{1}{p+q}}\right)\right\rangle$$

$$=\left\langle\left(\frac{1}{p+q}(pa+qa),\frac{1}{p+q}\frac{\sigma}{a}(pa+qa)\right),\left(1-\left(1-\left(1-(1-T)^{p+q}\right)\right)^{\frac{1}{p+q}},\left(1-\left(1-I^{p+q}\right)\right)^{\frac{1}{p+q}},\left(1-\left(1-F^{p+q}\right)\right)^{\frac{1}{p+q}}\right)\right\rangle$$

$$=\langle(a,\sigma),(T,I,F)\rangle=\tilde{a}$$

Which complete the proof of the theorem 19.

We will discuss some special cases of the NNNIGGWHM operator according to the parameters p and q.

(1) when $p=0$, then

$$NNNIGGWHM^{0,q}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left\langle\left(\frac{1}{q}\prod_{i=1}^{n}\prod_{j=i}^{n}(qa_j)^{H},\frac{1}{p}\prod_{i=1}^{n}\prod_{j=i}^{n}(qa_j)^{H}\sqrt{\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{H(\sigma_j)^{2}}{(a_j)^{2}}}\right),\right.$$

$$\left.\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-T_j)^{q}\right)^{H}\right)^{\frac{1}{q}},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-I_j^{q}\right)^{H}\right)^{\frac{1}{q}},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-F_j^{q}\right)^{H}\right)^{\frac{1}{q}}\right)\right\rangle \quad (55)$$

(2) when $q=0$, then

$$NNNIGGWHM^{p,0}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left\langle\left(\frac{1}{p}\prod_{i=1}^{n}\prod_{j=i}^{n}(pa_i)^H,\frac{1}{p}\left(\prod_{i=1}^{n}\prod_{j=i}^{n}(pa_i)^H\right)\sqrt{\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{H\sigma_i^2}{a_i^2}}\right),\right.$$

$$\left.\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-T_i)^p\right)^H\right)^{\frac{1}{p}},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-I_i^{\,p}\right)^H\right)^{\frac{1}{p}},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-F_i^{\,p}\right)^H\right)^{\frac{1}{p}}\right)\right\rangle$$

$$=\left\langle\left(\frac{1}{p}\prod_{i=1}^{n}(pa_i)^{\frac{2(n+1-i)}{n(n+1)}},\frac{1}{p}\left(\prod_{i=1}^{n}(pa_i)^{\frac{2(n+1-i)}{n(n+1)}}\right)\sqrt{\sum_{i=1}^{n}\frac{2(n+1-i)}{n(n+1)}\frac{\sigma_i^2}{a_i^2}}\right),\right.$$

$$\left.\left(1-\left(1-\prod_{i=1}^{n}\left(1-(1-T_i)^p\right)^{\frac{2(n+1-i)}{n(n+1)}}\right)^{\frac{1}{p}},\left(1-\prod_{i=1}^{n}\left(1-I_i^{\,p}\right)^{\frac{2(n+1-i)}{n(n+1)}}\right)^{\frac{1}{p}},\left(1-\prod_{i=1}^{n}\left(1-F_i^{\,p}\right)^{\left(1-F_i^{\,p}\right)}\right)^{\frac{1}{p}}\right)\right\rangle \quad (56)$$

Obviously, when $q=0$, $NNNIGGWHM^{p,0}$ does not have any relationship with w

(3)when $q=p=1$, then

$$NNNIGGWHM^{1,1}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left\langle\left(\frac{1}{2}\prod_{i=1}^{n}\prod_{j=i}^{n}(a_i+q_j)^H,\frac{1}{2}\left(\prod_{i=1}^{n}\prod_{j=i}^{n}(a_i+a_j)^H\right)\sqrt{\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{H(\sigma_i+\sigma_j)^2}{(a_i+a_j)^2}}\right),\right.$$

$$\left.\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-T_i)(1-T_j)\right)^H\right)^{\frac{1}{2}},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-I_iI_j\right)^H\right)^{\frac{1}{2}},\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-F_iF_j\right)^H\right)^{\frac{1}{2}}\right)\right\rangle \quad (57)$$

4. The group decision-making methods based on the NNNIGWHM operator and NNNIGGWHM operator

In this part, we use the normal neutrosophic number improved generalized weighted Heronian mean operator and the normal neutrosophic number improved generalized geometric weighted Heronian mean operator to deal with the multiple attribute group decision making (MAGDM) problems in which the attribute values take the form of NNNs.

For a multiple attribute decision making problem, suppose we have known that $A=\{A_1,A_2,\cdots,A_m\}$ is the collection of alternatives, $C=\{C_1,C_2,\cdots,C_n\}$ is the collection of attributes, and $e=\{e_1,e_2,\cdots,e_d\}$ is the collection of decision makers. Supposed the attributes are independent of each other, and the evaluation of the alternative A_i with respect to the criterion C_j given by the decision maker e_t is $r_{ij}^t=\left\langle(a_{ij}^t,\sigma_{ij}^t),(T_{ij}^t,I_{ij}^t,F_{ij}^t)\right\rangle$ which is represented by the form of NNN, where $T_{ij}^t,I_{ij}^t,F_{ij}^t\in[0,1]$ and $T_{ij}^t+I_{ij}^t+F_{ij}^t\le 3$. The weight vector of the criteria is

$w=(w_1,w_2,\cdots,w_n)$, which satisfying $w_j \in [0,1], \sum_{j=1}^{n} w_j = 1$. Let $\omega = (\omega_1,\omega_2,...,\omega_d)$ be the vector of decision makers, and $\omega_t \in [0,1], \sum_{t=1}^{d} \omega_t = 1$

Then, we give the steps of decision making method based on the NNNIGWHM operator and NNNIGGWHM operator. So, the procedures are shown as follows:

Step 1. Utilize the NNNIGWHM operator

$$r_i^t = \left\langle \left(a_i^t, \sigma_i^t\right), \left(T_i^t, I_i^t, F_i^t\right) \right\rangle = NNNIGWHM\left(r_{i1}^t, r_{i2}^t, \ldots, r_{in}^t\right) \tag{58}$$

or NNNIGGWHM operator

$$r_i^t = \left\langle \left(a_i^t, \sigma_i^t\right), \left(T_i^t, I_i^t, F_i^t\right) \right\rangle = NNNIGGWHM\left(r_{i1}^t, r_{i2}^t, \ldots, r_{in}^t\right) \tag{59}$$

to get the comprehensive attribute values for each decision-maker of each alternative.

Step 2. Utilize the NNNIGWHM operator

$$r_i = \left\langle \left(a_i, \sigma_i\right), \left(T_i, I_i, F_i\right) \right\rangle = NNNIGWHM\left(r_i^1, r_i^2, ..., r_i^d\right) \tag{60}$$

or NNNIGGWHM operator

$$r_i = \left\langle \left(a_i, \sigma_i\right), \left(T_i, I_i, F_i\right) \right\rangle = NNNIGGWHM\left(r_i^1, r_i^2, ..., r_i^d\right) \tag{61}$$

to get the collective values for each alternative.

Step 3. Calculate the value $s_1(r_i), s_2(r_i), h_1(r_i), h_2(r_i)$ of r_i.

Step 4. Rank all the alternatives $\{A_1, A_2, ..., A_m\}$ according to the theorem 1.

5. A numerical example

In this section, we provide an example to illustrate the application of NNNIGWHM and NNNIGGWHM operators. Suppose that an investment company wants to choose a company as the partner. There are four companies A_i $(i=1,2,3,4)$ evaluated by three decision makers $\{D_1, D_2, D_3\}$.

The weight vector of the decision makers is $\lambda = (0.314, 0.355, 0.331)^T$, and the attributes include: C_1 (the risk index) , C_2 (the growth index), and C_3 (the social-political impact index). Suppose the attribute weight vector is $w = (0.4, 0.20, 0.40)^T$. The three decision makers $\{D_1, D_2, D_3\}$ evaluate the four companies A_i $(i=1,2,3,4)$ with respect to the attributes C_j $(j=1,2,3)$. The values of evaluation information can be expressed by NNNs. According to the evaluation from three decision makers, we construct three decision matrices $R^t = [r_{ij}^{\,t}]_{4\times 3}$ $(t=1,2,3)$ which are listed in Tables 1-3.

Table 1. Decision matrix R^1.

	C1	C2	C3
A1	<(3,0.4),(0.265,0.350,0.385)>	<(7,0.6),(0.330,0.390,0.280)>	<(7,0.6),(0.245,0.275,0.480)>

A2	<(4,0.2),(0.345,0.245,0.410)>	<(8,0.4),(0.430,0.290,0.280)>	<(5,0.3),(0.245,0.375,0.380)>
A3	<(3,0.2),(0.365,0.300,0.335)>	<(6,0.2),(0.480,0.315,0.205)>	<(6,0.4),(0.340,0.370,0.290)>
A4	<(5,0.5),(0.430,0.300,0.270)>	<(7,0.5),(0.460,0.245,0.295)>	<(7,0.2),(0.310,0.520,0.170)>

Table 2. Decision matrix R^2.

	C1	C2	C3
A1	<(3,0.4),(0.125,0.470,0.405)>	<(5,0.4),(0.220,0.420,0.360)>	<(5,0.3),(0.345,0.490,0.165)>
A2	<(4,0.3),(0.355,0.315,0.330)>	<(6,0.7),(0.300,0.370,0.330)>	<(7,0.6),(0.205,0.630,0.165)>
A3	<(3,0.2),(0.315,0.380,0.305)>	<(5,0.6),(0.330,0.565,0.105)>	<(7,0.2),(0.280,0.520,0.200)>
A4	<(5,0.5),(0.365,0.365,0.270)>	<(4,0.5),(0.355,0.320,0.325)>	<(6,0.4),(0.425,0.485,0.090)>

Table 3. Decision matrix R^3.

	C1	C2	C3
A1	<(3,0.4),(0.260,0.425,0.315)>	<(7,0.6),(0.220,0.450,0.330)>	<(5,0.4),(0.255,0.500,0.245)>
A2	<(4,0.2),(0.270,0.370,0.360)>	<(8,0.4),(0.320,0.215,0.465)>	<(6,0.7),(0.135,0.575,0.290)>
A3	<(4,0.5),(0.245,0.465,0.290)>	<(6,0.2),(0.250,0.570,0.180)>	<(5,0.6),(0.175,0.660,0.165)>
A4	<(5,0.6),(0.390,0.340,0.270)>	<(7,0.5),(0.305,0.475,0.220)>	<(7,0.5),(0.465,0.485,0.050)>

5.1 The MAGDM method based on NNNIGWHM operator

(1) Calculate the comprehensive evaluation values r_i^t $(i=1,2,3,4;t=1,2,3)$ of each decision maker by formula (58) of the NNNIGWHM operator (suppose $q=p=1$),we can get

$r_1^1=\langle(5.450,0.522),(0.269,0.324,0.401)\rangle$, $r_2^1=\langle(5.156,0.279),(0.322,0.303,0.373)\rangle$,

$r_3^1=\langle(4.826,0.289),(0.330,0.291,0.376)\rangle$, $r_4^1=\langle(6.198,0.387),(0.369,0.229,0.388)\rangle$,

$r_1^2=\langle(4.208,0.361),(0.469,0.283,0.240)\rangle$, $r_2^2=\langle(5.636,0.502),(0.438,0.253,0.287)\rangle$,

$r_3^2=\langle(5.093,0.281),(0.466,0.218,0.303)\rangle$, $r_4^2=\langle(5.247,0.461),(0.403,0.188,0.389)\rangle$,

$r_1^3=\langle(4.583,0.438),(0.459,0.287,0.251)\rangle$, $r_2^3=\langle(5.573,0.461),(0.412,0.347,0.227)\rangle$,

$r_3^3=\langle(4.777,0.501),(0.558,0.214,0.218)\rangle$, $r_4^3=\langle(6.198,0.543),(0.420,0.144,0.408)\rangle$,

(2) Calculate the collective overall values r_i $(i=1,2,3,4)$ by formula (60) of the NNNIGWHM operator (suppose $q=p=1$), we can get

$r_1=\langle(4.678,0.433),(0.250,0.422,0.323)\rangle$, $r_2=\langle(5.416,0.419),(0.276,0.389,0.323)\rangle$

$r_3 = \langle (4.860, 0.358), (0.298, 0.453, 0.243) \rangle$, $r_4 = \langle (5.804, 0.463), (0.391, 0.402, 0.189) \rangle$

(3)Calculate the score functions $s_1(r_i)$, $s_2(r_i)$ and the accuracy function $h_1(r_i)$, $h_2(r_i)$ for all i (i=1,2,3,4) by formulas (17-18), we can get

$s_1(r_1) = 7.041$, $s_1(r_2) = 8.470$, $s_1(r_3) = 7.787$, $s_1(r_4) = 10.451$

$s_2(r_1) = 0.652$, $s_2(r_2) = 0.656$, $s_2(r_3) = 0.573$, $s_2(r_4) = 0.833$

$h_1(r_1) = 10.062$, $h_1(r_2) = 11.968$, $h_1(r_3) = 10.145$, $h_1(r_4) = 12.639$

$h_2(r_1) = 0.932$, $h_2(r_2) = 0.927$, $h_2(r_3) = 0.746$, $h_2(r_4) = 1.008$

(4) According to the score functions $s_1(r_i)$ $(i = 1,2,3,4)$, we can rank the alternatives A_1, A_2, A_3, A_4 show as follows

$$A_4 \succ A_2 \succ A_3 \succ A_1$$

So, we get the best alternatives is A_4 .

5.2 The MAGDM method based on NNNIGGWHM operator

(1)Calculate the comprehensive evaluation values r_i^t $(i = 1, 2, 3, 4; t = 1, 2, 3)$ of each decision maker by formula (59) of the NNNIGGWHM operator (suppose $q = p = 1$),we can get

$r_1^1 = \langle (5.342, 0.540), (0.274, 0.333, 0.394) \rangle$, $r_2^1 = \langle (5.325, 0.286), (0.326, 0.308, 0.365) \rangle$,

$r_3^1 = \langle (4.791, 0.279), (0.385, 0.331, 0.284) \rangle$, $r_4^1 = \langle (6.249, 0.425), (0.390, 0.373, 0.240) \rangle$,

$r_1^2 = \langle (4.222, 0.389), (0.221, 0.464, 0.307) \rangle$, $r_2^2 = \langle (5.576, 0.512), (0.279, 0.459, 0.270) \rangle$,

$r_3^2 = \langle (4.863, 0.342), (0.306, 0.488, 0.217) \rangle$, $r_4^2 = \langle (5.071, 0.477), (0.384, 0.400, 0.220) \rangle$,

$r_1^3 = \langle (4.667, 0.466), (0.247, 0.461, 0.293) \rangle$, $r_2^3 = \langle (5.700, 0.452), (0.225, 0.413, 0.364) \rangle$,

$r_3^3 = \langle (4.880, 0.495), (0.218, 0.572, 0.216) \rangle$, $r_4^3 = \langle (6.249, 0.557), (0.393, 0.435, 0.178) \rangle$,

(2) Calculate the collective overall values r_i $(i = 1,2,3,4)$ by formula (61) of the NNNIGGWHM operator (suppose $q = p = 1$), we can get

$r_1 = \langle (4.723, 0.462), (0.247, 0.422, 0.332) \rangle$, $r_2 = \langle (5.533, 0.422), (0.275, 0.397, 0.333) \rangle$

$r_3 = \langle (4.845, 0.378), (0.298, 0.470, 0.239) \rangle$, $r_4 = \langle (5.829, 0.490), (0.389, 0.403, 0.213) \rangle$

(3)Calculate the score functions $s_1(r_i)$, $s_2(r_i)$ and the accuracy function $h_1(r_i)$, $h_2(r_i)$ for all i (i=1,2,3,4) by formula (17-18), we can get

$s_1(r_1) = 7.047$, $s_1(r_2) = 8.549$, $s_1(r_3) = 7.696$, $s_1(r_4) = 10.333$

$s_2(r_1) = 0.689$, $s_2(r_2) = 0.652$, $s_2(r_3) = 0.600$, $s_2(r_4) = 0.869$

$h_1(r_1) = 10.186$, $h_1(r_2) = 12.238$, $h_1(r_3) = 10.015$, $h_1(r_4) = 12.820$

$h_2(r_1) = 0.997$, $h_2(r_2) = 0.934$, $h_2(r_3) = 0.781$, $h_2(r_4) = 1.078$

(4) According to the score functions $s_1(r_i)$ $(i = 1,2,3,4)$, we can rank the alternatives A_1, A_2, A_3, A_4 show as follows

$$A_4 \succ A_2 \succ A_3 \succ A_1$$

So, we get the best alternatives is A_4 .

5.3 Analysis on the effect of the parameters p, q

Table 4 Ordering of the alternatives by the different parameters *p* and *q* in NNNIGWHM operator

p, q	$s_1(r_i)$	ranking
$p=0, q=1$	$s_1(r_1)=7.376, s_1(r_2)=8.635$ $s_1(r_3)=7.790, s_1(r_4)=10.029$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=0, q=10$	$s_1(r_1)=9.917, s_1(r_2)=11.626$ $s_1(r_3)=10.507, s_1(r_4)=12.725$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=1, q=0$	$s_1(r_1)=6.210, s_1(r_2)=8.047$ $s_1(r_3)=7.052, s_1(r_4)=9.775$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=10, q=0$	$s_1(r_1)=9.745, s_1(r_2)=11.534$ $s_1(r_3)=10.289, s_1(r_4)=11.715$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=1, q=1$	$s_1(r_1)=7.041, s_1(r_2)=8.470$ $s_1(r_3)=7.787, s_1(r_4)=10.451$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=1, q=10$	$s_1(r_1)=9.635, s_1(r_2)=11.380$ $s_1(r_3)=10.308, s_1(r_4)=12.502$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=10, q=1$	$s_1(r_1)=9.758, s_1(r_2)=11.394$ $s_1(r_3)=10.285, s_1(r_4)=11.758$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=10, q=10$	$s_1(r_1)=10.430, s_1(r_2)=12.126$ $s_1(r_3)=10.979, s_1(r_4)=12.558$	$A_4 \succ A_2 \succ A_3 \succ A_1$

Table 5 Ordering of the alternatives by the different parameters *p* and *q* in NNNIGGWHM operator

p, q	$s_1(r_i)$	ranking
$p=0, q=1$	$s_1(r_1)=7.334, s_1(r_2)=8.362$ $s_1(r_3)=7.799, s_1(r_4)=10.879$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=0, q=10$	$s_1(r_1)=6.720, s_1(r_2)=7.324$ $s_1(r_3)=7.127, s_1(r_4)=9.977$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=1, q=0$	$s_1(r_1)=6.337, s_1(r_2)=8.463$ $s_1(r_3)=7.111, s_1(r_4)=9.813$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=10, q=0$	$s_1(r_1)=7.334, s_1(r_2)=8.362$ $s_1(r_3)=6.406, s_1(r_4)=9.161$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=1, q=1$	$s_1(r_1)=7.047, s_1(r_2)=8.549$ $s_1(r_3)=7.696, s_1(r_4)=10.333$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=1, q=10$	$s_1(r_1)=6.732, s_1(r_2)=7.475$ $s_1(r_3)=7.167, s_1(r_4)=9.937$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=10, q=1$	$s_1(r_1)=6.110, s_1(r_2)=7.406$ $s_1(r_3)=6.550, s_1(r_4)=9.288$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=10, q=10$	$s_1(r_1)=6.403, s_1(r_2)=7.210$ $s_1(r_3)=6.793, s_1(r_4)=9.471$	$A_4 \succ A_2 \succ A_3 \succ A_1$

Obviously, in this example, the ranking of the alternatives is unchanged regardless of the parameters *p* and *q*. In general, we can use the parameters $p=1, q=1$ in real applications because it is simple and it can also consider the correlations of inputs.

6. Conclusions

The multiple attribute group decision making problems are an important research topic, and they have the wide applications in real world. However, the attribute values in MAGDM problems are often uncertain, and they are difficult to be expressed by crisp numbers. In this paper, we proposed the normal neutrosophic numbers (NNNs) which are the generalization of FS, IFS, NS, NFN, and so on, and it not only can handle incompleteness, indeterminacy and inconsistency of evaluation information but also can handle the information of normal distribution. Obviously, NNNs can provide an easier way to express the uncertain information. In addition, Heronian mean (HM) operator has the characteristic of capturing the correlations of the aggregated arguments, and the traditional HM operator cannot process the NNNs, So, we extend HM operator to process the information with NNNs, and propose some normal neutrosophic number Heronian mean (NNNHM) operators, including the normal neutrosophic number improved generalized weighted Heronian mean (NNNIGWHM) operator and the normal neutrosophic number improved generalized geometric weighted Heronian mean (NNNIGGWHM) operator. Furthermore, we propose two multiple attribute group decision making methods respectively based on the NNNIGWHM and NNNIGGWHM operators, which have the advantages that they can take the correlations of the aggregated attributes into consideration. Finally, we give an illustrative example to demonstrate the practicality and effectiveness of the two methods, and analyze the influence of the parameters p and q on the two orderings. In the further research, we will continue studying the applications of two new methods, or some new aggregation operators for NNNs, such as power operator and priority operator for NNNs, etc.

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