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Characterization of Weak Bi-Ideals in Bi-Near Rings

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Abstract: *In this paper, with a new idea, we define weak bi-ideal and investigate some of its properties. We characterize weak bi-ideal by bi-ideals of bi-near ing .In the case of left self-distributive S-bi-near ring We establish necessary and sufficient condition for weak bi-ideal to be bi-ideal and strong bi-ideal.This concept motivates the study of different kinds of new biregular bi-near rings in algebraic theory especially regularity in quad near ring and fuzzy logic.A bisubgroup B of (N, +) is said to be a weak bi-ideal if $B^3 \subseteq B$. In the first section, we prove that the two concepts of bi-ideals and weak bi-ideals are equivalent in a left self-distributive S-bi-near ring. In the second section of this paper, we obtain equivalent conditions for a weak bi-ideal to be a bi-near field.*

1. Introduction

In mathematics, a bi-near ring is an algebraic structure similar to a near ring but satisfying fewer axioms. Near-rings arise naturally from functions on groups although bi-near-rings are a well-developed branch of algebra; little effort has been spent on treating near-rings by the help of a computer. This is most regrettable in the view of the research in both the theory and the applications of near-rings. Since good algorithms for computing with groups, in particular permutation groups, are now available, i.e. invented and implemented, my first approach to computing with near-rings was to reduce the near-ring problems to group-theoretic problems and to solve those using GAP.For basic definition one may refer to Pilz[1].Motivated by the study of bi-idels in " A Study on Regularities in Near-Rings" by S.Jayalakshmi and also motivated by the study of bi-near rings in " Bialgebraic structures and Smarandache bialgebraic Structures, American Research Press, 2003" by W. B. Vasantha Kandasamy, the new concepts 'weak bi-ideal' in bi- near ring is introduced.

2. Preliminaries

Definition: 2.1

Let (N, +, .) be a non-empty set. We call N a **binear-ring** if $N = N_1 \cup N_2$ where N_1 and N_2 are proper subsets of N i.e. $N_1 \not\subset N_2$ or $N_2 \not\subset N_1$ satisfying the following conditions:
At least one of $(N_i, +, .)$ is a right near-ring i.e. for preciseness we say
i). $(N_1, +, .)$ is a near-ring
ii.) $(N_2, +, .)$ is a ring.
We say that even if both $(N_i, +, .)$ are right near-rings still we call (N, +, .) to be a Bi- near ring. By default of notation by binear-ring we mean only right bi-near ring Unless explicitly stated.

Remark 2.1.1:

Throughout this paper, by a bi-near ring, we mean only a right bi-near ring. The symbol N stands for a bi-near ring (N, +, .) with at least two elements. 0 denotes the identity element of the bigroup (N, +) . L and E denotes the set of all nilpotent and idempotent elements of N.C(N) denotes the centre of N. We write xy for x.y for any two elements x, y of N. It can be easily proved that 0a=0 and (–a)b=–ab for all $a,b \in N_1 \cup N_2$. N must naturally be bi-regular and zero symmetric.

Definition 2.2:

$N_0 = \{ \forall n \in N_1 \cup N_2 / n0 = 0\}$ is called the **zero – symmetric part** of the bi-near ring N. A bi-near ring N is called **zero–symmetric**,if $N = N_0$. i.e., $N_1 = N_{1_0}$ and $N_2 = N_{2_0}$

Definition 2.3:

$N_d = \{ \forall nx,y \in N_1 \cup N_2 / n(x+y) = nx+ny\}$ is the set of all distributive elements of a bi-near ring N. A bi-near ring N is called **distributive**, if $N=N_d$ i.e. $N_1 = N_{1_d}$ and $N_2 = N_{2_d}$.

Definition: 2.4

A bi-near ring N is said to be a **S(S')-bi-near ring** if $x \in Nx (x \in xN)$ for all $x \in N_1 \cup N_2$

Definition: 2.5

A non-empty subset M of a bi-near ring $N = N_1 \cup N_2$ is said to be bisubgroup of (N, +) if
(i) $M = M_1 \cup M_2$
(ii) $(M_1,+)$ is a subgroup of $(N_1, +)$
(iii) $(M_2,+)$ is a subgroup of $(N_2, +)$.
But M is not a subgroup.

Definition: 2.6

A S-bi near ring N is said to be a $\overline{S}$ **-bi near ring** if x∈ for all a, $b\in N_1\cup N_2$

Definition: 2.7

A bi-near ring N is said to be a **P(m,n)(P'(m,n))-bi-near ring** if there exist a positive integer m,n such tht $xN = x^mNx^n(Nx = x^mNx^n)$ for all $x\in N_1\cup N_2$

Definition: 2.8

A bi-near ring N is said to be a **$P_k(P_k')$-bi-near ring** if there exist a positive integer k such that $x^kN = xNx(Nx^k = xNx)$ for all $x\in N_1\cup N_2$

Definition: 2.9

A bisubgroup A of a bi-near ring (N,+,.) is said to be **Left(right)N-bisubgroup** of N if $NA\subseteq A(AN\subseteq A)$

Definition: 2.10

A bisubgroup M of a bi-near ring N is called a **bi-sub near-ring** of N if $MM\subseteq M$.

Definition: 2.11

A bi-near ring N is said to have **property(α),** if xN is a bisubgroup of (N,+) for every $x\in N_1\cup N_2$

Definition: 2.12

Let (N,+,.) be a binear-ring. A bisubgroup B of (N,+) is called **bi-ideal** of N if $BNB\subseteq B$. i.e., $BN_1B\subseteq B$(or $BN_2B\subseteq B$).

Definition: 2.13

A bi-ideal B of a bi-near ring (N,+) is called **Strong bi-ideal** of N if $NB^2\subseteq B$.i.e., $N_1B^2\subseteq B$(or $N_2B^2\subseteq B$).

Definition: 2.14

A bisubgroup B of (N, +) is said to be a **Generalized (m,n) bi-ideal** of N if $B^mNB^n\subseteq B$i.e., $B^mN_1B^n \subseteq B$ (or $B^mN_2B^n \subseteq B$) where m and n are positive integers.

Definition: 2.15

Let (N,+,.) be a binear-ring. A bi subgroup Q of (N,+) is called a **Quasi-ideal** of N if $QN\cap NQ\subseteq Q$ i.e., $QN_1\cap N_1Q\subseteq Q$(or $QN_2\cap N_2Q\subseteq Q$)

Definition: 2.10

A bi-near ring N is said to be **bi-regular** if for any $a\in N_1\cup N_2$ there exists $b\in N_1\cup N_2$with aba = a .

Definition: 2.16

A bi-near ring N is said to be **Self distributive bi-near ring** if abc = abac for all $a,b,c\in N_1\cup N_2$

Definition: 2.17

A bi-near ring N is said to be **Strict weakly biregular** if A^2= A for every left N-bisubgroup of N.

Definition: 2.18

A bi-near ring N is called **strongly biregular** if for each $a\in N_1\cup N_2$, there exists $b\in N_1\cup N_2$ such that $a = ba^2$.

Definition: 2.19

A bi-near ring N is called **left bi-potent** if $Na^2 = Na$ all $a\in N_1\cup N_2$

Definition: 2.20

A bi-near ring N is called **left simple** if Na = N all $a\in N_1\cup N_2$.

Definition: 2.21

A bi-near-ring N is called **CPNS bi-near ring** if NxNy = NyNx for all $x, y\in N_1\cup N_2$

Definition: 2.22

A bi-near ring N is said to be **Sub commutative** if Nx= xN for all $x\in N_1\cup N_2$.

Definition: 2.23

A bi-near ring N is said to be **Stable** if Nx = xNx = Nx for all $x\in N_1\cup N_2$

Definition: 2.24

For $A\subseteq N$,we define **Radical** $\sqrt{A}$ of A to be { $x\in N_1\cup N_2$/ $x^k\in A$ for some positive integer k }. Clearly $A\subseteq\sqrt{A}$

Definition: 2.25

A bi-near ring N is called a **bi-near field,** if the set of all non-zero elements of N is a group under multiplication.

Definition: 2.26

A bi-near ring N is called a **Generalized Bi-near-Field** (GNF) if for each $a\in N_1\cup N_2$ there exists a unique $b\in N_1\cup N_2$ such that aba = a and b = bab.

Definition 2.27 :

A bi-near ring N is called **Boolean** if $x^2 = x$ for all $x\in N_1\cup N_2$

Lemma 2.28: Let N be a zero-symmetric bi-near ring and if N is strongly bi-regular, then N is a bi-regular bi-near ring.

Lemma 2.29:

Let N be a bi-regular bi-near ring. Then any left N-bisubgroup M of a bi-near ring N is an idempotent bi-near ring.

Theorem 2.30:

The following conditions are equivalent for a S-bi-near ring N.

(i) N is strict weakly bi-regular.

(ii) For every $a\in N$, $a\in(Na)^2$.

(iii) For any two left N-bi-subgroups S_1 and S_2 such that $S_1 \subseteq S_2$ we have $S_2S_1 = S_1$.

Result 2.31:

If $x^2 = 0 \Rightarrow x = 0$ for all $x\in N_1\cup N_2$, then a bi-near ring N has no non-zero nilpotent elements.

Lemma 2.32:

Let N be a zero-symmetric bi-near ring. If L = {0}, then en = ene for $0 \neq e\in E$ and for all $n\in N_1\cup N_2$.

Lemma 2.33:

If a bi-near ring N is a zero-symmetric with $E\subseteq N_d$ and L={0}, then ne = ene for all $e\in E$ and for all $n\in N_1\cup N_2$.

Theorem 2.34

The following are equivalent in a bi-near ring.

(i) A bi-near ring N is GNF.
(ii) A bi-near ring N is bi-regular and each idempotent is central.
(iii) N is bi-regular and sub commutative bi-near ring.

Lemma 2.35:
If a bi-near ring N has the condition, eN = eNe = Ne for all $e \in E$, then $E \subseteq C(N)$.

Theorem 2.36:
Let N be a bi-regular bi-near ring.Then N is stable a bi-near ring if and only if it is a sub commutative bi-near ring

Theorem 2.37:
Let N be a bi-regular bi-near ring.Then N is a stable bi-near ring iff $E\subseteq C(N)$.

Proposition 2.38:
Let N be a bi-near ring then the following are equivalent.

(i) N is a B-bi-regular bi-near ring
(ii) $RL = R\cap L$ for every left N-bi-subgroup L of N and for every right N-bi-subgroup R of a bi-near ring N
(iii) For every pair of elements a,b of a bi-near ring N, $(a)_r\cap(b)_l=(a)_r(b)_l$.
(iv) For any element a of a bi-near ring N, $(a)_r\cap(a)_l=(a)_r(a)_l$.

Theorem 2.39:
Let N be a bi-regular bi-near ring. Then if N has (Ps), then xN= xNx for all $x,n\in N_1 \cup N_2$.

Proposition 2.40:
Every bi-regular bi-near ring is a B-biregular bi-near ring

Proposition 2.41
Let N be a $\overline{S}$ bi-near ring with property (α). Then the following are equivalent.

(i) N is a B-biregular bi-near ring
(ii) N is a bi-regular bi-near ring
(iii) For every quasi-ideal Q, QPQ = Q for some subset P of a bi-near ring N.

Theorem 2.42
Let a bi-near ring N(=No) is bi-regular. Then the following statements are equivalent.

(i) N is an N.S.I. bi-near ring.
(ii) xN = xNx for all x in N.
(iii) For all N-bisubgroups M_1 and M_2 of N, $M_1 \cap M_2 = M_1 M_2$
(iv) $N_x \cap N_y = Nxy$ for all x,y in N.
(v) Every N-bisubgroup of a bi-near ring N is a completely semi - prime ideal.
(vi) A bi-near ring N has property P_4.
(vii) A bi-near ring N has strong IFP.

Proposition 1.1.66:
A sub commutative S-bi-near ring is left bi-potent if and only if B = BNB for every bi-ideal B of N.

3.Weak Bi-ideals

Definition 3.1.1:
A bisubgroup B of a bi-near ring (N, +) is said to be a weak bi-ideal if $B^3\subseteq B$.

Remark 3.1.2:
Every bi-ideal B of a bi-near ring is a weak bi-ideal of a bi-nearr ing, but the converse is not true. For, consider the bi-near ring $N = N_1 \cup N_2$ where N_1={0, a, b, c} according to the scheme (0,0,2,1) (p. 408 Pilz [1]).

.	**0**	**a**	**b**	**c**
0	0	0	0	0
a	0	0	a	a
b	0	0	c	b
c	0	0	b	c

Or N_2={0, a,b,c} according to the scheme(0,4, 1, 1) (p. 408 Pilz [1]).

.	**0**	**a**	**b**	**c**
0	0	0	0	0
a	0	b	a	a
b	0	c	b	b
c	0	a	c	c

In this bi-near ring one can check that {0, b} and {0, c} are weak bi-ideals of N_1 (or N_2). However {0, b}N_1{0, b} = {0, c, b}$\not\subset${0, b}(or{0, b}N_2{0, b} = {0, c, b}$\not\subset${0, b}) and hence {0, b} is not a bi-ideal of N_1(or N_2).Therefore {0, b} is a weak bi-ideal of a bi-near ring N but not a bi-ideal of a bi-near ring N.

Proposition 3.1.3:
Any homomorphic image of a weak bi-ideal is also a weak bi-ideal of a bi-near ring N

Proof:
Let $f: N \to N'$ be a homomorphism and B a weak bi-ideal of N where $N = N_1 \cup N_2$ and $N' = N_1' \cup N_2'$.Given B is a weak bi-ideal of a bi-near ring $N = N_1 \cup N_2$ implies $B^3\subseteq B$. Let $B'=f(B)$. Let $b'\in f(B)$. Then $b'^3 = f(b)^3 = f(b^3) \in f(B^3) \subseteq f(B) = B'$.i.e., $B'^3\subseteq B'$. Thus every homomorphic image of a weak bi-ideal is also a weak bi-ideal of $N = N_1 \cup N_2$. Hence any homomorphic image of a weak bi-ideal is also a weak bi-ideal of a bi-near ring N. □

Proposition 3.1.4:

The set of all weak bi-ideals of a bi-near ring N form a Moore system on N.

Proof:

Let $\{B_i\}_{i\in I}$ be a weak bi-ideal of a bi-near ring N. To prove $B=\bigcap_{i\in i} B_i$ be a weak bi-ideal of N. If B_i be a weak bi-ideal of a bi-near ring N implies B_i be a weak bi-ideal of $N = N_1 \cup N_2$. Clearly $B \subseteq B_i$ implies $B^3 \subseteq B_i^3 \subseteq B_i \subseteq B \Rightarrow B^3 \subseteq B$. Hence $B=\bigcap_{i\in i} B_i$ be a weak bi-ideal of $N = N_1 \cup N_2$ and hence weak bi-ideal of a bi-near ring N.

Proposition 3.1.5:

If B is a weak bi-ideal of a bi-near ring N and S is a sub bi-near ring of N, then B∩S is a weak bi-ideal of a bi-near ring N.

Proof:

Let C = B∩S. To prove C is a weak bi-ideal of a bi-near ring N.Given B is a weak bi-ideal of a bi-near ring $B^3 \subseteq B$.Now C^3 =(B∩S)((B∩S)(B∩S)) ⊆(B∩S)(BB∩SS)⊆(B∩S)BB∩(B∩S)SS⊆ BBB∩SSS=$B^3 \cap S^3 \subseteq$B∩S $(\because B^3 \subseteq B \,\&\, SS \subseteq S)$.

Hence $C^3 \subseteq$ B∩S = C. i.e. $C^3 \subseteq$ C.Therefore C is a weak bi-ideal of N and hence C = B∩S is a weak bi-ideal of a bi-near ring N. □

Proposition 3.1.6:

Let B be a weak bi-ideal of a bi-near ring N.Then Bb and b′B are the weak bi-ideals of a bi-near ring N where b,b′∈B and b′ is a distributive element of a bi-near ring N.

Proof :

Let B be a weak bi-ideal of a bi-near ring $N = N_1 \cup N_2$ implies $B^3 \subseteq$ B.Clearly Bb is a bisubgroup of (N, +).Then $(Bb)^3$ = BbBbBb ⊆ BBBb ⊆ B^3b ⊆ Bb.$(Bb)^3 \subseteq$Bb then Bb is a weak bi-ideal of $N = N_1 \cup N_2$. Hence Bb is a weak bi-ideal of the bi-near ring N. Now to prove b′B is a weak bi-ideals of a bi-near ring N. i.e.. to prove b′B is a weak bi-ideals of $N = N_1 \cup N_2$. i.e.. to prove $(b'B)^3 \subseteq$ b′B.Since b′ is distributive, b′B is a bisubgroup of (N,+).Now $(b'B)^3$ = (b′B)(b′B)(b′B) ⊆ b′BBB = b′$B^3 \subseteq$ b′B $(\because B^3 \subseteq B)$. Therefore b′B are weak bi-ideals of a bi-near ring N and Hence Bb and b′B are weak bi-ideals of a bi-near ring N.

Corollary 3.1.7:

Let B be a weak bi-ideal of a bi-near ring N. For b, c ∈ B, if b is distributive, then bBc is a weak bi-ideal of a bi-near ring N.

Proof :

Let B be a weak bi-ideal of a bi-near ring N. If c ∈ B then by the proposition 3.1.6 (Bc) is a weak bi-ideal of a bi-near ring N.Now Bc is a weak bi-ideal of $N = N_1 \cup N_2$ and b ∈ B, if b is distributive then by the proposition 3.1.6, bBc is a weak bi-ideal of a bi-near ring N.Hence the proof.

Proposition 3.1.8:

Let N be a left self-distributive S-bi-near ring. Then B^3 = B for every weak bi-ideal B of a bi-near ring N if and only if N is a strongly bi-regular bi-near ring.

Proof:

Let B be a weak bi-ideal of a bi-near ring N implies B be a weak bi-ideal of $N = N_1 \cup N_2$ and hence $B^3 \subseteq$ B.Assume that N is strongly bi-regular bi-near ring, then by the Lemma 2.28, N is bi-regular bi-near ring. Now to prove B⊆B^3. Let b∈B then b∈ $N_1 \cup N_2$.Since N is a bi-regular bi-near ring, b = bab for some a ∈ $N_1 \cup N_2$.By our assumption that N is left self-distributive bi-near ring, we have bab = babb for all a,b∈ $N = N_1 \cup N_2$.Thus b = bab = babb = bab^2 = $b^2 = b^3 \in B^3$. i.e. b∈B^3 implies B⊆ B^3 Hence B = B^3 for every weak bi-ideal B of a bi-near ring N.Conversely assume that B = B^3 for every weak bi-ideal B of a bi-near ring N.To prove N is strongly bi-regular bi-near ring. i.e., To prove for each a∈ $N_1 \cup N_2$,there exist b∈ $N_1 \cup N_2$ such that a = ba^2.let a∈ $N_1 \cup N_2$ then by the proposition 3.1.6,Na is a weak bi-ideal of a bi-near ring N and $(Na)^3 \subseteq$ (Na). Also N is a S-bi-near ring, we get a∈Na = $(Na)^3$ = NaNaNa ⊆ NaNa. That is a=n_1an_1a. Since N is left self-distributive, a = $n_1an_2a^2$ = ba^2 where b = n_1an_2∈ $N_1 \cup N_2$. Hence a = ba^2 for a,b∈ $N_1 \cup N_2$. Hence N is a strongly bi-regular bi-near ring.

Remark 3.1.9:

Let N be a left self - distributive S-bi-near ring. If B = B^3 for every weak bi-ideal B of a bi-near ring N, then L = {0}.

Proposition 3.1.10:

Let N be a left self-distributive S-bi-near ring. Then B = NB^2 for every strong bi-ideal B of a bi-near ring N if and only if N is a strongly bi-regular bi-near ring.

Proof:

Assume that B = NB^2 for every strong bi-ideal B of a bi-near ring N. To prove N is a strongly bi-regular bi-near ring. i.e., To prove for each a∈ $N_1 \cup N_2$, there exist b∈ $N_1 \cup N_2$ such that a = ba^2. let a∈ $N = N_1 \cup N_2$ implies Na is a strong bi-ideal of a bi-near ring N and N is S-bi-near ring, we have a∈Na= $N(Na)^2$ = NNaNa ⊆ NaNa.

i.e. $a = n_1an_2a$. Since N is a left self-distributive bi-near ring, $a= n_1an_2a = n_1an_2a^2 \in Na^2$. i.e. $a = n_1an_2 a^2 = ba^2$ where $b = n_1an_2 \in N_1 \cup N_2$. Hence $a=ba^2$ where $a,b \in N_1 \cup N_2$.Thus N is strongly bi-regular bi-near ring.Conversely, assume that N is strongly bi-regular bi-near ring.To prove $B =NB^2$ where B is a strong bi-ideal of a bi-near ring N.By definition of strong bi-ideal $NB^2 \subseteq B$.Since N is strongly bi-regular bi-near ring,for $b\in B$ then $b\in N_1 \cup N_2$ there exist $n\in N_1 \cup N_2$ such that $b=nb^2 \in NB^2$ implies $B \subseteq NB^2$. Hence $NB^2 = B$ for every strong bi-ideal B of a bi-near ring N.

Theorem 3.1.11:

Let N be a left self-distributive S-bi-near ring. Then $B^3 = B$ for every weak bi-ideal B of N if and only if $NB^2 = B$ for every strong bi-ideal B of a bi-near ring N.

Proof:

Let N be a left self-distributive S-bi-near ring. Assume that $B^3 = B$ for every weak bi-ideal B of a bi-near ring N then by Proposition 3.1.8 N is strongly bi-regular bi-near ring.Now by Proposition 3.1.10 $NB^2 = B$ for every strong bi-ideal B of a bi-near ring N. Conversely assume that $NB^2 = B$ for every strong bi-ideal B of a bi-near ring N then by Proposition 3.1.10 N is strongly bi-regular bi-near ring.Now by Proposition 3.1.8 $B^3 = B$ for every weak bi-ideal B of a bi-near ring N.

Proposition 3.1.12:

Let N be a left self-distributive S-bi-near ring.Then B = BNB for every bi-ideal B of a bi-near ring N if and only if N is a bi-regular bi-near ring.

Proof :

Assume that B = BNB for every bi-ideal B of a bi-near ring N. To prove N is bi-regular bi-near ring. let $a\in N$ then Na is a bi-ideal of N. Since N is a S-bi-near ring, we get $a\in Na=NaNNa$. i.e., $a = n_1an_2a$ for some $n_1,n_2\in N_1 \cup N_2$. Since N is a left self-distributive bi-near ring, $a = n_1an_2a^2 \in Na^2$. Therefore for $a,b\in N_1 \cup N_2$, $a= n_1an_2a^2 = ba^2$ where $b = n_1an_2$. Hence N is a strongly bi-regular bi-near ring and so by the Lemma 2.28, N is a bi-regular bi-near ring. Conversely, Assume that N is a bi-regular bi-near ring and B is a bi-ideal of a bi-near ring N implies $BNB\subseteq B$.Now to prove $B\subseteq BNB$. Let $b\in B$ then $b\in N_1 \cup N_2$. Since N is a bi-regular bi-near ring, for each $b\in N_1 \cup N_2$,there exist $a\in N_1 \cup N_2$ such that $b = bab \in BN_1B$(or BN_2B) implies $b\in BNB$.Hence $B\subseteq BNB$ andso B = BNB for every bi-ideal B of a bi-near ring N.

Proposition 3.1.13:

Let N be a left self-distributive S-bi-near ring. Then $B = B^3$ for every weak bi-ideal B of a bi-near ring N if and only if $L_1 \cap L_2 = L_1L_2$ for any two left N-bisubgroups of a bi-near ring N.

Proof :

Assume that $B = B^3$ for every weak bi-ideal B of a bi-near ring N. By the Proposition 3.1.8, N is a strongly bi-regular bi-near ring.Every bi-ideal B of a bi-near ring is also a weak bi-ideal of a bi-near ring then by proposition 3.1.12 N is a bi-regular bi-near ring.Let L_1 and L_2 be any two left N-bisubgroups of a bi-near ring N. Let $x\in L_1\cap L_2$ implies $x\in L_1$ and $x\in L_2$. Since N is a bi-regular bi-near ring, for $x\in N_1 \cup N_2$ there exist some $a\in N_1 \cup N_2$ such that x = xax.Therefore $x=xax\in L_1NL_2\subseteq L_1L_2$ since $NL_2 \subseteq L_2$ which implies that $L_1\cap L_2\subseteq L_1 L_2$.On the other hand, let $x \in L_1 L_2$. Since N is a strongly bi-regular bi-near ring, L={0} and so en=ene for all $e\in E$.Since $x\in L_1L_2$, trivially $x\in L_2$.Then $x=yz\in L_1L_2$ with $y\in L_1$ and $z\in L_2$.Now x=yz=(yby)z. Since by is an idempotent (by)z = (by)z(by). Thus x = yz = $y(by)z = y(by)z(by)\in N_1L_1$(or N_2L_1)$\subseteq L_1$ implies $x\in L_1$.Thus $x\in L_1\cap L_2$. From the two inclusions proved above, we get that $L_1L_2=L_1\cap L_2$.Conversely assume that $L_1\cap L_2=L_1L_2$ for any two left N-bisubgroups of a bi-near ring N.To prove $B = B^3$ for every weak bi-ideal B of a bi-near ring N. Let $a\in N_1 \cup N_2$ then Na is a left N-bisubgroup of a bi-near ring N.Now, by our assumption we get that $Na=Na\cap Na=NaNa$. But $Na=Na\cap N=NaN$ implies that Naa=NaNa. Therefore $Na = NaNa = Naa = Na^2$.Since N is a S-bi-near ring,$a\in Na=Na^2$. i.e., for $a\in N_1 \cup N_2$, $a\in Na=Na^2$ implies N is a strongly bi-regular bi-near ring. By the Proposition 3.1.8, $B = B^3$ for every weak bi-ideal B of a bi-near ring N.

Proposition 3.1.14:

Let N be a left self - distributive S-bi-near ring. Then $B = B^3$ for every weak bi-ideal B of a bi-near ring N if and only if $\sqrt{A} = A$ for a left N-bisubgroup A of a bi-near ring N.

Proof :

Assume $B = B^3$ for every weak bi-ideal B of a bi-near ring N. By the Proposition 3.1.8, N is a strongly bi-regular bi-near ring. Let A be a left N-bisubgroup of a bi-near ring N. To Prove $\sqrt{A}=A$. But obviously $A\subseteq\sqrt{A}$.Its enough to prove $\sqrt{A} \subseteq A$ .Let $a\in \sqrt{A}$, then $a^n\in A$ for some positive integer n.Since N is strongly bi-regular bi-near ring,for each $a\in N_1 \cup N_2$ there exist $b\in N_1 \cup N_2$ such that

$a=ba^2$= baa.Since N is left self-distributive bi-near ring we get that $a=ba^2$ = baa = baba = b(aba) = b(abaa) = $baba^2$=…= $baba^n \in N_1A$ (or N_2A)$\subseteq$A. i.e.,a$\in$A implies $\sqrt{A} \subseteq A$.Therefore A=$\sqrt{A}$ for every left N-bisubgroup A of a bi-near ring N. Conversely assume that A = $\sqrt{A}$ for every left N-bisubgroup A of a bi-near ring N.Let a$\in N_1 \cup N_2$ then $a^3 \in Na^2$ and hence a $\in \sqrt{Na^2} = Na^2$.Thus for all a$\in N_1 \cup N_2$,a$\in Na^2$ implies N is a strongly bi-regular bi-near ring. Therefore by the Proposition 3.1.8, B = B^3 for every weak bi- ideal B of a bi-near ring N.

§ 3.2 In this section, we prove certain results for left self distributive bi-near ring.

Remark 3.2.1 :

We can see that left permutable and left self - distributive are not equivalent we give the following two examples.

Exam ple 3.2.2 :

For, consider the bi-near ring $N = N_1 \cup N_2$ where (N_1,+) be the group of integers modulo 8 Define '.' as per the (scheme (0,6,4,6,0,6,4,6) p.414, Pilz[1]).

.	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	6	4	6	0	6	4	6
2	0	4	0	4	0	4	0	4
3	0	2	4	2	0	2	4	2
4	0	0	0	0	0	0	0	0
5	0	6	4	6	0	6	4	6
6	0	4	0	4	0	4	0	4
7	0	2	4	2	0	2	4	2

where (N_2,+) be the group of integers modulo 8 Define '.' as per the (scheme (0,2,4,2,0,2,4,2) p.414, Pilz[1]).

.	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	2	4	2	0	2	4	2
2	0	4	0	4	0	4	0	4
3	0	6	4	6	0	6	4	6
4	0	0	0	0	0	0	0	0
5	0	2	4	2	0	2	4	2
6	0	4	0	4	0	4	0	4
7	0	6	4	6	0	6	4	6

One can see that this $N = N_1 \cup N_2$ is left permutable, but not left self-distributive since 135≠1315(or 155≠1515)

Example 3.2.2.1 :

For, consider the bi-near ring $N = N_1 \cup N_2$ where (N_1,+) defined on the Klein's four group (N_1,+) with N_1 ={0,a,b,c} where the semigroup operation '.' is defined as follows (scheme (7,7,7,7) p.408, Pilz[1]).

.	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	0
c	a	a	a	a

Or N_2 ={0,a,b,c} where the semigroup operation '.' is defined as follows (scheme (7,7,1,7) p.408, Pilz[1]).

.	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	b	0
c	a	a	c	a

This is left self- distributive but not left permutable bi-near ring since abc $\neq$ bac for a,b,c$\in N_1 \cup N_2$.

Proposition 3.2.3:

Let N be a left self – distributive S-bi-near ring. Then the following conditions are equivalent.

(i) B = B^3 for every weak bi-ideal B of a bi-near ring N.

(ii) N is a bi-regular and CPNS bi-near ring.

(iii) Nx $\cap$ Ny = Nxy for all x,y$\in N = N_1 \cup N_2$.

(iv) N is a left bi-potent bi-near ring.

(v) N is a Boolean bi-near ring.

Proof :

To prove (i) ⇒ (ii)

Assume that B = B^3 for every weak bi-ideal B of a bi-near ring N.By the Proposition 3.1.8, N is a strongly bi-regular bi-near ring and so N is a bi-regular bi-near ring. It is enough to show that N is CPNS bi-near ring.Since N is bi-regular bi-near ring then by the Proposition 3.1.13, A $\cap$ B = AB for two left N-bisub groups A and B of a bi-near ring N. Let x, y $\in N_1 \cup N_2$ then Nx and Ny are left N-bisub groups of N. Now by the Proposition 3.1.13, we get that NxNy = Nx$\cap$Ny = Ny$\cap$Nx = NyNx.Therefore NxNy = NyNx for all x,y$\in N_1 \cup N_2$. Hence N is a CPNS bi-near ring. Hence N is bi-regular and CPNS bi-near ring.

To prove (ii)⇒(iii)

Let x,y$\in$N. As N is a bi-regular bi-near ring, by the Lemma 2.29,A = A^2 for every left N-bisubgroup A of a bi-near ring N.Since Nx $\cap$ Ny is a left N-bisub group of a bi-near ring N,Nx $\cap$ Ny = $(Nx \cap Ny)^2 \subseteq$ NxNy $\subseteq$ Ny.Again since N is a CPNS bi-near ring,NxNy=NyNx$\subseteq$Nx.Therefore Nx$\cap$Ny = NxNy. Now Nx = Nx$\cap$N = NxN implies Nxy = NxNy. Therefore Nxy = Nx$\cap$Ny for all x, y$\in N_1 \cup N_2$. Hence Nx $\cap$ Ny = Nxy for all x, y $\in N = N_1 \cup N_2$.

To prove (iii)$\Rightarrow$ (iv)

Assume that Nx$\cap$Ny = Nxy for all x, y $\in N = N_1 \cup N_2$. To Prove N is a left bi-potent bi-near ring.i.e. To Prove Na=Na^2 for all a$\in N_1 \cup N_2$. Let a$\in N_1 \cup N_2$ then Na = Na $\cap$ Na = Naa = Na^2.i.e., Na = Na^2 for all a$\in N_1 \cup N_2$. Hence N is a left bi-potent bi-near ring.

To prove (iv) $\Rightarrow$ (v)

Assume that N is a left bi-potent bi-near ring. To Prove N is a Boolean bi-near ring. i.e)To Prove a^2 = a for all a$\in N_1 \cup N_2$. Let a$\in N_1 \cup N_2$, by the assumption that a $\in$ Na=Na^2.Therefore a$\in$Na implies a$\in$Na^2 for all a$\in N_1 \cup N_2$. Hence N is a strongly bi-regular bi-near ring and so N is a bi-regular bi-near ring.Then for a$\in N_1 \cup N_2$ there exist b$\in N_1 \cup N_2$ such that a = aba =abaa = a^2. i.e., N is a Boolean bi-near ring.

To prove (v) $\Rightarrow$ (i)

Assume that N is a Boolean bi-near ring .Let B be a weak bi-ideal of a bi-near ring N then to prove B = B^3. B be a weak bi-ideal of a bi-near ring N implies B^3$\subseteq$B. Let x $\in$ B. By the assumption, x = x^2 = x^3 $\in$ B^3 implies B $\subseteq$ B^3 hence B = B^3 for every weak bi-ideal B of a bi-near ring N.

Theorem 3.2.4

Let N be a left self - distributive S-bi-near ring. Then the following conditions are equivalent.

(i) Q = QNQ for every quasi-ideal Q of a bi-near ring N.
(ii) B^3 = B for every weak bi-ideal B of a bi-near ring N.
(iii) NB^2 = B for every strong bi-ideal B of a bi-near ring N.
(iv) N is a bi-regular bi-near ring.
(v) B$_1$$\cap$B$_2$ = B$_1$B$_2$ $\cap$B$_2$B$_1$ for every pair of bi-ideals B$_1$, B$_2$ of a bi-near ring N.
(vi) Q$_1$$\cap$Q$_2$ = Q$_1$Q$_2$ $\cap$Q$_2$Q$_1$ for every pair of quasi-ideals Q$_1$,Q$_2$ of a bi-near ring N
(vii) Q^2 = Q for every quasi-ideal Q of a bi-near ring N.
(viii) B^2 = B for every bi-ideal B of a bi-near ring N.
(ix) N is strict weakly bi-regular bi-near ring.
(x) N is a strongly bi-regular bi-near ring.
(xi) N is a left bi-potent bi-near ring.
(xii) B = BNB for every bi-ideal B of a bi-near ring N.

Proof :

Let N be a left self - distributive S-bi-near ring

To prove (i) $\Rightarrow$ (ii)

Assume that Q = QNQ for every quasi-ideal Q of a bi-near ring N. To prove B^3 = B for every weak bi-ideal B of a bi-near ring N. Let a$\in N_1 \cup N_2$ then Na is a quasi-ideal of N then by our assumption Na = NaNNa. Since N is a S-bi-near ring, we have a$\in$Na=NaNNa$\subseteq$NaNa.i.e., a= n_1an_2a.Since N is left self-distributive, a = n_1an_2a = n_1an_2aa = $n_1an_2a^2$$\in$Na^2.Therefore a$\in$Na^2 for all a$\in N_1 \cup N_2$. Hence N is strongly bi-regular bi-near ring. Therefore, by Proposition 3.1.8, B^3 = B for every weak bi-ideal B of a bi-near ring N.

To prove (ii) $\Rightarrow$ (iii)

Assume that B^3 = B for every weak bi-ideal B of a bi-near ring N.Let B be a strong bi-ideal of a bi-near ring N then NB^2$\subseteq$B. Every strong bi-ideal of a bi-near ring is a bi-ideal of a bi-near ring and so weak bi-ideal.By the assumption B=B^3= BBB = BB^2 $\subseteq$ N$_1$B^2(or B$\subseteq$ N$_2$B^2)$\subseteq$ NB^2. i.e., B $\subseteq$ NB^2 and so B = NB^2 for every strong bi-ideal B of a bi-near ring N.

To prove (iii) $\Rightarrow$ (iv)

Assume that NB^2 = B for every strong bi-ideal B of a bi-near ring N.By the Proposition 3.1.10, N is a strongly bi-regular bi-near ring then by Lemma 2.28, N is a bi-regular bi-near ring.

To prove (iv) $\Rightarrow$ (v)

Assume that N is a bi-regular bi-near ring. Let B$_1$ and B$_2$ be a pair of bi-ideals of a bi-near ring N. Then prove that B$_1$$\cap$B$_2$ = B$_1$B$_2$ $\cap$B$_2$B$_1$ for every pair of bi-ideals B$_1$, B$_2$ of a bi-near ring N. Let x $\in$ B$_1$B$_2$ $\cap$ B$_2$B$_1$. Then x = b_1b_2 and x = $b_2'b_1'$. Now $b_1 = b_1a_1b_1$ and $b_2 = b_2a_2b_2$ for some $a_1,a_2$$\in N_1 \cup N_2$. Because N is a bi-regular bi-near ring.From this x = b_1b_2 = $b_1a_1b_1b_2$= $b_1a_1b_2'b_1'$$\in$ B$_1$NB$_1$ $\subseteq$ B$_1$. i.e., B$_1$B$_2$$\cap$B$_2B_1$$\subseteq$B$_1$.Similarly we can prove B$_1$B$_2$ $\cap$ B$_2$B$_1$ $\subseteq$ B$_2$. Hence B$_1$B$_2$ $\cap$ B$_2$B$_1$ $\subseteq$ B$_1$$\cap$B$_2$.On the other hand if x $\in$ B$_1$$\cap$B$_2$, then x = b_1 = b_2 for some b_1 $\in$B$_1$ and $b_2$$\in$ B$_2$.Since B$_1$ is a bi-ideal of a bi-near ring N and N is a bi-regular bi-near ring which gives B$_1$=B$_1$N$_1$B$_1$(or B$_1$=B$_1$N$_2$B$_1$)=BNB, and so $b_1 = b_1nb_1$ for some n$\in N_1 \cup N_2$. Since N is a left self- distributive bi-near ring, x = b_1 = b_1nb_1 = $b_1nb_1b_1$ = $b_1nb_1b_2$ $\in$ B$_1$N$_1$B$_1$B$_2$(or $\in$B$_1$N$_2$B$_1$B$_2$) $\subseteq$ B$_1$B$_2$.Therefore B$_1$$\cap$B$_2$$\subseteq$B$_1B_2$. Similarly we can prove that B$_1$$\cap$B$_2$ $\subseteq$ B$_2$B$_1$.Hence B$_1$$\cap$B$_2$ = B$_1$B$_2$ $\cap$ B$_2$B$_1$ for every pair of bi-ideals B$_1$, B$_2$ of a bi-near ring N.

To prove (v) $\Rightarrow$ (vi)

Assume that B$_1$$\cap$B$_2$ = B$_1$B$_2$ $\cap$ B$_2$B$_1$ for every pair of quasi-ideals B$_1$ and B$_2$ of a bi-near ring N. Since every quasi-ideal of a bi-near ring is also a bi-ideal of a bi-near ring N, we have Q$_1$$\cap$Q$_2$

$= Q_1Q_2 \cap Q_2Q_1$ for every pair of quasi-ideals Q_1 and Q_2 of a bi-near ring N.

To prove (vi) ⇒ (vii)

Assume that $Q_1 \cap Q_2 = Q_1Q_2 \cap Q_2Q_1$ for every pair of quasi-ideals Q_1 and Q_2 of a bi-near ring N.Take $Q_1 = Q_2 = Q$.By the assumption $Q = Q \cap Q = Q^2 \cap Q^2 = Q^2$. Hence $Q^2 = Q$ for every quasi-ideal Q of a bi-near ring N.

To prove (vii) ⇒ (viii)

Assume that $Q^2 = Q$ for every quasi-ideal Q of a bi-near ring N.To Prove $B^2 = B$ for every bi-ideal B of a bi-near ring N. Let $x \in B$ then Nx is a quasi-ideal of a bi-near ring N. If Nx is a quasi-ideal of N then $x \in Nx = (Nx)^2 = NxNx$($\because N$ is S-bi-near ring).i.e., $x = n_1xn_2x$ for some $n_1, n_2 \in N_1 \cup N_2$.Since N is a left self-distributive bi-near ring, $x = n_1xn_2x = n_1xn_2x^2 \in Nx^2$. From this $x \in Nx^2$ for all $x \in N_1 \cup N_2$.Hence N is a strongly bi-regular bi-near ring and so N is a bi-regular bi-near ring.Let $a \in BN \cap NB$. Then $a = xn = n_1x_1$ for some $x, x_1 \in B$ then $x, x_1 \in N_1 \cup N_2$ and $n, n_1 \in N_1 \cup N_2$. Since x is regular, $x = xyx$ for some $y \in N_1 \cup N_2$.Hence $a = xn = (xyx)n = (xy)(n_1x_1) \in BN_1B(\text{or}BN_2B) \subseteq BNB \subseteq B$.That is $BN \cap NB \subseteq B$. Therefore B is a quasi-ideal of a bi-near ring N and so $B^2 = B$ for every bi-ideal B of bi-near ring N.

To prove (viii) ⇒ (ix)

Assume that $B^2 = B$ for every bi-ideal B of a bi-near ring N.To prove N is a strict weakly bi-regular bi-near ring. Let $x \in N_1 \cup N_2$ then Nx is a bi-ideal of a bi-near ringN.Since N is a S-bi-near ring $x \in Nx = (Nx)^2$.Therefore $x \in N$ implies $x \in Nx = (Nx)^2$. Hence by the Theorem 2.30, N is a strict weakly bi-regular bi-near ring.

To prove (ix) ⇒ (x)

Assume that N is a strict weakly bi-regular bi-near ring.Let $a \in N_1 \cup N_2$ implies $a \in Na$($\because$ N is S-bi-near ring). If $a \in Na$ then $a \in Na = (Na)^2 = (Na)(Na)$. i.e., $a = n_1an_2a$ for some $n_1, n_2 \in N_1 \cup N_2$. Since N is a left self-distributive bi-near ring,$a = n_1an_2a = n_1an_2a^2 \in Na^2$. Hence for all $a \in N_1 \cup N_2$ implies $a \in Na^2$.Hence N is a strongly bi-regular bi-near ring.

To prove (x) ⇒ (xi)

Assume that N is strongly bi-regular bi-near ring then N is a bi-regular bi-near ring.Thus for every $a \in N_1 \cup N_2$ there exist $b \in N_1 \cup N_2$ such that $a = aba$. Let $x \in Na$ then $x = n_1a = n_1aba = n_1abaa \in Na^2$.From this $x \in Na$ implies $x \in Na^2$ for all $x \in N_1 \cup N_2$. Thus $Na = Na^2$ for all $a \in N_1 \cup N_2$ i.e., N is a left bi-potent bi-near ring.

To prove (xi) ⇒ (xii)

Assume that N is a left bi-potent bi-near ring.Let $a \in N_1 \cup N_2$ then $a \in Na = Na^2$ ($\because$ N is S-bi-near ring).Hence $a \in N = N_1 \cup N_2$ implies $a \in Na^2$ and so N is strongly bi-regular bi-near ring implies N is a bi-regular bi-near ring. Then by proposition 3.1.12 B = BNB for every bi-ideal B of a bi-near ring N.

To prove (xii) ⇒ (i)

Let Q be a quasi-ideal of a bi-near ring N. Since every quasi-ideal of a bi-near ring is a bi-ideal of a bi-near ring N, then Q = QNQ for every quasi-ideal Q of a bi-near ring N.

Proposition 3.2.5:

Let N be a left self – distributive S-bi-near ring and $B = B^3$ for every weak bi-ideal B of a bi-near ring N. Then a bi-near ring N has no non-zero zero-divisors if and only if N is a left simple bi-near ring.

Proof :

Let N be a left self-distributive S-bi-near ring and $B = B^3$ for every weak bi-ideal B of a bi-near ring N.Assume that N has no non-zero zero divisors.Since $B = B^3$ for every weak bi-ideal B of a bi-near ring N, then by the Proposition 3.1.17, N is a Boolean bi-near ring. i.e., For all $x \in N_1 \cup N_2$, $Nx = Nx^2$.Since N has no non-zero divisors, it follows that N=Nx for all $x \in (N_1 \cup N_2)^*$. Hence a bi-near ring N has no non-trivial left N-bisubgroup.Therefore a bi-near ring N is simple.Conversely assume that N is a left simple bi-near ring then by remark 2.31, a bi-near ring N has no non-zero zero-divisors.

Proposition 3.2.6:

Let N be a left self - distributive $\overline{S}$ - bi-near ring. Then $B = B^3$ for every weak bi-ideal B of a bi-near ring N if and only if xN = xNx for all $x \in N$.

Proof :

Assume that $B = B^3$ for every weak bi-ideal B of a bi-near ring N then by the Proposition 3.1.8, N is a strongly bi-regular bi-near ring and so N is a bi-regular bi-near ring. Hence by the Result 2.31 and by the Lemma 2.32,en = ene for $e \in E$ and $n \in N_1 \cup N_2$. Let $y \in xN$ then $y = xn_1 = xaxn_1$.($\because$ N is bi-regular).Since e = ax is an idempotent, $axn_1 = axn_1ax$ for all $n_1 \in N_1 \cup N_2$.Thus $y = xn_1 = xaxn_1 = x(axn_1ax) \in xNx$ and so $xN \subseteq xNx$. Hence $y \in xN$ implies $xN \subseteq xNx$.But trivially $xNx \subseteq xN$ and thus xN = xNx for $x \in N$.Conversely assume that xN = xNx for all $x \in N_1 \cup N_2$. since N is a $\overline{S}$ -bi-near ring, for $x \in N_1 \cup N_2$,$x \in$ xN= xNx which implies that x is regular. Therefore N is a bi-regular bi-near ring. Therefore by the Theorem 3.1.18, $B = B^3$ for every weak bi-ideal B of a bi-near ring N.

Theorem 3.2.7:

Let N be a left self-distributive S-bi-near ring. Then the following conditions are equivalent.

(i) $B = B^3$ for every weak bi-ideal B of a bi-near ring N with $E \subseteq N_d$.
(ii) $aNa = Na = Na^2$ for every $a \in N_1 \cup N_2$.
(iii) N is a S_k' and P_k bi-near ring for any positive integer k with $E \subseteq N_d$.
(iv) N is a stable bi-near ring.
(v) N is a P(1, 2) bi-near ring and S′ - bi-near ring.
(vi) N is a P(m,n) and bi-regular bi-near ring.
(vii) $B=B^mNB^n$ for every generalized(m,n) bi-ideal B of a bi-near ring N.

Proof :

To Prove (i) ⇒ (ii)

Assume that $B = B^3$ for every weak bi-ideal B of a bi-near ring N with $E \subseteq N_d$. Then by the Proposition 3.1.8, N is a strongly bi-regular bi-near ring. From this L={0}. By the Lemma 2.32, en = ene for all $e \in E$ & $n \in N_1 \cup N_2$.Since $E \subseteq N_d$, by the Lemma 2.33,ne=ene for all $e \in E$ and $n \in N_1 \cup N_2$. Hence $E \subseteq C(N)$. From this N is a GNF.Let $a \in N_1 \cup N_2$.Now N is strongly bi-regular bi-near ring then by the Proposition 2.34, N is a sub commutative bi-near ring.Therefore $aNa = Naa = Na^2$ for all $a \in N_1 \cup N_2$. By the Theorem 3.1.18, N is left bi-potent and so $Na = Na^2$ for all $a \in N_1 \cup N_2$. Thus $aNa = Na = Na^2$ for all $a \in N_1 \cup N_2$. Hence $aNa = Na = Na^2$ for all $a \in N_1 \cup N_2$.

To Prove (ii) ⇒ (iii)

Assume that $aNa = Na = Na^2$ for all $a \in N_1 \cup N_2$.Since N is a S-bi-near ring, $a \in Na = Na^2$ for all $a \in N_1 \cup N_2$.Therefore N is a strongly bi-regular bi-near ring. Therefore N contains no non-zero nilpotent elements. Thus eN = eNe. By the assumption eNe=Ne. Therefore eN = eNe = Ne and so by the Lemma 2.35, $E \subseteq C(N)$. Therefore by the Theorem 2.36, N is a P_k bi-near ring.Since $E \subseteq C(N)$, $E \subseteq N_d$.Let $x \in N_1 \cup N_2$. Since N is bi-regular bi-near ring and left self - distributive $x = xax = xax^2 = ... = x\,ax^k$ for all $x \in N_1 \cup N_2 \in Nx^k$ and so N is a S_k' bi-near ring.

To Prove (iii)⇒ (iv)

Assume that N is a S_k' and P_k bi-near ring for any positive integer k with $E \subseteq N_d$. Let $x \in N_1 \cup N_2$.Since N is a S_k'-P_k bi-near-ring, $x \in x^kN = xNx$, $\forall\, x \in N_1 \cup N_2$ and so N is a bi-regular bi-near ring. As in our assumption N is a P_k bi-near ring with $E \subseteq N_d$, using the Theorem 2.37,□and Corollary 2.38, $E \subseteq C(N)$. Therefore by the Theorem 2.39, N is a stable bi-near ring.

To Prove (iv)⇒ (v)

Assume that N is a stable bi-near ring. Let $x \in N_1 \cup N_2$.Since N is stable and a S-bi-near ring, $x \in Nx = xNx$ for all $x \in N_1 \cup N_2$. i.e., N is a bi-regular a bi-near ring.Then by the Theorem 2.37, $E \subseteq C(N)$. Again by Theorem 2.40, we get that N is a P(1,2) bi-near ring. Since N is a bi-regular bi-near ring, we get that N is a S′ -bi-near ring.

To Prove (v)⇒ (vi)

Assume that N is a S′and a P(1,2) bi-near ring,then $a \in aN = Na^2$ $\forall\, a \in N_1 \cup N_2$.Therefore N is a strongly bi-regular bi-near ring and so N is a bi-regular bi-near ring. By the Theorem 2.41, N is a P(m,n) bi-near ring.

To Prove (vi)⇒ (vii)

Assume that N is a bi-regular and P(m,n) bi-near ring, Then by the Theorem 2.42, and by the Theorem 2.41,$E \subseteq C(N)$. Let B be a generalized (m,n) bi-ideal of a bi-near ring N implies $B^mNB^n \subseteq B$.Let $x \in B$. Since N is a bi-regular bi-near ring, then for $x \in N_1 \cup N_2$ there exist some $y \in N_1 \cup N_2$ such that $x = xyx = xaxyxax = xa\,(xyx)\,ax = (xa)^m\,(xyx)\,(ax)^n = x^ma^m\,(xyx)\,a^nx^n \in x^m\,Nx^n \in B^mNB^n$. i.e.,$B \subseteq B^m\,N\,B^n$. Therefore $B = B^m\,N\,B^n$ for every generalized (m, n) bi – ideal B of a bi-near ring N.

To Prove (vii)⇒ (i)

Assume that $B = B^m\,N\,B^n$ for every generalized (m, n) bi – ideal B of a bi-near ring N. Since every bi-ideal is a generalized (m,n) bi-ideal of a bi-near ring N, $B = B^mNB^n \subseteq BNB$. Thus B=BNB forevery bi–ideal B of a bi-near ring N.By Proposition 3.1.18, $B = B^3$ for every weak bi- ideal B of bi-near ring N.

Proposition 3.2.8:

Let N be a left self - distributive S-bi-near ring and $B = B^3$ for every weak bi-ideal B of N. Then the following conditions are true.

(i) $B \cap R = BR \cap RB$ for every bi-ideal B of N and for every right N-bisubgroup R of a bi-near ring N.
(ii) $B \cap L = BL \cap LB$ for every bi-ideal B of N and for every left N-bisubgroup L of a bi-near ring N.
(iii) $Q \cap L = QL \cap LQ$ for every quasi-ideal Q of N and for every left N-bisubgroup L of a bi-near ring N.

(iv) $Q \cap R = QR \cap RQ$ for every quasi-ideal Q of N and for every right N-bisubgroup R of a bi-near ring N.
(v) $R \cap L = RL \cap LR = RL$ for every right and left N-bisubgroups R and L of a bi-near ring N.

Proof:

(i) Let B be a bi-ideal of a bi-near ring N. If $x \in BR \cap RB \subseteq BN \cap NB$, then $x = bn = n_1b_1$, where $b, b_1 \in B$ and $n, n_1 \in N_1 \cup N_2$. By the Proposition 3.1.8, N is a strongly bi-regular bi-near ring.Therefore N is a bi-regular bi-near ring. Let $b \in B$.Then $b = bab$ for some $a \in N_1 \cup N_2$.Therefore $x=bn=babn= ban_1b_1 \in BNB \subseteq B$. Also $x \in BR \cap RB \subseteq NR \cap RN \subseteq RN \subseteq R$. Therefore $BR \cap RB \subseteq B \cap R$.On the other hand let $x \in B \cap R$.Then $x=b=r$ where $b \in B$ and $r \in R$. If $b \in B$, then $b=bab$ for some $a \in N_1 \cup N_2$ and $r=rcr$ for some $c \in N_1 \cup N_2$.Now $x = b = bab = babr = br \in BR$. Similarly $x = r = rcr = rcrr = rcrb = rb \in RB$ and so $B \cap R \subseteq BR \cap RB$.Therefore $BR \cap RB = B \cap R$ for every bi–ideal B of a bi-near ring N and for every right N-bisubgroup R of a bi-near ring N.

(ii) Proof follows as in **(i).**

(iii) and **(iv)** Let Q be a quasi - ideal of a bi-near ring N. Every quasi - ideal a bi-near ring is also a bi - ideal of a bi-near ring N. From (i) and (ii), (iii) and (iv) are true.

(v) Clearly $RL \cap LR \subseteq R \cap L$. If $x \in R \cap L$,then $x = r = l$ for some $r \in R$ and $l \in L$. Since N is a bi-regular bi-near ring $r = rar$ and $l = lbl$ for some $a,b \in N_1 \cup N_2$. Then $x = r = rar = rarr = rarl = rl \in RL$.Similarly $x \in LR$.Therefore $R \cap L = RL \cap LR$. By the assumption,N is a bi-regular a bi-near ring. Every bi-regular bi-near ring is also B-biregular a bi-near ring. Therefore by the Proposition 2.43, $R \cap L = RL$ for every left and right N - bisubgroups L and R of a bi-near ring N.

The biggest application that we can enter now is the building of a large steel rusting bi-nearring near our institute. We think it's cool. It's also the reason that I have used the "open-top" bi-nearring symbol for this collection of pages, since this is the largest (3 metres plus, that's 10 feet for the imperialists) application of nearring. I respect it's orientation. It can become one more religious war, like left-right-ness or the existance of hyphens.One element that has appeared recently as an application of nearrings is the use of planar and other nearrings to develop designs and codes. Roland Eggestberger, Gerhard Wagner, Peter Fuchs, Gunter Pilz all worked on some projects in this direction. Other people who have been major movers in this direction are Jim Clay and We.F. Ke.

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