Sedeonic equations for electromagnetic field in anisotropic media

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In the present paper we develop the description of electromagnetic field in an anisotropic medium using the sedeonic wave equations based on sedeonic potentials and space-time operators.

1. Introduction

For a description of the electromagnetic field in the medium we widely used the equations obtained by averaging the Maxwell equations describing electromagnetic field in a vacuum [1]. The resulting system connects the vectors of electric field strength and induction ($\boldsymbol{E}$ and $\boldsymbol{D}$) and magnetic field strength and induction ($\boldsymbol{H}$ и $\boldsymbol{B}$):

$$\begin{aligned} &\operatorname{div} \boldsymbol{D} = 4\pi\rho, \\ &\operatorname{div} \boldsymbol{B} = 0, \\ &\operatorname{curl} \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \\ &\operatorname{curl} \boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} + \frac{4\pi}{c}\boldsymbol{j}, \end{aligned} \tag{1.1}$$

where ρ is a volume density of electric charge, $\boldsymbol{j}$ is a volume density of electric current. The field strengths and inductions are connected by the following relations:

$$\begin{aligned} &\boldsymbol{D} = \boldsymbol{E} + 4\pi \boldsymbol{P}, \\ &\boldsymbol{B} = \boldsymbol{H} + 4\pi \boldsymbol{M}, \end{aligned} \tag{1.2}$$

where $\boldsymbol{P}$ is a vector of electric polarization, $\boldsymbol{M}$ is a vector of magnetization. The equations (1.1) describe the electromagnetic field generated by free and bound charges and currents:

$$\begin{aligned} &\operatorname{div} \boldsymbol{E} = 4\pi\rho_e + 4\pi\rho_{eb}, \\ &\operatorname{div} \boldsymbol{H} = 4\pi\rho_{mb}, \\ &\operatorname{curl} \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}\boldsymbol{j}_{mb}, \\ &\operatorname{curl} \boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}\boldsymbol{j}_e + \frac{4\pi}{c}\boldsymbol{j}_{eb}, \end{aligned} \tag{1.3}$$

where volume densities of bound electric ρ_{eb} and magnetic ρ_{mb} charges, electric $\boldsymbol{j}_{eb}$ and magnetic $\boldsymbol{j}_{mb}$ currents are expressed through the vectors $\boldsymbol{P}$ and $\boldsymbol{M}$ as follows:

$$\begin{aligned} &\rho_{eb} = -\operatorname{div} \boldsymbol{P}, \\ &\rho_{mb} = -\operatorname{div} \boldsymbol{M}, \\ &\boldsymbol{j}_{eb} = \frac{\partial \boldsymbol{P}}{\partial t}, \\ &\boldsymbol{j}_{mb} = \frac{\partial \boldsymbol{M}}{\partial t}. \end{aligned} \tag{1.4}$$

However, expressions for the effective sources (1.4) do not account the electric and magnetic currents

$$\begin{aligned} &\boldsymbol{j}_{eb} = \operatorname{curl} \boldsymbol{M}, \\ &\boldsymbol{j}_{mb} = \operatorname{curl} \boldsymbol{P}, \end{aligned} \tag{1.5}$$

associated with domain boundaries in ferroelectrics and ferromagnetic materials as well as vortex distributions of polarization and magnetization generated by vortex $\boldsymbol{E}$ and $\boldsymbol{H}$ fields.

On the other hand, the averaging over a macroscopic volume of the medium in the equations (1.1) is not taken into account the fact that the speed of electromagnetic waves propagation in a medium n environment different from the speed of light in a vacuum.

In this paper, we develop a phenomenological approach to the description of the electromagnetic field in a medium based on sedeonic wave equations for the potentials. The basic assumption is that the speed of electromagnetic waves propagation in an anisotropic medium is a tensor of the second rank, which completely determines all features of the electrodynamics of continuous media.

2. Algebra of space-time sedeons

The algebra of sedeons [2-5] encloses four groups of values, which are differed with respect to spatial and time inversion.

- Absolute scalars (V) and absolute vectors ($\vec{V}$) are not transformed under spatial and time inversion.
- Time scalars ($V_{\mathbf{t}}$) and time vectors ($\vec{V}_{\mathbf{t}}$) are changed (in sign) under time inversion and are not transformed under spatial inversion.
- Space scalars ($V_{\mathbf{r}}$) and space vectors ($\vec{V}_{\mathbf{r}}$) are changed under spatial inversion and are not transformed under time inversion.
- Space-time scalars ($V_{\mathbf{tr}}$) and space-time vectors ($\vec{V}_{\mathbf{tr}}$) are changed under spatial and time inversion.

Here indexes $\mathbf{t}$ and $\mathbf{r}$ indicate the transformations ($\mathbf{t}$ for time inversion and $\mathbf{r}$ for spatial inversion), which change the corresponding values. All introduced values can be integrated into one space-time sedeon $\tilde{\mathbf{V}}$, which is defined by the following expression:

$$\tilde{\mathbf{V}} = V + \vec{V} + V_{\mathbf{t}} + \vec{V}_{\mathbf{t}} + V_{\mathbf{r}} + \vec{V}_{\mathbf{r}} + V_{\mathbf{tr}} + \vec{V}_{\mathbf{tr}} . \tag{2.1}$$

Let us introduce a scalar-vector basis $\mathbf{a}_0$, $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$, where the element $\mathbf{a}_0$ is an absolute scalar unit ($\mathbf{a}_0 \equiv 1$), and the values $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ are absolute unit vectors generating the right Cartesian basis. Further we will indicate the absolute unit vectors by symbols without arrows as $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. We also introduce the four space-time units $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, where $\mathbf{e}_0$ is an absolute scalar unit ($\mathbf{e}_0 \equiv 1$); $\mathbf{e}_1$ is a time scalar unit ($\mathbf{e}_1 \equiv \mathbf{e}_{\mathbf{t}}$); $\mathbf{e}_2$ is a space scalar unit ($\mathbf{e}_2 \equiv \mathbf{e}_{\mathbf{r}}$); $\mathbf{e}_3$ is a space-time scalar unit ($\mathbf{e}_3 \equiv \mathbf{e}_{\mathbf{tr}}$). Using space-time basis $\mathbf{e}_\alpha$ and scalar-vector basis $\mathbf{a}_\beta$ (Greek indexes $\alpha, \beta = 0,1,2,3$), we can introduce unified sedeonic components $V_{\alpha\beta}$ in accordance with following relations:

$$\begin{aligned}
V &= \mathbf{e}_0 V_{00} \mathbf{a}_0 , \\
\vec{V} &= \mathbf{e}_0 \left(V_{01}\mathbf{a}_1 + V_{02}\mathbf{a}_2 + V_{03}\mathbf{a}_3 \right), \\
V_{\mathbf{t}} &= \mathbf{e}_1 V_{10} \mathbf{a}_0 , \\
\vec{V}_{\mathbf{t}} &= \mathbf{e}_1 \left(V_{11}\mathbf{a}_1 + V_{12}\mathbf{a}_2 + V_{13}\mathbf{a}_3 \right), \\
V_{\mathbf{r}} &= \mathbf{e}_2 V_{20} \mathbf{a}_0 , \\
\vec{V}_{\mathbf{r}} &= \mathbf{e}_2 \left(V_{21}\mathbf{a}_1 + V_{22}\mathbf{a}_2 + V_{23}\mathbf{a}_3 \right), \\
V_{\mathbf{tr}} &= \mathbf{e}_3 V_{30} \mathbf{a}_0 , \\
\vec{V}_{\mathbf{tr}} &= \mathbf{e}_3 \left(V_{31}\mathbf{a}_1 + V_{32}\mathbf{a}_2 + V_{33}\mathbf{a}_3 \right).
\end{aligned} \tag{2.2}$$

Then sedeon (2.1) can be written in the following expanded form:

$$\begin{aligned}
\tilde{\mathbf{V}} = {} & \mathbf{e}_0 \left(V_{00}\mathbf{a}_0 + V_{01}\mathbf{a}_1 + V_{02}\mathbf{a}_2 + V_{03}\mathbf{a}_3 \right) \\
& + \mathbf{e}_1 \left(V_{10}\mathbf{a}_0 + V_{11}\mathbf{a}_1 + V_{12}\mathbf{a}_2 + V_{13}\mathbf{a}_3 \right) \\
& + \mathbf{e}_2 \left(V_{20}\mathbf{a}_0 + V_{21}\mathbf{a}_1 + V_{22}\mathbf{a}_2 + V_{23}\mathbf{a}_3 \right) \\
& + \mathbf{e}_3 \left(V_{30}\mathbf{a}_0 + V_{31}\mathbf{a}_1 + V_{32}\mathbf{a}_2 + V_{33}\mathbf{a}_3 \right).
\end{aligned} \tag{2.3}$$

The sedeonic components $V_{\alpha\beta}$ are numbers (complex in general). Further we will omit units $\mathbf{a}_0$ and $\mathbf{e}_0$ for the simplicity. The important property of sedeons is that the equality of two sedeons means the equality of all sixteen components $V_{\alpha\beta}$.

Let us consider the multiplication rules for the basis elements $\mathbf{a_n}$ and $\mathbf{e_k}$ (Latin indexes **n**, **k** = 1, 2, 3). The unit vectors $\mathbf{a_n}$ have the following multiplication and commutation rules:

$$\mathbf{a_n a_n} = \mathbf{a_n^2} = 1, \tag{2.4}$$

$$\mathbf{a_n a_k} = -\mathbf{a_k a_n} \text{ (for } \mathbf{n} \neq \mathbf{k}\text{)}, \tag{2.5}$$

$$\mathbf{a}_1\mathbf{a}_2 = i\mathbf{a}_3, \quad \mathbf{a}_2\mathbf{a}_3 = i\mathbf{a}_1, \quad \mathbf{a}_3\mathbf{a}_1 = i\mathbf{a}_2, \tag{2.6}$$

while the space-time units $\mathbf{e_k}$ satisfy the following rules:

$$\mathbf{e_k e_k} = \mathbf{e_k^2} = 1, \tag{2.7}$$

$$\mathbf{e_n e_k} = -\mathbf{e_k e_n} \text{ (for } \mathbf{n} \neq \mathbf{k}\text{)}, \tag{2.8}$$

$$\mathbf{e}_1\mathbf{e}_2 = i\mathbf{e}_3, \quad \mathbf{e}_2\mathbf{e}_3 = i\mathbf{e}_1, \quad \mathbf{e}_3\mathbf{e}_1 = i\mathbf{e}_2. \tag{2.9}$$

Here and further the value i is imaginary unit $(i^2 = -1)$. The multiplication and commutation rules for sedeonic absolute unit vectors $\mathbf{a_n}$ and space-time units $\mathbf{e_k}$ can be presented for obviousness as the tables 1 and 2.

Table 1. Multiplication rules for absolute unit vectors $\mathbf{a_n}$.

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$
$\mathbf{a}_1$	1	$i\mathbf{a}_3$	$-i\mathbf{a}_2$
$\mathbf{a}_2$	$-i\mathbf{a}_3$	1	$i\mathbf{a}_1$
$\mathbf{a}_3$	$i\mathbf{a}_2$	$-i\mathbf{a}_1$	1

Table 2. Multiplication rules for space-time units $\mathbf{e_k}$.

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_1$	1	$i\mathbf{e}_3$	$-i\mathbf{e}_2$
$\mathbf{e}_2$	$-i\mathbf{e}_3$	1	$i\mathbf{e}_1$
$\mathbf{e}_3$	$i\mathbf{e}_2$	$-i\mathbf{e}_1$	1

Note that units $\mathbf{e_k}$ commute with vectors $\mathbf{a_n}$:

$$\mathbf{a_n e_k} = \mathbf{e_k a_n} \tag{2.10}$$

for any $\mathbf{n}$ and $\mathbf{k}$.

In sedeonic algebra we assume the Clifford multiplication of vectors. The sedeonic product of two vectors $\vec{A}$ and $\vec{B}$ can be presented in the following form:

$$\vec{A}\vec{B} = \left(\vec{A}\cdot\vec{B}\right) + \left[\vec{A}\times\vec{B}\right]. \tag{2.11}$$

Here we denote the sedeonic scalar multiplication of two vectors (internal product) by symbol "$\cdot$" and round brackets

$$\left(\vec{A}\cdot\vec{B}\right) = A_1B_1 + A_2B_2 + A_3B_3, \tag{2.12}$$

and sedeonic vector multiplication (external product) by symbol "$\times$" and square brackets

$$\left[\vec{A}\times\vec{B}\right] = i\left(A_2B_3 - A_3B_2\right) + i\left(A_3B_1 - A_1B_3\right) + i\left(A_1B_2 - A_2B_1\right). \tag{2.13}$$

Note that in sedeonic algebra the expression for the vector product differs from analogous expression in Gibbs vector algebra. For the transition from sedeons to the common used Gibbs-Heaviside vector algebra the change

$$i\left[\vec{\nabla}\times\vec{A}\right] \Rightarrow -\nabla\times\boldsymbol{A} = -\mathrm{curl}\,\boldsymbol{A} \tag{2.14}$$

should be made in all vector expressions. Here ∇ and $\boldsymbol{A}$ are Gibbs-Heaviside vectors.

3. Sedeonic equations for electromagnetic field in crystals

In crystals the speed of light is a tensor of the second rank $\hat{s}$. Therefore, the Einstein relation for energy and momentum of the electromagnetic field can be presented in the following sedeonic form:

$$\left(i\mathbf{e}_t E + \mathbf{e}_r \hat{s}\vec{p}\right)\left(i\mathbf{e}_t E + \mathbf{e}_r \hat{s}\vec{p}\right) = 0\,. \tag{3.1}$$

Here the expression $\hat{s}\vec{p}$ is a convolution of the tensor with the vector. Let us introduce the operators

$$\begin{aligned} \partial &= \frac{\partial}{\partial t}, \\ \vec{\nabla} &= \frac{\partial}{\partial x}\mathbf{a}_1 + \frac{\partial}{\partial y}\mathbf{a}_2 + \frac{\partial}{\partial z}\mathbf{a}_3. \end{aligned} \tag{3.2}$$

Then sedeonic wave equation for the electromagnetic field in a crystal can be written as

$$\left(i\mathbf{e}_1\partial - \mathbf{e}_2\hat{s}\vec{\nabla}\right)\left(i\mathbf{e}_1\partial - \mathbf{e}_2\hat{s}\vec{\nabla}\right)\tilde{\mathbf{W}} = \tilde{\mathbf{J}}\,. \tag{3.3}$$

The potential of the electromagnetic field is

$$\tilde{\mathbf{W}} = i\mathbf{e}_1\varphi_e - i\mathbf{e}_2\varphi_m + \mathbf{e}_1\vec{A}_m + \mathbf{e}_2\vec{A}_e\,, \tag{3.4}$$

where φ_e is an electric scalar potential; φ_m is a magnetic scalar potential; $\vec{A}_e$ is an electric vector potential; $\vec{A}_m$ is a magnetic vector potential. A field source $\tilde{\mathbf{J}}$ can be written as

$$\tilde{\mathbf{J}} = -i\mathbf{e}_1 4\pi\hat{s}^2\rho_e + i\mathbf{e}_2 4\pi\hat{s}^2\rho_m - \mathbf{e}_2 4\pi\hat{s}\vec{j}_e - \mathbf{e}_1 4\pi\hat{s}\vec{j}_m\,, \tag{3.5}$$

where ρ_e is a volume density of electric charge; $\vec{j}_e$ is a volume density of electric current; ρ_m is a volume density of magnetic charge; $\vec{j}_m$ is a volume density of magnetic current. Thus, the inhomogeneous wave equation for the electromagnetic field in a crystal can be represented as follows:

$$\begin{aligned} &\left(i\mathbf{e}_1\partial - \mathbf{e}_2\hat{s}\vec{\nabla}\right)\left(i\mathbf{e}_1\partial - \mathbf{e}_2\hat{s}\vec{\nabla}\right)\left(i\mathbf{e}_1\varphi_e - i\mathbf{e}_2\varphi_m + \mathbf{e}_1\vec{A}_m + \mathbf{e}_2\vec{A}_e\right) \\ &= -i\mathbf{e}_1 4\pi\hat{s}^2\rho_e + i\mathbf{e}_2 4\pi\hat{s}^2\rho_m - \mathbf{e}_2 4\pi\hat{s}\vec{j}_e - \mathbf{e}_1 4\pi\hat{s}\vec{j}_m. \end{aligned} \tag{3.6}$$

The equation (3.6) is a compact and universal relation, which can be represented either as a system of wave equations for the field potentials, either in the form of Maxwell's equations for the field strengths. Let us introduce the scalar and vector inductions of electromagnetic field:

$$\begin{aligned} d_s &= \partial\varphi_e + \left(\hat{s}\vec{\nabla}\cdot\vec{A}_e\right), \\ b_s &= \partial\varphi_m + \left(\hat{s}\vec{\nabla}\cdot\vec{A}_m\right), \\ \vec{D}_s &= -\partial\vec{A}_e - \hat{s}\vec{\nabla}\varphi_e + i\left[\hat{s}\vec{\nabla}\times\vec{A}_m\right], \\ \vec{B}_s &= -\partial\vec{A}_m - \hat{s}\vec{\nabla}\varphi_m - i\left[\hat{s}\vec{\nabla}\times\vec{A}_e\right]. \end{aligned} \tag{3.7}$$

Then

$$\left(i\mathbf{e}_1\partial - \mathbf{e}_2\hat{s}\vec{\nabla}\right)\left(i\mathbf{e}_1\varphi_e - i\mathbf{e}_2\varphi_m + \mathbf{e}_1\vec{A}_m + \mathbf{e}_2\vec{A}_e\right) = -d_s + i\mathbf{e}_3 b_s + \mathbf{e}_3\vec{D}_s - i\vec{B}_s\,, \tag{3.8}$$

and the equation (3.6) can be rewritten as

$$\left(i\mathbf{e}_1\partial - \mathbf{e}_2\hat{s}\vec{\nabla}\right)\left(-d_s + i\mathbf{e}_3 b_s + \mathbf{e}_3\vec{D}_s - i\vec{B}_s\right) = -i\mathbf{e}_1 4\pi\hat{s}^2\rho_e + i\mathbf{e}_2 4\pi\hat{s}^2\rho_m - \mathbf{e}_2 4\pi\hat{s}\vec{j}_e - \mathbf{e}_1 4\pi\hat{s}\vec{j}_m. \tag{3.9}$$

From the equation (3.9) we obtain the Maxwell equations for inductions

$$
\begin{aligned}
&\partial d_s + \left(\hat{s}\vec{\nabla}\cdot\vec{D}_s\right) = 4\pi\hat{s}^2\rho_e,\\
&\partial b_s + \left(\hat{s}\vec{\nabla}\cdot\vec{B}_s\right) = 4\pi\hat{s}^2\rho_m,\\
&-i\left[\hat{s}\vec{\nabla}\times\vec{D}_s\right] + \partial\vec{B}_s + \hat{s}\vec{\nabla}b_s = -4\pi\hat{s}\vec{j}_m.\\
&+i\left[\hat{s}\vec{\nabla}\times\vec{B}_s\right] + \partial\vec{D}_s + \hat{s}\vec{\nabla}d_s = -4\pi\hat{s}\vec{j}_e,
\end{aligned}
\tag{3.10}
$$

This equation can be simplified. Taking into account the Lorentz gauge we can take scalar fields equal to zero

$$
\begin{aligned}
d_s &= \partial\varphi_e + (\hat{s}\vec{\nabla}\cdot\vec{A}_e) = 0,\\
b_s &= \partial\varphi_m + (\hat{s}\vec{\nabla}\cdot\vec{A}_m) = 0.
\end{aligned}
\tag{3.11}
$$

In addition, we can take the magnetic charges and currents equal to zero ($\rho_m = 0$, $\vec{j}_m = 0$). Then (3.9) takes the following form

$$
\left(i\mathbf{e}_1\partial - \mathbf{e}_2\hat{s}\vec{\nabla}\right)\left(\mathbf{e}_3\vec{D} - i\vec{B}\right) = -i\mathbf{e}_1 4\pi\hat{s}^2\rho_e - \mathbf{e}_2 4\pi\hat{s}\vec{j}_e. \tag{3.12}
$$

Performing the action of the operator on the left side (3.12), and separating the values with different space-time properties, we obtain a system of first-order equations

$$
\begin{aligned}
&\left(\hat{s}\vec{\nabla}\cdot\vec{D}_s\right) = 4\pi\hat{s}^2\rho_e,\\
&\left(\hat{s}\vec{\nabla}\cdot\vec{B}_s\right) = 0,\\
&-i\left[\hat{s}\vec{\nabla}\times\vec{B}_s\right] = \partial\vec{D}_s + 4\pi\hat{s}\vec{j}_e,\\
&-i\left[\hat{s}\vec{\nabla}\times\vec{D}_s\right] = -\partial\vec{B}_s.
\end{aligned}
\tag{3.13}
$$

This is the system of Maxwell equations for anisotropic matter. We can specify the relationship between the vectors of the induction on the one hand, and the magnetic and electric fields and vectors of polarization and magnetization on the other hand:

$$
\begin{aligned}
\vec{D}_s &= \vec{E}_s + 4\pi\vec{P}_s,\\
\vec{B}_s &= \vec{H}_s + 4\pi\vec{M}_s.
\end{aligned}
\tag{3.14}
$$

Then equations (3.13) can be represented as:

$$
\begin{aligned}
&\left(\hat{s}\vec{\nabla}\cdot\vec{E}_s\right) = 4\pi\hat{s}^2\rho_e - 4\pi\left(\hat{s}\vec{\nabla}\cdot\vec{P}_s\right),\\
&\left(\hat{s}\vec{\nabla}\cdot\vec{H}_s\right) = -\,4\pi\left(\hat{s}\vec{\nabla}\cdot\vec{M}_s\right),\\
&-i\left[\hat{s}\vec{\nabla}\times\vec{H}_s\right] = 4\pi\hat{s}\vec{j}_e + \partial\vec{E}_s + 4\pi\partial\vec{P}_s + i4\pi\left[\hat{s}\vec{\nabla}\times\vec{M}_s\right],\\
&-i\left[\hat{s}\vec{\nabla}\times\vec{E}_s\right] = -\partial\vec{H}_s - 4\pi\partial\vec{M}_s + i4\pi\left[\hat{s}\vec{\nabla}\times\vec{P}_s\right].
\end{aligned}
\tag{3.15}
$$

On the other hand, applying to the both sides of equation (3.9) operator

$$
\left(i\mathbf{e}_1\partial - \mathbf{e}_2\hat{s}\vec{\nabla}\right), \tag{3.16}
$$

we get wave equation for the inductions of electromagnetic field

$$
\begin{aligned}
&\left(i\mathbf{e}_1\partial - \mathbf{e}_2\hat{s}\vec{\nabla}\right)\left(i\mathbf{e}_1\partial - \mathbf{e}_2\hat{s}\vec{\nabla}\right)\left(-d_s + i\mathbf{e}_3 b_s + \mathbf{e}_3\vec{D}_s - i\vec{B}_s\right)\\
&= \left(i\mathbf{e}_1\partial - \mathbf{e}_2\hat{s}\vec{\nabla}\right)\left(-i\mathbf{e}_1 4\pi\hat{s}^2\rho_e + i\mathbf{e}_2 4\pi\hat{s}^2\rho_m - \mathbf{e}_2 4\pi\hat{s}\vec{j}_e - \mathbf{e}_1 4\pi\hat{s}\vec{j}_m\right).
\end{aligned}
\tag{3.17}
$$

4. Field equations for isotropic medium

For an isotropic medium the speed of light is a spherical tensor

$$
\hat{s} = s\delta_{ik}\,, \tag{4.1}
$$

where δ_{ik} is Kronecker symbol, s is a modulus of light speed

$$s = \frac{c}{n}, \tag{4.2}$$

n is a refractive index, c is the speed of light in a vacuum. Then for isotropic medium the equations (3.13) can be represented as

$$\begin{aligned}
&\left(s\vec{\nabla}\cdot\vec{D}_s\right) = 4\pi s^2 \rho_e,\\
&\left(s\vec{\nabla}\cdot\vec{B}_s\right) = 0,\\
&-i\left[s\vec{\nabla}\times\vec{B}_s\right] = \partial\vec{D}_s + 4\pi s\vec{j}_e,\\
&-i\left[s\vec{\nabla}\times\vec{D}_s\right] = -\partial\vec{B}_s.
\end{aligned} \tag{4.3}$$

Introducing new inductions

$$\begin{aligned}
&\vec{D} = -\frac{1}{s}\partial\vec{A}_e - \vec{\nabla}\varphi_e + i\left[\vec{\nabla}\times\vec{A}_m\right],\\
&\vec{B} = -\frac{1}{s}\partial\vec{A}_m - \vec{\nabla}\varphi_m - i\left[\vec{\nabla}\times\vec{A}_e\right],
\end{aligned} \tag{4.4}$$

the system (4.3) can be rewritten as

$$\begin{aligned}
&\left(\vec{\nabla}\cdot\vec{D}\right) = 4\pi\rho_e,\\
&\left(\vec{\nabla}\cdot\vec{B}\right) = 0,\\
&-i\left[\vec{\nabla}\times\vec{B}\right] = \frac{1}{s}\partial\vec{D} + \frac{4\pi}{s}\vec{j}_e,\\
&-i\left[\vec{\nabla}\times\vec{D}\right] = -\frac{1}{s}\partial\vec{B}.
\end{aligned} \tag{4.5}$$

We can specify similar relationships between the vectors of the inductions, field strengths, polarization and magnetization:

$$\begin{aligned}
&\vec{D} = \vec{E} + 4\pi\vec{P},\\
&\vec{B} = \vec{H} + 4\pi\vec{M}.
\end{aligned} \tag{4.6}$$

Then equations (4.5) can be represented as

$$\begin{aligned}
&\left(\vec{\nabla}\cdot\vec{E}\right) = 4\pi\rho_e - 4\pi\left(\vec{\nabla}\cdot\vec{P}\right),\\
&\left(\vec{\nabla}\cdot\vec{H}\right) = -4\pi\left(\vec{\nabla}\cdot\vec{M}\right),\\
&-i\left[\vec{\nabla}\times\vec{H}\right] = \frac{4\pi}{s}\vec{j}_e + \frac{1}{s}\partial\vec{E} + 4\pi\frac{1}{s}\partial\vec{P} + i4\pi\left[\vec{\nabla}\times\vec{M}\right],\\
&-i\left[\vec{\nabla}\times\vec{E}\right] = -\frac{1}{s}\partial\vec{H} - 4\pi\frac{1}{s}\partial\vec{M} + i4\pi\left[\vec{\nabla}\times\vec{P}\right].
\end{aligned} \tag{4.7}$$

These equations coincide with widely used Maxwell equations if we take the following conditions:

$$\begin{aligned}
&\left[\vec{\nabla}\times\vec{M}\right] = 0,\\
&\left[\vec{\nabla}\times\vec{P}\right] = 0,
\end{aligned} \tag{4.8}$$

and in the following approximation:

$$s \Rightarrow c. \tag{4.9}$$

Taking into account (4.8) and (4.9) we get

$$
\begin{aligned}
&\left(\vec{\nabla}\cdot\vec{D}\right)=4\pi\rho_e,\\
&\left(\vec{\nabla}\cdot\vec{B}\right)=0,\\
&-i\left[\vec{\nabla}\times\vec{H}\right]=\frac{1}{c}\partial\vec{D}+\frac{4\pi}{c}\vec{j}_e,\\
&-i\left[\vec{\nabla}\times\vec{E}\right]=-\frac{1}{c}\partial\vec{B},
\end{aligned}
\tag{4.10}
$$

that coincide with well known Maxwell's equations [1,6].

5. Pointing theorem for anisotropic medium

Let us multiply the equation (3.12) on the sedeon $\left(\mathbf{e}_3\vec{D}_s-i\vec{B}_s\right)$ from the left. Then we get

$$
\left(\mathbf{e}_3\vec{D}-i\vec{B}\right)\left(i\mathbf{e}_1\partial-\mathbf{e}_2\hat{c}\vec{\nabla}\right)\left(\mathbf{e}_3\vec{D}-i\vec{B}\right)=\left(\mathbf{e}_3\vec{D}-i\vec{B}\right)\left(-i\mathbf{e}_1 4\pi\hat{c}^2\rho_e-\mathbf{e}_2 4\pi\hat{c}\vec{j}_e\right).
\tag{5.1}
$$

Separating the values with different space-time properties we obtain following expressions:

$$
-i\mathbf{e}_1\left(\vec{B}_s\cdot\partial\vec{B}_s\right)-i\mathbf{e}_1\left(\vec{D}_s\cdot\partial\vec{D}_s\right)-\mathbf{e}_1\left(\vec{B}_s\cdot\left[\hat{s}\vec{\nabla}\times\vec{D}_s\right]\right)+\mathbf{e}_1\left(\vec{D}_s\cdot\left[\hat{s}\vec{\nabla}\times\vec{B}_s\right]\right)=i\mathbf{e}_1 4\pi\left(\vec{D}_s\cdot\hat{s}\vec{j}_e\right),
\tag{5.2}
$$

$$
-i\mathbf{e}_2\left(\vec{B}_s\cdot\partial\vec{D}_s\right)+i\mathbf{e}_2\left(\vec{D}_s\cdot\partial\vec{B}_s\right)+\mathbf{e}_2\left(\vec{B}_s\cdot\left[\hat{s}\vec{\nabla}\times\vec{B}_s\right]\right)+\mathbf{e}_2\left(\vec{D}_s\cdot\left[\hat{s}\vec{\nabla}\times\vec{D}_s\right]\right)=i\mathbf{e}_2 4\pi\left(\vec{B}_s\cdot\hat{s}\vec{j}_e\right),
\tag{5.3}
$$

$$
\begin{aligned}
&-i\mathbf{e}_1\left[\vec{B}_s\times\partial\vec{B}_s\right]-i\mathbf{e}_1\left[\vec{D}_s\times\partial\vec{D}_s\right]-\mathbf{e}_1\vec{B}_s\left(\hat{s}\vec{\nabla}\cdot\vec{D}_s\right)+\mathbf{e}_1\vec{D}_s\left(\hat{s}\vec{\nabla}\cdot\vec{B}_s\right)\\
&-\mathbf{e}_1\left[\vec{B}_s\times\left[\hat{s}\vec{\nabla}\times\vec{D}_s\right]\right]+\mathbf{e}_1\left[\vec{D}_s\times\left[\hat{s}\vec{\nabla}\times\vec{B}_s\right]\right]\\
&=-\mathbf{e}_1 4\pi\vec{B}_s\hat{s}^2\rho_e+i\mathbf{e}_1 4\pi\left[\vec{D}_s\hat{s}\vec{j}_e\right],
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
&-i\mathbf{e}_2\left[\vec{B}_s\times\partial\vec{D}_s\right]+i\mathbf{e}_2\left[\vec{D}_s\times\partial\vec{B}_s\right]+\mathbf{e}_2\vec{B}_s\left(\hat{s}\vec{\nabla}\cdot\vec{B}_s\right)+\mathbf{e}_2\vec{D}_s\left(\hat{s}\vec{\nabla}\cdot\vec{D}_s\right)\\
&+\mathbf{e}_2\left[\vec{B}_s\times\left[\hat{s}\vec{\nabla}\times\vec{B}_s\right]\right]+\mathbf{e}_2\left[\vec{D}_s\times\left[\hat{s}\vec{\nabla}\times\vec{D}_s\right]\right]\\
&=\mathbf{e}_2 4\pi\vec{D}_s\hat{s}^2\rho_e+i\mathbf{e}_2 4\pi\left[\vec{B}_s\hat{s}\vec{j}_e\right].
\end{aligned}
\tag{5.5}
$$

It is clearly seen that the equation (5.2) is the Pointing theorem for anisotropic medium

$$
\frac{1}{8\pi}\partial\left(\vec{D}_s^{\,2}+\vec{B}_s^{\,2}\right)-\frac{i}{4\pi}\left(\hat{c}\vec{\nabla}\cdot\left[\vec{D}_s\times\vec{B}_s\right]\right)+\left(\vec{D}_s\cdot\hat{c}\vec{j}_e\right)=0\,.
\tag{5.6}
$$

6. Plane waves in dielectric crystals

Let us consider the plane electromagnetic waves in homogeneous dielectric crystals. The equations (3.13) without sources take the following form:

$$
\begin{aligned}
&\left(\hat{s}\vec{\nabla}\cdot\vec{D}_s\right)=0,\\
&\left(\hat{s}\vec{\nabla}\cdot\vec{B}_s\right)=0,\\
&-i\left[\hat{s}\vec{\nabla}\times\vec{B}_s\right]=\partial\vec{D}_s,\\
&-i\left[\hat{s}\vec{\nabla}\times\vec{D}_s\right]=-\partial\vec{B}_s.
\end{aligned}
\tag{6.1}
$$

Substituting in (6.1) the plane waves:

$$
\begin{aligned}
&\vec{D}_s=\vec{D}_0\exp\left(i\omega t-i(\vec{k}\cdot\vec{r})\right),\\
&\vec{B}_s=\vec{B}_0\exp\left(i\omega t-i(\vec{k}\cdot\vec{r})\right),
\end{aligned}
\tag{6.2}
$$

we obtain:

$$-i\left[i\hat{s}\vec{k}\times\vec{B}_0\right]=i\omega\vec{D}_0,$$
$$-i\left[i\hat{s}\vec{k}\times\vec{D}_0\right]=-i\omega\vec{B}_0. \tag{6.3}$$

Here ω is the frequency and $\vec{k}$ is a wave vector. Let us introduce a unit vector $\vec{m}$ according the following relation:

$$\vec{k}=\frac{\omega}{c}\vec{m}, \tag{6.4}$$

and normalized tensor of light speed $\hat{\alpha}$ according

$$\hat{\alpha}=\frac{\hat{s}}{c}, \tag{6.5}$$

which has the sense of inverse tensor of refractive index. Then we obtain

$$-i\left[\hat{\alpha}\vec{m}\times\vec{B}_0\right]=\vec{D}_0,$$
$$-i\left[\hat{\alpha}\vec{m}\times\vec{D}_0\right]=-\vec{B}_0. \tag{6.6}$$

Excluding vector $\vec{B}_0$ we get

$$\hat{\alpha}\vec{m}\left(\hat{\alpha}\vec{m}\cdot\vec{D}_0\right)-\vec{D}_0\left(\hat{\alpha}\vec{m}\cdot\hat{\alpha}\vec{m}\right)=-\vec{D}_0. \tag{6.7}$$

In the principal axes of tensor $\hat{\alpha}$ this equation gives us the following system:

$$\begin{aligned}
&\left(1-\left(\alpha_{yy}m_y\right)^2-\left(\alpha_{zz}m_z\right)^2\right)D_{0x}+\left(\alpha_{xx}m_x\right)\left(\alpha_{yy}m_y\right)D_{0y}+\left(\alpha_{xx}m_x\right)\left(\alpha_{zz}m_z\right)D_{0z}=0,\\
&\left(\alpha_{xx}m_x\right)\left(\alpha_{yy}m_y\right)D_{0x}+\left(1-\left(\alpha_{xx}m_x\right)^2-\left(\alpha_{zz}m_z\right)^2\right)D_{0y}+\left(\alpha_{yy}m_y\right)\left(\alpha_{zz}m_z\right)D_{0z}=0,\\
&\left(\alpha_{xx}m_x\right)\left(\alpha_{zz}m_z\right)D_{0x}+\left(\alpha_{zz}m_z\right)\left(\alpha_{yy}m_y\right)D_{0y}+\left(1-\left(\alpha_{yy}m_y\right)^2-\left(\alpha_{xx}m_x\right)^2\right)D_{0z}=0.
\end{aligned} \tag{6.8}$$

The determinant of this system should be zero:

$$\begin{vmatrix}
\left(1-\left(\alpha_{yy}m_y\right)^2-\left(\alpha_{zz}m_z\right)^2\right) & \left(\alpha_{xx}m_x\right)\left(\alpha_{yy}m_y\right) & \left(\alpha_{xx}m_x\right)\left(\alpha_{zz}m_z\right)\\
\left(\alpha_{xx}m_x\right)\left(\alpha_{yy}m_y\right) & \left(1-\left(\alpha_{xx}m_x\right)^2-\left(\alpha_{zz}m_z\right)^2\right) & \left(\alpha_{yy}m_y\right)\left(\alpha_{zz}m_z\right)\\
\left(\alpha_{xx}m_x\right)\left(\alpha_{zz}m_z\right) & \left(\alpha_{zz}m_z\right)\left(\alpha_{yy}m_y\right) & \left(1-\left(\alpha_{yy}m_y\right)^2-\left(\alpha_{xx}m_x\right)^2\right)
\end{vmatrix}=0. \tag{6.7}$$

After simplification of (6.7) we obtain the following equation:

$$\left\{\left(\alpha_{xx}m_x\right)^2+\left(\alpha_{yy}m_y\right)^2+\left(\alpha_{zz}m_z\right)^2-1\right\}^2=0. \tag{6.8}$$

The solution of this equation is

$$\left(\alpha_{xx}m_x\right)^2+\left(\alpha_{yy}m_y\right)^2+\left(\alpha_{zz}m_z\right)^2-1=0. \tag{6.9}$$

If the wave propagates along the x-axis then

$$\left(\alpha_{xx}m_x\right)^2-1=0, \tag{6.7}$$

and

$$m_x=\pm\frac{1}{\alpha_{xx}}=\pm\sqrt{\varepsilon_{xx}}, \tag{6.8}$$

where ε_{ik} is the tensor of dielectric permeability.

Let us consider the wave propagating along the arbitrary Z direction. In this case the wave vector $\vec{m}$ has only z-component and $\vec{D}_0$ has only x and y components. From (6.7) we have

$$\begin{gathered}\left(1-\left(\alpha_{yz}m_z\right)^2-\left(\alpha_{zz}m_z\right)^2\right)D_x+\left(\alpha_{xz}m_z\right)\left(\alpha_{yz}m_z\right)D_y=0,\\ \left(\alpha_{yz}m_z\right)\left(\alpha_{xz}m_z\right)D_x+\left(1-\left(\alpha_{xz}m_z\right)^2-\left(\alpha_{zz}m_z\right)^2\right)D_y=0.\end{gathered} \tag{6.9}$$

The determinant of this system should be zero and we obtain the following equation:

$$\left[\left(\alpha_{zz}m_z\right)^2-1\right]\left[\left(\alpha_{zz}m_z\right)^2+\left(\alpha_{yz}m_z\right)^2+\left(\alpha_{xz}m_z\right)^2-1\right]=0. \tag{6.10}$$

The expression (6.10) splits into two equations for the ordinary and the extraordinary waves:

$$\begin{gathered}\left[\left(\alpha_{zz}m_z\right)^2-1\right]=0,\\ \left[\left(\alpha_{zz}m_z\right)^2+\left(\alpha_{yz}m_z\right)^2+\left(\alpha_{xz}m_z\right)^2-1\right]=0.\end{gathered} \tag{6.11}$$

For the ordinary wave we have

$$m_z=\pm\frac{1}{\alpha_{zz}}, \tag{6.12}$$

and for the extraordinary wave we get

$$m_z=\pm\frac{1}{\sqrt{\left(\alpha_{zz}\right)^2+\left(\alpha_{yz}\right)^2+\left(\alpha_{xz}\right)^2}}. \tag{6.13}$$

If the Z direction coincides with one of principal axes of tensor $\hat{\alpha}$, then the speed of propagation for ordinary and the extraordinary waves are the same.

7. Concluding remarks

Thus, we have developed the description of electromagnetic field in an anisotropic medium using the sedeonic wave equations based on sedeonic potentials and space-time operators. The central assumption of this theory is the hypothesis that the electromagnetic phenomena in the environment are defined by dependence of propagation velocity of the electromagnetic wave from the properties of the medium. This allowed the use of sedeonic algebra formalism to obtain the Maxwell equations and Poynting theorem, as well as to describe the propagation of waves in crystals.

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