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Cross-Entropy and Prioritized Aggregation Operator with Simplified Neutrosophic Sets and Their Application in Multi-Criteria Decision-Making Problems

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Abstract Simplified neutrosophic sets (SNSs) can effectively solve the uncertainty problems, especially those involving the indeterminate and inconsistent information. Considering the advantages of SNSs, a new approach for multi-criteria decision-making (MCDM) problems is developed under the simplified neutrosophic environment. First, the prioritized weighted average operator and prioritized weighted geometric operator for simplified neutrosophic numbers (SNNs) are defined, and the related theorems are also proved. Then two novel effective cross-entropy measures for SNSs are proposed, and their properties are proved as well. Furthermore, based on the proposed prioritized aggregation operators and cross-entropy measures, the ranking methods for SNSs are established in order to solve MCDM problems. Finally, a practical MCDM example for coping with supplier selection of an automotive company is used to demonstrate the effectiveness of the developed methods. Moreover, the same example-based comparison analysis of between the proposed methods and other existing methods is carried out.

Keywords Simplified neutrosophic sets · Prioritized aggregation operator · Cross-entropy · Multi-criteria decision-making

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1 Introduction

Fuzzy set (FS) theory was introduced by Zadeh [1] and used as a key method to solve multi-criteria decision-making (MCDM) problems [2], and pattern recognition [3]. But, some issues, where the membership degree is difficult to be defined by one specific value, cannot be well dealt with by FSs. In order to overcome the shortcomings of Zadeh's FS theory, Atanassov [4] introduced intuitionistic fuzzy sets (IFSs) and Gau and Buehrer [5] defined vague sets, but in fact, IFSs and vague sets are mathematically equivalent collections. Because of the advantages that an IFS considers the membership-degree, non-membership degree and hesitation degree simultaneously, it is more flexible and useful to describe the uncertain information than a traditional FS. Thus, many methods based on IFSs have been put forward and widely applied to solve MCDM problems [6–12], medical diagnosis [13, 14], pattern recognition [15, 16], stock market prediction [17, 18], and marketing strategy selection [19]. However, in some real situations, the membership degree, non-membership degree and hesitation degree may be difficultly given by specific numbers; hence, interval-valued intuitionistic fuzzy sets (IVIFSs) [20] are developed and applied to solve such problems [21–25]. In addition, Torra and Narukawa [26] proposed hesitant fuzzy sets (HFSs) to deal with the hesitant situation when people express their preferences for objects in a decision-making process. Since then, many researches on HFSs and their extensions have been carried out. Chen et al. [27] proposed interval-valued hesitant fuzzy sets (IVHFSs) and verified the effectiveness in solving MCDM problems. Wang et al. introduced several hesitant fuzzy linguistic aggregation operator-based methods [28, 29], and Zhou et al. [30] proposed a linguistic hesitant fuzzy decision-making method based on evidential

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reasoning to solve MCDM problems. Moreover, Wang et al. [31] studied hesitant fuzzy linguistic term sets, Tian et al. [32] introduced gray linguistic sets based on gray sets and linguistic term sets, and Peng et al. [33] proposed intuitionistic hesitant fuzzy sets based on HFSs and IFSs.

Although FSs have been extended and generalized, they still cannot handle all types of problems with uncertainty in reality, especially those of the indeterminate and inconsistent information [34]. For example, when an expert is asked for the opinion about a certain statement, he or she may say the possibility that the statement is true is 0.5, the possibility that the statement is false is 0.6, and the degree that he or she is not sure is 0.2. Such issues cannot be properly solved using HFSs and IFSs. Thus, a new theory is required.

Smarandache [35] proposed neutrosophic logic and neutrosophic sets, and then several researchers have made their efforts to enrich NSs [36–42]. Recently, some methods on simplified neutrosophic sets (SNSs) and interval neutrosophic sets (INSs) have been put forward and used to solve MCDM problems [43–51]. For example, Ye [43] defined the operational rules of SNSs and proposed a method with simplified neutrosophic information based on the weighted arithmetic average operator and the weighted geometric average operator. Ye [43, 44] proposed different methods based on single valued neutrosophic measures: one is the cosine similarity-based measure method, and another is the logarithm–based cross-entropy measure method. The effectiveness of both methods for MCDM problems have been proved through the same illustrative example. However, Peng et al. [50, 51] pointed out some limitations of previous research papers for SNSs [43–45], including the lacks of the SNS operation and cross-entropy measure, and brought forward an improved method of SNSs. In a word, it has been demonstrated that neutrosophic set-based methods are effective tools to handle indeterminate and inconsistent information, which cannot be achieved using HFSs and IFSs.

In this paper, in order to overcome the lacks of previous proposed methods [43–45], the prioritized weighted average operator (SNNPWA) and prioritized weighted geometric operator (SNNPWG) for SNS are defined and two novel cross-entropy measures are proposed. Moreover, based on the proposed operators and measures, the ranking methods are established. Then the assessment information of alternatives with respect to criteria is given by truth-membership degree, indeterminate-membership degree, and falsity-membership degree under simplified neutrosophic environment, and then the ranking of all alternatives is obtained using the developed approach.

The paper is organized as follows. Some concepts of NSs, SNSs, prioritized aggregation (PA) operator, and cross-entropy are introduced in Sect. 2. In Sect. 3, the SNNPWA and SNNPWG operators are defined and proved, two novel cross-entropy measures are proposed and their effectiveness is verified. Section 4 provides the ranking method for MCDM problems with simplified neutrosophic information. Section 5 shows the illustration of our approaches and the comparison analysis between the proposed methods and other existing methods. Finally, conclusions are drawn in Sect. 6.

2 Preliminaries

In this section, some basic concepts and definitions of NSs, SNSs, PA operator, cross-entropy, and cosine similarity measure are briefly reviewed.

2.1 NS and SNSs

In this subsection, the definitions and operations of NSs and SNSs are introduced.

Definition 1 [35] Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy- membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]0^-,1^+[$, that is, $T_A(x):X-\to]0^-,1^+[$, $I_A(x):X-\to]0^-,1^+[$, and $F_A(x):X-\to]0^-,1^+[$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^-\le\sup T_A(x)+\sup I_A(x)+\sup F_A(x)\le 3^+$.

Definition 2 [35] A neutrosophic set A is contained in the other neutrosophic set B, denoted by $A\subseteq B$ if and only if $\inf T_A(x)\le\inf T_B(x)$, $\sup T_A(x)\ge\sup T_B(x)$, $\inf I_A(x)\ge\inf I_B(x)$, $\sup I_A(x)\ge\sup I_B(x)$, $\inf F_A(x)\ge\inf F_B(x)$, and $\sup F_A(x)\ge\sup F_B(x)$ for every x in X.

Since it is hard to use NSs to solve practical problems, so Ye [43] reduced NSs of nonstardard intervals into a kind of SNSs of standard intervals.

Definition 3 [43] Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, a indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. If the functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are singleton subintervals/subsets in the real standard $[0,1]$, that is, $T_A(x):X-\to[0,1]$, $I_A(x):X-\to[0,1]$ and $F_A(x):X-\to[0,1]$. Then, a simplification of the neutrosophic set A is denoted by $A=\{\langle x,T_A(x),I_A(x),F_A(x)\rangle|x\in X\}$ which is called a SNS. It is a subclass of NSs.

Definition 4 [43]. A SNS A is contained in the other SNS B, denoted by $A \subseteq B$ if and only if $T_A(x) \le T_B(x)$, $I_A(x) \ge I_B(x)$ and $F_A(x) \ge F_B(x)$, for any $x \in X$. Especially, $A = B$ if $A \subseteq B$ and $B \subseteq A$. The complement set of A denoted by A^C is defined as $A^C = \{ < x, F_A(x), I_A(x), T_A(x) > | x \in X\}$.

Definition 5 [43] Let A and B are two SNSs, the operations of SNSs are defined as follows.

(1) $A + B = \langle T_A(x) + T_B(x) - T_A(x)T_B(x), I_A(x) + I_B(x) - I_A(x)I_B(x), F_A(x) + F_B(x) - F_A(x)F_B(x)\rangle$;
(2) $A \cdot B = \langle T_A(x)T_B(x), I_A(x)I_B(x), F_A(x)F_B(x)\rangle$;
(3) $\lambda A = \left\langle 1 - (1 - T_A(x))^{\lambda}, 1 - (1 - I_A(x))^{\lambda}, 1 - (1 - F_A(x))^{\lambda}\right\rangle$, $\lambda > 0$;
(4) $A^{\lambda} = \left\langle (T_A(x))^{\lambda}, (I_A(x))^{\lambda}, (F_A(x))^{\lambda}\right\rangle, \lambda > 0$.

However, Peng et al. [50, 51] pointed out there are still some lacks in Definition 5. In some cases, the operations such as $A + B$ and $A \cdot B$ might be impractical as presented in Example 1.

Example 1 Let $A = \{\langle x, 0.5, 0.5, 0.5\rangle\}$ and $B = \{\langle x, 1, 0, 0\rangle\}$ be two SNSs. Obviously, $B = \{\langle x, 1, 0, 0\rangle\}$ is the largest SNSs. Theoretically, the sum of an arbitrary value and the maximum value should be equal to the maximum value. However, according to Definition 5, $A + B = \{\langle x, 1, 0.5, 0.5, 0.5\rangle\} \ne B$. Thus, the operation "+" cannot be accepted. Similar contradictions exist in other operations of Definition 5, and thus the operations of SNSs need to be redefined.

Definition 6 [50, 51] Let A and B be two SNSs, and the operations of SNSs can be defined as follows:

(1) $A + B = \langle T_A(x) + T_B(x) - T_A(x)T_B(x), I_A(x)I_B(x), F_A(x)F_B(x)\rangle$;
(2) $A \cdot B = \langle T_A(x)T_B(x), I_A(x) + I_B(x) - I_A(x)I_B(x), F_A(x) + F_B(x) - F_A(x)F_B(x)\rangle$;
(3) $\lambda A = \left\langle 1 - (1 - T_A(x))^{\lambda}, (I_A(x))^{\lambda}, (F_A(x))^{\lambda}\right\rangle, \lambda > 0$;
(4) $A^{\lambda} = \left\langle (T_A(x))^{\lambda}, 1 - (1 - I_A(x))^{\lambda}, 1 - (1 - F_A(x))^{\lambda}\right\rangle$, $\lambda > 0$.

2.2 Prioritization Aggregation Operator

The prioritization aggregation (PA) operator was originally introduced by Yager [52], and is shown as follows.

Definition 7 [52] Let $G = \{G_1, G_2, \cdots, G_n\}$ be a collection of criteria and there is a prioritization between the criteria expressed by the linear ordering $G_1 \succ G_2 \succ G_3 \succ \cdots \succ G_n$, which indicates the criteria G_j has a higher priority than G_k, if $j < k$. $G_j(x)$ is an evaluation value denoting the performance of the alternative x under the criteria G_j, and satisfies $G_j \in [0, 1]$, thus

$$\mathrm{PA}(G_j(x)) = \sum_{j=1}^{n} W_j G_j(x),$$

where $W_j = \frac{T_j}{\sum_{i=1}^{n} T_i}$, $T_1 = 1$ and $T_j = \prod_{k=1}^{j-1} G_j(x)$ $(j = 2, \cdots, n)$. Then PA is called the prioritized aggregation operator.

2.3 Cross-Entropy of FSs and SNSs

The cross-entropy measure was introduced by Kullback [53] and its definition is shown as follows.

Definition 8 [53] Let $P = \{p_1, p_2, \cdots, p_n\}$ and $Q = \{q_1, q_2, \cdots, q_n\}$ be two given probability distributions, where $p_i \ge 0$, $\sum_{i=1}^{n} p_i = 1$, $q_i \ge 0$ and $\sum_{i=1}^{n} q_i = 1$ for $i = (1, 2, \cdots, n)$, the cross-entropy measure of P to Q is defined as

$$H(P, Q) = \sum_{i=1}^{n} p_i \times \ln\frac{p_i}{q_i}.$$

Based on Kullback's entropy definition, Shang and Jiang [54] proposed the cross-entropy measure between two FSs.

Definition 9 [54]. Assume that $A = \{A(x_1), A(x_2), \cdots, A(x_n)\}$ and $B = \{B(x_1), B(x_2), \cdots, B(x_n)\}$ are two FSs in the universe of discourse $X = \{x_1, x_2, \cdots, x_n\}$, and the fuzzy cross-entropy of A from B is defined as follows:

$$H(A, B) = \sum_{i=1}^{n}\left(A(x_i)\log_2^{\frac{A(x_i)}{\frac{1}{2}(A(x_i)+B(x_i))}} + (1 - A(x_i))\log_2^{\frac{1-A(x_i)}{1-\frac{1}{2}(A(x_i)+B(x_i))}}\right),$$

which indicates the degree of discrimination of A from B.

However, $H(A, B)$ is not symmetric with respect to its arguments. Shang and Jiang [54] proposed a symmetric discrimination information measure $I(A, B) = H(A, B) + H(B, A)$.

Similarly, considering the indeterminacy-membership and falsity-membership functions, Ye [44] proposed the cross-entropy measure of SNSs as follows:

$$\begin{aligned} &E(A,B)\\ &= \sum_{i=1}^{n}\left(T_A(x_i)\log_2^{\frac{T_A(x_i)}{\frac{1}{2}(T_A(x_i)+T_B(x_i))}} + (1 - T_A(x_i))\log_2^{\frac{1-T_A(x_i)}{1-\frac{1}{2}(T_A(x_i)+T_B(x_i))}}\right)\\ &+ \sum_{i=1}^{n}\left(I_A(x_i)\log_2^{\frac{I_A(x_i)}{\frac{1}{2}(I_A(x_i)+I_B(x_i))}} + (1 - I_A(x_i))\log_2^{\frac{1-I_A(x_i)}{1-\frac{1}{2}(I_A(x_i)+I_B(x_i))}}\right)\\ &+ \sum_{i=1}^{n}\left(F_A(x_i)\log_2^{\frac{F_A(x_i)}{\frac{1}{2}(F_A(x_i)+F_B(x_i))}} + (1 - F_A(x_i))\log_2^{\frac{1-F_A(x_i)}{1-\frac{1}{2}(F_A(x_i)+F_B(x_i))}}\right), \end{aligned}$$

which also indicates the discrimination degree of the SNSs A from B. Moreover, it can be easily proved that

$E(A,B) \geq 0$ and $E(A,B) = 0$ if and only if $T_A(x_i) = T_B(x_i)$, $I_A(x_i) = I_B(x_i)$, and $F_A(x_i) = F_B(x_i)$ for any $x_i \in (X)$. $E(A^C, B^C) = E(A,B)$, where A^C and B^C are the complement of SNSs A and B, respectively. Then, $E(A,B)$ is not symmetric, and similarly, it could be revised to a symmetric discrimination information measure for SNSs as $D(A,B) = E(A,B) + E(B,A)$.

The larger the difference between A and B is, the larger $D(A,B)$ is.

2.4 Cosine Similarity Measure of SNSs

The cosine similarity measure of SNSs was introduced by Ye [43], which was induced from the correlation coefficient of Ye [45]. To rank the alternatives in the decision-making process, Ye [43] defined the SNS value of ideal alternative as $\alpha^* = \langle 1,0,0\rangle$, and the cosine similarity measure between SNSs $\alpha^i (i = 1,2,\cdots,n)$ and α^* is defined as follows:

$$S_i(\alpha^i, \alpha^*) = \frac{t_i t^* + i_i i^* + f_i f^*}{\sqrt{t_i^2 + i_i^2 + f_i^2}\sqrt{t^{*2} + i^{*2} + f^{*2}}} = \frac{t_i}{\sqrt{t_i^2 + i_i^2 + f_i^2}}.$$

The bigger the measure value $S_i(\alpha^i, \alpha^*)$ $(i = 1,2,\cdots,n)$ is, the better alternative A_i is.

However, the cosine measure above has the lacks when it is used in a real situation as demonstrated in Example 2.

Example 2 Let $A_1 = \{\langle x, 0.8, 0, 0\rangle\}$ and $A_2 = \{\langle x, 0.2, 0, 0\rangle\}$ be two SNSs. Obviously, A_1 is superior than A_2, that is, $S_1(\alpha^1, \alpha^*) > S_2(\alpha^2, \alpha^*)$. However, according to the cosine measure of Ye [43], $S_1(\alpha^1, \alpha^*) = S_2(\alpha^2, \alpha^*) = 1$.

Therefore, the results in Example 2 cannot be accepted, and the measure given in [43] needs to be improved.

3 SNNPWA and SNNPWG Operators and Cross-Entropy Measure for SNSs

3.1 SNNPWA and SNNPWG Operators

In this subsection, the score function of a simplified neutrosophic number (SNN) is first defined. Then, the SNNPWA and SNNPWG operators are defined, and their relative theorems are proved.

From the intuitive judgment, A SNN A, which is closer to the ideal SNN $A^+ = \langle 1,0,0\rangle$, should possess a higher score, thus, the score function $S(A)$ can be defined as follows:

Definition 10 Let $A = \langle T_A, I_A, F_A\rangle$ be a SNN, and the score function $S(A)$ is represented as follows:

$$S(A) = \frac{T_A + 1 - I_A + 1 - F_A}{3} \tag{1}$$

Example 3 If $A = \langle 0.8, 0.2, 0.2\rangle$, by applying Eq. (1), then $S(A) = \frac{0.8+1-0.2+1-0.2}{3} = 0.8$.

In the following, the prioritized weighted average operator and prioritized weighted geometric operator under simplified neutrosophic environment are defined, and their related theorems are given.

Definition 11 Let $A_j = \langle T_{A_j}, I_{A_j}, F_{A_j}\rangle$ $(j = 1,2,\cdots,n)$ be a collection of SNNs, and the SNNPWA operator can be defined as follows:

$$\begin{aligned}\text{SNNPWA}(A_1, A_2, \cdots, A_n) &= \frac{T_1}{\sum_{i=1}^n T_i}A_1 + \frac{T_2}{\sum_{i=1}^n T_i}A_2 + \cdots + \frac{T_n}{\sum_{i=1}^n T_i}A_n \\ &= \sum_{j=1}^n \frac{T_j A_j}{\sum_{i=1}^n T_i},\end{aligned}$$

where $T_j = \Pi_{k=1}^{j-1} S(A_k)$ $(j = 2,\cdots,n)$, $T_1 = 1$ and $\mathrm{S}(A_k)$ is the score function of A_k.

Theorem 1 *For the collection of SNNs $A = \{A_j | j = 1,2,\cdots,n\}$, the following aggregated results will be obtained by using the SNNPWA operator:*

$$\text{SNNPWA}(A_1, A_2, \cdots, A_n) = \left\langle 1 - \prod_{j=1}^n (1 - T_{A_j})^{\frac{T_j}{\sum_{i=1}^n T_i}}, \prod_{j=1}^n (I_{A_j})^{\frac{T_j}{\sum_{i=1}^n T_i}}, \prod_{j=1}^n (F_{A_j})^{\frac{T_j}{\sum_{i=1}^n T_i}} \right\rangle, \tag{2}$$

where $T_j = \Pi_{k=1}^{j-1} S(A_k)$ $(j = 2,\cdots,n)$, $T_1 = 1$ and $S(A_k)$ is the score function of A_k.

Proof Clearly, according to Definition 11 and the operation of SNSs defined in Definition 6, Eq. (2) can be proven by utilizing mathematic induction.

(1) When $n = 2$, we have

$$\frac{T_1}{\sum_{i=1}^n T_i}A_1 = \left\langle 1 - (1 - T_{A_1})^{\frac{T_1}{\sum_{i=1}^n T_i}}, I_{A_1}^{\frac{T_1}{\sum_{i=1}^n T_i}}, F_{A_1}^{\frac{T_1}{\sum_{i=1}^n T_i}} \right\rangle,$$

$$\frac{T_2}{\sum_{i=1}^n T_i}A_2 = \left\langle 1 - (1 - T_{A_2})^{\frac{T_2}{\sum_{i=1}^n T_i}}, I_{A_2}^{\frac{T_2}{\sum_{i=1}^n T_i}}, F_{A_2}^{\frac{T_2}{\sum_{i=1}^n T_i}} \right\rangle.$$

Then

$$
\begin{aligned}
&\mathrm{SNNPWA}(A_1,A_2)\\
&\quad=\left\langle 1-(1-T_{A_1})^{\frac{1}{\sum_{i=1}^{2}T_i}},(I_{A_1})^{\frac{1}{\sum_{i=1}^{2}T_i}},(F_{A_1})^{\frac{1}{\sum_{i=1}^{2}T_i}}\right\rangle\\
&\quad+\left\langle 1-(1-T_{A_2})^{\frac{T_2}{\sum_{i=1}^{2}T_i}},(I_{A_2})^{\frac{T_2}{\sum_{i=1}^{2}T_i}},(F_{A_2})^{\frac{1}{\sum_{i=1}^{2}T_i}}\right\rangle\\
&=\left\langle 1-(1-T_{A_1})^{\frac{1}{\sum_{i=1}^{2}T_i}}+1-(1-T_{A_2})^{\frac{T_2}{\sum_{i=1}^{2}T_i}}\right.\\
&\quad-(1-(1-T_{A_1})^{\frac{1}{\sum_{i=1}^{2}T_i}})\\
&\quad\times\ (1-(1-T_{A_2})^{\frac{T_2}{\sum_{i=1}^{2}T_i}}),((I_{A_1})^{\frac{1}{\sum_{i=1}^{2}T_i}}\\
&\quad\left.\times(I_{A_2})^{\frac{T_2}{\sum_{i=1}^{2}T_i}},(F_{A_1})^{\frac{1}{\sum_{i=1}^{2}T_i}}\times(F_{A_2})^{\frac{1}{\sum_{i=1}^{2}T_i}}\right\rangle\\
&=\left\langle 1-\prod_{j=1}^{2}(1-T_{A_j}(x))^{\frac{T_j}{\sum_{i=1}^{2}T_i}},\prod_{j=1}^{2}(I_{A_j}(x))^{\frac{T_j}{\sum_{i=1}^{2}T_i}},\right.\\
&\quad\left.\times\prod_{j=1}^{2}(F_{A_j}(x))^{\frac{T_j}{\sum_{i=1}^{2}T_i}}\right\rangle. \qquad (3)
\end{aligned}
$$

(2) When $n=k$, the following results can be obtained by applying Eq. (2)

$$
\begin{aligned}
&\mathrm{SNNPWA}(A_1,A_2,\cdots,A_k)\\
&=\left\langle 1-\prod_{j=1}^{k}(1-T_{A_j})^{\frac{T_j}{\sum_{i=1}^{n}T_i}},\prod_{j=1}^{k}(I_{A_j})^{\frac{T_j}{\sum_{i=1}^{n}T_i}},\prod_{j=1}^{k}(F_{A_j})^{\frac{T_j}{\sum_{i=1}^{n}T_i}}\right\rangle. \qquad (4)
\end{aligned}
$$

When $n=k+1$, by using Eqs. (3) and (4), we can obtain

$$
\begin{aligned}
&\mathrm{SNNPWA}(A_1,A_2,\cdots,A_{k+1})\\
&=\left\langle 1-\prod_{j=1}^{k}(1-T_{A_j})^{\frac{T_j}{\sum_{i=1}^{n}T_i}},\prod_{j=1}^{k}(I_{A_j})^{\frac{T_j}{\sum_{i=1}^{n}T_i}},\prod_{j=1}^{k}(F_{A_j})^{\frac{T_j}{\sum_{i=1}^{n}T_i}}\right\rangle\\
&+\left\langle 1-(1-T_{A_{k+1}})^{\frac{T_{k+1}}{\sum_{i=1}^{n}T_i}},(I_{A_{k+1}})^{\frac{T_{k+1}}{\sum_{i=1}^{n}T_i}},(F_{A_{k+1}})^{\frac{T_{k+1}}{\sum_{i=1}^{n}T_i}}\right\rangle\\
&=\left\langle 1-\prod_{j=1}^{k+1}(1-T_{A_j})^{\frac{T_j}{\sum_{i=1}^{n}T_i}},\prod_{j=1}^{k+1}(I_{A_j})^{\frac{T_j}{\sum_{i=1}^{n}T_i}},\prod_{j=1}^{k+1}(F_{A_j})^{\frac{T_j}{\sum_{i=1}^{n}T_i}}\right\rangle.
\end{aligned}
$$

The proof that Eq. (2) holds for any n is completed now.

Property 1 (Boundedness). *Let* $A=\{Aj|j=1,2,\cdots,n\}$ *be a collection of SNNs.* $A^-=\langle \mathrm{Min}(T_{A_j}),\mathrm{Max}(I_{A_j}),\mathrm{Max}(F_{A_j})\rangle$, $A^+=\langle \mathrm{Max}(T_{A_j}),\mathrm{Min}(I_{A_j}),\mathrm{Min}(F_{A_j})\rangle$, *and then* $A^-\subseteq \mathrm{SNNPWA}\ (A_1,A_2,\cdots,A_n)\subseteq A^+$.

Proof Since

$$
\mathrm{SNNPWA}\,(A_1,A_2,\cdots,A_n)=\left\langle 1-\prod_{j=1}^{n}(1-T_{A_j})^{\frac{T_j}{\sum_{i=1}^{n}T_i}},\prod_{j=1}^{n}(I_{A_j})^{\frac{T_j}{\sum_{i=1}^{n}T_i}},\prod_{j=1}^{n}(F_{A_j})^{\frac{T_j}{\sum_{i=1}^{n}T_i}}\right\rangle.
$$

Then, we have

$$
\begin{aligned}
1-\prod_{j=1}^{n}(1-T_{A_j})^{\frac{T_j}{\sum_{i=1}^{n}T_i}}&\geq 1-\prod_{j=1}^{n}(1-\mathrm{Min}(T_{A_j}))^{\frac{T_j}{\sum_{i=1}^{n}T_i}}\geq 1\\
&-(1-\mathrm{Min}(T_{A_j}))\\
&=\mathrm{Min}(T_{A_j}),
\end{aligned}
$$

$$
\prod_{j=1}^{n}(I_{A_j})^{\frac{T_j}{\sum_{i=1}^{n}T_i}}\leq\prod_{j=1}^{n}(\mathrm{Max}(I_{A_j}))^{\frac{T_j}{\sum_{i=1}^{n}T_i}}\leq \mathrm{Max}(I_{A_j}),
$$

$$
\prod_{j=1}^{n}(F_{A_j})^{\frac{T_j}{\sum_{i=1}^{n}T_i}}\leq\prod_{j=1}^{n}(\mathrm{Max}(F_{A_j}))^{\frac{T_j}{\sum_{i=1}^{n}T_i}}\leq \mathrm{Max}(F_{A_j}).
$$

According to Definition 4 and the induced result above, $A^-\subseteq \mathrm{SNNPWA}\,(A_1,A_2,\cdots,A_n)$ holds. By the similar induction process, we obtain

$$
\begin{aligned}
1-\prod_{j=1}^{k}(1-T_{A_j})^{\frac{T_j}{\sum_{i=1}^{n}T_i}}&\leq 1\\
&-\prod_{j=1}^{k}(1-\mathrm{Max}(T_{A_j}))^{\frac{T_j}{\sum_{i=1}^{n}T_i}}\leq 1-(1-\mathrm{Max}(T_{A_j}))\\
&=\mathrm{Max}(T_{A_j}),
\end{aligned}
$$

$$
\prod_{j=1}^{n}(I_{A_j})^{\frac{T_j}{\sum_{i=1}^{n}T_i}}\geq\prod_{j=1}^{n}(\mathrm{Min}(I_{A_j}))^{\frac{T_j}{\sum_{i=1}^{n}T_i}}\geq \mathrm{Min}((I_{A_j}),
$$

$$
\prod_{j=1}^{n}(F_{A_j})^{\frac{T_j}{\sum_{i=1}^{n}T_i}}\geq\prod_{j=1}^{n}(\mathrm{Min}(F_{A_j}))^{\frac{T_j}{\sum_{i=1}^{n}T_i}}\geq \mathrm{Min}(F_{A_j}).
$$

In accordance with Definition 4, $\mathrm{SNNPWA}\,(A_1,A_2,\cdots,A_n)\subseteq A^+$ holds.

Thus, $A^-\subseteq \mathrm{SNNPWA}\,(A_1,A_2,\cdots,A_n)\subseteq A^+$ holds.

Property 2 (Idempotency). *Let* $A=\{A_j|j=1,2,\cdots,n\}$ *be a collection of SNNs. If* $A_j=B=\langle T_B,I_B,F_B\rangle$, *then* $\mathrm{SNNPWA}\,(A_1,A_2,\cdots,A_n)=B$.

Proof Utilizing Eq. (2), we have

$$
\begin{aligned}
&\mathrm{SNNPWA}\,(A_1,A_2,\cdots,A_n)\\
&=\left\langle 1-\prod_{j=1}^{n}(1-T_B)^{\frac{T_j}{\sum_{i=1}^{n}T_i}},\prod_{j=1}^{n}(I_B)^{\frac{T_j}{\sum_{i=1}^{n}T_i}},\prod_{j=1}^{n}(F_B)^{\frac{T_j}{\sum_{i=1}^{n}T_i}}\right\rangle\\
&=\langle 1-(1-T_B),I_B,F_B\rangle=\langle T_B,I_B,F_B\rangle=B.
\end{aligned}
$$

Thus, Property 2 is proved.

Property 3 (Monotonity). *If* $A = \{A_j | j = 1, 2, \cdots, n\}$ *and* $A^* = \{A_j^* | j = 1, 2, \cdots, n\}$ *are two collections of SNNs. If* $A_j \subseteq A_j^*$ *for* $j = 1, 2, \cdots, n$, *then* $\mathrm{SNNPWA}(A_1, A_2, \cdots, A_n) \subseteq \mathrm{SNNPWA}(A_1^*, A_2^*, \cdots, A_n^*)$.

Proof By applying Eq. (2), the following result can be obtained:

$$\mathrm{SNNPWA}(A_1, A_2, \cdots, A_n) = \left\langle 1 - \prod_{j=1}^{n} (1 - T_{A_j})^{\frac{T_j}{\sum_{i=1}^{n} T_i}}, \prod_{j=1}^{n} (I_{A_j})^{\frac{T_j}{\sum_{i=1}^{n} T_i}}, \prod_{j=1}^{n} (F_{A_j})^{\frac{T_j}{\sum_{i=1}^{n} T_i}} \right\rangle,$$

$$\mathrm{SNNPWA}(A_1^*, A_2^*, \cdots, A_n^*) = \left\langle 1 - \prod_{j=1}^{n} (1 - T_{A_j}^*)^{\frac{T_j}{\sum_{i=1}^{n} T_i}}, \prod_{j=1}^{n} (I_{A_j}^*)^{\frac{T_j}{\sum_{i=1}^{n} T_i}}, \prod_{j=1}^{n} (F_{A_j}^*)^{\frac{T_j}{\sum_{i=1}^{n} T_i}} \right\rangle.$$

Because of $Aj \subseteq Aj^*$, we obtain $T_{A_j} < T_{A_j}^*$, $I_{A_j} > I_{A_j}^*$, $F_{A_j} > F_{A_j}^*$ in accordance with Definition 4. Obviously, the inequality $1 - T_{A_j} > 1 - T_{A_j}^*$ holds, and $1 - \prod_{j=1}^{n} (1 - T_{A_j}^*)^{\frac{T_j}{\sum_{i=1}^{n} T_i}} > 1 - \prod_{j=1}^{n} (1 - T_{A_j})^{\frac{T_j}{\sum_{i=1}^{n} T_i}}$, $\prod_{j=1}^{n} (I_{A_j}^*)^{\frac{T_j}{\sum_{i=1}^{n} T_i}} < \prod_{j=1}^{n} (I_{A_j})^{\frac{T_j}{\sum_{i=1}^{n} T_i}}$ and $\prod_{j=1}^{n} (F_{A_j}^*)^{\frac{T_j}{\sum_{i=1}^{n} T_i}} < \prod_{j=1}^{n} (F_{A_j})^{\frac{T_j}{\sum_{i=1}^{n} T_i}}$ hold as well.

According to Definition 4, clearly, $\mathrm{SNNPWA}(A_1, A_2, \cdots, A_n) \subseteq \mathrm{SNNPWA}(A_1^*, A_2^*, \cdots, A_n^*)$.

Definition 12 Let $A = \{A_j | j = 1, 2, \cdots, n\}$ be a collection of SNNs, and then the SNNPWG operator can be defined as follows:

$$\mathrm{SNNPWG}(A_1, A_2, \cdots, A_n) = A_1^{\frac{T_1}{\sum_{i=1}^{n} T_i}} \times A_2^{\frac{T_2}{\sum_{i=1}^{n} T_i}} \times \cdots \times A_n^{\frac{T_n}{\sum_{i=1}^{n} T_i}} = \prod\nolimits_{j=1}^{n} A_j^{\frac{T_j}{\sum_{i=1}^{n} T_i}}, \tag{5}$$

where $T_j = \Pi_{k=1}^{j-1} S(A_k) (j = 2, \cdots, n)$, $T_1 = 1$, and $S(A_k)$ is the score function of A_k.

Theorem 2 *Let* $A = \{A_j | j = 1, 2, \cdots, n\}$ *be a SNS, and the following can be obtained by using Eq.* (5).

$$\mathrm{SNNPWG}(A_1, A_2, \cdots, A_n) = \left\langle \prod_{j=1}^{n} T_{A_j}^{\frac{T_j}{\sum_{i=1}^{n} T_i}}, 1 - \prod_{j=1}^{n} (1 - I_{A_j})^{\frac{T_j}{\sum_{i=1}^{n} T_i}}, 1 - \prod_{j=1}^{n} (1 - F_{A_j})^{\frac{T_j}{\sum_{i=1}^{n} T_i}} \right\rangle. \tag{6}$$

Proof According to Definition 12 and Definition 6, the proof of Eq. (6) can be proved in a similar proof manner.

(1) For $n = 2$, by using the operation defined in Definition 6, we have

$$\begin{aligned} &\mathrm{SNNPWG}(A_1, A_2) \\ &= \left\langle T_{A_1}^{\frac{T_1}{\sum_{i=1}^{2} T_i}}, 1 - (1 - I_{A_1})^{\frac{T_1}{\sum_{i=1}^{2} T_i}}, 1 - (1 - F_{A_1})^{\frac{T_1}{\sum_{i=1}^{2} T_i}} \right\rangle \\ &\times \left\langle T_{A_2}^{\frac{T_2}{\sum_{i=1}^{2} T_i}}, 1 - (1 - I_{A_2})^{\frac{T_2}{\sum_{i=1}^{2} T_i}}, 1 - (1 - F_{A_2})^{\frac{T_2}{\sum_{i=1}^{2} T_i}} \right\rangle \\ &= \left\langle \prod_{j=1}^{2} T_{A_j}^{\frac{T_j}{\sum_{i=1}^{2} T_i}}, 1 - \prod_{j=1}^{2} (1 - I_{A_j})^{\frac{T_j}{\sum_{i=1}^{2} T_i}}, 1 - \prod_{j=1}^{2} (1 - F_{A_j})^{\frac{T_j}{\sum_{i=1}^{2} T_i}} \right\rangle. \end{aligned} \tag{7}$$

For $n=k$, if Eq. (6) holds, we can get

$$\mathrm{SNNPWG}(A_1, A_2, \cdots, A_n) = \left\langle \prod_{j=1}^{n} T_{A_j}^{\frac{T_j}{\sum_{i=1}^{n} T_i}}, 1 - \prod_{j=1}^{n} (1 - I_{A_j})^{\frac{T_j}{\sum_{i=1}^{n} T_i}}, 1 - \prod_{j=1}^{n} (1 - F_{A_j})^{\frac{T_j}{\sum_{i=1}^{n} T_i}} \right\rangle. \tag{8}$$

(2) When $n = k + 1$, by applying Eqs. (7) and (8), we can obtain

$$\begin{aligned} &\mathrm{SNNPWG}(A_1, A_2, \cdots, A_{k+1}) \\ &= \left\langle \prod_{j=1}^{k} T_{A_j}^{\frac{T_j}{\sum_{i=1}^{n} T_i}}, 1 - \prod_{j=1}^{k} (1 - I_{A_j})^{\frac{T_j}{\sum_{i=1}^{n} T_i}}, 1 - \prod_{j=1}^{k} (1 - F_{A_j})^{\frac{T_j}{\sum_{i=1}^{n} T_i}} \right\rangle \\ &\times \left\langle T_{A_{k+1}}^{\frac{T_{k+1}}{\sum_{i=1}^{n} T_i}}, 1 - (1 - I_{A_{k+1}})^{\frac{T_{k+1}}{\sum_{i=1}^{n} T_i}}, 1 - (1 - F_{A_{k+1}})^{\frac{T_{k+1}}{\sum_{i=1}^{n} T_i}} \right\rangle \\ &= \left\langle \prod_{j=1}^{k+1} T_{A_j}^{\frac{T_j}{\sum_{i=1}^{n} T_i}}, 1 - \prod_{j=1}^{k+1} (1 - I_{A_j})^{\frac{T_j}{\sum_{i=1}^{n} T_i}}, 1 - \prod_{j=1}^{k+1} (1 - F_{A_j})^{\frac{T_j}{\sum_{i=1}^{n} T_i}} \right\rangle. \end{aligned}$$

Equation (6) holds for $n = k + 1$, so Eq. (6) holds for any n, and the proof is completed.

Obviously, the SNNPWG operator has the following properties:

Property 4 (Boundedness). *Let* $A = \{A_j | j = 1, 2, \cdots, n\}$ *be a collection of SNNs.* $A^- = \langle \mathrm{Min}(T_{A_j}), \mathrm{Max}(I_{A_j}), \mathrm{Max}(F_{A_j}) \rangle$, $A^+ = \langle \mathrm{Max}(T_{A_j}), \mathrm{Min}(I_{A_j}), \mathrm{Min}(F_{A_j}) \rangle$, *and then* $A^- \subseteq \mathrm{SNNPWA}(A_1, A_2, \cdots, A_n) \subseteq A^+$.

Property 5 (Idempotency). *Let* $A = \{A_j | j = 1, 2, \cdots, n\}$ *be a collection of SNNs. If* $A_j = B = \langle T_B, I_B, F_B \rangle$, *then* $\text{SNNPWA}(A_1, A_2, \cdots, A_n) = B$.

Property 6 (Monotonity). *If* $A = \{A_j | j = 1, 2, \cdots, n\}$ *and* $A^* = \{A_j^* | j = 1, 2, \cdots, n\}$ *are two collections of SNNs. If* $A_j \subseteq A_j^*$ *for any j, then* $\text{SNNPWA}(A_1, A_2, \cdots, A_n) \subseteq \text{SNNPWA}(A_1^*, A_2^*, \cdots, A_n^*)$.

By a similar proof manner, the properties above can be proved.

3.2 Cross-Entropy Measure for SNSs

The cross-entropy measure for SNSs was proposed by Ye [44], but it cannot be accepted in some specific cases, as shown in the example given by Peng et al. [50].

Example 4 Let $A_1 = \{\langle x, 0.1, 0, 0 \rangle\}$ and $A_2 = \{\langle x, 0.9, 0, 0 \rangle\}$ be two SNSs, and $B = \{\langle x, 1, 0, 0 \rangle\}$ be the largest SNS. According to the cross-entropy measure for SNSs [44], $S_1(A_1, B) = S_2(A_2, B) = 1$ can be obtained, which indicates that A_1 is equal to A_2. However, it is not possible to discern which one is the best. As $T_{A_2}(x) > T_{A_1}(x)$, $I_{A_2}(x) = I_{A_1}(x)$ and $F_{A_2}(x) = F_{A_1}(x)$ for any x in X, it is clear that A_2 is superior to A_1.

In order to overcome the shortcomings mentioned above, in this subsection, two new cross-entropy measures for SNSs are defined. Before defining the new cross-entropy measures, the following definition is required to be introduced to help us obtain the proof of the properties of the proposed cross-entropy measures later.

Definition 13 A SNS A is greater than or equal to the other SNS B, denoted by $A \geq B$ if and only if $T_A \leq T_B$, $I_A \geq I_B$ and $F_A \geq F_B$.

Next, the novel cross-entropy measures are defined.

Definition 14 Let A and B be two SNSs, and then the cross-entropy between A and B can be defined as:

$$(1) I_{\text{SNS}_1}(A, B) = \sum_{i=1}^{n} [\sin T_A(x_i) \times \sin(T_A(x_i) - T_B(x_i)) + \sin I_A(x_i) \times \sin(I_A(x_i) - I_B(x_i)) + \sin F_A(x_i) \times \sin(F_A(x_i) - F_B(x_i))],$$

and

$$(2)\ I_{\text{SNS}_2}(A, B) = \sum_{i=1}^{n} [\tan T_A(x_i) \times \tan(T_A(x_i) - T_B(x_i)) + \tan I_A(x_i) \times \tan(I_A(x_i) - I_B(x_i)) + \tan F_A(x_i) \times \tan(F_A(x_i) - F_B(x_i))],$$

which can indicate the degree of discrimination of A from B. However, $I_{\text{SNS}_1}(A, B)$ and $I_{\text{SNS}_2}(A, B)$ is not symmetric with respect to its argument. Therefore, a modified cross-entropy measure based on $I_{\text{SNS}_1}(A, B)$ and $I_{\text{SNS}_2}(A, B)$ can be defined as follows:

(1) $D_{\text{SNS}_1}(A, B) = I_{\text{SNS}_1}(A, B) + I_{\text{SNS}_1}(B, A)$;
(2) $D_{\text{SNS}_2}(A, B) = I_{\text{SNS}_2}(A, B) + I_{\text{SNS}_2}(B, A)$.

Property 7 *Let A and B be two SNSs. Define the degree of discrimination of A from B as* $D_{\text{SNS}}(A, B)$, *and then the following properties hold:*

(1) $D_{\text{SNS}_1}(A, B) = D_{\text{SNS}_1}(B, A)$ *and* $D_{\text{SNS}_2}(A, B) = D_{\text{SNS}_2}(B, A)$;
(2) $D_{\text{SNS}_1}(A, B) = D_{\text{SNS}_1}(A^C, B^C)$, *and* $D_{\text{SNS}_2}(A, B) = D_{\text{SNS}_2}(A^C, B^C)$, *where* A^C *and* B^C *are the complement sets of A and B, respectively, as defined in Definition* 4;
(3) $D_{\text{SNS}_1}(A, B) \geq 0$ ($D_{\text{SNS}_1}(A, B) = 0$ *if and only if* $A = B$) *and* $D_{\text{SNS}_2}(A, B) \geq 0$ ($D_{\text{SNS}_2}(A, B) = 0$ *if and only if* $A = B$);
(4) *The larger the difference between A and B, the larger* $D_{\text{SNS}_1}(A, B)$ *or* $D_{\text{SNS}_2}(A, B)$ *will be.*

Proof Obviously, it can be easily verified that (1) and (2) hold. Next, the proofs of (3) and (4) are shown in the following.

(3) Now, let us consider the following functions:

$$f_1(x, y) = \sin(x) \times \sin(x - y) + \sin(y) \times \sin(y - x) \quad \text{and} \\ = \sin(x - y) \times (\sin(x) - \sin(y)) \tag{9}$$

$$f_2(x, y) = \tan(x) \times \tan(x - y) + \tan(y) \times \tan(y - x) \\ = \tan(x - y) \times (\tan(x) - \tan(y)) \tag{10}$$

where $x \in [0, 1]$ and $y \in [0, 1]$. Obviously, whether $x \geq y$ or $x \leq y$, the function $f_1(x, y) \geq 0$ and $f_2(x, y) \geq 0$ always hold.

According to Definition 14, the following Equation can be concluded:

$$\begin{aligned} & D_{\text{SNS}_1}(A, B) \\ & = I_{\text{SNS}_1}(A, B) + I_{\text{SNS}_1}(B, A) \\ & = \sum_{i=1}^{n} [\sin T_A(x_i) \times \sin(T_A(x_i) - T_B(x_i)) + \sin I_A(x_i) \times \sin(I_A(x_i) - I_B(x_i)) \\ & \quad + \sin F_A(x_i) \times \sin(F_A(x_i) - F_B(x_i))] + \sum_{i=1}^{n} [\sin T_B(x_i) \times \sin(T_B(x_i) - T_A(x_i)) \\ & \quad + \sin I_B(x_i) \times \sin(I_B(x_i) - I_A(x_i)) + \sin F_B(x_i) \times \sin(F_B(x_i) - F_A(x_i))] \\ & = \sum_{i=1}^{n} [(\sin T_A(x_i) - \sin T_B(x_i)) \times \sin(T_A(x_i) - T_B(x_i)) + (\sin I_A(x_i) - \sin I_B(x_i)) \\ & \quad \times \sin(I_A(x_i) - I_B(x_i)) + (\sin F_A(x_i) - \sin F_B(x_i)) \times \sin(F_A(x_i) - F_B(x_i))] \end{aligned} \tag{11}$$

Because $\forall (T_A(x_i), T_B(x_i), I_A(x_i), I_B(x_i), F_A(x_i), F_B(x_i)) \in [0, 1]$ and $f_1(x, y) \geq 0$ according to Eq. (9), $(\sin T_A(x_i) - \sin T_B(x_i)) \times \sin(T_A(x_i) - T_B(x_i)) \geq 0$, $(\sin I_A(x_i) -$

$\sin I_B(x_i)) \times \sin(I_A(x_i) - I_B(x_i)) \geq 0$, and $(\sin F_A(x_i) - \sin F_B(x_i)) \times \sin(F_A(x_i) - F_B(x_i)) \geq 0$. Hence, $D_{\mathrm{SNS}_1}(A, B) \geq 0$ holds. Especially, $D_{\mathrm{SNS}_1}(A, B) = 0$ holds if and only if $T_A(x_i) = T_B(x_i)$, $I_A(x_i) = I_B(x_i)$, and $F_A(x_i) = F_B(x_i)$, namely, $A = B$.

In accordance with Definition 14, we can obtain

$$\begin{aligned}
&D_{\mathrm{SNS}_2}(A,B)\\
&= I_{\mathrm{SNS}_2}(A,B) + I_{\mathrm{SNS}_2}(B,A)\\
&= \sum_{i=1}^{n} [\tan T_A(x_i) \times \tan(T_A(x_i) - T_B(x_i)) + \tan I_A(x_i) \times \tan(I_A(x_i) - I_B(x_i))\\
&\quad + \tan F_A(x_i) \times \tan(F_A(x_i) - F_B(x_i))] + \sum_{i=1}^{n} [\tan T_B(x_i) \times \tan(T_B(x_i) - T_A(x_i))\\
&\quad + \tan I_B(x_i) \times \tan(I_B(x_i) - I_A(x_i)) + \tan F_B(x_i) \times \tan(F_B(x_i) - F_A(x_i))]\\
&= \sum_{i=1}^{n} [(\tan T_A(x_i) - \tan T_B(x_i)) \times \tan(T_A(x_i) - T_B(x_i)) + (\tan I_A(x_i) - \tan I_B(x_i))\\
&\quad \times \tan(I_A(x_i) - I_B(x_i)) + (\tan F_A(x_i) - \tan F_B(x_i)) \times \tan(F_A(x_i) - F_B(x_i))].
\end{aligned} \tag{12}$$

Similarly, $\forall (T_A(x_i), T_B(x_i), I_A(x_i), I_B(x_i), F_A(x_i), F_B(x_i)) \in [0, 1]$, and $f_2(x, y) \geq 0$ according to Eq. (10); thus, $(\tan T_A(x_i) - \tan T_B(x_i)) \times \tan(T_A(x_i) - T_B(x_i)) \geq 0$, $(\tan I_A(x_i) - \tan I_B(x_i)) \times \tan(I_A(x_i) - I_B(x_i)) \geq 0$, and $(\tan F_A(x_i) - \tan F_B(x_i)) \times \tan(F_A(x_i) - F_B(x_i)) \geq 0$. Therefore, $D_{\mathrm{SNS}_2}(A, B) \geq 0$ holds. Especially, $D_{\mathrm{SNS}_2}(A, B) = 0$ holds if and only if $T_A(x_i) = T_B(x_i)$, $I_A(x_i) = I_B(x_i)$,and $F_A(x_i) = F_B(x_i)$, namely, $A = B$.

(4) Let $A = \langle T_A, I_A, F_A \rangle$, $B = \langle T_B, I_B, F_B \rangle$, and $C = \langle T_C, I_C, F_C \rangle$ be three SNSs. Assume $A \geq B \geq C$. then according to Definition 13, we have $T_A \geq T_B \geq T_C$, $I_A \leq I_B \leq I_C$, and $F_A \leq F_B \leq F_C$.By using Eq. (11), we obtain

$$\begin{aligned}
D_{\mathrm{SNS}_1}(A, C) = {} & \sin(T_A - T_C) \times (\sin T_A - \sin T_C)\\
& + \sin(I_A - I_C)\\
& \times (\sin I_A - \sin I_C) + \sin(F_A - F_C)\\
& \times (\sin F_A - \sin F_C);
\end{aligned}$$

$$\begin{aligned}
D_{\mathrm{SNS}_1}(A, B) = {} & \sin(T_A - T_B) \times (\sin T_A - \sin T_B)\\
& + \sin(I_A - I_B)\\
& \times (\sin I_A - \sin I_B) + \sin(F_A - F_B)\\
& \times (\sin F_A - \sin F_B);
\end{aligned}$$

$$\begin{aligned}
D_{\mathrm{SNS}_1}(B, C) = {} & \sin(T_B - T_C) \times (\sin T_A - \sin T_C)\\
& + \sin(I_B - I_C)\\
& \times (\sin I_B - \sin I_C) + \sin(F_B - F_C)\\
& \times (\sin F_B - \sin F_C).
\end{aligned}$$

Moreover, it is easy to conclude that the following inequality is correct.

$$\sin(T_A - T_B) \times (\sin T_A - \sin T_B) \leq \sin(T_A - T_C) \times (\sin T_A - \sin T_C);$$

$$\sin(I_A - I_C) \times (\sin I_A - \sin I_C) \geq \sin(I_A - I_B) \times (\sin I_A - \sin I_B);$$

$$\sin(F_A - F_C) \times (\sin F_A - \sin F_C) \geq \sin(F_A - F_B) \times (\sin F_A - \sin F_B).$$

Clearly, $D_{\mathrm{SNS}_1}(A, C) \geq D_{\mathrm{SNS}_1}(A, B)$ holds. Similarly, $D_{\mathrm{SNS}_1}(A, C) \geq D_{\mathrm{SNS}_1}(B, C)$ holds

The similar proof can also be given for D_{SNS_2}.

In order to verify the effectiveness of the two proposed cross-entropy measures, the data of Example 4 is used again, and the results are obtained as follows:.

$D_{\mathrm{SNS}_1}(A_1, B) = 0.58094$, $D_{\mathrm{SNS}_1}(A_2, B) = 0.00580$, $D_{\mathrm{SNS}_2}(A_1, B) = 1.83614$, and $D_{\mathrm{SNS}_2}(A_2, B) = 0.02982$.

Clearly, the problem pointed out by Peng et al. [50] can be solved by using the proposed cross-entropy measures.

4 The Ranking Method for MCDM Problems with Simplified Neutrosophic Information

The ranking method based on the SNNPWA and SNNPWG operators and the cross-entropy measure under simplified neutrosophic environment is presented to deal with MCDM problems.

Let $A = \{A_1, A_2, \cdots, A_m\}$ be a set of m alternatives and $C = \{C_1, C_2, \cdots, C_n\}$ be a set of n criteria. Assume that criterion prioritization relationships are $C_1 \succ C_2 \succ \cdots \succ C_n$, and if $i < j$, then the priority of C_i is higher than that of C_j. The assessment value of the alternative A_i on the criterion C_j can be expressed in the following form: $A_i(C_j) = \left\{ \left\langle C_j, T_{A_i}(C_j), I_{A_i}(C_j), F_{A_i}(C_j) \right\rangle \middle| C_j \in C \right\}$. Suppose that $D = \beta_{ij} = (A_i(C_j))_{m \times n}$ is a simplified neutrosophic decision matrix.

To rank the alternatives, we define a positive ideal solution and a negative ideal solution for SNNs denoted by A^+ and A^- , and they are $A^+ = \langle 1, 0, 0 \rangle$ and $A^- = \langle 0, 0, 1 \rangle$.

The decision process procedure of the proposed method is summarized as follows.

Step 1 Normalize the decision matrix.

First, the decision-making information $\widehat{\beta}_{ij}$ in the matrix $\bar{D} = (\widehat{\beta}_{ij})_{m \times n}$ must be normalized. The criteria can be classified into the benefit and cost types. The evaluation information does not need to be changed for the benefit-type criteria; however, for the cost-type criteria, it must be transformed with the complement set.

The normalization of the decision matrices can be expressed as $\begin{cases} \beta_{ij} = \widehat{\beta}_{ij}\ , C_j \in B_T \\ \beta_{ij} = \widehat{\beta}_{ij}^{\,c}\ , C_j \in C_T \end{cases}$,where B_T is the set of

benefit-type criteria, and C_T is the set of cost-type criteria, and $\widehat{\beta}_{ij}^{C}$ is complement set of $\widehat{\beta}_{ij}$.
The normalized decision matrix can be denoted by $D = (\beta_{ij})_{m \times n}$.

Step 2 Compute the aggregation results of each alternative A_i
Compute the aggregation values of each alternative A_i by using Eq. (2) or (6), and then, the SNNPWA or SNNPWG operator aggregation values are obtained.

Step 3 Determine the cross-entropy and ranking value S_{β_i} of each alternative A_i
The cross-entropy values of each alternative A_i from the positive ideal solution A^+ and the negative solution A^- are calculated by using Eqs. (11) and (12). Then S_{β_i} can be obtained based on the following equation:

$$S_{\beta_i} = \frac{D_{\mathrm{SNS}}(A_i, A^+)}{(D_{\mathrm{SNS}}(A_i, A^+) + D_{\mathrm{SNS}}(A_i, A^-))} \quad (13)$$

Step 4 Select the best alternative by the ranking value S_{β_i}
The smaller S_{β_i} is, the better the alternative is. According to S_{β_i}, the ranking of all alternatives is obtained and the best alternative is chosen.

5 Illustrated Example

The MCDM problem with simplified neutrosophic information example is used to demonstrate the application of the proposed approach and the relative comparison analysis.

5.1 An Illustrative Example with Simplified Neutrosophic Information

The illustrative example is the supplier-selection problem of an automotive company in reality. Actually, supplier selection of an automotive company is quite complex, and the number of the related criteria is up to fourteen [55], including quality, delivery, reputation, risk, security, service, and so forth. Furthermore, normally, considering real situations of a specific automotive company from different aspects of company strategies, product features, etc., different criteria sets should be constructed for different situations. In order to verify the effectiveness of the proposed method on the representative supplier-selection problems, a simplified supplier selection of an automotive company [56] with four essential criteria is adopted.

Suppose that for an automotive company, which expects to select the most appropriate key components supplier, after first round assessment, five suppliers $A_i (i = 1, 2, \cdots, 5)$ have been selected as alternatives for the final evaluation. During evaluation, four criteria are chosen, including product quality (c_1), relationship closeness (c_2), price (c_3), and delivery performance (c_4). The prioritization relationship of the criteria is $C_1 \succ C_2 \succ C_3 \succ C_4$. The decision makers gave the evaluation values of all alternatives for each criterion with simplified netursophic information, and then, a simplified neutrosophic decision matrix $\bar{D}$ is provided as follows:

$$\bar{D} = (\widehat{\beta}_{ij})_{5 \times 4} = \begin{bmatrix} \langle 0.7, 0.0, 0.1 \rangle \langle 0.6, 0.1, 0.2 \rangle \langle 08, 0.7, 0.6 \rangle \langle 0.5, 0.2, 0.3 \rangle \\ \langle 0.4, 0.2, 0.3 \rangle \langle 0.7, 0.1, 0.0 \rangle \langle 0.1, 0.1, 0.6 \rangle \langle 0.5, 0.3, 0.6 \rangle \\ \langle 0.5, 0.2, 0.2 \rangle \langle 0.4, 0.1, 0.2 \rangle \langle 0.1, 0.1, 0.4 \rangle \langle 0.4, 0, 1, 0.2 \rangle \\ \langle 0.7, 0.3, 0.2 \rangle \langle 0.5, 0.3, 0.2 \rangle \langle 0.3, 0.2, 0.5 \rangle \langle 0.6, 0.1, 0.1 \rangle \\ \langle 0.6, 0.5, 0.1 \rangle \langle 0.7, 0.1, 0.1 \rangle \langle 0.1, 0.2, 0.9 \rangle \langle 0.8, 0.1, 0.0 \rangle \end{bmatrix}.$$

The following shows the decision-making procedure by means of the SNNPWA operator.

Step 1 Normalize the decision matrix.
The price (c_3) is considered as cost-criterion, others are considered as benefit-criteria. Therefore, the decision martrix $\bar{D} = (\widehat{\beta}_{ij})_{5 \times 4}$ can be normalized as

$$D = (\beta_{ij})_{5 \times 4} = \begin{bmatrix} \langle 0.7, 0.0, 0.1 \rangle \langle 0.6, 0.1, 0.2 \rangle \langle 0.6, 0.7, 0.8 \rangle \langle 0.5, 0.2, 0.3 \rangle \\ \langle 0.4, 0.2, 0.3 \rangle \langle 0.7, 0.1, 0.0 \rangle \langle 0.6, 0.1, 0.1 \rangle \langle 0.5, 0.3, 0.6 \rangle \\ \langle 0.5, 0.2, 0.2 \rangle \langle 0.4, 0.1, 0.2 \rangle \langle 0.4, 0.1, 0.1 \rangle \langle 0.4, 0, 1, 0.2 \rangle \\ \langle 0.7, 0.3, 0.2 \rangle \langle 0.5, 0.3, 0.2 \rangle \langle 0.5, 0.2, 0.3 \rangle \langle 0.6, 0.1, 0.1 \rangle \\ \langle 0.6, 0.5, 0.1 \rangle \langle 0.7, 0.1, 0.1 \rangle \langle 0.9, 0.2, 0.1 \rangle \langle 0.8, 0.1, 0.0 \rangle \end{bmatrix}.$$

Step 2 Compute the aggregation results of each alternative A_i
The SNNPWA operator aggregation values β_i for each alternative A_i are obtained by applying Eq. (2). The results are: $\beta_1 = \langle 0.64167, 0.0, 0.180645 \rangle$, $\beta_2 = \langle 0.60339, 0.12859, 0.0 \rangle$, $\beta_3 = \langle 0.42117, 0.11463, 0.16456 \rangle$, $\beta_4 = \langle 0.60639, 0.23802, 0.19470 \rangle$ and $\beta_5 = \langle 0.76401, 0.19314, 0.0 \rangle$.

Step 3 Determine the cross-entropy and ranking value S_{β_i} of each alternative A_i
Calculate the cross-entropy D_{SNS} of $\beta_i (i = 1, 2, \cdots, 5)$ from $A^+ = \langle 1, 0, 0 \rangle$ and $A^- = \langle 0, 0, 1 \rangle$ by applying Eqs. (11) and (12), and obtain the S_{β_i} value by utilizing Eq. (13). The results are shown in Tables 1 and 2.

Step 4 Select the best alternative by the ranking value S_{β_i}.
According to the S_{β_i} value in Tables 1 and 2, the ranking of five alternatives is

Table 1 The cross-entropy D_{SNS_1} and the ranking values S_{β_i}

	$D_{\mathrm{SNS}_1}(\beta_i, A^+)$	$D_{\mathrm{SNS}_1}(\beta_i, A^-)$	S_{β_i}
β_1	0.11748	0.84182	0.12246
β_2	0.12230	1.0465	0.10463
β_3	0.27660	0.68276	0.28831
β_4	0.19718	0.84762	0.18872
β_5	0.07183	1.22353	0.05545

Table 2 The cross-entropy D_{SNS_2} and the ranking values S_{β_i}

	$D_{\mathrm{SNS}_2}(\beta_i, A^+)$	$D_{\mathrm{SNS}_2}(\beta_i, A^-)$	S_{β_i}
β_1	0.33681	2.02965	0.14233
β_2	0.38036	2.91714	0.11535
β_3	0.76585	1.75200	0.30417
β_4	0.45649	1.95535	0.18927
β_5	0.18236	3.38175	0.05117

$$A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3.$$

In the following, we utilize the SNNPWG operator in the decision-making procedure.

Step 1 Normalize the decision matrix.

This step is the same as that of the decision-making procedure by means of the SNNPWA operator and thus omitted here.

Step 2 Compute the aggregation results of each alternative A_i.

Compute the SNNPWG operator aggregation values β_i for the alternative A_i by applying Eq. (5), and the result is shown in the following. $\beta_1 = \langle 0.63309, 0.18638, 0.29952\rangle$, $\beta_2 = \langle 0.57704, 0.14348\rangle$, $\beta_3 = \langle 0.41797, 0.12064, 0.17304\rangle$, $\beta_4 = \langle 0.58833, 0.25650, 0.20565\rangle$, and $\beta_5 = \langle 0.72538, 0.27040, 0.08057\rangle$.

Step 3 Determine the cross-entropy and ranking values S_{β_i} of each alternative A_i.

By applying Eqs. (11) and (12), calculate the cross-entropy D_{SNS} of $\beta_i (i = 1, 2, \cdots, 5)$ from $A^+ = \langle 1, 0, 0\rangle$ and $A^- = \langle 0, 0, 1\rangle$, and obtain the value of S_{β_i} by applying Eq. (13). The results are shown in Tables 3 and 4.

Step 4 Select the best alternative by the ranking value S_{β_i}.

According to the S_{β_i} value shown in Tables 3 and 4, the ranking of all alternatives is

$$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3.$$

From the above results, we can see that the best alternative is A_5, but the worst alternative is A_3 no matter which proposed aggregation operator is used.

5.2 A Comparison Analysis

In order to verify the effectiveness of proposed method, a comparison analysis is carried out with other four representative methods [43–45, 57] by using the same illustrative example and the same weight. Among four representative methods, three methods are proposed by Ye [43–45], the other one is proposed by Liu and Wang [57]. Meanwhile, the weight for each criterion in the illustrative example is still calculated using the PA operator.

Given the same decision information on the simplified supplier-selection problem under simplified neutrosophic environment, the final results of all compared methods is shown in Table 5. If the aggregation operators proposed by Ye [43] are used, for F_w, the final ranking is $A_5 \succ A_2 \succ A_3 \succ A_4 \succ A_1$; for G_w, the final raking is $A_2 \succ A_5 \succ A_1 \succ A_3 \succ A_4$. Clearly, the best alternative is A_5 or A_2, and the worst alternative is A_1 or A_4. If the methods of Ye [44, 45] are used, the final rankings are $A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$ or $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$, and the best alternative is A_5 while the worst alternative is A_1 or A_3. However, if the proposed methods and Liu and Wang's methods [57] are utilized, the best alternative and worst alternative are the same, that is, A_5 and A_3, but the final rankings are slightly different.

There are three reasons why different rankings exist in the proposed method and other previous methods:

(1) The operations of SNSs [43] conflicts with the theory that the sum of an arbitrary value and the maximum value should be equal to the maximum one, as explained in Example 1. And the cross-entropy measure given in [44] has the lacks as discussed in

Table 3 The cross-entropy D_{SNS_1} and ranking values S_{β_i}

	$D_{\mathrm{SNS}_1}(\beta_i, A^+)$	$D_{\mathrm{SNS}_1}(\beta_i, A^-)$	S_{β_i}
β_1	0.21103	0.73658	0.22269
β_2	0.17410	0.80195	0.17837
β_3	0.28356	0.67176	0.29682
β_4	0.22070	0.82700	0.21065
β_5	0.12612	1.11666	0.10148

Table 4 The cross-entropy D_{SNS_2} and ranking values S_{β_i}

	$D_{\mathrm{SNS}_2}(\beta_i, A^+)$	$D_{\mathrm{SNS}_2}(\beta_i, A^-)$	S_{β_i}
β_1	0.44743	1.62680	0.21571
β_2	0.46216	1.91710	0.19424
β_3	0.77785	1.71456	0.31209
β_4	0.50100	1.88707	0.20979
β_5	0.27234	2.79991	0.08865

Table 5 The results of different methods for the illustrated example

Method	The final ranking
Aggregation result F_w of Ye [43]	$A_5 \succ A_2 \succ A_3 \succ A_4 \succ A_1$
Aggregation result G_w of Ye [43]	$A_2 \succ A_5 \succ A_1 \succ A_3 \succ A_4$
Ye [44]	$A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$
Ye [45]	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
Liu and Wang [57]	$A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$
The proposed approach based on the SNNPWA operator	$A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$
The proposed approach based on the SNNPWG operator	$A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$

Example 4, that is, the cross- entropy of two different SNNs to the same SNN may be equal.

(2) The cosine similarity measure [43] between SNSs has the lacks as discussed in Example 2, that is, the similarity measure only considers the comparison with the positive ideal solution $\langle 1, 0, 0 \rangle$ and ignores the negative ideal solution $\langle 0, 0, 1 \rangle$.

(3) The previous methods were established by combining different operations of SNSs [43] with cross-entropy measure [44], correlation coefficients [45], and aggregation operators [43, 57].

Besides, the proposed method and the method of Liu and Wang [57] can obtain the same best and worst alternatives, but the final raking slightly varied. This is because the SNNPWA operator emphasizes the overall truth-membership of criteria, and the SNNPWG operator emphasizes the overall indeterminacy-membership and falsity- membership of criteria.

In summary, from the above analysis, it is concluded that the proposed method is more reasonable and reliable than the existing methods; meanwhile, the proposed method has several advantages: (1) The new proposed cross-entropy measures can overcome the shortcoming of the cross-entropy measure of Ye [44]. (2) The SNNPWA and SNNPWG aggregation operators can compute the weighted vector of criteria, but they do not need to give the values by decision-maker in advance. (3) The improved operation of SNSs [50, 51] is adopted to effectively make up the previous method's shortcomings [43].

6 Conclusions

SNSs can be utilized to solve the indeterminate and inconsistent information that exists in the real world but which FSs and IFSs cannot deal with. Considering the advantages of SNSs, several methods of SNSs were put forward and used to solve MCDM problems. However, there are some shortcomings in those methods [43, 44]. Therefore, two novel cross-entropy measures are put forward to overcome the shortcomings of the previously proposed cross-entropy measure [44]. Based on the SNNPWA and SNNPWG aggregation operators, a MCDM method was established. Utilizing the proposed method, the best and the worst alternative can be identified easily.

In this paper, the main contributions are two novel cross-entropy measure that were put forward to overcome the shortcomings of the existing methods as discussed by Peng et al. [50, 51], and the SNNPWA and SNNPWG operators that were inferred from the PA operator to solve the MCDM problems with incomplete weight information. Finally, the comparison results produced by different methods can show the effectiveness of the proposed method.

In the future, according to the different requirements in the real-world applications, how to optimize the score function could form the scope of discussion and further detailed study.

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