Underlying symmetry among the quark and lepton mixing angles (Nine year update)

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In 2007 a single mathematical model encompassing both quark and lepton mixing was described. This model exploited the fact that when a 3×3 rotation matrix whose elements are squared is subtracted from its transpose, a matrix is produced whose non-diagonal elements have a common absolute value, where this value is an intrinsic property of the rotation matrix. For the traditional CKM quark mixing matrix with its second and third rows interchanged (i.e., c - t interchange) this value equals one-third the corresponding value for the leptonic matrix (roughly, 0.05 versus 0.15). This model is distinguished by three such constraints on mixing. As nine years have elapsed since its introduction, it is timely to assess the accuracy of the model's six mixing angles. In 2012 a large experimental conflict with leptonic angle θ_{13} required toggling the sign of one of the model's integer exponents; this change did not significantly impair the model's economy, where it is just this economy that makes the model notable. There followed a nearly fourfold improvement in the accuracy of the measurement of leptonic θ_{13}. Despite this much-improved measurement, and despite much-improved measurements for three other mixing angles since the model's introduction in 2007, no other conflicts have emerged. The model's mixing angles in degrees are 45, 33.210911, 8.034394 (originally 0.013665) for leptons; and 12.920966, 2.367442, 0.190986 for quarks.

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Contents

I. INTRODUCTION

A mathematical model encompassing both quark and lepton mixing was introduced in 2007 [1] and extended to include CP-violating phases in 2011 [2]. As nine years have elapsed since its introduction it is timely to issue an update to review the status of the model's predictions. Five-year and seven-year updates were made in 2012 [3] and 2014 [4], respectively.

II. FINE STRUCTURE CONSTANT DERIVED FROM $g_{12} = 1/10$ AND $g_{13} = 1/30\,000$

Experiment reveals that the quark and lepton mixing angles occupy a wide range [5, 6]

$$\sim\!45^\circ > \sim\!33^\circ > \sim\!13^\circ > \sim\!8^\circ > \sim\!2^\circ > \sim\!0.2^\circ \quad .$$

In order to produce model angles fitting these experimental angles we begin by defining

$$\left.\begin{aligned} g_{12} &= \frac{1}{10} \\ g_{13} &= \frac{1}{30\,000} \end{aligned}\right\} \quad . \tag{1a}$$

These definitions partly derive from this fine structure constant inverse α^{-1} approximation (accurate to within about six parts per billion)

$$\left[\frac{1}{3g_{12}} - \frac{g_{13}}{3}\right]^3 + \left[\frac{1}{g_{12}} - g_{13}\right]^2 =$$
$$\left[\frac{10}{3} - \frac{1}{3\times 30\,000}\right]^3 + \left[10 - \frac{1}{30\,000}\right]^2 = \alpha^{-1} = 137.036\,000\,0023\ldots \quad ,$$

where the 2014 CODATA value for α^{-1} equals 137.035 999 139 (31) [7]. Importantly, it has been shown that this equation — including 137.036 — occurs naturally in connection with the solution to the following nonstandard cubic equation: $(m+x)^3/3m + (m+x)^2 = Z$ [8, 9]. This underscores that the above values for g_{12} and g_{13} were not conveniently "chosen to fit the data." Moreover, an important new result shows that this nonstandard cubic equation relates to the mixing matrices in a way that is mathematically interesting in its own right [10]. This result concludes a nine-year effort by the author to tie the matrix algebra of the model to the above fine structure constant equation in a way that is of interest to mathematicians.

TABLE I: The six angles below are constrained by the requirement that (a) the values of the first two rows sum to equal the values of the third; (b) the values of each row fulfill Eq. (1b); (c) the values of rows one, two, and three produce the identities of Eqs. (2d), (3d), and (4d), respectively. With the exception of θ_{13}^{L} the angles of row three are the predicted mixing angles.

g_{12}	g_{13}	θ_{23}^{L}	θ_{13}^{L} [a]	θ_{12}^{L}	θ_{23}^{Q}	θ_{13}^{Q}	θ_{12}^{Q}
1/10 [b]	0	$45° + 90°$	0°	33.210 911°	$+90°$	0°	12.920 966°
0 [c]	1/30 000	$-90°$	0.013 665°	0°	2.367 442°	0.190 986°	0°
1/10 [d]	1/30 000	45°	0.013 665°	33.210 911°	$2.367\,442° + 90°$	0.190 986°	12.920 966°

[a] But it is $\varphi_{13}^{L} = 8.034\,394°$ that is expected to match experiment. See Sec. VI
[b] This row's values match Eq. (2a) and produce Eq. (2d).
[c] This row's values match Eq. (3a) and produce Eq. (3d).
[d] This row's values match Eq. (4a) and produce Eq. (4d).

TABLE II: In Eqs. (2d), (3d), and (4d) the nonzero quark matrix elements equal $1/3^{\text{rd}}$ the nonzero leptonic elements.

Identity	Quark matrix elements	Leptonic matrix elements	g_{12}	g_{13}
Eq. (2d)	0.05	0.15	1/10	0
Eq. (3d)	$1.895\,936 \times 10^{-8}$	$5.687\,808 \times 10^{-8}$	0	1/30 000
Eq. (4d)	0.049 963 56	0.149 8907	1/10	1/30 000

The above definitions, in turn, aid in the definition of the four '12" and "13" mixing angles of Table I

$$\left.\begin{aligned} \sin\theta_{12}^{L} &= \sqrt{3g_{12}} \\ \sin\theta_{13}^{Q} &= \sqrt{g_{13}/3} \\ \sin\theta_{12}^{Q} &= \sqrt{g_{12}} \times \sin\theta_{23\ \text{offset}}^{L} \\ \sin\theta_{13}^{L} &= \sqrt{g_{13}} \times \sin^{+1}\theta_{23\ \text{offset}}^{Q} \end{aligned}\right\} \tag{1b}$$

where g_{12} helps define the "12" mixing angles, and g_{13} the "13" mixing angles. Now let

$$\sin\varphi_{13}^{L} = \sqrt{g_{13}} \times \sin^{-1}\theta_{23\ \text{offset}}^{Q} \quad , \tag{1c}$$

where it will be φ_{13}^{L} that will actually fit the smallest neutrino mixing angle. Note that the definitions of θ_{13}^{L} and φ_{13}^{L} differ just slightly: in the sign of an exponent (±1).

At this point the reader perhaps has noticed that to calculate all four "12" and "13" angles from Eqs. (1a) and (1b) we need only also know the two "23" angles. Their values will be

$$\left.\begin{aligned} \theta_{23\ \text{offset}}^{L} &= 45° \\ \theta_{23\ \text{offset}}^{Q} &= 2.367\,442° \end{aligned}\right\} \quad . \tag{1d}$$

We now have enough information to deduce all six angles of Table I.

But how to justify this odd value for $\theta_{23\ \text{offset}}^{Q}$ and the peculiar form of Eq. (1b)? In fact, neither is freely chosen. As will now be shown, the mixing angles of Table I produce leptonic matrices having a property that is three times larger than the corresponding property for the quark matrices *in three closely-related ways.* This property mirrors the way that the sum of the charges of the leptons

$$-1 + 0 = -1$$

is threefold larger than the sum of the charges of the quarks

$$-\frac{2}{3} + \frac{1}{3} = -\frac{1}{3} \quad .$$

Importantly, it is the restrictions posed by Eq. (1b) that automatically produce this hidden threefold symmetry for various g_{12}, g_{13}, and $\theta_{23\ \text{offset}}^{Q}$. This threefold symmetry, in turn, underscores that Eq. (1b) was not merely "*designed* to fit the data." The next three sections will examine these three types symmetry in detail, while Tables I and II will summarize these results in their three rows.

III. MIXING MATRICES DERIVED FROM $g_{12} = 1/10$ AND $g_{13} = 0$

Define the usual CKM mixing matrix [5], *but without its phase*, as

$$V = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{bmatrix} ,$$

where $s_{12} \equiv \sin\theta^Q_{12}$, $c_{12} \equiv \cos\theta^Q_{12}$, etc., and where θ^Q_{23}, θ^Q_{13}, and θ^Q_{12} are the CKM mixing angles. And define the usual leptonic mixing matrix [6], *also without its phase*, as

$$U = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{bmatrix} ,$$

where $s_{12} \equiv \sin\theta^L_{12}$, $c_{12} \equiv \cos\theta^L_{12}$, etc., and where θ^L_{23}, θ^L_{13}, and θ^L_{12} are the leptonic mixing angles. Now consider that the following matrix

$$\begin{array}{c} \\ u \\ t \\ c \end{array} \begin{array}{c} \begin{array}{ccc} d & s & b \end{array} \\ \begin{bmatrix} 0.95 & 0.05 & 0 \\ 0 & 0 & 1.00 \\ 0.05 & 0.95 & 0 \end{bmatrix} \end{array}$$

results if the above CKM matrix *with its elements squared* has its angles determined by these values

$$\left.\begin{aligned} g_{12} &= 1/10 \\ g_{13} &= 0 \\ \theta^L_{23} &= \theta^L_{23\ \mathrm{offset}} + 90^\circ \\ \theta^Q_{23} &= 90^\circ \end{aligned}\right\} \textbf{Row one of Table I} \tag{2a}$$

and by Eq. (1b). Observe that the above matrix's second and third rows (i.e., its c- and t-quarks) are interchanged relative to convention, a consequence of $\theta^Q_{23} = 90^\circ$. Subtracting this matrix from its transpose gives

$$\begin{bmatrix} 0.95 & 0.05 & 0 \\ 0 & 0 & 1.00 \\ 0.05 & 0.95 & 0 \end{bmatrix} - \begin{bmatrix} 0.95 & 0 & 0.05 \\ 0.05 & 0 & 0.95 \\ 0 & 1.00 & 0 \end{bmatrix} = \begin{bmatrix} 0 & +0.05 & -0.05 \\ -0.05 & 0 & +0.05 \\ +0.05 & -0.05 & 0 \end{bmatrix} . \tag{2b}$$

Now consider that the following matrix

$$\begin{array}{c} \\ \nu_e \\ \nu_\tau \\ \nu_\mu \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{bmatrix} 0.70 & 0.30 & 0 \\ 0.15 & 0.35 & 0.50 \\ 0.15 & 0.35 & 0.50 \end{bmatrix} \end{array}$$

results if the above leptonic matrix *also with its elements squared* has its angles also determined by **row one of Table I** and Eq. (1b). Observe that the above matrix's second and third rows (i.e., ν_μ and ν_τ) also are interchanged relative to convention, but this time it is a consequence of $\theta^L_{23} = \theta^L_{23\ \mathrm{offset}} + 90^\circ$. Subtracting the above matrix from its transpose gives

$$\begin{bmatrix} 0.70 & 0.30 & 0 \\ 0.15 & 0.35 & 0.50 \\ 0.15 & 0.35 & 0.50 \end{bmatrix} - \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.30 & 0.35 & 0.35 \\ 0 & 0.50 & 0.50 \end{bmatrix} = \begin{bmatrix} 0 & +0.15 & -0.15 \\ -0.15 & 0 & +0.15 \\ +0.15 & -0.15 & 0 \end{bmatrix} . \tag{2c}$$

Now the right-hand-sides of Eqs. (2b) and (2c) combine to form the identity

$$3 \times \begin{bmatrix} 0 & +0.05 & -0.05 \\ -0.05 & 0 & +0.05 \\ +0.05 & -0.05 & 0 \end{bmatrix} = \begin{bmatrix} 0 & +0.15 & -0.15 \\ -0.15 & 0 & +0.15 \\ +0.15 & -0.15 & 0 \end{bmatrix} , \tag{2d}$$

where the nonzero matrix elements on the quark side are exactly one-third those of the leptonic side. This is the first of the three key constraints distinguishing the mixing model. These values occupy **row one of Table II**.

IV. MIXING MATRICES DERIVED FROM $g_{12} = 0$ AND $g_{13} = 1/30\,000$

Consider that the following matrix

$$\begin{array}{c} \\ u \\ c \\ t \end{array} \begin{array}{c} \begin{array}{ccc} d & s & b \end{array} \\ \begin{bmatrix} 9.999\,889 \times 10^{-1} & 0 & 1.111\,111 \times 10^{-5} \\ 1.895\,936 \times 10^{-8} & 9.982\,937 \times 10^{-1} & 1.706\,323 \times 10^{-3} \\ 1.109\,215 \times 10^{-5} & 1.706\,342 \times 10^{-3} & 9.982\,826 \times 10^{-1} \end{bmatrix} \end{array}$$

results if the earlier CKM matrix *with its elements squared* has its angles determined by

$$\left.\begin{array}{l} g_{12} = 0 \\ g_{13} = 1/30\,000 \\ \theta^L_{23} = -90^\circ \\ \theta^Q_{23} = \theta^Q_{23\ \mathrm{offset}} \end{array}\right\} \textbf{Row two of Table I} \tag{3a}$$

and by Eq. (1b). Subtracting the above matrix from its transpose gives

$$\begin{aligned} &\begin{bmatrix} 9.999\,889 \times 10^{-1} & 0 & 1.111\,111 \times 10^{-5} \\ 1.895\,936 \times 10^{-8} & 9.982\,937 \times 10^{-1} & 1.706\,323 \times 10^{-3} \\ 1.109\,215 \times 10^{-5} & 1.706\,342 \times 10^{-3} & 9.982\,826 \times 10^{-1} \end{bmatrix} \\ &- \begin{bmatrix} 9.999\,889 \times 10^{-1} & 1.895\,936 \times 10^{-8} & 1.109\,215 \times 10^{-5} \\ 0 & 9.982\,937 \times 10^{-1} & 1.706\,342 \times 10^{-3} \\ 1.111\,111 \times 10^{-5} & 1.706\,323 \times 10^{-3} & 9.982\,826 \times 10^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1.895\,936 \times 10^{-8} & +1.895\,936 \times 10^{-8} \\ +1.895\,936 \times 10^{-8} & 0 & -1.895\,936 \times 10^{-8} \\ -1.895\,936 \times 10^{-8} & +1.895\,936 \times 10^{-8} & 0 \end{bmatrix} . \end{aligned} \tag{3b}$$

Now consider that the following matrix

$$\begin{array}{c} \\ \nu_e \\ \nu_\tau \\ \nu_\mu \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{bmatrix} 9.999\,999 \times 10^{-1} & 0 & 5.687\,808 \times 10^{-8} \\ 5.687\,808 \times 10^{-8} & 0 & 9.999\,999 \times 10^{-1} \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

results if the earlier leptonic matrix *also with its elements squared* has its angles also determined by **row two of Table I** and Eq. (1b). Observe that the above matrix's second and third rows (i.e., ν_μ and ν_τ) are interchanged relative to convention, a consequence of $\theta^L_{23} = -90^\circ$. Subtracting the above matrix from its transpose gives

$$\begin{aligned} &\begin{bmatrix} 9.999\,999 \times 10^{-1} & 0 & 5.687\,808 \times 10^{-8} \\ 5.687\,808 \times 10^{-8} & 0 & 9.999\,999 \times 10^{-1} \\ 0 & 1 & 0 \end{bmatrix} \\ &- \begin{bmatrix} 9.999\,999 \times 10^{-1} & 5.687\,808 \times 10^{-8} & 0 \\ 0 & 0 & 1 \\ 5.687\,808 \times 10^{-8} & 9.999\,999 \times 10^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -5.687\,808 \times 10^{-8} & +5.687\,808 \times 10^{-8} \\ +5.687\,808 \times 10^{-8} & 0 & -5.687\,808 \times 10^{-8} \\ -5.687\,808 \times 10^{-8} & +5.687\,808 \times 10^{-8} & 0 \end{bmatrix} . \end{aligned} \tag{3c}$$

Now the right-hand-sides of Eqs. (3b) and (3c) combine to form the identity

$$\begin{aligned} 3 \times &\begin{bmatrix} 0 & -1.895\,936 \times 10^{-8} & +1.895\,936 \times 10^{-8} \\ +1.895\,936 \times 10^{-8} & 0 & -1.895\,936 \times 10^{-8} \\ -1.895\,936 \times 10^{-8} & +1.895\,936 \times 10^{-8} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -5.687\,808 \times 10^{-8} & +5.687\,808 \times 10^{-8} \\ +5.687\,808 \times 10^{-8} & 0 & -5.687\,808 \times 10^{-8} \\ -5.687\,808 \times 10^{-8} & +5.687\,808 \times 10^{-8} & 0 \end{bmatrix} , \end{aligned} \tag{3d}$$

where the nonzero matrix elements on the quark side are (again) exactly one-third those of the leptonic side. This is the second of the three key constraints distinguishing the mixing model. These values occupy **row two of Table II**.

V. MIXING MATRICES DERIVED FROM $g_{12} = 1/10$ AND $g_{13} = 1/30\,000$

Consider that the following matrix

$$\begin{array}{c} \\ u \\ t \\ c \end{array} \begin{array}{c} \begin{array}{ccc} d & s & b \end{array} \\ \begin{bmatrix} 9.499\,894 \times 10^{-1} & 4.999\,944 \times 10^{-2} & 1.111\,111 \times 10^{-5} \\ 3.588\,691 \times 10^{-5} & 1.681\,548 \times 10^{-3} & 9.982\,826 \times 10^{-1} \\ 4.997\,467 \times 10^{-2} & 9.483\,190 \times 10^{-1} & 1.706\,323 \times 10^{-3} \end{bmatrix} \end{array}$$

results if the CKM matrix *with its elements squared* has its angles determined by

$$\left.\begin{array}{l} g_{12} = 1/10 \\ g_{13} = 1/30\,000 \\ \theta^L_{23} = \theta^L_{23\text{ offset}} \\ \theta^Q_{23} = \theta^Q_{23\text{ offset}} + 90^\circ \end{array}\right\} \textbf{Row three of Table I} \tag{4a}$$

and by Eq. (1b). Observe that the above matrix's second and third rows (i.e., its c- and t-quarks) are interchanged relative to convention, a consequence of $\theta^Q_{23} = \theta^Q_{23\text{ offset}} + 90^\circ$. Subtracting the above matrix from its transpose gives

$$\begin{aligned} &\begin{bmatrix} 9.499\,894 \times 10^{-1} & 4.999\,944 \times 10^{-2} & 1.111\,111 \times 10^{-5} \\ 3.588\,691 \times 10^{-5} & 1.681\,548 \times 10^{-3} & 9.982\,826 \times 10^{-1} \\ 4.997\,467 \times 10^{-2} & 9.483\,190 \times 10^{-1} & 1.706\,323 \times 10^{-3} \end{bmatrix} \\ &- \begin{bmatrix} 9.499\,894 \times 10^{-1} & 3.588\,691 \times 10^{-5} & 4.997\,467 \times 10^{-2} \\ 4.999\,944 \times 10^{-2} & 1.681\,548 \times 10^{-3} & 9.483\,190 \times 10^{-1} \\ 1.111\,111 \times 10^{-5} & 9.982\,826 \times 10^{-1} & 1.706\,323 \times 10^{-3} \end{bmatrix} \\ &= \begin{bmatrix} 0 & +4.996\,356 \times 10^{-2} & -4.996\,356 \times 10^{-2} \\ -4.996\,356 \times 10^{-2} & 0 & +4.996\,356 \times 10^{-2} \\ +4.996\,356 \times 10^{-2} & -4.996\,356 \times 10^{-2} & 0 \end{bmatrix} . \end{aligned} \tag{4b}$$

Now consider that the following matrix

$$\begin{array}{c} \\ \nu_e \\ \nu_\mu \\ \nu_\tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{bmatrix} 6.999\,999\,602 \times 10^{-1} & 2.999\,999\,829 \times 10^{-1} & 5.687\,808\,086 \times 10^{-8} \\ 1.501\,093\,103 \times 10^{-1} & 3.498\,907\,181 \times 10^{-1} & 4.999\,999\,716 \times 10^{-1} \\ 1.498\,907\,295 \times 10^{-1} & 3.501\,092\,990 \times 10^{-1} & 4.999\,999\,716 \times 10^{-1} \end{bmatrix} \end{array}$$

results if the earlier leptonic matrix *also with its elements squared* has its angles also determined by **row three of Table I** and Eq. (1b). Subtracting the above matrix from its transpose gives

$$\begin{aligned} &\begin{bmatrix} 6.999\,999\,602 \times 10^{-1} & 2.999\,999\,829 \times 10^{-1} & 5.687\,808\,086 \times 10^{-8} \\ 1.501\,093\,103 \times 10^{-1} & 3.498\,907\,181 \times 10^{-1} & 4.999\,999\,716 \times 10^{-1} \\ 1.498\,907\,295 \times 10^{-1} & 3.501\,092\,990 \times 10^{-1} & 4.999\,999\,716 \times 10^{-1} \end{bmatrix} \\ &- \begin{bmatrix} 6.999\,999\,602 \times 10^{-1} & 1.501\,093\,103 \times 10^{-1} & 1.498\,907\,295 \times 10^{-1} \\ 2.999\,999\,829 \times 10^{-1} & 3.498\,907\,181 \times 10^{-1} & 3.501\,092\,990 \times 10^{-1} \\ 5.687\,808\,086 \times 10^{-8} & 4.999\,999\,716 \times 10^{-1} & 4.999\,999\,716 \times 10^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & +1.498\,906\,726 \times 10^{-1} & -1.498\,906\,726 \times 10^{-1} \\ -1.498\,906\,726 \times 10^{-1} & 0 & +1.498\,906\,726 \times 10^{-1} \\ +1.498\,906\,726 \times 10^{-1} & -1.498\,906\,726 \times 10^{-1} & 0 \end{bmatrix} . \end{aligned} \tag{4c}$$

Now the right-hand-sides of Eqs. (4b) and (4c) combine to form the identity

$$\begin{aligned} &3 \times \begin{bmatrix} 0 & +4.996\,356 \times 10^{-2} & -4.996\,356 \times 10^{-2} \\ -4.996\,356 \times 10^{-2} & 0 & +4.996\,356 \times 10^{-2} \\ +4.996\,356 \times 10^{-2} & -4.996\,356 \times 10^{-2} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & +1.498\,907 \times 10^{-1} & -1.498\,907 \times 10^{-1} \\ -1.498\,907 \times 10^{-1} & 0 & +1.498\,907 \times 10^{-1} \\ +1.498\,907 \times 10^{-1} & -1.498\,907 \times 10^{-1} & 0 \end{bmatrix} , \end{aligned} \tag{4d}$$

where the nonzero matrix elements on the quark side are (again) exactly one-third those of the leptonic side. This is the third of the three key constraints distinguishing the mixing model. These values occupy **row three of Table II**.

VI. PREDICTED VALUE FOR θ_{13}^L

As promised earlier, the angles θ_{13}^L, θ_{12}^L, etc. have been shown to possess the same property in three closely-related ways. This justifies the value for $\theta_{23\text{ offset}}^Q$ — and the form of Eq. (1b) — introduced at the outset. Given that $g_{12} = 1/10$ and $g_{13} = 1/30\,000$ originated in connection with the cubic equation and 137.036 [8, 9], this leaves precious little wiggle room for fitting the precisely-measured mixing angles turned up by experiment — and yet the model does fit these angles.

It only remains to calculate φ_{13}^L to complete the list of *predicted* mixing angles. Substituting the earlier values for g_{13} and $\theta_{23\text{ offset}}^Q$ into Eq. (1c) gives

$$\begin{aligned}\sin^2\varphi_{13}^L &= g_{13} \times \frac{1}{\sin^2\theta_{23\text{ offset}}^Q} \\ &\approx \frac{1}{30\,000} \times \frac{1}{\sin^2 2.367\,442^\circ} \\ &\approx 0.0195 \quad , \end{aligned} \tag{5}$$

so that $\varphi_{13}^L \approx 8.034\,394^\circ$.

At this point the reader may object that the above definition of φ_{13}^L was arbitrarily chosen in 2012 to fit the (then new) $\sim 9^\circ$ Daya Bay measurement of the smallest leptonic mixing angle [11]. But φ_{13}^L is interesting and economical in its own right, as it neatly combines with the other mixing angles to produce this α^{-1} approximation

$$\begin{aligned}&\left[\frac{1}{\sin^2\theta_{12}^L} \qquad\qquad - \sin^2\theta_{13}^Q\right]^3 \\ +&\left[\frac{1}{\sin^2\theta_{12}^Q} \times \sin^2\theta_{23\text{ offset}}^L - \sin^2\varphi_{13}^L \times \sin^{+2}\theta_{23\text{ offset}}^Q\right]^2 \\ &= 137.036\,000\,0023\ldots \quad . \end{aligned} \tag{6a}$$

In this way the 2012 mixing model retains the original model's ability to produce α^{-1} from the sines squared of the predicted mixing angles, where the 2007 method instead used θ_{13}^L as follows

$$\begin{aligned}&\left[\frac{1}{\sin^2\theta_{12}^L} \qquad\qquad - \sin^2\theta_{13}^Q\right]^3 \\ +&\left[\frac{1}{\sin^2\theta_{12}^Q} \times \sin^2\theta_{23\text{ offset}}^L - \sin^2\theta_{13}^L \times \sin^{-2}\theta_{23\text{ offset}}^Q\right]^2 \\ &= 137.036\,000\,0023\ldots \quad . \end{aligned} \tag{6b}$$

Observe that these two equations are equally simple, where it is only the differing exponents (± 2) in the expressions in light blue that caused $\sin^2\varphi_{13}^L$ and $\sin^2\theta_{13}^L$ to be defined differently in Eqs. (1c) and (1b), respectively. The 2012 mixing model is, therefore, only a slightly modified version of the 2007 model, retaining five of six of its predictions, while constraining φ_{13}^L in *almost* the identical way that θ_{13}^L was constrained in 2007 (see Eq. (9) in [1]).

VII. HOW HAVE THE PREDICTIONS FARED?

In order to compare the mixing model predictions against experiment it is helpful to know that the angles of Table I produce these CKM matrix elements:

$$\begin{aligned}|V_{us}| &\approx \sin 12.920\,966^\circ \times \cos 0.190\,986^\circ \approx 0.223\,61 \\ |V_{ub}| &\approx \sin 0.190\,986^\circ \approx 0.003\,333 \\ |V_{cb}| &\approx \sin 2.367\,442^\circ \times \cos 0.190\,986^\circ \approx 0.041\,31\end{aligned}$$

TABLE III: Model predictions from 2007 compared against CKM mixing data.

Year	$\lvert V_{us}\rvert$	$\lvert V_{ub}\rvert$	$\lvert V_{cb}\rvert$
2007 Prediction	0.22361	0.003333	0.04131
2016 [a]	$0.225\,06^{+0.000\,50}_{-0.000\,50}$	$0.00357^{+0.000\,15}_{-0.000\,15}$	$0.0411^{+0.0013}_{-0.0013}$
Error in SD	2.9	1.6	0.2
2014 [b]	$0.225\,36^{+0.000\,61}_{-0.000\,61}$	$0.00355^{+0.000\,15}_{-0.000\,15}$	$0.0414^{+0.0012}_{-0.0012}$
Error in SD	2.9	1.4	0.1
2012 [c]	$0.225\,34^{+0.000\,65}_{-0.000\,65}$	$0.00351^{+0.000\,15}_{-0.000\,14}$	$0.0412^{+0.0011}_{-0.0005}$
Error in SD	2.7	1.3	0.2
2010 [d]	$0.225\,30^{+0.000\,70}_{-0.000\,70}$	$0.00347^{+0.000\,16}_{-0.000\,12}$	$0.0410^{+0.0011}_{-0.0007}$
Error in SD	2.4	1.1	0.3
2008 [e]	$0.225\,70^{+0.001\,00}_{-0.001\,00}$	$0.00359^{+0.000\,16}_{-0.000\,16}$	$0.0415^{+0.0010}_{-0.0011}$
Error in SD	2.1	1.6	0.2
2006 [f]	$0.227\,20^{+0.001\,00}_{-0.001\,00}$	$0.00396^{+0.000\,09}_{-0.000\,09}$	$0.04221^{+0.0001}_{-0.0008}$
Error in SD	3.6	7.0	1.1

[a]Ref. [5]. Particle Data Group 1σ global fit.
[b]Ref. [12]. Particle Data Group 1σ global fit.
[c]Ref. [13]. Particle Data Group 1σ global fit.
[d]Ref. [14]. Particle Data Group 1σ global fit.
[e]Ref. [15]. Particle Data Group 1σ global fit.
[f]Ref. [16]. Particle Data Group 1σ global fit.

TABLE IV: Model predictions from 2007 (and 2012) compared against leptonic mixing data. Normal hierarchy.

Year	$\sin^2\theta^L_{23}$	$\sin^2\theta^L_{13}$	$\sin^2\theta^L_{12}$
2007 (2012) Prediction	0.5	(0.0195)	0.3
2016 [a]	$0.441^{+0.027}_{-0.021}$	$0.02166^{+0.00075}_{-0.00075}$	$0.306^{+0.012}_{-0.012}$
Error in SD	2.2	2.8	0.5
2014 [b]	$0.452^{+0.052}_{-0.028}$	$0.0218^{+0.0010}_{-0.0010}$	$0.304^{+0.013}_{-0.012}$
Error in SD	0.9	2.3	0.3
2012 [c]	$0.427^{+0.034}_{-0.027}$ [d]	$0.0246^{+0.0029}_{-0.0028}$	$0.320^{+0.016}_{-0.017}$
Error in SD	2.1	1.8	1.2
2010 [e]	$0.50^{+0.07}_{-0.06}$	$0.013^{+0.013}_{-0.009}$	$0.318^{+0.019}_{-0.016}$
Error in SD	0	0.5	1.1
2008 [f]	$0.50^{+0.07}_{-0.06}$	$0.010^{+0.016}_{-0.011}$	$0.304^{+0.022}_{-0.016}$
Error in SD	0	0.6	0.25
2006 [g]	$0.50^{+0.08}_{-0.07}$	≤ 0.025 [h]	$0.300^{+0.020}_{-0.030}$
Error in SD	0		0

[a]Ref. [6]. A 1σ global fit.
[b]Ref. [17]. A 1σ global fit.
[c]Ref. [18]. A 1σ global fit.
[d]Ref. [18]. One of two minima, the other being $0.613^{+0.022}_{-0.040}$.
[e]Ref. [19]. A 1σ global fit. This source includes 2008 and 2010 data.
[f]Ref. [19]. A 1σ global fit. This source includes 2008 and 2010 data.
[g]Ref. [20]. A 1σ global fit.
[h]Ref. [20]. A 2σ global fit.

Tables III and IV also help in the comparison of mixing model predictions against experiment:

- In 2007 the model's value for $|V_{us}|$ had a 3.6σ disagreement with experiment. This value is now off by 2.9σ, its absolute error having been reduced by 60%.
- In 2007 the model's value for $|V_{ub}|$ had an improbable 7.0σ disagreement with experiment (naively assuming a Gaussian probability distribution). This value is now off by 1.6σ, its absolute error having been reduced by 62%.
- The 2012 value for $\sin^2\theta^L_{13}$ had a 1.8σ disagreement with experiment. Despite a nearly fourfold improvement of the accuracy of this measurement since the 2012 prediction, its value is now off by only 2.8σ, its absolute error having been reduced by 58%.
- The values $|V_{cb}|$ and $\sin^2\theta^L_{12}$, which posed no early conflicts with experiment, are all within 1.0σ.

There is a pattern to the above results: All three predictions with a 1.8σ or greater conflict with experiment when first proposed have benefited from an over 50% reduction in absolute error; the remaining three predictions had no early conflict with experiment — nor do they now. But it must be admitted that for the first time in nine years there are three values with more than a 2σ error.

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