n-FOLD FILTERS IN SMARANDACHE RESIDUATED LATTICES, PART (I)

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Abstract. In this paper we introduce the notions of n-fold B_L-Smarandache positive implicateve filter and n-fold B_L-Smarandache implicateve filter in Smarandache residuated lattices and study the relations among them. And we also introduce the notions of n-fold Smarandache positive implicateve B_L-residuated lattice, n-fold Smarandache implicateve B_L-residuated lattice and investigate its properties.

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1. Introduction

BL-algebras (basic logic algebras) are the algebraic structures for Hájek basic logic [3], in order to investigate many valued logic by algebraic means. Residuated lattices play an important role in the study of fuzzy logic and filters are basic concepts in residuated lattices and other algebraic structures. A Smarandache structure on a set L means a weak structure W on L such that there exists a proper subset B of L which is embedded with a strong structure S. In [9], Vasantha Kandasamy studied the concept of Smarandache groupoids, subgroupoids, ideal of groupoids and strong Bol groupoids and obtained many interesting results about them. It will be very interesting to study the Smarandache structure in these algebraic structures. A BL-algebra is a weaker structure than residuated lattice, then we can consider in any residuated lattice a weaker structure as BL-algebra.

The concept of Smarandache residuated lattice, Smarandache (positive) implicative filters and Smarandache fantastic filters defined in [1]. In [4, 8] the authors defined the notion of n-fold (positive) implicative filters, n-fold fantastic filters, n-fold obstinate filters in BL-algebras and studied the relation among many type of n-fold filters in BL-algebra. The aim of this paper is to extend this research to Smarandache residuated lattices.

2. Preliminaries

[2] A residuated lattice is an algebra $L = (L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ equipped with an order $\leq$ satisfying the following:

(LR_1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,

(LR_2) $(L, \odot, 1)$ is a commutative ordered monoid,

(LR_3) $\odot$ and $\rightarrow$ form an adjoint pair i.e, $c \leq a \rightarrow b$ if and only if $a \odot c \leq b$ for all $a, b, c \in L$.

A BL-algebra is a residuated lattice L if satisfying the following identity, for all $a, b \in L$:

(BL_1) $(a \rightarrow b) \vee (b \rightarrow a) = 1$,

(BL_2) $a \wedge b = a \odot (a \rightarrow b)$.

Theorem 2.1. [2, 5, 6, 7] *Let L be a residuated lattice. Then the following properties hold, for all $x, y, z \in L$:*

(lr_1) $1 \rightarrow x = x$, $x \rightarrow x = 1$,

(lr_2) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y) \leq z \rightarrow (x \rightarrow y)$,

(lr_3) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$ and $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$,

(lr_4) $x \leq y \Leftrightarrow x \rightarrow y = 1$, $x \leq y \rightarrow x$,

(lr_5) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) = (x \odot y) \rightarrow z$,

(lr_6) $x \odot (x \rightarrow y) \leq y, x \leq y \rightarrow (x \odot y)$ and $y \leq (y \rightarrow x) \rightarrow x$,

(lr_7) If $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$, $z \rightarrow x \leq z \rightarrow y$ and $y^* \leq x^*$,

(lr_8) $x \leq y$ and $z \leq w$ then $x \odot z \leq y \odot w$,

(lr_9) $x \leq x^{**}$, $x^{***} = x^*$,

(lr_{10}) $x^* \odot y^* \leq (x \odot y)^*$ (so, $(x^*)^n \leq (x^n)^*$ for every $n \geq 1$),

(lr_{11}) $x^{**} \odot y^{**} \leq (x \odot y)^{**}$ (so, $(x^{**})^n \leq (x^n)^{**}$ for every $n \geq 1$).

The following definitions and theorems are stated from [1].

A Smarandache B_L-residuated lattice is a residuated lattice L in which there exists a proper subset B of L such that:

(1) $0, 1 \in B$ and $|B| > 2$,

(2) B is a BL- algebra under the operations of L.

From now on $L_B = (L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a Smarandache B_L-residuated lattice and $B = (B, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL- algebra unless otherwise specified.

Definition 2.2. A nonempty subset F of L_B is called a

(i) Smarandache deductive system of L_B related to B (or briefly B_L-Smarandache deductive system of L_B) if it satisfies:

(DB_1) $1 \in F$,

(DB_2) If $x \in F$, $y \in B$ and $x \rightarrow y \in F$ then $y \in F$.

(ii) Smarandache filter of L_B related to B (or briefly B_L-Smarandache filter of L_B) if satisfies:

(FB_1) if $x, y \in F$, then $x \odot y \in F$,

(FB_2) if $x \in F$, $y \in B$ and $x \leq y$, then $y \in F$.

Theorem 2.3. *(i) Let F be a B_L-Smarandache filter of L_B, then F is a B_L-Smarandache deductive system of L_B.*

(ii) Let F be a B_L-Smarandache deductive system of L_B. If $F \subseteq B$, then F is a B_L-Smarandache filter of L_B.

Definition 2.4. Let F be a subset of L_B and $1 \in F$.

- ⋇ F is called a Smarandache implicative filter of L_B related to B (or briefly B_L-Smarandache implicative filter of L_B) if $z \in F$, $x, y \in B$ and $z \to ((x \to y) \to x) \in F$, then $x \in F$,
- ⋇ F is called a Smarandache positive implicative filter of L_B related to B (or briefly B_L-Smarandache positive implicative filter of L_B) if $x, y, z \in B$, $z \to (x \to y) \in F$ and $z \to x \in F$, then $z \to y \in F$.

Now, unless mentioned otherwise, $n \geq 1$ will be an integer.

3. n-Fold B_L-Smarandache (Positive) Implicative Filters

Definition 3.1. A subset F of L_B is called an n-fold Smarandache positive implicative filter of L_B related to B (or briefly n-fold B_L-Smarandache positive implicative filter of L_B) if it satisfies in the following conditions:

(i) $1 \in F$,

(ii) for all $x, y, z \in B$, if $x^n \to (y \to z) \in F$ and $x^n \to y \in F$, then $x^n \to z \in F$.

Example 3.2. Let $L = \{0, a, b, c, d, 1\}$ be a residuated lattice such that $0 < a < c < d < 1$ and $0 < b < c < d < 1$. We define

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	c	1	1	1
b	c	c	1	1	1	1
c	c	c	c	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	a	a
b	0	0	0	0	b	b
c	0	0	0	0	c	c
d	0	a	b	c	d	d
1	0	a	b	c	d	1

We can see that $L = (L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice, in which $B = \{0, a, c, 1\}$ is a BL-algebra which properly contained in L. Then L is a Smarandache B_L-residuated lattice. $F = \{d, 1\}$ is an n-fold B_L-Smarandache positive implicative filter of L_B, for $n \geq 2$. But F is not 1-fold B_L-Smarandache positive implicative filter, since $a \to (a \to a^2) = 1 \in F$ and $a \to a = 1 \in F$, while $a \to a^2 = c \notin F$.

Theorem 3.3. *n-fold B_L-Smarandache positive implicative filters are B_L-Smarandache deductive systems.*

Proof. Let F be an n-fold B_L-Smarandache positive implicative filter of L_B. Suppose $z \in B$, such that $y, y \to z \in F$. We have $1^n \to y, 1^n \to (y \to z) \in F$, these imply $z = 1^n \to z \in F$. Hence F is a B_L-Smarandache deductive system of L_B.

□

Theorem 3.4. *Let F be a B_L-Smarandache deductive system of L_B. Then for all $x, y, z \in B$, the following conditions are equivalent:*

(i) F is an n-fold B_L-Smarandache positive implicative filter of L_B,

(ii) $x^n \to x^{2n} \in F$, for all $n \in N$,

(iii) if $x^{n+1} \to y \in F$, then $x^n \to y \in F$, for all $n \in N$,

(iv) if $x^n \to (y \to z) \in F$, then $(x^n \to y) \to (x^n \to z) \in F$, for all $n \in N$,

(v) $A_a = \{b \in B : a^n \to b \in F\}$ is a B_L-Smarandache deductive system of L_B, for any $a \in B$.

Proof. $(i) \Rightarrow (ii)$ We have $x^n \to (x^n \to x^{2n}) = x^n \odot x^n \to x^{2n} = x^{2n} \to x^{2n} = 1 \in F$ and $x^n \to x^n = 1 \in F$. Since F is an n-fold B_L-Smarandache positive implicative filter of L_B, we get $x^n \to x^{2n} \in F$.

$(ii) \Rightarrow (i)$ Let $x^n \to (y \to z) \in F$ and $x^n \to y \in F$. By Theorem 2.1 we have $x^n \odot (x^n \to (y \to z)) \leq y \to z$ and $x^n \odot (x^n \to y) \leq y$. Hence by (lr_8), $(x^n \odot (x^n \to (y \to z))) \odot (x^n \odot (x^n \to y)) \leq y \odot (y \to z)$. Then by (lr_6), we have $(x^n \to (y \to z)) \odot (x^n \to y) \odot x^{2n} \leq z$. So by (LR_3), we get $(x^n \to (y \to z)) \odot (x^n \to y) \leq x^{2n} \to z$, (I). Since $x^n \to (y \to z) \in F$ and $x^n \to y \in F$, then we get $(x^n \to (y \to z)) \odot (x^n \to y) \in F$. Hence by (I) we get $x^{2n} \to z \in F$. By Theorem 2.1, we have $x^n \to x^{2n} \leq (x^{2n} \to z) \to (x^n \to z)$. So by hypothesis we get $(x^{2n} \to z) \to (x^n \to z) \in F$. Then by the fact that $x^{2n} \to z \in F$, we obtain $x^n \to z \in F$. Therefore F is an n-fold B_L-Smarandache positive implicative filter of L_B.

$(i) \Rightarrow (iii)$ By Theorem 2.1 we have $x^n \to (x \to y) = x^{n+1} \to y \in F$ and $x^n \to x = 1 \in F$. So by (i) we get $x^n \to y \in F$.

$(iii) \Rightarrow (ii)$ We have $x^{n+1} \to (x^{n-1} \to x^{2n}) = x^{2n} \to x^{2n} = 1 \in F$. From this and the fact that (iii) holds, we also have $x^n \to (x^{n-1} \to x^{2n}) \in F$. But $x^{n+1} \to (x^{n-2} \to x^{2n}) = x^n \to (x^{n-1} \to x^{2n}) \in F$. From this and the fact that (iii) holds, we have $x^n \to (x^{n-2} \to x^{2n}) \in F$. By repeating the process n times, we get $x^n \to (x^{n-n} \to x^{2n}) = x^n \to x^{2n} \in F$.

$(ii) \Rightarrow (iv)$ Assume that $x^n \to (y \to z) \in F$. By Theorem 2.1 we have $y \to z \leq (x^n \to y) \to (x^n \to z)$ and

$$\begin{aligned} x^n \to (y \to z) &\leq x^n \to ((x^n \to y) \to (x^n \to z)); \\ &= x^n \to (x^n \to ((x^n \to y) \to z)); \\ &= x^{2n} \to ((x^n \to y) \to z). \end{aligned}$$

Hence $x^{2n} \to ((x^n \to y) \to z) \in F$. We get

$$\begin{aligned} x^{2n} \to ((x^n \to y) \to z) &\leq (x^n \to x^{2n}) \to (x^n \to ((x^n \to y) \to z)); \\ &= (x^n \to x^{2n}) \to ((x^n \to y) \to (x^n \to z)). \end{aligned}$$

So $(x^n \to x^{2n}) \to ((x^n \to y) \to (x^n \to z)) \in F$. Then by (ii) we get $(x^n \to y) \to (x^n \to z) \in F$.

$(iv) \Rightarrow (ii)$ Since $x^n \to (x^n \to x^{2n}) = x^{2n} \to x^{2n} = 1 \in F$, by (iv) we have $(x^n \to x^n) \to (x^n \to x^{2n}) \in F$, i.e. $x^n \to x^{2n} \in F$.

$(i) \Rightarrow (v)$ Let F be an n-fold B_L-Smarandache positive implicative filter of L_B. Since $a^n \to 1 = 1 \in F$, we have $1 \in A_a$. Let $x, x \to y \in A_a$. Then $a^n \to x \in F$ and $a^n \to (x \to y) \in F$. Since F is an n-fold B_L-Smarandache positive implicative filter of L_B, $a^n \to y \in F$, hence $y \in A_a$. Therefore A_a is a B_L-Smarandache deductive system of L_B.

$(v) \Rightarrow (i)$ Let $x, y, z \in B$, such that $x^n \to (y \to z) \in F$ and $x^n \to y \in F$. We get $y, y \to z \in A_x$, by the hypothesis A_x is a B_L-Smarandache deductive system of L_B, so $z \in A_x$ hence $x^n \to z \in F$. Therefore F is an n-fold B_L-Smarandache positive implicative filter of L_B .

□

Proposition 3.5. *Any n-fold B_L-Smarandache positive implicative filter is an $(n+1)$-fold B_L-Smarandache positive implicative filter.*

Proof. Let F be an n-fold B_L-Smarandache positive implicative filter. Let $x, y \in B$ such that $x^{n+2} \to y \in F$. By Theorem 2.1, $x^{n+1} \to (x \to y) = x^{n+2} \to y \in F$. Since F is an n-fold B_L-Smarandache positive implicative filter and $x^{n+1} \to x = 1 \in F$ we get $x^{n+1} \to y \in F$. Therefore by Theorem 3.4 ($(iii) \Rightarrow (i)$), F is an $(n+1)$-fold B_L-Smarandache positive implicative filter.

□

Theorem 3.6. *Let F and G be B_L-Smarandache deductive systems of L_B such that $F \subseteq G$. If F is an n-fold B_L-Smarandache positive implicative filter, then G is an n-fold B_L-Smarandache positive implicative filter.*

Proof. By Theorem 3.4 ($(ii) \Leftrightarrow (i)$), the proof is clear.

□

Definition 3.7. An n-fold Smarandache positive implicative B_L-residuated lattice L_B is a Smarandache B_L-residuated lattice which $x^{n+1} = x^n$ for all $x \in B$.

Example 3.8. (i) Let $L = \{0, a, b, c, d, 1\}$ be a residuated lattice such that $0 < b < a < 1$ and $0 < d < a, c < 1$. We define

$\odot$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	a	b	b	d	0	0
b	b	b	b	0	0	0
c	c	d	0	c	d	0
d	d	0	0	d	0	0
0	0	0	0	0	0	0

$\to$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice, in which $B = \{0, b, c, 1\}$ is a BL-algebra which properly contained in L. Then L is a Smarandache B_L-residuated lattice. Clearly L is an n-fold Smarandache positive implicative B_L-residuated lattice.

(ii) Consider the Smarandache B_L-residuated lattice L in Example 3.2. L_B is not a 1-fold Smarandache positive implicative B_L-residuated lattice, since $a \in B$ and $0 = a^{1+1} \neq a^1$.

Corollary 3.9. *Let L_B be an n-fold Smarandache positive implicative B_L-residuated lattice. The following conditions are equivalent:*

(i) F is an n-fold B_L-Smarandache positive implicative filter,

(ii) F is a B_L-Smarandache deductive system.

Proof. $(i) \Rightarrow (ii)$ By Theorem 3.3, the proof is clear.

$(ii) \Rightarrow (i)$ By Definition 3.7 and Theorem 3.4 ($(iii) \Rightarrow (i)$), the proof is clear.

□

Theorem 3.10. *Let F be a B_L-Smarandache deductive system of L_B. The following conditions are equivalent:*

(i) L_B is an n-fold Smarandache positive implicative B_L-residuated lattice,

(ii) every B_L-Smarandache deductive system of L_B is an n-fold B_L-Smarandache positive implicative filter of L_B,

(iii) $\{1\}$ is an n-fold B_L-Smarandache positive implicative filter of L_B,

(iv) $x^n = x^{2n}$, for all $x \in B$.

Proof. $(i) \Rightarrow (ii)$ Follows from Corollary 3.9.

$(ii) \Rightarrow (iii)$ It is clear.

$(iii) \Rightarrow (iv)$ Assume that $\{1\}$ is an n-fold B_L-Smarandache positive implicative filter of L_B. So from Theorem 3.4, we have $x^n \rightarrow x^{2n} = 1$, for all $x \in B$. Then by Theorem 2.1, $x^n = x^{2n}$ for all $x \in B$.

$(iv) \Rightarrow (i)$ Let $x^n = x^{2n}$, for all $x \in B$. Then $x^n \rightarrow x^{2n} = 1 \in \{1\}$, for all $x \in B$. Then by Theorem 3.4, $\{1\}$ is an n-fold B_L-Smarandache positive implicative filter of L_B. Since $x^n \rightarrow (x^n \rightarrow x^{n+1}) = x^{2n} \rightarrow x^{n+1} = 1 \in \{1\}$ and $x^n \rightarrow x^n = 1 \in \{1\}$, we get $x^n \rightarrow x^{n+1} \in \{1\}$, that is $x^{n+1} = x^n$, for all $x \in B$. Therefore L_B is an n-fold Smarandache positive implicative B_L-residuated lattice.

□

Corollary 3.11. *Let F be a proper B_L-Smarandache deductive system of L_B. Then the following statements are equivalent:*

(i) F is an n-fold B_L-Smarandache positive implicative filter (in short n-fold B_L-SPIF).

(ii) L_B/F is an n-fold Smarandache positive $B_{L/F}$-residuated lattice (in short n-fold $B_{L/F}$-SPRL).

Proof. Let F be a B_L-Smarandache deductive system of L_B. By Theorems 3.4 and 3.10 we get:

$$\begin{aligned}
F \text{ is an } n\text{-fold } B_L\text{-SPIF} &\Leftrightarrow x^n \to x^{2n} \in F,\ \forall\ x \in B,\\
&\Leftrightarrow (x^n \to x^{2n})/F = 1/F,\ \forall\ x/F \in B/F,\\
&\Leftrightarrow (x/F)^n \to (x/F)^{2n} = 1/F,\ \forall\ x/F \in B/F,\\
&\Leftrightarrow (x/F)^n \le (x/F)^{2n},\ \forall\ x/F \in B/F,\\
&\Leftrightarrow (x/F)^n = (x/F)^{2n},\ \forall\ x/F \in B/F,\\
&\Leftrightarrow L_B/F \text{ is an } n\text{-fold } B_{L/F}\text{-SPIRL}.
\end{aligned}$$

□

Definition 3.12. A subset F of L_B is called an n-fold Smarandache implicative filter of L_B related to B (or briefly n-fold B_L-Smarandache implicative filter of L_B) if it satisfies in the following conditions:

(i) $1 \in F$,

(ii) For all $y, z \in B$ and $x \in F$, if $x \to ((y^n \to z) \to y) \in F$ then $y \in F$.

Example 3.13. (i) In Example 3.2, $F = \{d, 1\}$ is an n-fold B_L-Smarandache implicative filter, for $n \geq 2$.

(ii) Let $L = \{0, a, b, c, 1\}$ be a residuated lattice such that a, b, c are incomparable. We define

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	c	d	1	c	d
c	b	b	b	1	1
1	0	a	b	c	1

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	c	c	a
b	0	c	b	c	b
c	0	c	c	c	c
1	0	a	b	c	1

We can see that $L = (L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice, in which $B = \{0, a, 1\}$ is a BL-algebra which properly contained in L. Then L is a Smarandache B_L-residuated lattice. $F = \{b, 1\}$ is not an n-fold B_L-Smarandache implicative filter, since $b, b \to ((a^n \to 0) \to a) = 1 \in F$ but $a \notin F$, $(0, a \in B)$.

Theorem 3.14. *Every n-fold B_L-Smarandache implicative filter is a B_L-Smarandache deductive system.*

Proof. Let F be an n-fold B_L-Smarandache implicative filter of L_B. Let $x \in F$, $y \in B$ and $x \to y \in F$. We have $x \to ((y^n \to 1) \to y) = x \to y \in F$. Since F is an n-fold B_L-Smarandache implicative filter and $x \in F$, we get $y \in F$. So F is a B_L-Smarandache deductive system of L_B. □

Theorem 3.15. *The following conditions are equivalent for B_L-Smarandache deductive system F of L_B.*

(i) F is an n-fold B_L-Smarandache implicative filter,

(ii) for all $x, y \in B$, $(x^n \to y) \to x \in F$ *implies* $x \in F$,
(iii) for all $x \in B$, $(x^n)^* \to x \in F$ *implies* $x \in F$,
(iv) for all $x \in B$, $x \vee (x^n)^* \in F$.

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are clear, by Definition 3.12.

$(iii) \Rightarrow (i)$ Suppose that $x \to ((y^n \to z) \to y) \in F$ and $x \in F$. Then $(y^n \to z) \to y \in F$. By Theorem 2.1 we have $0 \leq z$ then $y^n \to 0 \leq y^n \to z$ and so $(y^n \to z) \to y \leq (y^n \to 0) \to y$. Hence $(y^n \to 0) \to y \in F$. We apply the hypothesis and obtain $y \in F$. Therefore F is an n-fold B_L-Smarandache implicative filter.

$(iii) \Rightarrow (iv)$ We know $x \leq x \vee (x^n)^*$, then $x^n \leq (x \vee (x^n)^*)^n$ and then $((x \vee (x^n)^*)^n)^* \leq (x^n)^* \leq (x^n)^* \vee x$. Hence $((x \vee (x^n)^*)^n)^* \to (x \vee (x^n)^*) = 1 \in F$. Hence by (iii) we get $x \vee (x^n)^* \in F$.

$(iv) \Rightarrow (iii)$ Assume that for all $x \in B$, $(x^n)^* \to x \in F$. By Theorem 2.1 we have $x \vee (x^n)^* \leq ((x^n)^* \to x) \to x$ so by (iv), $((x^n)^* \to x) \to x \in F$. By using the fact that $(x^n)^* \to x \in F$, we get $x \in F$. □

Theorem 3.16. *Let* F *be an* n*-fold* B_L*-Smarandache implicative filter of* L_B. *Then* $(x^n \to y) \to y \in F$ *implies* $(y \to x) \to x \in F$, *for all* $x, y \in B$.

Proof. Let F be an n-fold B_L-Smarandache implicative filter and $(x^n \to y) \to y \in F$. We have $x \leq (y \to x) \to x$ then by Theorem 2.1 we get $x^n \leq ((y \to x) \to x)^n$, for all $n \in N$. And so $((y \to x) \to x)^n \to y \leq x^n \to y$. So we have

$$\begin{aligned} (x^n \to y) \to y &\leq (y \to x) \to ((x^n \to y) \to y) \\ &= (x^n \to y) \to ((y \to x) \to x) \\ &\leq ([((y \to x) \to x)]^n \to y) \to ((y \to x) \to x). \end{aligned}$$

Thus $([((y \to x) \to x)]^n \to y) \to ((y \to x) \to x) \in F$. By the fact that F is an n-fold B_L-Smarandache implicative filter of L_B, we get $(y \to x) \to x \in F$. □

By Theorem 3.15 ($(i) \Leftrightarrow (iv)$) we have the following theorem.

Theorem 3.17. *Let* F *and* G *be two* B_L*-Smarandache deductive systems of* L_B *such that* $F \subseteq G$. *If* F *is an* n*-fold* B_L*-Smarandache implicative filter of* L_B, *then so is* G.

Theorem 3.18. *Let every Smarandache deductive system be a Smarandache filter. Then every* n*-fold Smarandache implicative filter is* n*-fold Smarandache positive implicative filter.*

Proof. Let F be an n-fold B_L-Smarandache implicative filter of L_B and $x^{n+1} \to y \in F$, where $x, y \in B$. By Theorem 2.1 we have the following

$$\begin{aligned}(x^{n+1} \to y)^n \quad &\to \quad (x^n \to y) = ((x^{n+1} \to y)^{n-1} \odot (x^{n+1} \to y)) \to (x^n \to y)\\
&= \quad ((x^{n+1} \to y)^{n-1}) \to ((x^{n+1} \to y) \to (x^n \to y))\\
&= \quad ((x^{n+1} \to y)^{n-1}) \to ((x^{n+1} \to y) \to (x^{n-1} \to (x \to y)))\\
&= \quad ((x^{n+1} \to y)^{n-1}) \to (x^{n-1} \to ((x^{n+1} \to y) \to (x \to y)))\\
&= \quad ((x^{n+1} \to y)^{n-1}) \to (x^{n-1} \to ((x \to (x^n \to y)) \to (x \to y))).\end{aligned}$$

So $(x^{n+1} \to y)^n \to (x^n \to y) = ((x^{n+1} \to y)^{n-1}) \to (x^{n-1} \to ((x \to (x^n \to y)) \to (x \to y)))$. (I) We have $(x^n \to y) \to y \leq ((x \to (x^n \to y)) \to (x \to y))$. By (lr_7) we get $((x^{n+1} \to y)^{n-1}) \to (x^{n-1} \to ((x^n \to y) \to y)) \leq ((x^{n+1} \to y)^{n-1}) \to (x^{n-1} \to ((x \to (x^n \to y)) \to (x \to y)))$. So by (I) we have $((x^{n+1} \to y)^{n-1}) \to (x^{n-1} \to ((x^n \to y) \to y)) \leq (x^{n+1} \to y)^n \to (x^n \to y)$. (II) By Theorem 2.1 we have

$$\begin{aligned}&((x^{n+1} \to y)^{n-1}) \to (x^{n-1} \to ((x^n \to y) \to y)) =\\
&((x^{n+1} \to y)^{n-1}) \to ((x^n \to y) \to (x^{n-1} \to y)) =\\
&((x^n \to y)) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y)).\end{aligned}$$

Hence by (II) we obtain $(x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y)) \leq (x^{n+1} \to y)^n \to (x^n \to y)$. (III) We have

$$\begin{aligned}&(x^n \to y) \odot (x^{n+1} \to y)^{n-1} \odot x^{n-1} =\\
&(x^n \to y) \odot (x^{n+1} \to y)^{n-2} \odot (x^{n+1} \to y) \odot x^{n-2} \odot x =\\
&(x^n \to y) \odot (x^{n+1} \to y)^{n-2} \odot x^{n-2} \odot x \odot (x^{n+1} \to y). \text{ (IV)}\end{aligned}$$

Also we have $x \odot (x^{n+1} \to y) = x \odot (x \to (x^n \to y)) \leq x^n \to y$. Therefore by (IV) we obtain $(x^n \to y) \odot (x^{n+1} \to y)^{n-2} \odot x^{n-2} \odot x \odot (x^{n+1} \to y) \leq (x^n \to y) \odot (x^{n+1} \to y)^{n-2} \odot x^{n-2} \odot (x^n \to y)$. So we have

$$(x^n \to y) \odot (x^{n+1} \to y)^{n-1} \odot x^{n-1} \leq (x^n \to y)^2 \odot (x^{n+1} \to y)^{n-2} \odot x^{n-2}.$$

Then by (lr_7) we get $((x^n \to y)^2 \odot (x^{n+1} \to y)^{n-2} \odot x^{n-2}) \to y \leq ((x^n \to y)^1 \odot (x^{n+1} \to y)^{n-1} \odot x^{n-1}) \to y$. So by (lr_5) we obtain

$$((x^n \to y)^2 \odot (x^{n+1} \to y)^{n-2}) \to (x^{n-2} \to y) \leq ((x^n \to y) \odot (x^{n+1} \to y)^{n-1}) \to$$

$(x^{n-1} \to y)$. Thus by (lr_5) we get $(x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y)) \leq (x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y))$. So by (III) we have $(x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y)) \leq (x^{n+1} \to y)^n \to (x^n \to y)$. By repeating n times, we obtain $(x^n \to y)^n \to ((x^{n+1} \to y)^0 \to (x^0 \to y)) \leq (x^{n+1} \to y)^n \to (x^n \to y)$. This implies $(x^n \to y)^n \to (1 \to (1 \to y)) \leq (x^{n+1} \to y)^n \to (x^n \to y)$. Hence $(x^n \to y)^n \to y \leq (x^{n+1} \to y)^n \to (x^n \to y)$. Then $((x^n \to y)^n \to y) \to ((x^{n+1} \to y)^n \to (x^n \to y)) = 1$. Hence by (lr_5), we have $(x^{n+1} \to y)^n \to (((x^n \to y)^n \to y) \to (x^n \to y)) = 1 \in F$, (V). Since $x^{n+1} \to y \in F$ and F is a B_L-Smarandache filter, we have $(x^{n+1} \to y)^n \in F$. By (V), we have $((x^n \to y)^n \to y) \to (x^n \to y) \in F$. Then by Theorem 3.15 ($(i) \Rightarrow (ii)$), we get $x^n \to y \in F$ and so by Theorem 3.4, F is an n-fold B_L-Smarandache positive implicative filter.

□

In the following example we show that the converse of Theorem 3.18 is not true in general.

Example 3.19. In Example 3.13 (ii), $F = \{b, 1\}$ is an n-fold B_L-Smarandache positive implicative filter, while it is not an n-fold B_L-Smarandache implicative filter.

Proposition 3.20. *Every n-fold B_L-Smarandache implicative filter is an $(n+1)$-fold B_L-Smarandache implicative filter.*

Proof. Let F be an n-fold B_L-Smarandache implicative filter of L_B and $(x^{n+1})^* \to x \in F$, where $x \in B$. By Theorem 2.1, $(x^n)^* \leq (x^{n+1})^*$ then $(x^{n+1})^* \to x \leq (x^n)^* \to x$. Then we get $(x^n)^* \to x \in F$. Since F is an n-fold B_L-Smarandache implicative filter of L_B, by Theorem 3.15, we obtain $x \in F$. So by Theorem 3.15, F is an $(n+1)$-fold B_L-Smarandache implicative filter of L_B. □

Definition 3.21. A Smarandache residuated lattice L_B is called n-fold Smarandache implicative B_L-residuated lattice if it satisfies $(x^n)^* \to x = x$, for each $x \in B$.

Example 3.22. In Example 3.2, L_B is a 2-fold Smarandache implicative B_L-residuated lattice and it is not a 1-fold Smarandache implicative B_L-residuated lattice. Since $(a^1)^* \to a = c \neq a$.

Lemma 3.23. *(i) 1-fold Smarandache implicative B_L-residuated lattices are n-fold Smarandache implicative B_L-residuated lattice.*

(ii) n-fold Smarandache implicative B_L-residuated lattices are $(n+1)$-fold Smarandache implicative B_L-residuated lattice.

Proof. (i) Let L_B be a 1-fold Smarandache implicative B_L-residuated lattice. Then $x^* \to x = x$ for each $x \in B$. By Theorem 2.1 we have $(x^n)^* \to x \leq x^* \to x$. Hence $(x^n)^* \to x \leq x$ and so $(x^n)^* \to x = x$, for each $x \in B$. Therefore L_B is an n-fold Smarandache implicative B_L-residuated lattice.

(ii) Let L_B be an n-fold Smarandache implicative B_L-residuated lattice. Then $(x^n)^* \to x = x$ for each $x \in B$. By Theorem 2.1 we have $(x^{n+1})^* \to x \leq (x^n)^* \to x$. Hence $(x^{n+1})^* \to x \leq x$ and so $(x^{n+1})^* \to x = x$, for each $x \in B$. Therefore L_B is an $(n+1)$-fold Smarandache implicative B_L-residuated lattice.

□

Proposition 3.24. *If L_B is an n-fold Smarandache implicative B_L-residuated lattice. Then the following statements are equivalent:*

(i) F is an n-fold B_L-Smarandache implicative filter.

(ii) F is a B_L-Smarandache deductive system.

Proof. By Theorem 3.14 and Theorem 3.15 $(iii) \Rightarrow (i)$, the proofs are clear.

□

Proposition 3.25. *The following conditions are equivalent:*

(i) L_B is an n-fold Smarandache implicative B_L-residuated lattice.

(ii) Every n-fold B_L-Smarandache deductive system F of L_B is an n-fold B_L-Smarandache implicative filter of L_B.

(iii) $(x^n \to y) \to x = x$, for all $x, y \in B$.

(iv) $\{1\}$ is an n-fold B_L-Smarandache implicative filter.

Proof. $(i) \Rightarrow (ii)$ Follows from Proposition 3.24.

$(ii) \Rightarrow (iv)$ The proof is easy.

$(iii) \Rightarrow (iv)$ The proof is obvious.

$(i) \Rightarrow (iii)$ By hypothesis (i) and Definition 3.21, we have $(x^n)^* \to x = x$, for all $x \in B$. So by Theorem 2.1, we have $0 \leq y$ then $(x^n \to y) \to x \leq (x^n)^* \to x = x$. Therefore $(x^n \to y) \to x = x$.

$(iv) \Rightarrow (i)$ Let $\{1\}$ be an n-fold B_L-Smarandache implicative filter. By Theorem 3.15, for all $x \in B$, $x \vee (x^n)^* = 1$. By Theorem 2.1, $x \vee (x^n)^* \leq ((x^n)^* \to x) \to x$. Hence $((x^n)^* \to x) \to x = 1$ or equivalently $(x^n)^* \to x \leq x$. Hence by (lr_4) we have $(x^n)^* \to x = x$, for all $x \in B$. So the proof is complete. □

Proposition 3.26. *n-fold Smarandache implicative B_L-residuated lattices are n-fold Smarandache positive implicative B_L-residuated lattices.*

Proof. Let L_B be an n-fold Smarandache implicative B_L-residuated lattice. Then $(x^n)^* \to x = x$ for each $x \in B$. Hence every B_L-Smarandache deductive system is an n-fold B_L-Smarandache implicative filter. So $\{1\}$ is an n-fold B_L-Smarandache implicative filter. We know $\{1\}$ is a B_L-Smarandache filter, hence by Theorem 3.18, $\{1\}$ is an n-fold B_L-Smarandache positive implicative filter. Since $\{1\}$ is a Smarandache deductive system, then by Theorem 3.6, every deductive system is an n-fold B_L-Smarandache positive implicative filter. Hence by Theorem 3.10, L_B is an n-fold Smarandache positive implicative B_L-residuated lattice. □

In the following example we show that the converse of above proposition is not true in general.

Example 3.27. Consider L_B in Example 3.13(ii). Clearly L_B is an n-fold Smarandache positive implicative B_L-residuated lattices, while it is not an n-fold Smarandache implicative B_L-residuated lattice, since $(a^n)^* \to a \neq a$.

Proposition 3.28. *Let F be a proper B_L-Smarandache deductive system of L_B. Then the following statements are equivalent:*

(i) F is an n-fold B_L-Smarandache implicative filter of L_B.

(ii) L_B/F is an n-fold Smarandache implicative $B_{L/F}$-residuated lattice.

Proof. $(i) \Rightarrow (ii)$ Let F be an n-fold B_L-Smarandache implicative filter of L_B. By Proposition 3.25, it is enough show that $\{1/F\}$ is an n-fold $B_{L/F}$-Smarandache implicative filter of L_B/F. Let $((x/F)^n)^* \to x/F \in \{1/F\}$, for all $x/F \in B/F$. Then $((x^n)^* \to x)/F = 1/F$, so $(x^n)^* \to x \in F$. Since F is an n-fold B_L-Smarandache implicative filter we get $x \in F$, for all $x \in B$.

And so $x/F = 1/F \in \{1/F\}$, for all $x/F \in B/F$, i.e. $\{1/F\}$ is an n-fold $B_{L/F}$-Smarandache implicative filter of L_B/F.

$(ii) \Rightarrow (i)$ Let L_B/F be an n-fold Smarandache implicative $B_{L/F}$-residated lattice and $(x^n)^* \to x \in F$, for all $x \in B$. We get $((x^n)^* \to x)/F = 1/F$, for all $x/F \in B/F$, this implies $((x/F)^n)^* \to x/F \in \{1/F\}$. Since L_B/F is an n-fold Smarandache implicative B_L-residuated lattice, by Proposition 3.25, $\{1/F\}$ is an n-fold $B_{L/F}$-Smarandache implicative filter of L_B/F. Hence $x/F \in \{1/F\}$ or equivalently $x \in F$. Then by Theorem 3.15, F is an n-fold B_L-Smarandache implicative filter of L_B. □

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