A new quantum mechanical formalism based on the probability representation of quantum states.

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Abstract: A new quantum mechanical formalism based on the probability representation of quantum states is proposed. This paper in particular deals with the special case of the measurement problem, known as Schrödinger's cat paradox. We pointed out that Schrödinger's cat demands to reconcile Born's rule. Using new quantum mechanical formalism we find the collapsed state of the Schrödinger's cat always shows definite and predictable outcomes even if cat also consists of a superposition

$$|\text{cat}\rangle = c_1|\text{live cat}\rangle + c_2|\text{death cat}\rangle$$
$$|c_1|^2 + |c_2|^2 = 1.$$

Using new quantum mechanical formalism the EPRB-paradox is considered successfully. We find that the EPRB-paradox can be resolved by nonprincipal and convenient relaxing of the Einstein's locality principle.

Content

Part I.

Schrödinger's cat paradox resolution using GRW collapse model. Von Neumann measurement postulate revisited.

I. Introduction

In his famous thought experiment,Schrôdinger(1935) imagined a cat that measures the value of an quantum mechanical observable with its life. Since Schrödinger's time, no any interpretations or modifications of quantum mechanics have been proposed which gives clear unambiguous answers to the questions posed by Schrödinger's cat of how long superpositions last and when (or whether) they collapse? In this paper appropriate modification of quantum mechanics are proposed. We claim that canonical interpretation of the wave function $\psi = c_1\psi_1 + c_2\psi_2$ is correct only when the supports the wave functions ψ_1 and ψ_2 essentially overlap. When the wave functions ψ_1 and ψ_2 have separated supports (as in the case of the experiment that we are considering in this paper) we claim that canonical interpretation of the wave function $\psi = c_1\psi_1 + c_2\psi_2$ is no longer valid for a such cat state. Possible solution of the Schrödinger's cat paradox are considered.We pointed out that the collapsed state of the cat always shows definite and predictable outcomes even if cat also consists of a superposition [16]-[17] :

$$|\text{cat}\rangle = c_1|\text{live cat}\rangle + c_2|\text{death cat}\rangle.$$

I.1.The canonical interpretations of the Schrödinger experiment.

As Weinberg recently reminded us [1], the measurement problem remains a fundamental conundrum. During measurement the state vector of the microscopic system collapses in a probabilistic way to one of a number of classical states, in a way that is unexplained, and cannot be described by the time-dependent Schrödinger equation [1].To review the essentials, it is sufficient to consider two-state systems. Suppose a nucleus $\mathbf{n}$, whose Hilbert space is spanned by orthonormal states $|s_i(t)\rangle,\ i = 1,2,$where $|s_1(t)\rangle = |\text{undecayed nucleus at instant } t\rangle$ and $|s_2(t)\rangle = |\text{decayed nucleus at instant } t\rangle$ is in the superposition state,

$$|\Psi_t\rangle_{\mathbf{n}} = c_1|s_1(t)\rangle + c_2|s_2(t)\rangle, |c_1|^2 + |c_2|^2 = 1. \tag{1.1.1}$$

A measurement apparatus A, which may be microscopic or macroscopic, is designed to distinguish between states $|s_i(t)\rangle$ by transitioning at each instant t into state $|a_i(t)\rangle$ if it finds $\mathbf{n}$ is in $|s_i(t)\rangle,\ i = 1,2.$ Assume the detector is reliable, implying the $|a_1(t)\rangle$ and $|a_2(t)\rangle$ are orthonormal at each instant t ,i.e., $\langle a_1(t)||a_2(t)\rangle = 0$ and that the measurement interaction does not disturb states $|s_i\rangle$ -i.e., the measurement is “ideal”. When A measures $|\Psi_t\rangle_{\mathbf{n}}$, the Schrödinger equation’s unitary time evolution then leads to the “measurement state” $|\Psi_t\rangle_{\mathbf{n}A}$:

$$|\Psi_t\rangle_{\mathbf{n}A} = c_1|a_1(t)\rangle + c_2|a_2(t)\rangle, |c_1|^2 + |c_2|^2 = 1. \tag{1.1.2}$$

of the composite system $\mathbf{n}A$ following the measurement.

Standard formalism of continuous quantum measurements [2],[3],[4],[5-10] leads to a definite but unpredictable measurement outcome, either $|a_1(t)\rangle$ or $|a_2(t)\rangle$ and that $|\Psi_t\rangle_{\mathbf{n}}$ suddenly “collapses” at instant t' into the corresponding state $|s_i(t')\rangle$. But unfortunately the equation (1.1.2) does not appear to resemble such a collapsed state at instant t'?.

The measurement problem is as follows:

(**I**) How do we reconcile canonical collapse models postulate’s

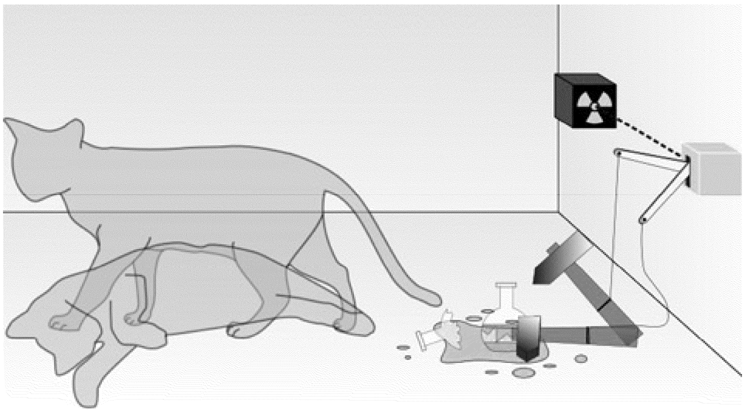
(**II**) How do we reconcile the measurement postulate’s definite outcomes with the

“measurement state” $|\Psi_t\rangle_{\mathbf{n}A}$ at each instant t and

(**III**) how does the outcome become irreversibly recorded in light of the Schrödinger

equation’s unitary and, hence, reversible evolution?

The Part I of this paper in particular deals with the special case of the measurement problem, known as Schrödinger’s cat paradox. For a good and complete explanation of this paradox see Albert [7], Leggett [11], Hobson [12] and Schrödinger[13], see also [14]-[16].



Pic.1.1.1.Schrödinger's generic cat.

In his famous thought experiment [11], Schrôdinger(1935) imagined a cat that measures the value of an quantum mechanical observable with its life. Adapted to the measurement of position of an alpha particle, the experiment is this. A cat, a flask of poison, and a radioactive source are placed in a sealed box. If an internal monitor detects radioactivity (i.e. a single atom decaying), the flask is shattered, releasing the poison that kills the cat. The Copenhagen interpretation of quantum mechanics implies that after a while, the cat is simultaneously alive and dead. Yet, when one looks in the box, one sees the cat either alive or dead, not both alive and dead.

This poses the question of when exactly quantum superposition ends and reality collapses into one possibility or the other?

Since Schrödinger's time, no any interpretations or extensions of quantum mechanics have been proposed which gives clear unambiguous answers to the questions posed by Schrödinger's cat of how long superpositions last and when (or whether) they collapse.

The canonical interpretations of the Schrödinger experiment.

I.1.1.The Copenhagen interpretation

The most commonly held interpretation of quantum mechanics is the Copenhagen interpretation.[14-15] In the Copenhagen interpretation, a system stops being a superposition of states and becomes either one or the other when an observation takes place. This thought experiment makes apparent the fact that the nature of measurement, or observation, is not well-defined in this interpretation. The experiment can be interpreted to mean that while the box is closed, the system simultaneously exists in a superposition of the states "decayed nucleus/dead cat" and "undecayed nucleus/living cat", and that only when the box is opened and an observation performed does the wave function collapse into one of the two states.

However, one of the main scientists associated with the Copenhagen interpretation, Niels Bohr, never had in mind the observer-induced collapse of the wave function, so that Schrödinger's cat did not pose any riddle to him. The cat would be either dead or alive long before the box is opened by a conscious observer [14-15]. Analysis of an actual experiment found that measurement alone (for example by a Geiger counter) is sufficient to collapse a quantum wave function before there is any conscious observation of the measurement.[15] The view that the "observation" is taken when a particle from the nucleus hits the detector can be developed into objective collapse theories. The thought experiment requires an "unconscious observation" by the detector in order for magnification to occur.

I.1.2.The Objective collapse theories

According to objective collapse theories, superpositions are destroyed spontaneously (irrespective of external observation) when some objective physical threshold (of time, mass, temperature, irreversibility, etc.) is reached. Thus, the cat would be expected to have settled into a definite state long before the box is opened. This could loosely be phrased as "the cat observes itself", or "the environment observes the cat".

Objective collapse theories require a modification of standard quantum mechanics to allow superpositions to be destroyed by the process of time evolution. This process, known as "decoherence", is among the fastest processes currently known to physics [5],[].

I.1.3.The Ensemble interpretation

The ensemble interpretation states that superpositions are nothing but subensembles of a larger statistical ensemble. The state vector would not apply to individual cat experiments, but only to the statistics of many similarly prepared cat experiments. Proponents of this interpretation state that this makes the Schrödinger's cat paradox a trivial matter, or a non-issue. This interpretation serves to discard the idea that a single physical system in quantum mechanics has a mathematical description that corresponds to it in any way.

Remark 1.1.1. Ensemble interpretation in a good agreement with a canonical interpretetion of the wave function (ψ-function) in canonical QM-measurement theory. However under rigorous consideration of the dinamics of the Schrödinger's cat this interpretation gives obviously unphysical result, see section III, Proposition 3.1.2.(ii).

I.2.The canonical collapse models.Quantum Mechanics with Spontaneous Localizations (QMSL)

We remind that Quantum Mechanics with Spontaneous Localizations [2],[3] is based on the following assumptions:

(1) Each particle of a system of n distinguishable particles experiences, with a mean rate λ_i, a sudden spontaneous localization process.

(2) In the time interval between two successive spontaneous processes the system evolves according to the usual Schrödinger equation.

(3) The sudden spontaneous process is a localization given by:

$$|\psi\rangle \overset{\textbf{localization}}{\longrightarrow} \frac{|\psi_{\mathbf{x}}^{i}\rangle}{\||\psi_{\mathbf{x}}^{i}\rangle\|}, \mathbf{x} \in \mathbb{R}^{3}, \tag{1.2.1}$$

where

$$|\psi_{\mathbf{x}}^{i}\rangle = L_{\mathbf{x}}^{i}|\psi\rangle. \tag{1.2.2}$$

Here $L_{\mathbf{x}}^{i}$ is a norm-reducing, positive, self-adjoint, linear operator in the n-particle projective Hilbert space $\mathbf{H}$, representing the localization of particle i around the point $\mathbf{x}$.

(4) The probability density for the occurrence of a localization at point $\mathbf{x}$ is assumed to be

$$P_{i}(\mathbf{x}) = \||\psi_{\mathbf{x}}^{i}\rangle\|^{2}. \tag{1.2.3}$$

Eq.(1.2.3) requires that $\int d^{3}x[L_{\mathbf{x}}^{i}]^{2} = 1$.

(5) The localization operators $L_{\mathbf{x}}^{i}$ have been chosen to have the form:

$$L_{\mathbf{x}}^{i} = \left(\frac{1}{\delta\pi}\right)^{3/4} \exp\left[-\frac{1}{2\delta}(\widehat{q}_{i} - \mathbf{x})^{2}\right], \tag{1.2.4}$$

$\widehat{q}_{i}$ being the position operator for particle i.

I.2.1.The classical GRW model

In order to appreciate how canonical collapse models work, and what they are able to achieve, we briefly review classical GRW model [2]. Let us consider a system of n particles which, only for the sake of simplicity, we take to be scalar and spinless; the GRW model is defined by the following postulates: (1) The state of the system is represented by a wave function $\psi_{t}(\mathbf{x}_{1},\mathbf{x}_{2},\ldots,\mathbf{x}_{n})$ belonging to the Hilbert space $\mathcal{L}_{2}(\mathbb{R}^{3n})$. (2) At random times, the wave function experiences a sudden jump of the form:

$$\begin{gathered}\psi_{t}(\mathbf{x}_{1},\mathbf{x}_{2},\ldots,\mathbf{x}_{n}) \to \psi_{t}(\mathbf{x}_{1},\mathbf{x}_{2},\ldots,\mathbf{x}_{n};\widetilde{\mathbf{x}}_{m}) = \\ \frac{\Re_{m}(\widetilde{\mathbf{x}}_{m})\psi_{t}(\mathbf{x}_{1},\mathbf{x}_{2},\ldots,\mathbf{x}_{n})}{\|\Re_{m}(\widetilde{\mathbf{x}}_{m})\psi_{t}(\mathbf{x}_{1},\mathbf{x}_{2},\ldots,\mathbf{x}_{n})\|_{2}},\end{gathered} \tag{1.2.5}$$

where $\psi_{t}(\mathbf{x}_{1},\mathbf{x}_{2},\ldots,\mathbf{x}_{n})$ is the state vector of the whole system at time t, immediately prior to the jump process and $\Re_{n}(\widetilde{\mathbf{x}}_{m})$ is a linear operator which is conventionally chosen equal to:

$$\Re_{m}(\widetilde{\mathbf{x}}_{m}) = (\pi r_{c}^{2})^{-3/4} \exp\left[-\frac{(\widehat{\mathbf{x}}_{m} - \widetilde{\mathbf{x}}_{m})^{2}}{2r_{c}^{2}}\right], \tag{1.2.6}$$

where r_{c} is a new parameter of the model which sets the width of the localization process, and $\widehat{\mathbf{x}}_{m}$ is the position operator associated to the m-th particle of the system and the random variable $\tilde{\mathbf{x}}_m$ corresponds to the place where the jump occurs. (3) It is assumed that the jumps are distributed in time like a Poissonian process with frequency $\lambda = \lambda_{GRW}$ this is the second new parameter of the model. (4) Between two consecutive jumps, the state vector evolves according to the standard Schrödinger equation.

Let us consider a single particle. Suppose it suffers a hitting process: its wave function $|\psi\rangle$ changes it into the new wavefunction $|\psi_x\rangle$. We do not know where the hitting occurs, but only the probability for it to occur around position $\mathbf{x}$. Accordingly, the pure state is transformed into the following statistical mixture:

$$|\psi\rangle\langle\psi| = \int d^3xP(\mathbf{x})\frac{|\psi_{\mathbf{x}}\rangle\langle\psi_{\mathbf{x}}|}{\| |\psi_{\mathbf{x}}\rangle \|^2} = \int d^3xL_{\mathbf{x}}|\psi\rangle\langle\psi|L_{\mathbf{x}} = T[|\psi\rangle\langle\psi|] \qquad (1.2.7)$$

Of course, if the initial state of the particle is not pure but a statistical mixture given by the operator ρ, the effect of a hitting process is the same as the one described above: ρ changes into $T[\rho]$. We derive now the evolution equation for $\rho(t)$. In a time interval dt, the statistical operator evolves in the following way: since the localization mechanism is Poissonian, there is a probability λdt for a hitting to occur during that time interval, in which case ρ changes to $T[\rho]$, and a probability $1 - \lambda dt$ for no hittings to occur so that the statistical operator evolves according to the usual Schrödinger equation:

$$\rho(t+dt) = (1-\lambda dt)\left[\rho(t) - \frac{i}{\hbar}\left[\widehat{\mathbf{H}}, \rho(t)\right]dt\right] + \lambda dt T[\rho(t)] \qquad (1.2.8)$$

Thus 1-particle master equation of the GRW model takes the form [2]-[5]

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}\left[\widehat{\mathbf{H}}, \rho(t)\right] - \lambda(\rho(t) - T[\rho(t)]). \qquad (1.2.9)$$

Here $\widehat{\mathbf{H}}$ is the standard quantum Hamiltonian of the particle, and $T[\cdot]$ represents the effect of the spontaneous collapses on the particle's wave function. In the position representation, this operator becomes:

$$\langle\mathbf{q}'|T[\rho(t)]|\mathbf{q}''\rangle = \exp\left[-\frac{(\mathbf{q}'-\mathbf{q}'')^2}{4r_c^2}\right]\langle\mathbf{q}'|\rho(t)|\mathbf{q}''\rangle. \qquad (1.2.10)$$

Since, owing to Eq.(1.2.10) $\langle\mathbf{q}|T[\rho(t)]|\mathbf{q}\rangle = \langle\mathbf{q}|\rho(t)|\mathbf{q}\rangle$, equation (1.2.9) is obviously trace preserving. Moreover, using equation (1.2.9), it can be proved that

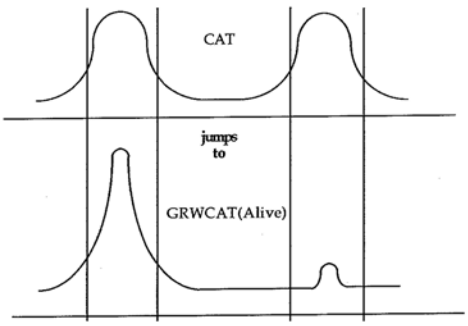
$$\frac{d}{dt}\mathbf{Tr}[\rho^2(t)] < 0. \qquad (1.2.11)$$

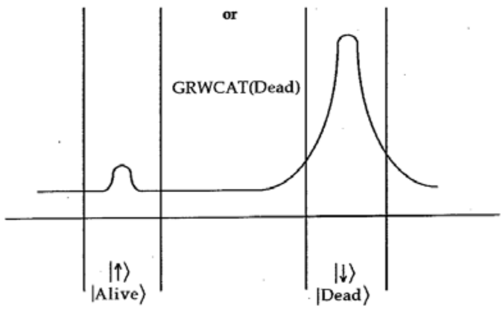
This implies that the dynamical evolution transforms pure states into statistical mixtures.

I.2.2. Tails of Schrödinger's cat.Schrödinger's cat demands to reconcile canonical collapse models

postulates

GRW collapse of the Schrödinger’s cat state considered in [7].The orthodox account attempts to solve the measurement problem by claiming that in measurement interactions the quantum state of the measured system plus measuring system does not evolve in accord with the Schrodinger equation but instead collapses into one of the states that is an eigenstate of the measurement observable (i.e. the observable that records the result of the measurement) with a probability proportional to the square of the coefficient of that state. For example, if Schrodinger’s cat measures the x-spin, then the post-measurement state $|CAT\rangle$ collapses either into $|\uparrow\rangle|Alive\rangle$ or $|\downarrow\rangle|Dead\rangle$ each with a probability proportional to the square of the coefficient associated with each state. So on the orthodox account, there are two fundamental laws that govern the evolution of quantum states. In non-measurement situations, Schrodinger,s deterministic law holds away. But in measurement situations, the collapse dynamics takes over.





Pic.1.2.1.(i) The GRW collapse of the Schrödinger’s cat state $|\mathbf{CAT}\rangle$: $|\mathbf{CAT}\rangle \overset{GRW\ \mathbf{collapse}}{\rightarrow} |\uparrow\rangle|Alive\rangle$

Pic.1.2.1.(ii)The GRW collapse of the Schrödinger’s cat state $|\mathbf{CAT}\rangle$: $|\mathbf{CAT}\rangle \overset{GRW\ \mathbf{collapse}}{\rightarrow} |\downarrow\rangle|Dead\rangle$ Adopted from [7].

There are difficulties, however. Look more dosely, for example, at a post GRW-collapse state like GRWCAT(Alive), illustrated in Pic.1.2.1(i)-(ii). Note that while most of GRWCAT(Alive)’s amplitude is indeed (as we mentioned above) concentrated in the "Alive" region of the state space, it also has non-zero tails which extend into the "Dead" region.

And so it follows from the eigenstate-eigenvalue rule that the cat is, as a matter of fact, not determinately alive (or dead), when GRWCAT(Alive) obtains, after all.

And so the GRW theory, as we have stated it above, patently fails to solve Schrodinger’s paradox.

Remark 1.2.1. In order to avoid the problem of the tails of Schrödinger’s cat we replace now postulate (5) by the next postulate:

($5'$) The localization operators $L^i_{\mathbf{x}}$ have been chosen to have the form:

$$L_{\mathbf{x}}^{i} = L_{\mathbf{x}}^{i}(\delta,\varepsilon) = \begin{cases} \left(\frac{1}{\delta\pi_\delta}\right)^{3/4} \exp\left[-\frac{1}{2\delta}(\widehat{q}_i - \mathbf{x})^2\right] \text{ iff } \|q_i - \mathbf{x}\| \le \varepsilon \ll 1, \\ 0 \text{ iff } \|q_i - \mathbf{x}\| > \varepsilon. \end{cases} \quad (1.2.12)$$

Here $\delta \in (0,1]$ and $\lim_{\delta\to 0}\pi_\delta = \pi$.

I.2.3. The stochastic nonlinear Schrödinger equation

Another modern approach to stochastic reduction is to describe it using a stochastic nonlinear Schrödinger equation [2]-[10],[46] an elegant simplied example of which is the following one particle case known as Quantum Mechanics with Universal Position Localization [QMUPL]:

$$d|\psi_t(x)\rangle = \left[-\frac{i}{\hbar}\widehat{\mathbf{H}} - \frac{k}{2}(\widehat{q} - \langle q_t\rangle)^2 dt\right]|\psi_t(x)\rangle dt + \sqrt{k}\,(\widehat{q} - \langle q_t\rangle)dW_t|\psi_t(x)\rangle. \quad (1.2.13)$$

Here $\widehat{q}$ is the position operator, $\langle q_t\rangle = \langle\psi_t|\widehat{q}|\psi_t\rangle$ it is its expectation value, and k is a constant, characteristic of the model, which sets the strength of the collapse mechanics, and it is chosen proportional to the mass m of the particle according to the formula: $k = (m/m_0)\lambda_0$, where m_0 is the nucleon's mass and λ_0 measures the collapse strength. It is easy to see that Eqn.(1.2.5) contains both non-linear and stochastic terms, which are necessary to induce the collapse of the wave function. For an examle let us consider a free particle ($\widehat{\mathbf{H}} = p^2/2m$), and a Gaussian state:

$$\psi_t(x) = \exp\left\{-a_t(x - \bar{x}_t)^2 + i\bar{k}_t x\right\}. \quad (1.2.14)$$

It is easy to see that $\psi_t(x)$ given by Eq.(1.2.14) is solution of Eq.(1.2.13), where

$$\frac{da_t}{dt} = k - \frac{2i\hbar}{m}a_t^2,\ \frac{d\bar{x}_t}{dt} = \frac{\hbar}{m}\bar{k}_t + \frac{\sqrt{k}}{2\,\mathrm{Re}(a_t)}\dot{W}_t,\ \frac{d\bar{k}_t}{dt} = -\sqrt{k}\,\frac{\mathrm{Im}(a_t)}{\mathrm{Re}(a_t)}\dot{W}_t. \quad (1.2.15)$$

The many-particle equation [2]:

$$\begin{aligned} d|\psi_t(x)\rangle = &\left[-\frac{i}{\hbar}\widehat{\mathbf{H}} - \sum_{i=1}^{N}\frac{k_i}{2}(\widehat{q}_i - \langle q_{i,t}\rangle)^2 dt\right]|\psi_t(x)\rangle dt + \\ &+\sum_{i=1}^{N}\sqrt{k_i}\,(\widehat{q}_i - \langle q_{i,t}\rangle)dW_{i,t}|\psi_t(x)\rangle, \end{aligned} \quad (1.2.16)$$

where $\widehat{\mathbf{H}}$ is the quantum Hamiltonian of the composite system, the operators $\widehat{q}_i$ ($i = 1,\ldots N$) are the position operators of the particles of the system, and $W_{i,t}$ (i = 1,...N) are N independent standard Wiener processes.

The CSL model is defined by the following stochastic differential equation in the Fock space [2]:

$$d|\psi_t(\mathbf{x})\rangle = \left[-\frac{i}{\hbar}\widehat{\mathbf{H}} - \frac{k}{2}\left(\widehat{M}(\mathbf{x}) - \langle M_t(\mathbf{x})\rangle\right)^2 dt\right]|\psi_t(\mathbf{x})\rangle dt + \\ + \sqrt{k}\left(\widehat{M}(\mathbf{x}) - \langle M_t(\mathbf{x})\rangle\right)dW_t(\mathbf{x})|\psi_t(\mathbf{x})\rangle. \tag{1.2.17}$$

The parameter k is a positive coupling constant which sets the strength of the collapse process, while $\widehat{\mathbf{M}}(\mathbf{x})$ is a smeared mass density operator:

$$\widehat{M}(\mathbf{x}) = \sum_j m_j N_j(\mathbf{x}), \\ N_j(\mathbf{x}) = \int d\mathbf{y} g(\mathbf{y}-\mathbf{x})\psi_j^{\dagger}(\mathbf{x})\psi_j(\mathbf{x}), \tag{1.2.18}$$

$\psi_j^{\dagger}(\mathbf{x}), \psi_j(\mathbf{x})$ being, respectively, the creation and annihilation operators of a particle of type j in the space point $\mathbf{y}$. The smearing function $g(\mathbf{x})$ is taken equal to

$$g(\mathbf{x}) = \left(\frac{1}{2\pi\delta}\right)^{3/4}\exp\left[-\frac{\mathbf{x}^2}{2\delta}\right] \tag{1.2.19}$$

where δ is the second phenomenological constant of the model. $W_t(\mathbf{x})$ is an ensemble of independent Wiener processes, one for each point in space.

I.2.4.The nonclassical collapse models with spontaneous localizations based on generalized measurement postulates

The nonclassical collapse models attempt to overcome the difficulties that standard quantum mechanics meets in accounting for the measurement (or macro-objectification) problem, an attempt based on the consideration of nonlinear and nonlocal stochastic modifications of the Schroedinger equation.The proposed new nonlocal dynamics is characterized by the feature of not contradicting any known fact about microsystems and of accounting, on the basis of a unique, universal dynamical principle, for wavepacket reduction and for the classical behavior of macroscopic systems.

Quantum Mechanics with Nonclassical Spontaneous Localizations is based on the following assumptions:

(1) Each particle of a system of n distinguishable particles experiences, with a mean rate λi, a sudden spontaneous localization process.

(2) In the time interval between two successive spontaneous processes the system evolves according to the usual Schrödinger equation.

(3) Let $|\psi\rangle_{cl}$ be the classical pure state correspond to an vector $|\psi\rangle_{cl} \in \mathbf{S}^{\infty} \subsetneq \mathbf{H}$

in a non projective Hilbert space $\mathbf{H}$, see Subsection I.7.1, Def I.7.1-I.7.2. Then the sudden spontaneous process is a localization given by:

$$|\psi\rangle_{cl} \overset{\delta,\varepsilon-\textbf{localization}}{\longrightarrow} \frac{|\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl}}{\| |\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl} \|}, \mathbf{x} \in \mathbb{R}^3, \qquad (1.2.20)$$

$$\delta \in (0,1], \varepsilon \ll 1,$$

where

$$|\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl} = \hat{L}^{i}_{\mathbf{x}}(\delta,\varepsilon)|\psi\rangle_{cl}. \qquad (1.2.21)$$

Here $\hat{L}^{i}_{\mathbf{x}}(\delta,\varepsilon)$ is a norm-reducing, positive, self-adjoint, linear operator with a symbol $L^{i}_{\mathbf{x}}(\delta,\varepsilon)$ in the n-particle non projective Hilbert space $\mathbf{H}$, representing the localization of particle i around the point $\mathbf{x}$.

Definition 1.2.1. Such localization is called δ,ε-localization or δ,ε-collapse of the state

$|\psi\rangle_{cl}$.

(4) The probability density $p_i(\mathbf{x},\delta,\varepsilon)$ for the occurrence of a localization at point $\mathbf{x}$ is assumed to be

$$p_i(\mathbf{x},\delta,\varepsilon) = \frac{\| |\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl} \|^2}{\iiint\limits_{\mathbb{R}^3} \| |\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl} \|^2 d^3x}. \qquad (1.2.22)$$

(5) Let $|\psi\rangle_{n.cl}$ be the nonclassical pure state correspond to an vector $|\psi^{\zeta}\rangle = \zeta|\psi\rangle \in \mathbf{H}\backslash\mathbf{S}^{\infty}$,

where $|\psi\rangle \in \mathbf{S}^{\infty}, |\zeta| \neq 1$, see Appendix C, Def.C.3. Then the sudden spontaneous process is a localization given by:

$$|\psi\rangle_{n.cl} \overset{\delta,\varepsilon-\textbf{localization}}{\longrightarrow} \frac{\zeta|\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{n.cl}}{\| |\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{n.cl} \|}, \mathbf{x} \in \mathbb{R}^3, \qquad (1.2.23)$$

where

$$|\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{n.cl} = \hat{L}^{i}_{\mathbf{x}}(\delta,\varepsilon)|\psi\rangle_{n.cl}. \qquad (1.2.24)$$

Definition 1.2.2. Such localization is called δ,ε-localization or δ,ε-collapse of the state

$|\psi\rangle_{n.cl}$.

(6) The probability density $p_i(\mathbf{x},\boldsymbol{\zeta},\delta,\varepsilon,)$ for the occurrence of a localization at point $\mathbf{x} \in \mathbb{R}^3$ in

acordance to postulate Q.IV.3 (see Subsection **I**.7.1, Eq.(1.7.8)) is assumed to be

$$p_i(\mathbf{x},\boldsymbol{\zeta},\delta,\varepsilon,) = \frac{|\zeta|^{-6} \left\| \left|\psi^{i}_{\delta,\varepsilon,|\zeta|^{-2}\mathbf{x}}\right\rangle_{n.cl} \right\|^2}{\iiint\limits_{\mathbb{R}^3} \| |\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl} \|^2 d^3x}. \qquad (1.2.25)$$

(7) The localization operators $\hat{L}^{i}_{\mathbf{x}}(\delta,\varepsilon)$ have been chosen to have the form:

$$\widehat{L}^i_{\mathbf{x}}(\delta,\varepsilon) = \begin{cases} \left(\frac{1}{\delta\pi_\delta}\right)^{3/4} \exp\left[-\frac{1}{2\delta}(\widehat{\mathbf{q}}_i - \mathbf{x})^2\right] \text{ iff } \|\mathbf{q}_i - \mathbf{x}\| \leq \varepsilon \ll 1, \\ 0 \text{ iff } \|\mathbf{q}_i - \mathbf{x}\| > \varepsilon. \end{cases} \quad (1.2.26)$$

Here $\delta \in (0,1]$ $\int d^3x[L^i_{\mathbf{x}}(\delta,\varepsilon)]^2 = 1$ and $\lim_{\delta\to 0}\pi_\delta = \pi$.

Remark 1.2.2. In one dimension case it follows that

$$\widehat{L}^i_x(\delta,\varepsilon) = \begin{cases} \left(\frac{1}{\delta\pi_\delta}\right)^{1/4} \exp\left[-\frac{1}{2\delta}(\widehat{q}_i - x)^2\right] \text{ iff } |q_i - x| \leq \varepsilon \ll 1, \\ 0 \text{ iff } |q_i - x| > \varepsilon. \end{cases} \quad (1.2.27)$$

Remark 1.2.3. Note that from Eq.(1.2.22) and Eq.(1.2.26) follows that a probability density $p_i(\mathbf{x},\zeta,\delta,\varepsilon,)$ for the occurrence of a localization inside sphere $S(\mathbf{x},\varepsilon) = \{\mathbf{q}_i \in \mathbb{R}^3 | \|\mathbf{q}_i - \mathbf{x}\| \leq \varepsilon\}$ is given by

$$p_i(\mathbf{x},\delta,\varepsilon) = \frac{\| |\psi^i_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl}\|^2}{\Omega(\delta,\varepsilon)}, \Omega(\delta,\varepsilon) = \iiint_{\mathbb{R}^3} \| |\psi^i_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl}\|^2 d^3x,$$
$$\| |\psi^i_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl}\|^2 = \left(\frac{1}{\delta\pi_\delta}\right)^{3/2} \int_{\|\mathbf{q}_i - \mathbf{x}\|\leq\varepsilon} d^3q_i \psi^i(\mathbf{q}_i) \exp\left[-\frac{1}{\delta}(\mathbf{q}_i - \mathbf{x})^2\right], \quad (1.2.28)$$
$$\psi^i(q_i) = \langle q_i \| \psi^i\rangle_{cl},$$

and therefore

$$p_i(\mathbf{x},\varepsilon) = \lim_{\delta\to 0} p_i(\mathbf{x},\delta,\varepsilon) =$$
$$\lim_{\delta\to 0} \Omega^{-1}(\delta,\varepsilon)\left(\frac{1}{\delta\pi_\delta}\right)^{3/2} \int_{\|\mathbf{q}_i - \mathbf{x}\|\leq\varepsilon} d^3q_i \psi^i(\mathbf{q}_i) \exp\left[-\frac{1}{\delta}(\mathbf{q}_i - \mathbf{x})^2\right] = \psi^i(\mathbf{x}). \quad (1.2.29)$$

Remark 1.2.4. In one dimension case it follows that a probability density $p_i(x,\delta,\varepsilon,)$ for the occurrence of a localization inside interval $[x-\varepsilon, x+\varepsilon]$ is given by

$$p_i(x,\delta,\varepsilon) = \| |\psi^i_{\delta,\varepsilon,x}\rangle_{cl}\|^2 = \left(\frac{1}{\delta\pi_\delta}\right)^{1/2} \int_{|q_i - x|\leq\varepsilon} d^3q_i \psi^i(q_i) \exp\left[-\frac{1}{\delta}(q_i - x)^2\right],$$
$$\psi^i(q_i) = \langle q_i \| \psi^i\rangle_{cl}, \quad (1.2.30)$$

and therefore

$$p_i(x,\varepsilon) = \lim_{\delta\to 0} p_i(x,\delta,\varepsilon) =$$
$$= \lim_{\delta\to 0} \Omega^{-1}(\delta,\varepsilon) \left(\frac{1}{\delta\pi_\delta}\right)^{1/2} \int_{|q_i - x|\leq\varepsilon} dq_i \psi^i(q_i) \exp\left[-\frac{1}{\delta}(q_i - x)^2\right] = \psi^i(x). \quad (1.2.31)$$

I.2.5.The generalization of nonclassical collapse models

(8) Let $|\psi_t\rangle_{cl}, t \in [0,T]$ be the classical pure states correspond to an vector-function $|\psi_t\rangle_{cl} : [0,T] \times \mathbf{S}^\infty \to \mathbf{S}^\infty$ such that $|\psi_t\rangle_{cl} \in \mathbf{S}^\infty \subsetneqq \mathbf{H}$ $, t \in [0,T]$, where is a non projective Hilbert space $\mathbf{H}$, see Subsection **I**.7.1, Def **I**.7.1-**I**.7.2. Then the sudden spontaneous process is a localization along classical trajectory $\mathbf{x}_t : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$|\psi_t\rangle_{cl} \overset{\delta,\varepsilon,\mathbf{x}_t-\textbf{localization}}{\longrightarrow} \frac{|\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl}}{\||\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl}\|}, \quad \delta \in (0,1], \varepsilon \ll 1, \mathbf{x}_t \in \mathbb{R}^3, t \in [0,T]. \tag{1.2.32}$$

where

$$|\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl} = \widehat{L}^i_{\mathbf{x}_t}(\delta,\varepsilon)|\psi_t\rangle_{cl}. \tag{1.2.33}$$

Here $\widehat{L}^i_{\mathbf{x}_t}(\delta,\varepsilon)$ is a norm-reducing, positive, self-adjoint, linear operator with a symbol $L^i_{\mathbf{x}_t}(\delta,\varepsilon)$ in the n-particle non projective Hilbert space $\mathbf{H}$, representing the localization of particle i at each instant $t \in [0,T]$ around the point $\mathbf{x}_t$.

Definition 1.2.3. Such localization as mentioned above is called $\delta,\varepsilon,\mathbf{x}_t$-localization or $\delta,\varepsilon,\mathbf{x}_t$-collapse of the state $|\psi_t\rangle_{cl}$.

(9) The probability density $p_i(t,\mathbf{x}_t,\delta,\varepsilon)$ for the occurrence of a localization at point $\mathbf{x}_t$ at instant t is assumed to be

$$p_i(t,\mathbf{x}_t,\delta,\varepsilon) = \frac{\||\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl}\|^2}{\Omega(t,\delta,\varepsilon)}, \quad \Omega(t,\delta,\varepsilon) = \iiint_{\mathbb{R}^3} \||\psi^i_{\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl}\|^2 d^3x. \tag{1.2.34}$$

(10) Let $|\psi_t\rangle_{n.cl}$ be the nonclassical pure state correspond to an vector-function $\left|\psi^\zeta_t\right\rangle = \zeta|\psi_t\rangle \in \mathbf{H}\backslash\mathbf{S}^\infty$, where $|\psi_t\rangle \in \mathbf{S}^\infty, |\zeta| \neq 1, t \in [0,T]$ see Subsection **I**.7.1, Def.**I**.7.3.

Then the sudden spontaneous process is a localization along classical trajectory $\mathbf{x}_t : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$|\psi_t\rangle_{n.cl} \overset{\delta,\varepsilon,\mathbf{x}_t-\textbf{localization}}{\longrightarrow} \frac{\zeta|\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{n.cl}}{\||\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{n.cl}\|}, \quad \mathbf{x}_t \in \mathbb{R}^3, t \in [0,T] \tag{1.2.35}$$

where

$$|\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{n.cl} = \widehat{L}^i_{\mathbf{x}_t}(\delta,\varepsilon)|\psi_t\rangle_{n.cl}. \tag{1.2.36}$$

Definition 1.2.4. Such localization is called $\delta,\varepsilon,\mathbf{x}_t$-localization or $\delta,\varepsilon,\mathbf{x}_t$-collapse of the state $|\psi\rangle_{n.cl}$.

(11) The probability density $p_i(t,\mathbf{x}_t,\boldsymbol{\zeta},\delta,\varepsilon,)$ for the occurrence of a localization at point

$\mathbf{x}_t \in \mathbb{R}^3$ at instant $t \in [0,T]$ in acordance to postulate Q.IV.3 (see Subsection **I**.7.1,

Eq.(1.7.8)) is assumed to be

$$p_i(t,\mathbf{x}_t,\boldsymbol{\zeta},\delta,\varepsilon,) = \frac{|\zeta|^{-6}\left\| \left|\psi^i_{t,\delta,\varepsilon,|\zeta|^{-2}\mathbf{x}_t}\right\rangle_{n.cl}\right\|^2}{\Omega(t,\delta,\varepsilon)},$$
$$\Omega(t,\delta,\varepsilon) = \iiint_{\mathbb{R}^3} \| |\psi^i_{\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl}\|^2 d^3x. \tag{1.2.37}$$

(12) The localization operators $\widehat{L}^i_{\mathbf{x}_t}(\delta,\varepsilon)$ have been chosen to have the form:

$$\widehat{L}^i_{\mathbf{x}_t}(\delta,\varepsilon) = \begin{cases} \left(\frac{1}{\delta\pi_\delta}\right)^{3/4} \exp\left[-\frac{1}{2\delta}(\widehat{\mathbf{q}}_i - \mathbf{x}_t)^2\right] \text{ iff } \|\mathbf{q}_i - \mathbf{x}_t\| \leq \varepsilon \ll 1, \\ 0 \text{ iff } \|\mathbf{q}_i - \mathbf{x}_t\| > \varepsilon. \end{cases} \tag{1.2.38}$$

Here $\delta \in (0,1]$ and $\lim_{\delta\to 0}\pi_\delta = \pi$.

Remark 1.2.5. In one dimension case it follows that

$$\widehat{L}^i_{x_t}(\delta,\varepsilon) = \begin{cases} \left(\frac{1}{\delta\pi_\delta}\right)^{1/4} \exp\left[-\frac{1}{2\delta}(\widehat{q}_i - x_t)^2\right] \text{ iff } |q_i - x_t| \leq \varepsilon \ll 1, \\ 0 \text{ iff } |q_i - x_t| > \varepsilon. \end{cases} \tag{1.2.39}$$

Remark 1.2.6. Note that from Eq.(1.2.34) and Eq.(1.2.38) follows that a probability density

$p_i(t,\mathbf{x}_t,\boldsymbol{\zeta},\delta,\varepsilon,)$ for the occurrence of a localization at instant t inside sphere $S(\mathbf{x}_t,\varepsilon) = \{\mathbf{q}_i \in \mathbb{R}^3 | \|\mathbf{q}_i - \mathbf{x}_t\| \leq \varepsilon\}$ is given by

$$p_i(t,\mathbf{x}_t,\delta,\varepsilon) = \frac{\| |\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl}\|^2}{\Omega(t,\delta,\varepsilon)}$$
$$\| |\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl}\|^2 = \left(\frac{1}{\delta\pi_\delta}\right)^{3/2} \int\limits_{\|\mathbf{q}_i - \mathbf{x}_t\|\leq\varepsilon} d^3q_i\psi^i_t(\mathbf{q}_i)\exp\left[-\frac{1}{\delta}(\mathbf{q}_i - \mathbf{x}_t)^2\right], \tag{1.2.40}$$
$$\psi^i_t(q_i) = \langle q_i \| \psi^i_t\rangle_{cl},$$

and therefore

$$p_i(t,x,\varepsilon) = \lim_{\delta\to 0} p_i(t,x,\delta,\varepsilon) =$$
$$= \lim_{\delta\to 0}\Omega^{-1}(t,\delta,\varepsilon)\left(\frac{1}{\delta\pi_\delta}\right)^{1/2} \int\limits_{|q_i - x|\leq\varepsilon} dq_i\psi^i(q_i)\exp\left[-\frac{1}{\delta}(q_i - x_t)^2\right] = \psi^i(x_t). \tag{1.2.41}$$

I.3.The nonlocal Schrödinger equations and nonlocal nature of the wave function collapse

In this subsection we introduce new additional QM-postulates namely nonlocal Schrödinger equations. This nonlocal equations in a good consent with nonlocal nature of the wave function collapse. We obtain main GRW postulates given by Eq.(1.2.20) and Eq.(1.2.32) using nonlocal Schrödinger equations (1.3.3) and (1.3.6).

Assumption 1.3.1.We assume now that a wawe function $\Psi(\mathbf{x},t) = \langle \mathbf{x}|\psi_t\rangle$ **collapses at instant**: $t = 0$.

Definition 1.3.1. Let us consider the time-dependent canonical Schrödinger equation:

$$i\hbar\frac{\partial\Psi(\mathbf{x},t)}{\partial t} = \widehat{\mathbf{H}}(t)\Psi(\mathbf{x},t), \tag{1.3.1}$$
$$t \in [0,T], \mathbf{x} = (\mathbf{x_1},\mathbf{x_2},\ldots,\mathbf{x_n}) \in \mathbb{R}^{3n}.$$

Let $\Psi(\mathbf{x},t)$ be a classical solution of the equation (1.3.1). The time-dependent Schrödinger equation (1.3.1) is a weakly well preserved by corresponding to $\Psi(\mathbf{x},t)$ collapsed along classical trajectory
$\widetilde{\mathbf{X}}_{m_1,\ldots,m_k}(t,t') = [\widetilde{\mathbf{x}}_{m_1}(t,t'),\ldots,\widetilde{\mathbf{x}}_{m_i}(t,t'),\ldots,\widetilde{\mathbf{x}}_{m_k}(t,t')], 1 \leq i \leq k$

$$\widetilde{\mathbf{x}}_{m_i}(t,t') = \begin{cases} \mathbf{x}_{m_i}(t) \text{ iff } t' < t \leq T \\ 0 \text{ iff } 0 \leq t \leq t' \end{cases}$$

wave function $\Psi^{\#}(\mathbf{x},t,t')$:

$$\Psi^{\#}(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n,t,t') =$$

$$\Psi^{\#}(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n,t,t';\widetilde{\mathbf{x}}_{m_1}(t,t'),\ldots,\widetilde{\mathbf{x}}_{m_k}(t,t')) =$$

$$\frac{\Re_{m_1,\ldots,m_k}(\widetilde{\mathbf{x}}_{m_1}(t,t'),\ldots,\widetilde{\mathbf{x}}_{m_k}(t,t'))\Psi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n,t)}{\|\Re_{m_1,\ldots,m_k}(\widetilde{\mathbf{x}}_{m_1}(t,t'),\ldots,\widetilde{\mathbf{x}}_{m_k}(t,t'))\Psi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n,t)\|_2},$$
$$\Re_{m_1,\ldots,m_k}(\widetilde{\mathbf{x}}_{m_1}(t,t'),\ldots,\widetilde{\mathbf{x}}_{m_k}(t,t')) = \prod_{i=1}^{k}\Re_{m_i}(\widetilde{\mathbf{x}}_{m_i}(t,t')), \tag{1.3.2}$$
$$\Re_{m_i}(\widetilde{\mathbf{x}}_{m_i}(t,t')) =$$

$$\begin{cases} (\pi\delta)^{-3/4}\exp\left[-\frac{(\widehat{\mathbf{x}}_{m_i}-\widetilde{\mathbf{x}}_{m_i}(t,t'))^2}{2\delta}\right] \text{ iff } |\mathbf{x}_{m_i}-\widetilde{\mathbf{x}}_{m_i}(t,t')| \leq \varepsilon, \\ 0 \text{ iff } \|\mathbf{x}_m-\widetilde{\mathbf{x}}_m(t,t')\| > \varepsilon. \end{cases}$$

in region $\Gamma \subseteq \mathbb{R}^{3d}$ if the estimate

$$\int_0^T \int_\Gamma \left\{ i\hbar \frac{\partial \Psi^{\#}(\mathbf{x},t,t')}{\partial t} - \widehat{\mathbf{H}}(t)\Psi^{\#}(\mathbf{x},t,t') \right\} d^{3n}x = O(\hbar^{\alpha}), 1/4 < \alpha < 1/2, \tag{1.3.3}$$

$$t \in [0,T], \mathbf{x} \in \mathbb{R}^{3n},$$

is satisfied.

Definition 1.3.2.(i) The time-dependent integral equation (1.3.3) is colled the time- dependent nonlocal Schrödinger equation of the order $\hbar^{\alpha}$.

(ii) Such collapsed wave function $\Psi^{\#}(\mathbf{x},t,t')$ as mentioned in Definition 1.3.1 is colled the

$\hbar^{\alpha}$- solution of the nonlocal Schrödinger equation (1.3.3) of the order α.

Definition 1.3.3. Let us consider the time-independent canonical Schrödinger equation:

$$\widehat{\mathbf{H}}\Psi(\mathbf{x}) = 0, \tag{1.3.4}$$

$$t \in [0,T], \mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}) \in \mathbb{R}^{3n}.$$

Let $\Psi(\mathbf{x})$ be a classical solution of the equation (1.3.4). The time-independent Schrödinger equation (1.3.4) is a weakly well preserved by corresponding to $\Psi(\mathbf{x})$ collapsed wave function $\Psi^{\#}(\mathbf{x})$:

$$\Psi^{\#}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) =$$

$$\Psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n; \widetilde{\mathbf{x}}_{m_1}, \ldots, \widetilde{\mathbf{x}}_{m_k}) =$$

$$= \frac{\Re_{m_1,\ldots,m_k}(\widetilde{\mathbf{x}}_{m_1}, \ldots, \widetilde{\mathbf{x}}_{m_k})\Psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)}{\|\Re_{m_1,\ldots,m_k}(\widetilde{\mathbf{x}}_{m_1}, \ldots, \widetilde{\mathbf{x}}_{m_k})\Psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)\|_2},$$

$$\Re_{m_1,\ldots,m_k}(\widetilde{\mathbf{x}}_{m_1}, \ldots, \widetilde{\mathbf{x}}_{m_k}) = \prod_{i=1}^{k} \Re_{m_i}(\widetilde{\mathbf{x}}_{m_i}), \tag{1.3.5}$$

$$\Re_m(\widetilde{\mathbf{x}}_m) = \begin{cases} (\pi\delta)^{-3/4} \exp\left[-\frac{(\widehat{\mathbf{x}}_m - \widetilde{\mathbf{x}}_m)^2}{2\delta} \right] \text{ iff } \|\mathbf{x}_m - \widetilde{\mathbf{x}}_m\| \leq \varepsilon, \\ 0 \text{ iff } \|\mathbf{x}_m - \widetilde{\mathbf{x}}_m\| > \varepsilon. \end{cases}$$

in region $\Gamma \subseteq \mathbb{R}^{3d}$ if the estimate

$$\int_\Gamma \widehat{\mathbf{H}}\Psi^{\#}(\mathbf{x}) d^{3d}x = O(\hbar^{\alpha}), 1/4 < \alpha < 1/2, \tag{1.3.6}$$

$$\mathbf{x} \in \mathbb{R}^{3n},$$

is satisfied.

Definition 1.3.4.(i) The stationary integral equation (1.3.6) is called nonlocal stationary Schrödinger equation of the order $\hbar^{\alpha}$.

(ii) Such collapsed wave function $\Psi^{\#}(\mathbf{x})$ as mentioned in Definition 1.3.3 is colled the

$\hbar^{\alpha}$- solution of the time-independent nonlocal Schrödinger equation (1.3.6) of the

order α.

Definition 1.3.5.The time-dependent integral equation

$$\int_0^T \int_\Gamma \left\{ i\hbar \frac{\partial \Psi^{\#}(\mathbf{x},t)}{\partial t} - \widehat{\mathbf{H}}(t)\Psi^{\#}(\mathbf{x},t) \right\} d^{3n}x = 0, \tag{1.3.7}$$

$$t \in [0,T], \mathbf{x} \in \mathbb{R}^{3n},$$

is colled the time-dependent nonlocal Schrödinger equation.

Definition 1.3.6.The stationary integral equation

$$\int_\Gamma \widehat{\mathbf{H}}\Psi^{\#}(\mathbf{x}) d^{3n}x = 0, \mathbf{x} \in \mathbb{R}^{3n}, \tag{1.3.8}$$

is called nonlocal stationary Schrödinger equation

Remark 1.3.1.We have introduced in consideration Eq.(1.3.3) and Eq.(1.3.6) in order to obtain good approximation of the solutions of the Eq.(1.3.7) and Eq.(1.3.8) correspondingly.

Lemma 1.3.1.[24].Let $\Phi(\lambda)$ be a function

$$\Phi(\lambda) = \int_0^a x^{\beta-1} \exp(-\lambda x^{\alpha}) f(x) dx, \tag{1.3.9}$$

where $\lambda \gg 1,\ 0 < a < \infty, 0 < \beta, 0 < \alpha$. Assume that $f(x)$ is continuous on $[0,a]$. Then

$$\Phi(\lambda) = \alpha^{-1}\Gamma\left(\frac{\beta}{\alpha}\right)[f(0) + o(1)]\lambda^{-\beta/\alpha} \tag{1.3.10}$$

Theorem 1.3.1. We assume now for a short that $m = n = 1$. Let $\Psi(x), x \in \mathbb{R}$ be a classical

solution of the equation (1.3.4), where

$$\widehat{\mathbf{H}} = \frac{\partial^2}{\partial x^2} + V(x). \tag{1.3.11}$$

Assume that

$$|\Psi(x)| = O(\hbar^{-1/4}),$$
$$|\partial\Psi(x)/\partial x| = O(\hbar^{-5/4}). \tag{1.3.12}$$

Then any collapsed wave function $\Psi^{\#}(x)$ given by Eq.(1.3.5) with

$\sqrt{\hbar/\delta} = \hbar^{\alpha}, 1/4 < \alpha < 1/2$ that is $\hbar^{\alpha}$-solution of the time-independent nonlocal Schrödinger equation (1.3.6) of the order α.

Proof. Note that

$$\Psi^{\#}(x) = \phi_{\delta}(x)\Psi(x),$$
$$\phi_{\delta}(x) = (\pi_{\delta}\delta)^{-1/4}\exp\left[-\frac{(x-\widetilde{x})^2}{2\delta}\right] \text{ iff } |x-\widetilde{x}| \leq \varepsilon, \tag{1.3.13}$$
$$\phi_{\delta}(x) = 0 \text{ iff } |x-\widetilde{x}| > \varepsilon.$$

From Eq.(1.3.13) one obtains

$$\frac{\partial\phi_{\delta}(x)}{\partial x} = -(\pi_{\delta}\delta)^{-1/4}\delta^{-1}(x-\widetilde{x})\exp\left[-\frac{(x-\widetilde{x})^2}{2\delta}\right]+$$
$$+([\phi_{\delta}]_{\widetilde{x}-\varepsilon})\delta(x-\widetilde{x}+\varepsilon)+([\phi_{\delta}]_{\widetilde{x}+\varepsilon})\delta(x-\widetilde{x}-\varepsilon) \text{ iff } |x-\widetilde{x}| \leq \varepsilon,$$
$$\frac{\partial\phi_{\delta}(x)}{\partial x} = 0 \text{ iff } |x-\widetilde{x}| > \varepsilon.$$
$$\frac{\partial^2\phi_{\delta}(x)}{\partial x^2} = -(\pi_{\delta}\delta)^{-1/4}\delta^{-1}\exp\left[-\frac{(x-\widetilde{x})^2}{2\delta}\right]+$$
$$+(\pi_{\delta}\delta)^{-1/4}\delta^{-2}(x-\widetilde{x})^2\exp\left[-\frac{(x-\widetilde{x})^2}{2\delta}\right]+ \tag{1.3.14}$$
$$\left(\left[\frac{\partial\phi_{\delta}}{\partial x}\right]_{\widetilde{x}-\varepsilon}\right)\delta(x-\widetilde{x}+\varepsilon)+\left(\left[\frac{\partial\phi_{\delta}}{\partial x}\right]_{\widetilde{x}+\varepsilon}\right)\delta(x-\widetilde{x}-\varepsilon)+$$
$$+([\phi_{\delta}]_{\widetilde{x}-\varepsilon})\delta'(x-\widetilde{x}+\varepsilon)+([\phi_{\delta}]_{\widetilde{x}+\varepsilon})\delta'(x-\widetilde{x}-\varepsilon) \text{ iff } |x-\widetilde{x}| \leq \varepsilon,$$
$$\frac{\partial^2\phi_{\delta}(x)}{\partial x^2} = 0 \text{ iff } |x-\widetilde{x}| > \varepsilon.$$

and

$$\frac{\partial^2\Psi^{\#}(x)}{\partial x^2} = \frac{\partial^2[\phi_{\delta}(x)\Psi(x)]}{\partial x^2} = \frac{\partial}{\partial x}\left[\Psi(x)\frac{\partial\phi_{\delta}(x)}{\partial x}+\phi_{\delta}(x)\frac{\partial\Psi(x)}{\partial x}\right] = \tag{1.3.15}$$
$$2\frac{\partial\Psi(x)}{\partial x}\frac{\partial\phi_{\delta}(x)}{\partial x}+\Psi(x)\frac{\partial^2\phi_{\delta}(x)}{\partial x^2}+\phi_{\delta}(x)\frac{\partial^2\Psi(x)}{\partial x^2}.$$

Substitution Eq.(1.3.11) and Eq.(1.3.15) into LHS of the Eq.(1.3.8) gives

$$\int_\Gamma \widehat{\mathbf{H}}\Psi^{\#}(x)dx = \int_\Gamma\left[\hbar^2\frac{\partial^2}{\partial x^2}\Psi^{\#}(x) + V(x)\Psi^{\#}(x)\right]dx =$$

$$= \int_\Gamma \phi_\delta(x)\left[\hbar^2\frac{\partial^2\Psi(x)}{\partial x^2} + V(x)\Psi(x)\right]dx + 2\hbar^2\int_\Gamma\left[\frac{\partial\Psi(x)}{\partial x}\frac{\partial\phi_\delta(x)}{\partial x}\right]dx + \tag{1.3.16}$$

$$+\hbar^2\int_\Gamma\left[\Psi(x)\frac{\partial^2\phi_\delta(x)}{\partial x^2}\right]dx =$$

$$2\hbar^2\int_\Gamma\left[\frac{\partial\Psi(x)}{\partial x}\frac{\partial\phi_\delta(x)}{\partial x}\right]dx + \hbar^2\int_\Gamma\left[\Psi(x)\frac{\partial^2\phi_\delta(x)}{\partial x^2}\right]dx.$$

From Eq.(1.3.16) and Eq.(1.3.12) one obtains

$$\left|\int_\Gamma \widehat{\mathbf{H}}\Psi^{\#}(x)dx\right| \leq 2\hbar^2\int_\Gamma\left[\left|\frac{\partial\Psi(x)}{\partial x}\right|\left|\frac{\partial\phi_\delta(x)}{\partial x}\right|\right]dx + \hbar^2\int_\Gamma\left[|\Psi(x)|\left|\frac{\partial^2\phi_\delta(x)}{\partial x^2}\right|\right]dx \leq$$

$$2\hbar^2 O(\hbar^{-5/4})\int_\Gamma\left|\frac{\partial\phi_\delta(x)}{\partial x}\right|dx + \hbar^2 O(\hbar^{-1/4})\int_\Gamma\left|\frac{\partial^2\phi_\delta(x)}{\partial x^2}\right|dx \leq$$

$$O(\hbar^{3/4})(\pi_\delta\delta)^{-1/4}\delta^{-1}\int_\Gamma|x-\widetilde{x}|\exp\left[-\frac{(x-\widetilde{x})^2}{2\delta}\right]dx + \tag{1.3.17}$$

$$+O(\hbar^{3/2})(\pi_\delta\delta)^{-1/4}\delta^{-1}\int_\Gamma\exp\left[-\frac{(x-\widetilde{x})^2}{2\delta}\right]dx +$$

$$+O(\hbar^{7/4})(\pi_\delta\delta)^{-1/4}\delta^{-2}\int_\Gamma(x-\widetilde{x})^2\exp\left[-\frac{(x-\widetilde{x})^2}{2\delta}\right]dx.$$

Having applied Lemma 1.3.1 to RHS of the (1.3.17) we have finalized the proof.

Theorem 1.3.2.We assume now for a short that $m = n = 1$. Let $\Psi(x,t), x \in \mathbb{R}$ be a classical
solution of the equation (1.3.1), where

$$\widehat{\mathbf{H}} = \hbar^2\frac{\partial^2}{\partial x^2} + V(x,t). \tag{1.3.18}$$

Assume that

$$\begin{aligned} |\Psi(x,t)| &= O(\hbar^{-1/4}), \\ |\partial\Psi(x,t)/\partial x| &= O(\hbar^{-5/4}), \\ |\partial\Psi(x,t)/\partial t| &= O(\hbar^{-5/4}). \end{aligned} \tag{1.3.19}$$

Then any collapsed wave function $\Psi^{\#}(x,t) = \Psi^{\#}(x,t,t' = 0)$ given by Eq.(1.3.3) with $t' = 0$ and $\sqrt{\hbar/\delta} = \hbar^\alpha, 1/4 < \alpha < 1/2$ that is $\hbar^\alpha$-solution of the time-dependent nonlocal Schrödinger equation (1.3.7) of the order α.

Proof. Note that

$$\Psi^{\#}(x,t) = \phi_\delta(x,t)\Psi(x,t),$$
$$\phi_\delta(x,t) = \begin{cases} (\pi_\delta\delta)^{-1/4}\eta(t)\exp\left[-\dfrac{(x-\widetilde{x}_t)^2}{2\delta}\right] \text{ iff } |x-\widetilde{x}_t| \leq \varepsilon, \\ 0 \text{ iff } |x-\widetilde{x}_t| > \varepsilon, \end{cases} \tag{1.3.20}$$
$$\eta(t) = (\|\Re(\widetilde{x}_t)\Psi(x,t)\|_2)^{-1}.$$

From Eq.(1.3.20) one obtains

$$\frac{\partial\phi_\delta(x,t)}{\partial x} = -(\pi_\delta\delta)^{-1/4}\delta^{-1}(x-\widetilde{x}_t)\eta(t)\exp\left[-\frac{(x-\widetilde{x}_t)^2}{2\delta}\right]+$$
$$+\left([\phi_\delta]_{\widetilde{x}_t-\varepsilon}\right)\delta(x-\widetilde{x}_t+\varepsilon)+\left([\phi_\delta]_{\widetilde{x}_t+\varepsilon}\right)\delta(x-\widetilde{x}_t-\varepsilon) \text{ iff } |x-\widetilde{x}_t| \leq \varepsilon,$$
$$\frac{\partial\phi_\delta(x,t)}{\partial x} = 0 \text{ iff } |x-\widetilde{x}_t| > \varepsilon.$$
$$\frac{\partial^2\phi_\delta(x,t)}{\partial x^2} = -(\pi_\delta\delta)^{-1/4}\delta^{-1}\eta(t)\exp\left[-\frac{(x-\widetilde{x}_t)^2}{2\delta}\right]+$$
$$+(\pi_\delta\delta)^{-1/4}\delta^{-2}(x-\widetilde{x}_t)^2\eta(t)\exp\left[-\frac{(x-\widetilde{x}_t)^2}{2\delta}\right]+$$
$$+\left(\left[\frac{\partial\phi_\delta}{\partial x}\right]_{\widetilde{x}_t-\varepsilon}\right)\delta(x-\widetilde{x}_t+\varepsilon)+\left(\left[\frac{\partial\phi_\delta}{\partial x}\right]_{\widetilde{x}_t+\varepsilon}\right)\delta(x-\widetilde{x}_t-\varepsilon) \tag{1.3.21}$$
$$+\left([\phi_\delta]_{\widetilde{x}_t-\varepsilon}\right)\delta'(x-\widetilde{x}_t+\varepsilon)+\left([\phi_\delta]_{\widetilde{x}_t+\varepsilon}\right)\delta'(x-\widetilde{x}_t-\varepsilon) \text{ iff } |x-\widetilde{x}_t| \leq \varepsilon,$$
$$\frac{\partial^2\phi_\delta(x,t)}{\partial x^2} = 0 \text{ iff } |x-\widetilde{x}_t| > \varepsilon.$$
$$\frac{\partial\phi_\delta(x,t)}{\partial t} = (\pi_\delta\delta)^{-1/4}\eta'(t)\exp\left[-\frac{(x-\widetilde{x}_t)^2}{2\delta}\right]-$$
$$-(\pi_\delta\delta)^{-1/4}\eta(t)\widetilde{x}'_t(x-\widetilde{x}_t)\exp\left[-\frac{(x-\widetilde{x}_t)^2}{2\delta}\right].$$

Note that

$$i\hbar\frac{\partial\Psi^{\#}(x,t)}{\partial t} - \widehat{\mathbf{H}}(t)\Psi^{\#}(x,t) = i\hbar\frac{\partial\Psi^{\#}(x,t)}{\partial t} - \left[\hbar^2\frac{\partial^2}{\partial x^2}\Psi^{\#}(x,t) + V(x,t)\Psi^{\#}(x,t)\right]. \tag{1.3.22}$$

Substitution Eq.(1.3.20) into LHS of the Eq.(1.3.22) gives

$$
\begin{aligned}
&i\hbar\frac{\partial\Psi^{\#}(x,t)}{\partial t}-\widehat{\mathbf{H}}(t)\Psi^{\#}(x,t) = i\hbar\frac{\partial\phi_{\delta}(x,t)}{\partial t}\Psi(x,t)-\\
&\hbar^{2}\left[2\frac{\partial\Psi(x,t)}{\partial x}\frac{\partial\phi_{\delta}(x,t)}{\partial x}+\Psi(x,t)\frac{\partial^{2}\phi_{\delta}(x)}{\partial x^{2}}\right]-\\
&\phi_{\delta}(x,t)\left[i\hbar\frac{\partial\Psi(x,t)}{\partial t}-\hbar^{2}\frac{\partial^{2}\Psi(x,t)}{\partial x^{2}}-V(x,t)\Psi(x,t)\right]=\\
&i\hbar\frac{\partial\phi_{\delta}(x,t)}{\partial t}\Psi(x,t)-\hbar^{2}\left[2\frac{\partial\Psi(x,t)}{\partial x}\frac{\partial\phi_{\delta}(x,t)}{\partial x}+\Psi(x,t)\frac{\partial^{2}\phi_{\delta}(x)}{\partial x^{2}}\right]
\end{aligned}
\tag{1.3.23}
$$

Substitution Eq.(1.3.23) into LHS of the Eq.(1.3.7) gives

$$
\begin{aligned}
&\int_{\Gamma}\left\{i\hbar\frac{\partial\Psi^{\#}(x,t)}{\partial t}-\widehat{\mathbf{H}}(t)\Psi^{\#}(x,t)\right\}dx=\\
&\int_{\Gamma}\left\{i\hbar\frac{\partial\phi_{\delta}(x,t)}{\partial t}\Psi(x,t)-\hbar^{2}\left[2\frac{\partial\Psi(x,t)}{\partial x}\frac{\partial\phi_{\delta}(x,t)}{\partial x}+\Psi(x,t)\frac{\partial^{2}\phi_{\delta}(x)}{\partial x^{2}}\right]\right\}dx.
\end{aligned}
\tag{1.3.24}
$$

From Eq.(1.3.24) we obtain

$$
\begin{aligned}
&\left|\int_{\Gamma}\left\{i\hbar\frac{\partial\Psi^{\#}(x,t)}{\partial t}-\widehat{\mathbf{H}}(t)\Psi^{\#}(x,t)\right\}dx\right|\leq\\
&\int_{\Gamma}\left\{\hbar\left|\frac{\partial\phi_{\delta}(x,t)}{\partial t}\right||\Psi(x,t)|+\hbar^{2}\left[2\left|\frac{\partial\Psi(x,t)}{\partial x}\right|\left|\frac{\partial\phi_{\delta}(x,t)}{\partial x}\right|+|\Psi(x,t)|\left|\frac{\partial^{2}\phi_{\delta}(x)}{\partial x^{2}}\right|\right]\right\}dx.
\end{aligned}
\tag{1.3.25}
$$

Substitution Eq.(1.3.19),Eq.(1.3.19) into RHS of the (1.3.25) gives

$$
\begin{aligned}
&\left|\int_{\Gamma}\left\{i\hbar\frac{\partial\Psi^{\#}(x,t)}{\partial t}-\widehat{\mathbf{H}}(t)\Psi^{\#}(x,t)\right\}dx\right|\leq\\
&\leq\hbar O(\hbar^{-1/4})\int_{\Gamma}\left|\frac{\partial\phi_{\delta}(x,t)}{\partial t}\right|dx+2\hbar^{2}O(\hbar^{-5/4})\int_{\Gamma}\left|\frac{\partial\phi_{\delta}(x,t)}{\partial x}\right|dx+\\
&+2\hbar^{2}O(\hbar^{-5/4})\int_{\Gamma}\left|\frac{\partial\phi_{\delta}(x,t)}{\partial x}\right|dx=\\
&O(\hbar^{3/4})\int_{\Gamma}\left|\frac{\partial\phi_{\delta}(x,t)}{\partial t}\right|dx+O(\hbar^{3/4})\int_{\Gamma}\left|\frac{\partial\phi_{\delta}(x,t)}{\partial x}\right|dx+O(\hbar^{3/4})\int_{\Gamma}\left|\frac{\partial^{2}\phi_{\delta}(x,t)}{\partial x^{2}}\right|dx
\end{aligned}
\tag{1.3.26}
$$

Substitution Eq.(1.3.20),Eq.(1.3.21) into RHS of the (1.3.26) gives

$$\left|\int_{\Gamma}\left\{i\hbar\frac{\partial\Psi^{\#}(x,t)}{\partial t}-\widehat{\mathbf{H}}(t)\Psi^{\#}(x,t)\right\}dx\right|\leq$$

$$O(\hbar^{3/4})\int_{\Gamma}\left|\frac{\partial\phi_{\delta}(x,t)}{\partial t}\right|dx+O(\hbar^{3/4})\int_{\Gamma}\left|\frac{\partial\phi_{\delta}(x,t)}{\partial x}\right|dx+O(\hbar^{3/4})\int_{\Gamma}\left|\frac{\partial^{2}\phi_{\delta}(x,t)}{\partial x^{2}}\right|dx\leq$$

$$O(\hbar^{3/4})(\pi_{\delta}\delta)^{-1/4}|\eta'(t)|\int_{\Gamma}dx\exp\left[-\frac{(x-\widetilde{x}_{t})^{2}}{2\delta}\right]+ \tag{1.3.27}$$

$$+O(\hbar^{3/4})(\pi_{\delta}\delta)^{-1/4}|\eta(t)|\left|\widetilde{x}'_{t}\right|\int_{\Gamma}dx|x-\widetilde{x}_{t}|\exp\left[-\frac{(x-\widetilde{x}_{t})^{2}}{2\delta}\right]+$$

$$O(\hbar^{1/2})(\pi_{\delta}\delta)^{-1/4}\delta^{-1}|\eta(t)|\int_{\Gamma}dx|x-\widetilde{x}_{t}|\exp\left[-\frac{(x-\widetilde{x}_{t})^{2}}{2\delta}\right]+$$

$$O(\hbar^{3/4})(\pi_{\delta}\delta)^{-1/4}\delta^{-2}|\eta(t)|\int_{\Gamma}dx(x-\widetilde{x}_{t})^{2}\exp\left[-\frac{(x-\widetilde{x}_{t})^{2}}{2\delta}\right].$$

Having applied Lemma 1.3.1 to RHS of the (1.3.27) we have finalized the proof.

Assumption 1.3.2.We assume now that a wawe function $\Psi(\mathbf{x},t)=\langle\mathbf{x}|\psi_{t}\rangle$ **collapses at instant**: $t'>0$.

Definition 1.3.7. Let $\Psi(\mathbf{x},t)$ be a classical solution of the equation (1.3.1). The time- dependent Schrödinger equation (1.3.1) is a weakly well preserved by corresponding to classical solution $\Psi(\mathbf{x},t)$ collapsed along classical trajectory $\widetilde{\mathbf{X}}_{m_1,\ldots,m_k}(t,t')=[\widetilde{\mathbf{x}}_{m_1}(t,t'),\ldots,\widetilde{\mathbf{x}}_{m_i}(t,t'),\ldots,\widetilde{\mathbf{x}}_{m_k}(t,t')],1\leq i\leq k$ a wave function $\Psi^{\#}(\mathbf{x},t,t')$ given by Eq.(1.3.2) with $t'>0$ in region $[0,T]\times\Gamma\subseteq[0,T]\times\mathbb{R}^{3n}$, if the estimate

$$\int_{\Gamma}d^{3n}x\int_{0}^{T}dt\left\{i\hbar\frac{\partial\Psi^{\#}(\mathbf{x},t,t')}{\partial t}-\widehat{\mathbf{H}}(t)\Psi^{\#}(\mathbf{x},t,t')\right\}=O(\hbar^{\alpha}),0<\alpha\leq 1, \tag{1.3.28}$$

$$t\in[0,T],\mathbf{x}\in\mathbb{R}^{3n},$$

is satisfied.

Definition 1.3.8.(i) The time-dependent integral equation (1.3.28) is colled the time- dependent nonlocal Schrödinger equation of the order $\hbar^{\alpha}$.

(ii) Such collapsed wave function $\Psi^{\#}(\mathbf{x},t,t')$ as mentioned in Definition 1.3.7 is colled the

$\hbar^{\alpha}$- solution of the nonlocal Schrödinger equation (1.3.28) of the order α.

Definition 1.3.9. The time-dependent integral equation

$$\int_{\Gamma} d^{3n}x \int_{0}^{T} dt \left\{ i\hbar \frac{\partial \Psi^{\#}(\mathbf{x},t,t')}{\partial t} - \widehat{\mathbf{H}}(t)\Psi^{\#}(\mathbf{x},t,t') \right\} = 0, \tag{1.3.29}$$
$$t \in [0,T], \mathbf{x} \in \mathbb{R}^{3n},$$

is colled the time-dependent nonlocal Schrödinger equation.

Remark 1.3.2.We have introduced in consideration Eq.(1.3.28) in order to obtain good approximation of the solutions of the time-dependent nonlocal Schrödinger equation (1.3.29).

Theorem 1.3.4.Then any collapsed wave function $\Psi^{\#}(\mathbf{x},t,t')$ given by Eq.(1.3.) that is $\hbar^{\alpha}$- solution of the time-independent nonlocal Schrödinger equation (1.3.29) of the order $\hbar^{\alpha}$.

Proof. The proof similarly as the proof of the Theorem 1.3.2.

I.4. The nonlocal evolution equation for the statistical operator

Let us consider the evolution equation for the statistical operator [25]:

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}\left[\widehat{\mathbf{H}}, \rho(t)\right]. \tag{1.4.1}$$

Here $\widehat{\mathbf{H}}$ is the standard quantum Hamiltonian.In the coordinate representation one has, according to Eq.(1.4.1)

$$\frac{d}{dt}\rho(t,\mathbf{q}',\mathbf{q}'') = -\frac{i}{\hbar}\left(\widehat{\mathbf{H}}_{\mathbf{q}'} - \left(\widehat{\mathbf{H}}_{\mathbf{q}''}\right)^{*}\right)\rho(t,\mathbf{q}',\mathbf{q}''),$$
$$\rho(0,\mathbf{q}',\mathbf{q}'') = \rho(\mathbf{q}',\mathbf{q}''), \tag{1.4.2}$$
$$\mathbf{q}',\mathbf{q}'' \in \mathbb{R}^{3d}.$$

Here

$$\rho(t,\mathbf{q}',\mathbf{q}'') = \Psi(t,\mathbf{q}')\Psi^{*}(t,\mathbf{q}'') \tag{1.4.3}$$

and $\Psi(t,\mathbf{q}')$ is the classical solution of the Schrödinger equation (1.3.1), $H_{\mathbf{q}'}$ is the Hamiltonian of the system acting on a functions of variable $\mathbf{q}'$ and $\widehat{H}_{\mathbf{q}''}$ is the same Hamiltonian of the system acting on a functions of variable $\mathbf{q}''$.

Definition 1.4.1.(i) Let $\rho(t,\mathbf{q}',\mathbf{q}'')$ be an classical solution of the equation (1.4.2).Collapsed statistical operator $\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')$ corresponding to statistical operator $\rho(t,\mathbf{q}',\mathbf{q}'')$ given by

$$\rho^{\#}(t,\mathbf{q}',\mathbf{q}'') = \Psi^{\#}(t,\mathbf{q}')\Psi^{\#*}(t,\mathbf{q}'') \tag{1.4.4}$$

and where $\Psi^{\#}(t,\mathbf{q}')$ given by Eq.(1.3.2) with $t' = 0$.

(ii) Let $\rho(t,\mathbf{q}',\mathbf{q}'')$ be classical solution of the equation (1.4.2). The time-dependent equation (1.4.5) is a weakly well preserved by corresponding to $\rho(t,\mathbf{q}',\mathbf{q}'')$ collapsed statistical operator $\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')$ in region $\Gamma \subseteq \mathbb{R}^{3d}$ if the estimate

$$\frac{d}{dt}\int_{\Gamma}\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')d^{3d}\mathbf{q}'d^{3d}\mathbf{q}'' =$$

$$-\frac{i}{\hbar}\int_{\Gamma}\left(H_{\mathbf{q}'}-\left(\widehat{H}_{\mathbf{q}''}\right)^{*}\right)\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')d^{3d}\mathbf{q}'d^{3d}\mathbf{q}''+O(\hbar^{\alpha}),$$

$$1/4<\alpha\leq 1/2, \tag{1.4.5}$$

$$\rho^{\#}(0,\mathbf{q}',\mathbf{q}'')=\rho^{\#}(\mathbf{q}',\mathbf{q}''),$$

$$\mathbf{q}',\mathbf{q}''\in\mathbb{R}^{3d}$$

is satisfied. Here

$$\rho^{\#}(\mathbf{q}',\mathbf{q}'')=\Psi^{\#}(0,\mathbf{q}')\Psi^{\#*}(0,\mathbf{q}'') \tag{1.4.6}$$

and a function $\Psi^{\#}(0,\mathbf{q}')$ is the solution of the nonlocal stationary Schrödinger equation (1.3.34).

Definition 1.4.2.(i) The integral equation (1.4.5) is called nonlocal equation of the order $\hbar^{\alpha}$

for the statistical operator $\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')$.

(ii) Such collapsed statistical operator $\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')$ as mentioned in Definition 1.4.1 is called

the $\hbar^{\alpha}$- solution of the time-dependent nonlocal equation (1.4.5) of the order α.

Definition 1.4.3. The time-dependent integral equation

$$\frac{d}{dt}\int_{\Gamma}\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')d^{3d}\mathbf{q}'d^{3d}\mathbf{q}'' =$$

$$-\frac{i}{\hbar}\int_{\Gamma}\left(H_{\mathbf{q}'}-\left(\widehat{H}_{\mathbf{q}''}\right)^{*}\right)\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')d^{3d}\mathbf{q}'d^{3d}\mathbf{q}'', \tag{1.4.7}$$

$$\rho^{\#}(0,\mathbf{q}',\mathbf{q}'')=\rho^{\#}(\mathbf{q}',\mathbf{q}''),$$

$$\mathbf{q}',\mathbf{q}''\in\mathbb{R}^{3d}$$

is called nonlocal equation for the statistical operator $\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')$.

Remark 1.4.1. We have introduced in consideration Eq.(1.4.5) in order to obtain good approximation of the solutions of the Eq.(1.4.7)

Definition 1.4.4. Let $\rho(t,\mathbf{q}_1',\mathbf{q}_2',\ldots,\mathbf{q}_n',\mathbf{q}_1'',\mathbf{q}_2'',\ldots,\mathbf{q}_n'')$ be an statistical operator.We define statistical operator $\rho^{\#}(t,t',\mathbf{q}_1',\mathbf{q}_2',\ldots,\mathbf{q}_n',\mathbf{q}_1'',\mathbf{q}_2'',\ldots,\mathbf{q}_n'')$ corresponding to statistical operator $\rho(t,\mathbf{q}_1',\mathbf{q}_2',\ldots,\mathbf{q}_n',\mathbf{q}_1'',\mathbf{q}_2'',\ldots,\mathbf{q}_n'')$ and collapsed along classical trajectories

$$\widetilde{\mathbf{X}}'_{m_1,\ldots,m_k}(t,t') = [\widetilde{\mathbf{x}}'_{m_1}(t,t'),\ldots,\widetilde{\mathbf{x}}'_{m_i}(t,t'),\ldots,\widetilde{\mathbf{x}}'_{m_k}(t,t')], 1 \le i \le k,$$

$$\widetilde{\mathbf{x}}'_{m_i}(t,t') = \begin{cases} \mathbf{x}'_{m_i}(t) \text{ iff } t' < t \le T \\ 0 \text{ iff } 0 \le t \le t' \end{cases}$$

$$\widetilde{\mathbf{X}}''_{m_1,\ldots,m_k}(t,t') = [\widetilde{\mathbf{x}}''_{m_1}(t,t'),\ldots,\widetilde{\mathbf{x}}''_{m_i}(t,t'),\ldots,\widetilde{\mathbf{x}}''_{m_k}(t,t')], 1 \le i \le k, \tag{1.4.8}$$

$$\widetilde{\mathbf{x}}''_{m_i}(t,t') = \begin{cases} \mathbf{x}''_{m_i}(t) \text{ iff } t' < t \le T \\ 0 \text{ iff } 0 \le t \le t' \end{cases}$$

by formulae

$$\rho^{\#}(t,t',\mathbf{q}'_1,\mathbf{q}'_2,\ldots,\mathbf{q}'_n,\mathbf{q}''_1,\mathbf{q}''_2,\ldots,\mathbf{q}''_n) =$$

$$\rho^{\#}(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n,t,t';\widetilde{\mathbf{x}}_{m_1}(t,t'),\ldots,\widetilde{\mathbf{x}}_{m_k}(t,t')) =$$

$$\Re_{m_1,\ldots,m_k}(\widetilde{\mathbf{x}}'_{m_1}(t,t'),\ldots,\widetilde{\mathbf{x}}'_{m_i}(t,t'),\ldots,\widetilde{\mathbf{x}}'_{m_k}(t,t')) \times$$
$$\Re_{m_1,\ldots,m_k}(\widetilde{\mathbf{x}}''_{m_1}(t,t'),\ldots,\widetilde{\mathbf{x}}''_{m_i}(t,t'),\ldots,\widetilde{\mathbf{x}}''_{m_k}(t,t'))\rho(t,\mathbf{q}'_1,\mathbf{q}'_2,\ldots,\mathbf{q}'_n,\mathbf{q}''_1,\mathbf{q}''_2,\ldots,\mathbf{q}''_n),$$

$$\Re_{m_1,\ldots,m_k}(\widetilde{\mathbf{x}}'_{m_1}(t,t'),\ldots,\widetilde{\mathbf{x}}'_{m_i}(t,t'),\ldots,\widetilde{\mathbf{x}}'_{m_k}(t,t')) = \prod_{i=1}^{k}\Re_{m_i}(\widetilde{\mathbf{x}}'_{m_i}(t,t')),$$

$$\Re_{m_1,\ldots,m_k}(\widetilde{\mathbf{x}}''_{m_1}(t,t'),\ldots,\widetilde{\mathbf{x}}''_{m_i}(t,t'),\ldots,\widetilde{\mathbf{x}}''_{m_k}(t,t')) = \prod_{i=1}^{k}\Re_{m_i}(\widetilde{\mathbf{x}}''_{m_i}(t,t')),$$

$$\Re_{m_i}(\widetilde{\mathbf{x}}'_{m_i}(t,t')) = \tag{1.4.9}$$

$$\begin{cases} (\pi\delta)^{-3/4}\exp\left[-\dfrac{\left(\widehat{\mathbf{q}}'_{m_i} - \widetilde{\mathbf{x}}'_{m_i}(t,t')\right)^2}{2\delta}\right] \text{iff } \|\mathbf{q}'_{m_i} - \widetilde{\mathbf{x}}'_{m_i}(t,t')\| \le \varepsilon, \\ 0 \text{ iff } \|\mathbf{q}'_{m_i} - \widetilde{\mathbf{x}}'_{m_i}(t,t')\| > \varepsilon. \end{cases}$$

$$\Re_{m_i}(\widetilde{\mathbf{x}}''_{m_i}(t,t')) =$$

$$\begin{cases} (\pi\delta)^{-3/4}\exp\left[-\dfrac{\left(\widehat{\mathbf{q}}''_{m_i} - \widetilde{\mathbf{x}}''_{m_i}(t,t')\right)^2}{2\delta}\right] \text{iff } \|\mathbf{q}''_{m_i} - \widetilde{\mathbf{x}}''_{m_i}(t,t')\| \le \varepsilon, \\ 0 \text{ iff } \|\mathbf{q}''_{m_i} - \widetilde{\mathbf{x}}''_{m_i}(t,t')\| > \varepsilon. \end{cases}$$

Let us consider now the 1-particle master equation of the classical GRW model

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}\left[\widehat{\mathbf{H}}, \rho(t)\right] - \lambda(\rho(t) - T[\rho(t)]). \tag{1.4.10}$$

Here $\widehat{\mathbf{H}}$ is the standard quantum Hamiltonian of the particle, and $T[\bullet]$ represents the effect of the spontaneous collapses on the particle's wave function. In the position representation, this equation becomes:

$$\begin{gathered}\frac{d}{dt}\rho(t,\mathbf{q}',\mathbf{q}'') = \\ -\frac{i}{\hbar}\left(\widehat{\mathbf{H}}_{\mathbf{q}'} - \left(\widehat{\mathbf{H}}_{\mathbf{q}''}\right)^*\right)\rho(t,\mathbf{q}',\mathbf{q}'') - \lambda\left\{1 - \exp\left[-\frac{(\mathbf{q}'-\mathbf{q}'')^2}{4r_c^2}\right]\right\}\rho(t,\mathbf{q}',\mathbf{q}''), \\ \rho(0,\mathbf{q}',\mathbf{q}'') = \rho(\mathbf{q}',\mathbf{q}''), \\ \mathbf{q}',\mathbf{q}'' \in \mathbb{R}^{3d}.\end{gathered} \tag{1.4.11}$$

Here $H_{\mathbf{q}'}$ is the Hamiltonian of the system acting on a functions of variable $\mathbf{q}'$ and $\widehat{H}_{\mathbf{q}''}$ is the same Hamiltonian of the system acting on a functions of variable $\mathbf{q}''$.

Definition 1.4.5. Let $\rho(t,\mathbf{q}',\mathbf{q}'')$ be classical solution of the equation (1.4.11). The time-dependent equation (1.4.11) is a weakly well preserved by corresponding to $\rho(t,\mathbf{q}',\mathbf{q}'')$ collapsed statistical operator $\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')$ given by (1.4.9) with $t' = 0$ in region $\Gamma \subseteq \mathbb{R}^{3d}$ if the estimate

$$\begin{gathered}\frac{d}{dt}\int_\Gamma \rho^{\#}(t,\mathbf{q}',\mathbf{q}'')d^{3d}\mathbf{q}'d^{3d}\mathbf{q}'' = \\ -\frac{i}{\hbar}\int_\Gamma\left(H_{\mathbf{q}'} - \left(\widehat{H}_{\mathbf{q}''}\right)^*\right)\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')d^{3d}\mathbf{q}'d^{3d}\mathbf{q}'' - \\ -\lambda\int_\Gamma\left\{1 - \exp\left[-\frac{(\mathbf{q}'-\mathbf{q}'')^2}{4r_c^2}\right]\right\}\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')d^{3d}\mathbf{q}'d^{3d}\mathbf{q}'' + O(\hbar^\alpha), \\ 1/4 < \alpha \leq 1/2, \\ \rho^{\#}(0,\mathbf{q}',\mathbf{q}'') = \rho^{\#}(\mathbf{q}',\mathbf{q}''), \\ \mathbf{q}',\mathbf{q}'' \in \mathbb{R}^{3d}\end{gathered} \tag{1.4.12}$$

is satisfied.

Definition 1.4.6. The time-dependent integral equation

$$\frac{d}{dt}\int_{\Gamma}\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')d^{3d}\mathbf{q}'d^{3d}\mathbf{q}'' =$$
$$-\frac{i}{\hbar}\int_{\Gamma}\left(H_{\mathbf{q}'}-\left(\widehat{H}_{\mathbf{q}''}\right)^{*}\right)\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')d^{3d}\mathbf{q}'d^{3d}\mathbf{q}'' -$$
$$-\lambda\int_{\Gamma}\left\{1-\exp\left[-\frac{(\mathbf{q}'-\mathbf{q}'')^{2}}{4r_{c}^{2}}\right]\right\}\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')d^{3d}\mathbf{q}'d^{3d}\mathbf{q}'', \tag{1.4.13}$$
$$1/4 < \alpha \leq 1/2,$$
$$\rho^{\#}(0,\mathbf{q}',\mathbf{q}'') = \rho^{\#}(\mathbf{q}',\mathbf{q}''),$$
$$\mathbf{q}',\mathbf{q}'' \in \mathbb{R}^{3d}$$

is called nonlocal master equation for the statistical operator $\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')$.

Remark 1.4.2.We have introduced in consideration Eq.(1.4.12) in order to obtain good approximation of the solutions of the Eq.(1.4.13)

Theorem 1.4.1. We assume now for a short that $m = n = 1$. Let $\Psi(x,t), x \in \mathbb{R}$ be a classical solution of the equation (1.3.1), where

$$\widehat{\mathbf{H}} = \hbar^{2}\frac{\partial^{2}}{\partial x^{2}} + V(x,t). \tag{1.4.14}$$

Assume that

$$|\Psi(x,t)| = O(\hbar^{-1/4}),$$
$$|\partial\Psi(x,t)/\partial x| = O(\hbar^{--5/4}), \tag{1.4.15}$$
$$|\partial\Psi(x,t)/\partial t| = O(\hbar^{--5/4}).$$

Then for any collapsed wave function $\Psi^{\#}(x,t) = \Psi^{\#}(x,t,t' = 0)$ given by Eq.(1.3.2) with $t' = 0$ and $\sqrt{\hbar/\delta} = \hbar^{\alpha}, 1/4 < \alpha < 1/2$ corresponding collapsed statistical operator $\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')$ given by Eq.(1.4.4) that is $\hbar^{\alpha}$-solution of the time-dependent nonlocal equation (1.4.5) of the order α.

Proof. The proof similarly as the proof of the Theorem 1.3.2.

Let us consider now the equation (1.4.11) in the case in which $\widehat{\mathbf{H}}$ is the Hamiltonian for a free particle; for simplicity we work in one dimension. In the coordinate representation we get

$$\frac{d}{dt}\rho(t,q',q'') =$$
$$\frac{i\hbar}{2m}\left(\frac{\partial^{2}}{\partial q'^{2}}-\frac{\partial^{2}}{\partial q''^{2}}\right)\rho(t,q',q'')-\lambda\left\{1-\exp\left[-\frac{(q'-q'')^{2}}{4r_{c}^{2}}\right]\right\}\rho(t,q',q''). \tag{1.4.16}$$

One can express the solution of the above equation satisfying given initial conditions in terms of the solution $\rho_{\mathbf{Sch}}(t,q',q'')$ of the usual Schrödinger equation ($\lambda = 0$) satisfying the same initial conditions, according to [3]:

$$\rho(t,q',q'') = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dyF(k,q'-q'',t)\rho_{Sch}(t,q'+y,q''+y)\exp\left(\frac{iky}{\hbar}\right),$$
$$F(k,q,t) = \exp\left[-\lambda t + \lambda\int_0^t d\tau \exp\left(-\frac{1}{4r_c^2}\left[q-\frac{k\tau}{m}\right]^2\right)\right]. \tag{1.4.17}$$

By Theorem 1.4.1 corresponding to $\rho(t,q',q'')$ given by

$$\rho^{\#}(t,q',q'') = \Re(\widetilde{q}'(t))\Re(\widetilde{q}''(t))\rho(t,q',q''),$$
$$\Re(\widetilde{q}'(t)) = \begin{cases} (\pi\delta)^{-1/4}\exp\left[-\frac{\left(\widehat{q}'-\widetilde{q}'(t)\right)^2}{2\delta}\right] \text{ iff } |q'-\widetilde{q}'(t)| \leq \varepsilon, \\ 0 \text{ iff } |q'-\widetilde{q}'(t)| > \varepsilon. \end{cases} \tag{1.4.18}$$
$$\Re(\widetilde{q}''(t)) = \begin{cases} (\pi\delta)^{-1/4}\exp\left[-\frac{\left(\widehat{q}''-\widetilde{q}''(t)\right)^2}{2\delta}\right] \text{ iff } |q''-\widetilde{q}''(t)| \leq \varepsilon, \\ 0 \text{ iff } |q''-\widetilde{q}''(t)| > \varepsilon. \end{cases}$$

Assumption 1.4.1.We assume now that a wawe function collapses at

instant $t' > 0$.

Definition 1.4.7. Let $\rho(t,\mathbf{q}',\mathbf{q}'')$ be an classical solution of the equation (1.4.2). The time-dependent equation (1.4.2) is a weakly well preserved by corresponding to $\rho(t,\mathbf{q}',\mathbf{q}'')$ collapsed statistical operator $\rho^{\#}(t,t',\mathbf{q}',\mathbf{q}'')$ in region $[0,T]\times\Gamma,\Gamma\subseteq\mathbb{R}^{3d}$ iff the estimate

$$\int_0^T dt\int_\Gamma \frac{d}{dt}\rho^{\#}(t,t',\mathbf{q}',\mathbf{q}'')d^{3d}\mathbf{q}'d^{3d}\mathbf{q}'' =$$
$$-\frac{i}{\hbar}\int_0^T dt\int_\Gamma\left(H_{\mathbf{q}'}-\left(\widehat{H}_{\mathbf{q}''}\right)^*\right)\rho^{\#}(t,t',\mathbf{q}',\mathbf{q}'')d^{3d}\mathbf{q}'d^{3d}\mathbf{q}'' + O_T(\hbar^\alpha), \tag{1.4.19}$$
$$1/4 < \alpha \leq 1/2,$$
$$\rho^{\#}(0,\mathbf{q}',\mathbf{q}'') = \rho^{\#}(\mathbf{q}',\mathbf{q}''),$$
$$\mathbf{q}',\mathbf{q}'' \in \mathbb{R}^{3d}$$

is satisfied.

Definition 1.4.8. The time-dependent integral equation

$$\int_0^T dt \int_\Gamma \frac{d}{dt} \rho^{\#}(t,t',\mathbf{q}',\mathbf{q}'') d^{3d}\mathbf{q}' d^{3d}\mathbf{q}'' =$$

$$-\frac{i}{\hbar} \int_0^T dt \int_\Gamma \left(H_{\mathbf{q}'} - \left(\widehat{H}_{\mathbf{q}''} \right)^* \right) \rho^{\#}(t,t',\mathbf{q}',\mathbf{q}'') d^{3d}\mathbf{q}' d^{3d}\mathbf{q}'', \quad (1.4.20)$$

$$\rho^{\#}(0,\mathbf{q}',\mathbf{q}'') = \rho^{\#}(\mathbf{q}',\mathbf{q}''),$$

$$\mathbf{q}',\mathbf{q}'' \in \mathbb{R}^{3d}$$

is called nonlocal master equation for the statistical operator $\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')$.

Definition 1.4.9.Let $\rho(t,\mathbf{q}',\mathbf{q}'')$ be an classical solution of the equation (1.4.13). The time-dependent equation (1.4.13) is a weakly well preserved by corresponding to $\rho(t,\mathbf{q}',\mathbf{q}'')$ collapsed statistical operator $\rho^{\#}(t,t',\mathbf{q}',\mathbf{q}'')$ given by (1.4.9) in region $[0,T] \times \Gamma, \Gamma \subseteq \mathbb{R}^{3d}$ if the estimate

$$\int_0^T dt \int_\Gamma \frac{d}{dt} \rho^{\#}(t,\mathbf{q}',\mathbf{q}'') d^{3d}\mathbf{q}' d^{3d}\mathbf{q}'' =$$

$$-\frac{i}{\hbar} \int_0^T dt \int_\Gamma \left(H_{\mathbf{q}'} - \left(\widehat{H}_{\mathbf{q}''} \right)^* \right) \rho^{\#}(t,\mathbf{q}',\mathbf{q}'') d^{3d}\mathbf{q}' d^{3d}\mathbf{q}'' -$$

$$-\lambda \int_0^T dt \int_\Gamma \left\{ 1 - \exp\left[-\frac{(\mathbf{q}'-\mathbf{q}'')^2}{4r_c^2} \right] \right\} \rho^{\#}(t,\mathbf{q}',\mathbf{q}'') d^{3d}\mathbf{q}' d^{3d}\mathbf{q}'' + O_T(\hbar^\alpha),, \quad (1.4.21)$$

$$1/4 < \alpha \leq 1/2,$$

$$\rho^{\#}(0,\mathbf{q}',\mathbf{q}'') = \rho^{\#}(\mathbf{q}',\mathbf{q}''),$$

$$\mathbf{q}',\mathbf{q}'' \in \mathbb{R}^{3d}$$

is satisfied.

Definition 1.4.10.The time-dependent integral equation

$$\int_0^T dt \int_\Gamma \frac{d}{dt} \rho^{\#}(t,\mathbf{q}',\mathbf{q}'') d^{3d}\mathbf{q}' d^{3d}\mathbf{q}'' =$$

$$-\frac{i}{\hbar} \int_0^T dt \int_\Gamma \left(H_{\mathbf{q}'} - \left(\widehat{H}_{\mathbf{q}''} \right)^* \right) \rho^{\#}(t,\mathbf{q}',\mathbf{q}'') d^{3d}\mathbf{q}' d^{3d}\mathbf{q}'' -$$

$$-\lambda \int_0^T dt \int_\Gamma \left\{ 1 - \exp\left[-\frac{(\mathbf{q}'-\mathbf{q}'')^2}{4r_c^2} \right] \right\} \rho^{\#}(t,\mathbf{q}',\mathbf{q}'') d^{3d}\mathbf{q}' d^{3d}\mathbf{q}'', \quad (1.4.22)$$

$$1/4 < \alpha \leq 1/2,$$

$$\rho^{\#}(0,\mathbf{q}',\mathbf{q}'') = \rho^{\#}(\mathbf{q}',\mathbf{q}''),$$

$$\mathbf{q}',\mathbf{q}'' \in \mathbb{R}^{3d}$$

is called nonlocal master equation for the statistical operator $\rho^{\#}(t,\mathbf{q}',\mathbf{q}'')$.

I.5. The nonlocal stochastic nonlinear Schrödinger equation

Let us consider the stochastic nonlinear Schrödinger equation [9],[10],[46]:

$$d|\psi_t(\mathbf{x})\rangle = \left[-\frac{i}{\hbar}\widehat{\mathbf{H}} - \frac{k}{2}(\widehat{q} - \langle q_t\rangle)^2 dt\right]|\psi_t(\mathbf{x})\rangle dt + \sqrt{k}\,(\widehat{q} - \langle q_t\rangle)dW_t(\omega)|\psi_t(\mathbf{x})\rangle. \tag{1.5.1}$$

Here $\mathbf{x} \in \mathbb{R}^r, \widehat{q}$ is the position operator, $\langle q_t\rangle = \langle\psi_t|\widehat{q}|\psi_t\rangle$ it is its expectation value, and k is a constant.

Definition 1.5.1.Let $(w_{\epsilon,t}(\omega))_\epsilon, \epsilon \in (0,1]$ be smoothed with respect to $\mathbb{R}^r$ white noise, (see [20],[21]) and let $(W_{\epsilon,t}(\omega))_\epsilon$ be Colombeau-Wiener process $(\dot{W}_{\epsilon,t}(\omega))_\epsilon = (w_{\epsilon,t}(\omega))_\epsilon$.

We rewrite now Eq.(1.5.1) in Colombeau-Ito form

$$\begin{gathered}(d|\psi_{\epsilon,t}(\mathbf{x})\rangle)_\epsilon = \\ \left(\left[-\frac{i}{\hbar}\widehat{\mathbf{H}}_\epsilon - \frac{k}{2}(\widehat{q} - \langle q_{\epsilon,t}\rangle)^2 dt\right]|\psi_{\epsilon,t}(\mathbf{x})\rangle dt + \sqrt{k}\,(\widehat{q} - \langle q_{\epsilon,t}\rangle)dW_{\epsilon,t}(\omega)|\psi_{\epsilon,t}(\mathbf{x})\rangle\right)_\epsilon.\end{gathered} \tag{1.5.2}$$

Here $\mathbf{x} \in \mathbb{R}^r, \widehat{q}$ is the position operator, $(\langle q_{\epsilon,t}\rangle)_\epsilon = (\langle\psi_{\epsilon,t}|\widehat{q}|\psi_{\epsilon,t}\rangle)_\epsilon$ it is its expectation value, and k is a constant.

Definition 1.5.2. Let $(\psi_{\epsilon,t}(\mathbf{x}))_\epsilon = (\psi_\epsilon(\mathbf{x},t))_\epsilon, \mathbf{x} =(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)$ be an Colombeau solution of the equation (1.5.2) and let

$$\begin{gathered}\left(\widetilde{\mathbf{X}}_{m_1,.,m_i,..,m_k}(t,t',\omega,\epsilon)\right)_\epsilon = \\ [(\widetilde{\mathbf{x}}_{m_1}(t,t',\omega,\epsilon))_\epsilon,\ldots,(\widetilde{\mathbf{x}}_{m_i}(t,t',\omega,\epsilon))_\epsilon,\ldots,(\widetilde{\mathbf{x}}_{m_k}(t,t',\omega,\epsilon))_\epsilon], 1 \leq i \leq k\end{gathered} \tag{1.5.3}$$

be Colombeau ctochastic trajectory. Then we define collapsed Colombeau solution $\left(\psi^{\#}_{\epsilon,t,t'}(\mathbf{x})\right)_\epsilon$ corresponding to Colombeau solution $(\psi_{\epsilon,t}(\mathbf{x}))_\epsilon$ by formula

$$\left(\psi^{\#}_{\epsilon,t,t'}(\mathbf{x})\right)_{\epsilon} = (\psi_{\epsilon,t}(\mathbf{x}))_{\epsilon} \text{ iff } t < t',$$

$$\text{iff } t \geq t' :$$

$$\left(\psi^{\#}_{\epsilon,t,t'}(\mathbf{x})\right)_{\epsilon} = (\psi^{\#}_{\epsilon}(\mathbf{x},t,t'))_{\epsilon} =$$

$$(\psi^{\#}_{\epsilon}(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n,t,t'))_{\epsilon} =$$

$$(\psi^{\#}_{\epsilon}(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n,t,t';\widetilde{\mathbf{x}}_{m_1}(t,t',\omega,\epsilon),\ldots,\widetilde{\mathbf{x}}_{m_k}(t,t',\omega,\epsilon)))_{\epsilon} =$$

$$\left(\frac{\Re_{m_1,\ldots,m_k}(\widetilde{\mathbf{x}}_{m_1}(t,t',\omega,\epsilon),\ldots,\widetilde{\mathbf{x}}_{m_k}(t,t',\omega,\epsilon))\psi_{\epsilon,t}(\mathbf{x})}{\|\Re_{m_1,\ldots,m_k}(\widetilde{\mathbf{x}}_{m_1}(t,t',\omega,\epsilon),\ldots,\widetilde{\mathbf{x}}_{m_k}(t,t',\omega,\epsilon))\psi_{\epsilon,t}(\mathbf{x})\|_2}\right)_{\epsilon}, \tag{1.5.4}$$

$$(\Re_{m_1,\ldots,m_k}(\widetilde{\mathbf{x}}_{m_1}(t,t',\omega,\epsilon),\ldots,\widetilde{\mathbf{x}}_{m_k}(t,t',\omega,\epsilon)))_{\epsilon} = \left(\prod_{i=1}^{k}\Re_{m_i}(\widetilde{\mathbf{x}}_{m_i}(t,t',\omega,\epsilon))\right)_{\epsilon},$$

$$(\Re_{m_i}(\widetilde{\mathbf{x}}_{m_i}(t,t',\omega,\epsilon)))_{\epsilon} =$$

$$\begin{cases} (\pi\delta)^{-3/4}\left(\exp\left[-\dfrac{(\widehat{\mathbf{x}}_{m_i}-\widetilde{\mathbf{x}}_{m_i}(t,t',\omega,\epsilon))^2}{2\delta}\right]\right)_{\epsilon} \text{ iff } (\|\mathbf{x}_{m_i}-\widetilde{\mathbf{x}}_{m_i}(t,t',\omega,\epsilon)\|)_{\epsilon} \leq \varepsilon, \\ 0 \text{ iff } (\|\mathbf{x}_{m_i}-\widetilde{\mathbf{x}}_{m_i}(t,t',\omega,\epsilon)\|)_{\epsilon} > \varepsilon. \end{cases}$$

Definition 1.5.3.Let $(\psi_{\epsilon,t}(\mathbf{x}))_{\epsilon}$ be an Colombeau solution of the equation (1.5.2). The time- dependent Colombeau-Schrödinger equation (1.5.2) is a weakly well preserved by corresponding to Colombeau solution $(\psi_{\epsilon,t}(\mathbf{x}))_{\epsilon}$ collapsed along Colombeau trajectory $\left(\widetilde{\mathbf{X}}_{m_1,..,m_i,..,m_k}(t,t',\omega,\epsilon)\right)_{\epsilon}, 1 \leq i \leq k$ a wave function $\left(\psi^{\#}_{\epsilon,t,t'}(\mathbf{x})\right)_{\epsilon}$ given by Eq.(1.5.4) with $t' > 0$ in region $[0,T]\times\Gamma \subseteq [0,T]\times\mathbb{R}^{3n}$, if the estimate

$$\left(\int_{\Gamma} d^{3n}x\int_{0}^{T} d\left|\psi^{\#}_{\epsilon,t,t'}(\mathbf{x},\omega)\right\rangle\right)_{\epsilon} =$$

$$\left(\int_{\Gamma} d^{3n}x\int_{0}^{T}\left[-\frac{i}{\hbar}\widehat{\mathbf{H}}_{\epsilon}-\frac{k}{2}(\widehat{q}-\langle q_{\epsilon,t,t'}\rangle)^2\right]\left|\psi^{\#}_{\epsilon,t,t'}(\mathbf{x},\omega)\right\rangle dt\right)_{\epsilon} + \tag{1.5.5}$$

$$\left(\sqrt{k}\int_{\Gamma} d^{3}x\int_{0}^{T}(\widehat{q}-\langle q_{\epsilon,t,t'}\rangle)dW_{\epsilon,t}(\omega)\left|\psi^{\#}_{\epsilon,t,t'}(\mathbf{x},\omega)\right\rangle\right)_{\epsilon} + (O_{\epsilon}(\hbar^{\alpha}))_{\epsilon},$$

$$1/4 < \alpha \leq 1/2$$

is satisfied a.s. Here $\mathbf{x} \in \mathbb{R}^r, \widehat{q}$ is the position operator, $(\langle q_{\epsilon,t}\rangle)_{\epsilon} = (\langle\psi_{\epsilon,t}|\widehat{q}|\psi_{\epsilon,t}\rangle)_{\epsilon}$ it is its expectation value, and k is a constant.

Definition 1.5.4.(i)The time-dependent integral equation (1.5.5) is colled the time-dependent nonlocal Schrödinger equation of the order $\hbar^{\alpha}$.

(i) A wave function $\left(\psi^{\#}_{\epsilon,t,t'}(\mathbf{x})\right)_{\epsilon}$ is called the $\hbar^{\alpha}$- solution of the time-dependent nonlocal equation (1.5.5) of the order α.

Theorem 1.5.1.Let $(\psi_{\epsilon,t,t'}(\mathbf{x}))_\epsilon, \mathbf{x} \in \mathbb{R}^{3n}$ be a Colombeau solution of the equation (1.5.2), where

$$\left(\widehat{\mathbf{H}}_\epsilon\right)_\epsilon = \hbar^2 \sum_{i=1}^{3n} \frac{\partial^2}{\partial x_i^2} + (V_\epsilon(\mathbf{x},t))_\epsilon. \tag{1.5.6}$$

Assume that

$$\begin{gathered}(|\psi_{\epsilon,t,t'}(\mathbf{x})|)_\epsilon = (O_\epsilon(\hbar^{-3n/4}))_\epsilon,\\ (|\partial\psi_{\epsilon,t,t'}(\mathbf{x})/\partial x_i|)_\epsilon = (O_\epsilon(\hbar^{-3n/4-1}))_\epsilon, i = 1,\ldots,3n,\\ (|\partial\psi_{\epsilon,t,t'}(\mathbf{x})/\partial t|)_\epsilon = (O_\epsilon(\hbar^{-3n/4-1}))_\epsilon.\end{gathered} \tag{1.5.7}$$

Then any collapsed Colombeau wave function $(|\psi_{\epsilon,t,t'}(\mathbf{x})|)_\epsilon$ given by Eq.(1.5.4) with $\sqrt{\hbar/\delta} = \hbar^\alpha, 1/4 < \alpha < 1/2$ that is $\hbar^\alpha$-solution of the time-dependent nonlocal Schrödinger equation (1.5.5) of the order α.

Proof. The proof similarly as the proof of the Theorem 1.3.2.

I.6. The reconciled Bohr rule. Schrödinger's cat through Stern-Gerlach experiment.Schrödinger's cat demands to reconcile Bohr rule.

Another known in literature special sort of the Schrödinger cat paradox can be simply illustrated with the famous Stern-Gerlach experiment (Fig.1.6.1). Silver atoms boiled off from a furnace are sent through a non-uniform magnetic field, and impinge on a photographic plate. Instead of a continuous distribution of spots, one sees two spots, corresponding to spin up and spin down relative to the magnetic field axis. Each atom goes up OR down, but one cannot predict which in any given run – the results of the experiment are probabilistic. There is a 50% chance of an atom going up, and a 50% chance that it will go down.

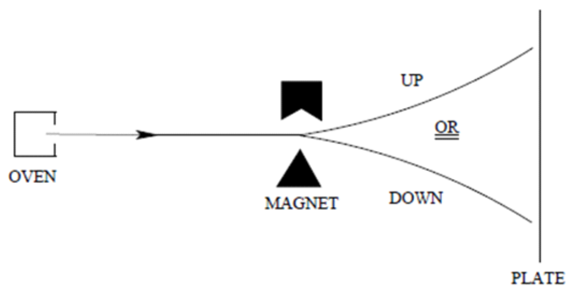


Fig.1.6.1.Stern-Gerlach experiment.

Adapted from [27].

Remark 1.6.1. We remind that from the point of view of the Schrödinger equation of quantum theory, this result has no any rigorous explanation.

In quantum theory, the state of the particle is described by its wave function, and the Schr¨odinger equation says that at a post-measurement final time $t_{\mathbf{f}}$, the wave function is related to that at a pre-measurement initial time $t_{\mathbf{i}}$, by known deterministic relation

$$\begin{aligned} \Psi(t_{\mathbf{f}}) &= U(t_{\mathbf{f}}, t_{\mathbf{i}})\Psi(t_{\mathbf{i}}), \\ U(t_{\mathbf{f}}, t_{\mathbf{i}}) &= \exp\left[i\widehat{H}(t_{\mathbf{f}} - t_{\mathbf{i}}) \right] \end{aligned} \tag{1.6.1}$$

with the transition unitary operator U completely specified by the Hamiltonian $\widehat{H}$. To explain what is observed, the Schr¨odinger equation must be supplemented by the reduction postulate and the Born rule. These state that the wave function only gives a description of probabilities when a measurement is made, with the probabilities for an "up" outcome and a "down" outcome given by the squares of the coefficients of the corresponding components in the initial wave function $\Psi(t_{\mathbf{i}})$,

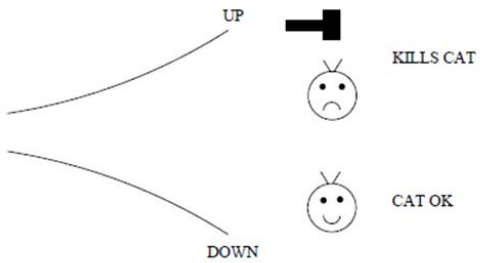


Fig.1.6.2.The Stern-Gerlach apparatus with
a Schrödinger cat as the outcome registration.
Adapted from [27].

Born's Rule for Probabilities

$$\begin{aligned} &\Psi(t_{\mathbf{i}}) = c_{\mathbf{up}}\psi_{\mathbf{up}} + c_{\mathbf{down}}\psi_{\mathbf{down}}, \\ &\mathbf{prob}(\psi_{\mathbf{up}}) = |c_{\mathbf{up}}|^2, \mathbf{prob}(\psi_{\mathbf{down}}) = |c_{\mathbf{down}}|^2, \\ &|c_{\mathbf{up}}|^2 + |c_{\mathbf{down}}|^2 = 1. \end{aligned} \tag{1.6.2}$$

with the sum of the up and down probabilities equal to one. The reduction postulate and Born rule are an add-on to the Schrödinger equation. According to the Copenhagen interpretation of quantum mechanics, the Schrödinger equation applies when a microscopic system, the silver atom, is time-evolving in isolation. But when the atom interacts with a macroscopic measuring apparatus, as in the Stern–Gerlach setup, you have to use the reduction postulate and Born rule.

Remark 1.6.2. This situation leads to puzzles that have been debated for over eighty years. If quantum mechanics describes the whole universe, then why can't one use the Schrödinger equation to describe the system consisting of the silver atom plus the measuring apparatus? But we never see a superposition state of the atom plus apparatus. This is Schrödinger's famous cat paradox. Arrange the experiment so that an "up" outcome triggers a mechanism that kills the cat, while a "down" outcome keeps the cat alive. Of course we don't do this, but if we were to do it, we would always see a live cat OR a dead one, never a superposition of the two (Fig. 1.6.2). So we have the problem of definite outcomes: where does the "either"–"or" dichotomy arise?

We remind now some fundamental notions from probability theory.

Definition 1.6.1. In probability theory, the **sample space** (observation space) of an experiment or random trial is the set of all possible outcomes or results of that experiment. A sample space is usually denoted using set notation, and the possible outcomes are listed as elements in the set. It is common to refer to a sample space by the label Ω.

Remark 1.6.3. A well-defined sample space (observation space) is one of three basic elements in a probabilistic model (a probability space $\Theta = \{\Omega, \Sigma, \mathbf{P}\}$); the other two are a well-defined set of possible events (a sigma-algebra Σ) and a probability assigned to each event (a probability measure function $\mathbf{P}$).

Remark 1.6.4. An simply example of a sample phase space and corresponding probability space closer to our Stern-Gerlach experiment, is a coin toss. Consider 1000 coin tosses. If the coin is tossed without bias, you will find close to 500 heads and 500 tails, corresponding to prob$_{heads} = 0.5$ and prob$_{tails} = 0.5$. Here the sample space consists of the 1000 detailed trajectories of the toss, which your eye cannot follow,

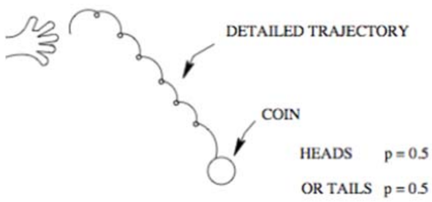


Fig.1.6.3. An sample space.Trajectories in a coin toss.Adapted from [27].

but which if analyzed by a very fast computer could predict which toss would give a head and which a tail (Fig.1.6. 3). Again, the probabilities are just reflections of our ignorance of the details, but the details are there. So we have the questions – are there hidden details underlying the probabilities in quantum mechanics? Is there a hidden sample space and corresponding probability space?

At a phenomenological level, there are very interesting models for the emergence of probabilities within the usual wave function formulation of nonrelativistic quantum theory, pioneered by Ghirardi, Rimini, and Weber [2]-[4]. These models postulate that space is filled with a very low level noise with a coupling to matter proportional to the imaginary unit i, rather than with a real-valued coupling (more technically, they couple through an anti-Hermitian Hamiltonian term). For example, there could be a small, rapidly fluctuating contribution to the gravitational potential or g_{00} metric component proportional to the imaginary unit i. If such a theory obeys two general properties, (1) the total probability of a particle being present remains one for all times (that is, the wave function normalization is preserved), and (2) there is no faster than light signaling, then the extra terms in the Schr¨odinger equation equation must have a special structure. This special structure allows one to prove definite outcomes obeying the Born rule!

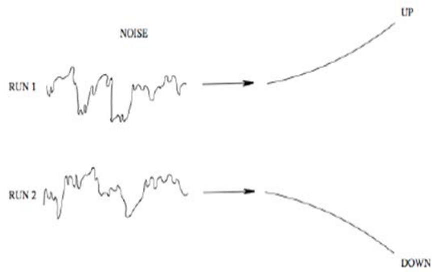


Fig.1.6.4.Different noise histories,in objective reduction models,can explain “up” and “down” registrations in the Stern-Gerlach experiment. Adapted from [27].

In these models, for each repetition of the Stern–Gerlach experiment, the noise variable takes different values. For a large apparatus, these have a measurable effect, whereas for an atom not interacting with an apparatus, the effect is not measurable. The noise leads to different outcomes for different runs, with probabilities given by the Born rule. The different noises for different runs of the experiment are analogous, in the coin toss example I gave earlier, to different details of the tumbling coin trajectories for the different coin tosses (Fig. 1.6.4).

In the de Broglie-Bohm interpretation: a particle has an initial position and follows a path whose velocity at each instant is given by an classical equation. On the basis of this assumption we conduct a simulation experiment by drawing random initial positions of the electrons in the initial wave acket ("quantum equilibrium hypothesis").

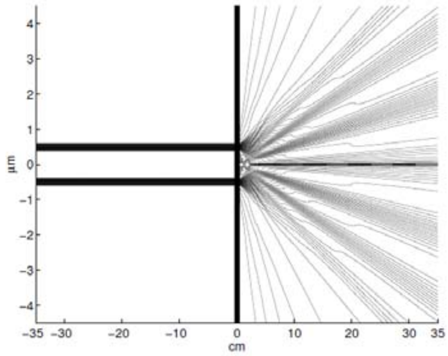


Fig.1.6.5. An sample space in Bohmian QM.
100 electron trajectories for the
Jönsson experiment. Adapted from [55].

Figure 1.6.5 shows, after its initial starting position,100 possible quantum trajectories of an electron passing through one of the two slits: We have not represented the paths of the electron when it is stopped by the first screen. Figure 1.6.6 shows a close-up of these trajectories just after they leave their slits.

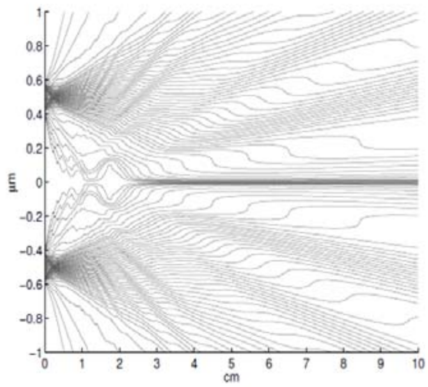


Fig.1.6.6.Close-up on the 100 trajectories of
the electrons just after the slits.
Adapted from [55].

Remark 1.6.5. The different trajectories explain both the impact of electrons on the detection screen and the interference fringes. This is the simplest and most natural interpretation to explain the impact positions: "The position of an impact is simply the position of the particle at the time of impact." This was the view defended by Einstein at the Solvay Congress of 1927. **The position is the only measured variable of the experiment**.

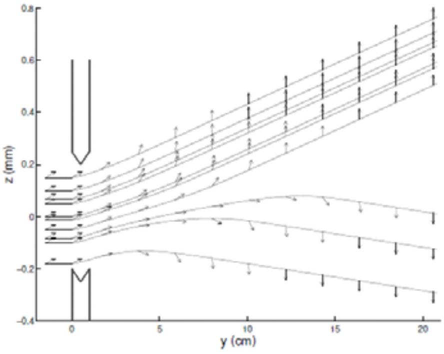


Fig.1.6.6.Ten silver atom trajectories within initial spin orientation $\theta_0 = \pi/3$ and initial position z_0; arrows represent the spin orientation $\theta(z,t)$ along the trajectories.Adapted from [55].

Figure 1.6.6. presents, for a silver atom with the initial spinor orientation ($\theta_0 = \pi 3, \phi_0 = 0$), a plot in the ($Oyz$) plane of a set of 10 trajectories whose initial position z_0 has been randomly chosen from a Gaussian distribution with standard deviation σ0. The spin orientations $\theta(z,t)$ are represented by arrows

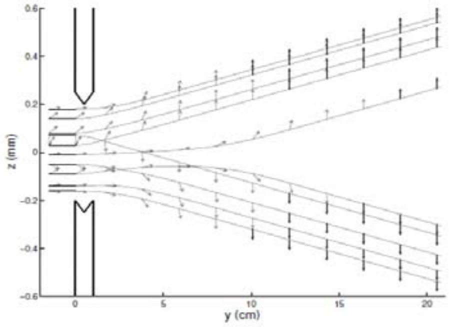


Fig.1.6.7.Ten silver atom trajectories where the initial orientation (θ_0, ϕ_0) has been randomly chosen;arrows represent the spin orientation $\theta(z,t)$ along the trajectories.Adapted from [55].

Now let us consider a mixture of pure states where the initial orientation (θ_0, ϕ_0) from the spinor has been randomly chosen. These are the conditions of the initial Stern and Gerlach experiment. Figure 1.6.7 represents a simulation of 10 quantum trajectories of silver atoms from which the initial positions z_0 are also randomly chosen.

Definition 1.6.2. A probability space consists of three parts:

1. A sample space (observation space) Ω, which is the set of all possible single outcomes

$\omega \in \Omega$.

2. A set of events Σ, where each event is a set containing $\varnothing$ or more outcomes.

3.The assignment of probabilities to the events; that is, a function $\mathbf{P}$ from events to

probabilities.

An outcome is the result of a single execution of the model. Since individual outcomes might be of little practical use, more complex events are used to characterize groups of outcomes. The collection of all such events is a σ-algebra Σ. Finally, there is a need to specify each event's likelihood of happening. This is done using the probability measure function, $\mathbf{P} : \mathbf{\Sigma} \to [0,1]$.

Remark 1.6.6. Note that:

(i) In conventional quantum mechanics we dealing with a probabilities without any

probability space $\Theta = \{\Omega, \Sigma, \mathbf{P}\}$.

(ii) However a wave function ψ in quantum mechanics is a description of the quantum state

$|\psi\rangle$ of a quantum system Ξ. The wave function is a complex-valued probability amplitude,

and the **probabilities for the possible results of measurements of an observable**

$Q = Q_\Xi$ (represented by oerator $\widehat{Q}$) made on the system Ξ in state $|\psi\rangle$ can be derived from

a wave function ψ.

(iii) From (ii) follows that there exist an probability space $\Theta_\Xi = \{\Omega_\Xi, \Sigma_\Xi, \mathbf{P}_\Xi\}$ and random

variable $Q_{\widehat{Q}|\psi\rangle} : \Omega_\Xi \to E_\Xi$, i.e. $X_{\widehat{Q}|\psi\rangle}$ is a measurable function from the set of possible

outcomes Ω_Ξ to some set E_Ξ.

Examle 1.6.1.We now, consider as an examle,the simple case of a non-relativistic single particle, without spin, in one spatial dimension.

Note that:

(i) The state of such a particle is completely described by its position-space wave function, $\psi(x)$ where x is position of a particle. This is a complex-valued function of real variable x. For one spinless particle in $1D$, if the wave function is interpreted as a probability amplitude, the square modulus of the wave function, the positive real number

$$|\psi(x)|^2 = \psi^*(x)\psi(x) = \rho(x)$$

is interpreted as the probability density that the particle is at x.

(ii) If the particle's position is measured, its location cannot be determined from the wave function, but is described by a probability distribution. The probability that its position x will be in the interval $a \leq x \leq b$ is the integral of the density over this interval:

$$P(a \leq x \leq b) = \int_a^b |\psi(x)|^2 dx.$$

This leads to the normalization condition

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$$

because if the particle is measured, there is 100% probability that it will be somewhere.

(iii) Assume that particle in state $|\psi\rangle$. From a statement (ii) follows that the coordinate x of

the particle wave function, $\psi(x)$ under measurement by an measuring device is a random

variable $x_\psi(\omega) \triangleq X_{\hat{x}|\psi\rangle}(\omega), X_{\hat{x}|\psi\rangle}\Omega_{|\psi\rangle} \to E_{|\psi\rangle}$ which well defined on an probability

space

$\Theta_{|\psi\rangle} = \{\Omega_{|\psi\rangle}, \Sigma_{|\psi\rangle}, \mathbf{P}\}$.

(iv) However in conventional quantum mechanics as mentioned above [see Remark

1.6.6(i)] such probability space $\Theta_{|\psi\rangle} = \{\Omega_{|\psi\rangle}, \Sigma_{|\psi\rangle}, \mathbf{P}\}$ is missing.

Remark 1.6.7. For a given system, the set of all possible normalizable wave functions (at any given time) forms an abstract mathematical vector space, meaning that it is possible to add together different wave functions, and multiply wave functions by complex numbers.

Note that:

(i) Technically, because of the normalization condition, wave functions form a **projective space** $\mathbf{H}_p$ rather than an ordinary infinite-dimensional vector space $\mathbf{H}$. Also $\mathbf{H}$ is a Hilbert space, because the inner product of two wave functions ψ_1 and ψ_2 can be defined as the complex number

$$(\psi_1, \psi_2) = \int_{-\infty}^{\infty} \psi_1^*(x)\psi_2(x) dx.$$

(ii) $\mathbf{H}_p = \mathbf{S}^\infty \subsetneqq \mathbf{H}$.

(iii) The all values of the wave function $\psi(x)$ are components of an vector $|\psi\rangle$. There are uncountably infinitely many of them and integration is used in place of summation. In Bra-ket notation, this vector is written

$$|\psi\rangle = \int_{-\infty}^{\infty} dx \psi(x)|x\rangle,$$

where $\langle x'|x\rangle = \delta(x' - x)$.

Let us consider QM system which consists of one particle with a wave function $\psi(x),\ x \in [a,b]$, such that $\mathrm{supp}(\psi(x)) \subseteq [a,b]$ and $\int_{\mathbb{R}} |\psi(x)|^2 dx = 1$. We go to construct now corresponding probability space $\Theta_{|\psi\rangle} = \{\Omega_{|\psi\rangle}, \Sigma_{|\psi\rangle}, \mathbf{P}\}$. In one dimension, the position x of a such particle can range over the values $a \leq x \leq b$.

Consider now measurement of coordinate of such QM particle. Obviously a sample space for such coordinate measurement is $\Omega_{|\psi\rangle} = \Omega_{a,b} = [a,b]$. Note that in practice observable x is measured to an accuracy δx determined by the measuring device. Thus $\forall x \forall \delta x_1 \forall \delta x_2 [(x - \delta x_1, x + \delta x_2) \subseteq [a,b] \to (x - \delta x_1, x + \delta x_2) \in \Sigma_{|\psi\rangle}]$ and therefore σ-algebra $\Sigma_{a,b} = B([a,b])$ is the Borel algebra on the set $[a,b]$. The probability measure function, $\mathbf{P} : \Sigma_{a,b} \to [0,1]$ we choose of the form

$$\mathbf{P}(A) = \int_A \sigma(x) d\mu(x), \tag{1.6.3}$$

where $A \in \Sigma_{a,b}$ and $d\mu(x)$ is the Lebesgue measure.

Definition 1.6.3.The probability measure $\mathbf{P}_{|\psi\rangle} : \Sigma_{a,b} \to [0,1]$ corresponding to a wave function $\psi(x), \|\psi(x)\|_2^2 = 1,$ we choose in the following form:

$$\mathbf{P}_{|\psi\rangle}(A) = \int_A |\psi(x)|^2 d\mu(x), \tag{1.6.4}$$

where $A \in \Sigma_{a,b}$ and $d\mu(x)$ is the Lebesgue measure.

Definition 1.6.4. A random variable $X_{|\psi\rangle} : \Omega_{|\psi\rangle} \to E_{|\psi\rangle}$ is a measurable function from the set of possible outcomes Ω to some set $E_{|\psi\rangle}$. The technical axiomatic definition requires $\Omega_{|\psi\rangle}$ to be a probability space and $E_{|\psi\rangle}$ to be a measurable space. Note that although $X_{|\psi\rangle}$ is usually a real-valued function $X_{|\psi\rangle} : \Omega_{|\psi\rangle} \to [a,b]$, it does not return a probability. The probabilities of different outcomes or sets of outcomes (events) in our case are already given by the probability measure $\mathbf{P}_{|\psi\rangle}$ with which $\Omega_{|\psi\rangle}$ is equipped above.

Definition 1.6.5.(Real-valued random variables) In a case mentioned above the observation space is a set $[a,b]$. Recall, $\{\Omega_{a,b}, \Sigma_{a,b}, \mathbf{P}\}$ is the probability space. For real observation space, the function $X_{|\psi\rangle} : \Omega_{a,b} \to [a,b]$ is a real-valued random variable, i.e. $\forall r[\{\omega : X_{|\psi\rangle}(\omega) \leq r\} \in \Sigma_{a,b}]$.

Assumption 1.6.1.We assume now that $\mathbf{P}_{|\psi\rangle} \ll \mathbf{P}$, i.e. $\mathbf{P}_{|\psi\rangle}$ is absolutely continuous with respect to $\mathbf{P}$. Then by Radon-Nicodym theorem we obtain for any $A \in \Sigma_{a,b}$:

$$\begin{aligned}
&\mathbf{P}_{|\psi\rangle}(A) = \int_{A^+} X^+_{|\psi\rangle}(\omega) d\mathbf{P} + \int_{A^-} X^-_{|\psi\rangle}(\omega) d\mathbf{P} = \mathbf{P}^+_{|\psi\rangle}(A) + \mathbf{P}^-_{|\psi\rangle}(A),\\
&\mathbf{P}^+_{|\psi\rangle}(A) = \int_{A^+} X^+_{|\psi\rangle}(\omega) d\mathbf{P}, \mathbf{P}^-_{|\psi\rangle}(A) = \int_{A^-} X^-_{|\psi\rangle}(\omega) d\mathbf{P},\\
&A^+ = A \cap [0,+\infty], A^+ = A \cap [-\infty, 0),\\
&X^+_{|\psi\rangle}(\omega) = \frac{d\mathbf{P}^+_{|\psi\rangle}}{d\mathbf{P}}, X^-_{|\psi\rangle}(\omega) = \frac{d\mathbf{P}^-_{|\psi\rangle}}{d\mathbf{P}}.
\end{aligned} \tag{1.6.5.a}$$

Using (1.6.5.a) we define random variable $X_{|\psi\rangle} : \Omega_{|\psi\rangle} \to [a,b]$ by formula

$$X_{|\psi\rangle} = X^+_{|\psi\rangle}(\omega) - X^-_{|\psi\rangle}(\omega). \tag{1.6.5.b}$$

Definition 1.6.6.The cumulative distribution function of a real-valued random variable

$X_{|\psi\rangle}(\omega)$ is the function given by

$$F_{X_{|\psi\rangle}}(x) = \mathbf{P}(\omega \in \Omega_{a,b} \big| X_{|\psi\rangle}(\omega) \leq x), \qquad (1.6.6)$$

where the right-hand side represents the probability that the random variable $X_{|\psi\rangle}(\omega)$ takes on a value less than or equal to x. The probability that $X_{|\psi\rangle}(\omega)$ lies in the semi-closed interval $(a_1, b_1] \subset [a,b]$, where $a_1 < b_1$, is therefore $\mathbf{P}_{|\psi\rangle}(a_1 < X_{|\psi\rangle} \leq b_1) = F_{X_{|\psi\rangle}}(b_1) - F_{X_{|\psi\rangle}}(a_1)$.

We remind that:

(i) The CDF of any continuous random variable $X_{|\psi\rangle} = X_{|\psi\rangle}(\omega)$ can be expressed as the
integral of its probability density function $p_{X_{|\psi\rangle}}(x)$ as follows:

$$F_{X_{|\psi\rangle}}(x) = \int_{-\infty}^{x} p_{X_{|\psi\rangle}}(t)dt = \int_{-\infty}^{x} |\psi(t)|^2 dt. \qquad (1.6.7.a)$$

(ii) In the case of any random variable $X_{|\psi\rangle}$ which has distribution having a discrete
component at a value b,

$$\mathbf{P}(\{X_{|\psi\rangle} = b\}) = F_{X_{|\psi\rangle}}(b) - \lim_{x \to b_-} F_{X_{|\psi\rangle}}(x). \qquad (1.6.7.b)$$

(iii) Every cumulative distribution function $F_{X_{|\psi\rangle}}(x)$ is non-decreasing and right-continuous,
which makes it a càdlàg function.

(iv)

$$\lim_{x \to -\infty} F_{X_{|\psi\rangle}}(x) = 0, \lim_{x \to +\infty} F_{X_{|\psi\rangle}}(x) = 1. \qquad (1.6.7.c)$$

The next result from probability theory it well known.

Theorem 1.6.1. (i) Every function $F_{X_{|\psi\rangle}}(x)$ with these properties (i)-(iv) is a CDF, i.e., for every such function, a random variable can be defined such that the function is the cumulative distribution function of that random variable.

(ii) If $X_{|\psi\rangle}$ is a purely discrete random variable, then it attains values $x_1, x_2, \ldots, x_i, \ldots$ with probability $p_i = \mathbf{P}(\{X_{|\psi\rangle} = x_i\})$, and the CDF of $\mathbf{X}^{\mathbf{B}}$ will be discontinuous at the points x_i and constant in between

$$F_{X_{|\psi\rangle}}(x) = \mathbf{P}(X_{|\psi\rangle} \leq x) = \sum_{x_i \leq x} \mathbf{P}(X_{|\psi\rangle} = x_i) = \sum_{x_i \leq x} p_{X_{|\psi\rangle}}(x_i) \qquad (1.6.7.d)$$

(iii) If the CDF $F_{X_{|\psi\rangle}}(x)$ of a real valued random variable $X_{|\psi\rangle}$ is continuous, then $X_{|\psi\rangle}$ is a continuous random variable.

(iv) If furthermore $F_{X_{|\psi\rangle}}(x)$ is absolutely continuous, then there exists a Lebesgue-integrable function $f(x)$ such that

$$F_{X_{|\psi\rangle}}(x) = \int_{-\infty}^{x} f(t)dt. \qquad (1.6.7.e)$$

In that caseы when $\neg(\mathbf{P}_{|\psi\rangle} \ll \mathbf{P})$ we define random variable $X_{|\psi\rangle} = X_{|\psi\rangle}(\omega)$ by using Theorem 1.6.1. Thus from Eq.(1.6.7.a) we obtain

$$\mathbf{E}_{\Omega_{a,b}}[X_{|\psi\rangle}(\omega)] = \int\limits_{\Omega_{a,b}} X_{|\psi\rangle}(\omega)d\mathbf{P} = \int\limits_{-\infty}^{+\infty} xp_{X_{|\psi\rangle}}(x)dx. \quad (1.6.8)$$

Using canonical QM-abbreviations (see subsection I.7.2)

$$|\psi\rangle = \int_{-\infty}^{+\infty}|x\rangle\langle x|\psi\rangle dx, \quad (1.6.9)$$

where $\langle x|\psi\rangle = \psi(x), |\psi\rangle \in \mathbf{S}^{\infty} \subsetneqq \mathbf{H}$, from Eq.(1.6.8)-Eq.(1.6.9) we obtain

$$\langle\psi|\hat{x}|\psi\rangle = \int\limits_{\Omega_{a,b}} X_{|\psi\rangle}(\omega)d\mathbf{P} = \int\limits_{-\infty}^{+\infty} xp_{X_{|\psi\rangle}}(x)dx. \quad (1.6.10)$$

Assumption 1.6.2. We assume now that:
(i) for any $\psi(x) \in \mathbf{H}$: (a) $\int\limits_{\Omega_{a,b}} |X_{|\psi\rangle}|(\omega)d\mathbf{P} < \infty$, (b) $\int\limits_{\Omega_{a,b}} X^2_{|\psi\rangle}(\omega)d\mathbf{P} < \infty$,
(ii) for any $\psi(x) \in \mathbf{H}$: $X_{|\psi\rangle} \in \mathcal{L}_{1,2}(d\mathbf{P}) = \mathcal{L}_1(d\mathbf{P}) \cap \mathcal{L}_2(d\mathbf{P})$.

Definition 1.6.7. We will write the Eq.(1.6.10) in the following form

$$\langle\psi|\hat{x}|\psi\rangle = \int\limits_{\Omega_{a,b}} X_{\hat{x}|\psi\rangle}(\omega)d\mathbf{P} = \int\limits_{-\infty}^{+\infty} xp_{X_{\hat{x}|\psi\rangle}}(x)dx, \quad (1.6.11)$$

where $\hat{x}|\psi\rangle = x|\psi\rangle$. This form remind that continuous random variable $X_{|\psi\rangle} = X_{|\psi\rangle}(\omega)$ corresponds to measurement of the coordinate of an particle with a state vector $|\psi\rangle$.

Remark 1.6.8. If $X_{|\psi\rangle} \in \mathcal{L}_{1,2}(d\mathbf{P})$ then for any $\eta \in \mathbb{R}_+$: $\eta X_{|\psi\rangle} \in \mathcal{L}_{1,2}(d\mathbf{P})$. From RHS of the Eq.(1.6.10) by change of random variable: $Y(\omega) = \eta X_{|\psi\rangle}(\omega)$ we obtain

$$\int\limits_{\Omega_{a,b}} \eta X_{|\psi\rangle}(\omega)d\mathbf{P} = \int\limits_{\Omega_{a,b}} Y(\omega)(\omega)d\mathbf{P} = \int\limits_{-\infty}^{+\infty} y\eta^{-1}p_{X_{|\psi\rangle}}(\eta^{-1}y)dy. \quad (1.6.12)$$

From LHS of the Eq.(1.6.10) and Eq.(1.6.12) for any $\xi \in \mathbb{C}\backslash\{0\}$, we obtain

$$\langle\xi\psi|\hat{x}|\xi\psi\rangle = \int\limits_{\Omega_{a,b}} |\xi|^2 X_{|\psi\rangle}(\omega)d\mathbf{P} = \int\limits_{\Omega_{a,b}} \eta X_{|\psi\rangle}(\omega)d\mathbf{P} = \int\limits_{-\infty}^{+\infty} y\eta^{-1}p_{X_{|\psi\rangle}}(\eta^{-1}y)dy, \quad (1.6.13)$$

where $\eta = |\xi|^2$.

Remark 1.6.9. Formula (1.6.13) allow us to change conventional projective Hilbert space $\mathbf{H}_p = \mathbf{S}^{\infty} \subsetneqq \mathbf{H}$ (see Remark 1.6.7) by full nonprojective Hilbert space $\mathbf{H}$.

Definition 1.6.8. The probability measure $\mathbf{P}_{\xi|\psi\rangle}:\Sigma_{a,b}\to[0,1]$ corresponding to a wave function $\xi\psi(x)$, where $|\xi|^2 \neq 1$, $\|\psi(x)\|_2^2 = 1$, we choose in the following form:

$$\mathbf{P}_{\xi|\psi\rangle}(A) = \int\limits_{A} \eta^{-1}|\psi(\eta^{-1}y)|^2 d\mu(y), \quad (1.6.14)$$

where $\eta = |\xi|^2, A \in \Sigma_{a,b}$ and $d\mu(x)$ is the Lebesgue measure.

Definition 1.6.9. (Real-valued random variables) In this case the observation space is a set $[a,b]$. Recall, $\{\Omega_{a,b},\Sigma_{a,b},\mathbf{P}\}$ is the probability space. For real observation space, the function $X_{\xi|\psi\rangle} : \Omega_{a,b} \to [a',b']$ is a real-valued random variable if $\forall r[\{\omega : X_{\xi|\psi\rangle}(\omega) \leq r\} \in \Sigma_{a,b}]$.

By Radon-Nicodym theorem we obtain for $A^{\pm} \in \Sigma_{a,b}$:

$$\mathbf{P}^{\pm}_{\xi|\psi\rangle}(A) = \int_{A^{\pm}} X^{\pm}_{\xi|\psi\rangle}(\omega)d\mathbf{P},$$
$$X^{\pm}_{\xi|\psi\rangle}(\omega) = \frac{d\mathbf{P}^{\pm}_{\xi|\psi\rangle}}{d\mathbf{P}}. \tag{1.6.15}$$

The CDF of a continuous random variable $X_{\xi|\psi\rangle} = X_{\xi|\psi\rangle}(\omega)$ can be expressed as the integral of its probability density function $p_{X_{\xi|\psi\rangle}}(y)$ as follows:

$$F_{X_{\xi|\psi\rangle}}(y) = \int_{-\infty}^{y} p_{X_{\xi|\psi\rangle}}(t)dt. \tag{1.6.16}$$

From Eq.(1.6.14)-Eq.(1.6.16) we obtain

$$\langle\xi\psi|\widehat{x}|\xi\psi\rangle = \mathbf{E}_{\Omega_{a,b}}[X_{\xi|\psi\rangle}(\omega)] = \int_{\Omega_{a,b}} X_{\xi|\psi\rangle}(\omega)d\mathbf{P} = \int_{-\infty}^{+\infty} yp_{X_{\xi|\psi\rangle}}(y)dy =$$
$$\int_{-\infty}^{+\infty} \eta^{-1}|\psi(\eta^{-1}y)|^2 yd\mu(y) = \int_{-\infty}^{+\infty} y\eta^{-1}p_{X_{|\psi\rangle}}(\eta^{-1}y)dy = \int_{\Omega_{a,b}} \eta^{-1}X_{|\psi\rangle}(\omega)d\mathbf{P} = \int_{\Omega_{a,b}} |\xi|^2 X_{|\psi\rangle}(\omega)d\mathbf{P}. \tag{1.6.17}$$

Definition 1.6.10. We will write the Eq.(1.6.17) in the following symbolical form

$$\langle\xi\psi|\widehat{x}|\xi\psi\rangle = \int_{\Omega_{a,b}} X_{\widehat{x}|\xi\psi\rangle}(\omega)d\mathbf{P} = \int_{-\infty}^{+\infty} xp_{X_{\widehat{x}|\xi\psi\rangle}}(x)dx =$$
$$\int_{\Omega_{a,b}} X_{\widehat{x}|\psi^{\xi}\rangle}(\omega)d\mathbf{P} = \int_{-\infty}^{+\infty} xp_{X_{\widehat{x}|\psi^{\xi}\rangle}}(x)dx. \tag{1.6.18}$$

This symbolical form remind that continuous random variable $X_{\xi\psi} = X_{\xi\psi}(\omega) \triangleq X_{\psi^{\xi}}(\omega)$ corresponds to measurement of the coordinate af an particle with a wave function $\psi(x)$.

The reconciled Bohr rule.

We choose now an constants $\xi_1,\xi_2,|\xi_1| \neq 1,|\xi_2| \neq 1$, and an wave functions $\psi_1(x)$ and $\psi_2(x)$ such that:

(i) $\|\psi_1(x)\|_2^2 = 1, \|\psi_2(x)\|_2^2 = 1$,

(ii) supp$(\psi_1(x)) \subset [a_1,b_1]$,supp$(\psi_2(x)) \subset [a_2,b_2]$,

(iii) $[a_1,b_1] \cap [a_2,b_2] = \varnothing$

(iv) we assum that: if $\psi_{1,2}(x_1,x_2)$ is a two particles wave function,then $\psi_{1,2}(x_1,x_2) = \psi_1(x_1)\psi_2(x_2)$.

Therefore we obtain

$$\langle\xi_1\psi_1 + \xi_2\psi_2|\widehat{x}|\xi_1\psi_1 + \xi_2\psi_2\rangle = \langle\xi_1\psi_1|\widehat{x}|\xi_1\psi_1\rangle + \langle\xi_2\psi_2|\widehat{x}|\xi_2\psi_2\rangle. \tag{1.6.17}$$

Substituting Eq.(1.6.16) into Eq.(1.6.17) gives

$$\begin{gathered}\langle\xi_1\psi_1+\xi_2\psi_2|\widehat{x}|\xi_1\psi_1+\xi_2\psi_2\rangle=\langle\xi_1\psi_1|\widehat{x}|\xi_1\psi_1\rangle+\langle\xi_2\psi_2|\widehat{x}|\xi_2\psi_2\rangle=\\ \int\limits_{\Omega_{a_1,b_1}}X_{\xi_1|\psi_1\rangle}(\omega)d\mathbf{P}+\int\limits_{\Omega_{a_2,b_2}}X_{\xi_2|\psi_2\rangle}(\omega)d\mathbf{P}=\\ \int\limits_{\Omega_{a_1,b_1}\bigcup\Omega_{a_2,b_2}}[X_{\xi_1|\psi_1\rangle}(\omega)+X_{\xi_2|\psi_2\rangle}(\omega)]d\mathbf{P}=\\ \int\limits_{-\infty}^{+\infty}x\left[p_{X_{\widehat{x}|\xi_1\psi_1\rangle}}(x)+p_{X_{\widehat{x}|\xi_2\psi_1\rangle}}(x)\right]dx=\int\limits_{-\infty}^{+\infty}x\left[p_{X_{\widehat{x}|\xi_1\psi_1\rangle}}(x)*p_{X_{\widehat{x}|\xi_2\psi_1\rangle}}(x)\right]dx=\\ \int\limits_{-\infty}^{+\infty}y\eta_1^{-1}\eta_2^{-1}\left[|\psi_1(\eta_1^{-1}y)|^2*|\psi_2(\eta_2^{-1}y)|^2\right]dy,\end{gathered}\tag{1.6.18}$$

where $\eta_1=|\xi_1|^2,\eta_2=|\xi_2|^2$. From Eq.(1.6.18) follows that if particle in a state $|\xi_1\psi_1+\xi_2\psi_2\rangle$ then the probability density $p(x)$ that the particle is at x is given by formula

$$p(x)=\eta_1^{-1}\eta_2^{-1}\left(|\psi_1(\eta_1^{-1}x)|^2*|\psi_2(\eta_2^{-1}x)|^2\right)\tag{1.6.19}$$

but not by conventional formula

$$p(x)=|\xi_1|^2|\psi_1|^2+|\xi_2|^2|\psi_2|^2\tag{1.6.20}$$

even if $|\xi_1|^2+|\xi_2|^2=1$.

Remark 1.6.10. Obviously formula (1.6.19) forced us to changed not only conventional

Born's rule (1.6.2) but even conventional interpretation of the wave function of the QM

particle must be changed if particle consist of superposition such that mentioned above.

Remark 1.6.11. An canonical quantum mechanical formalism based on the probability representation of states was also proposed S.Mancini and V.I.Man'ko in [28]-[37]. Note that the distribution $w(X,\mu,\nu)$ of the observable $\widehat{X}=\mu\widehat{q}+\nu\widehat{p}$ one can

expressed by the relation [32]:

$$w(X,\mu,\nu)=(2\pi)^{-2}\int dkdqdpW(q,p)\exp[-ik(X-\mu q-\nu p)],\tag{1.6.21}$$

where $W(q,p)$ that is a Wigner function, operators $\widehat{q}$ and $\widehat{p}$ that is operator of coordinate and momentum operator correspondingly.Note that [32]:

$$W(q,p)=(2\pi)^{-2}\int d\mu d\nu dXw(X,\mu,\nu)\exp[-ik(\mu q+\nu p-X)].\tag{1.6.22}$$

Because the Wigner function completely determains the quantum system and the distribution $w(X,\mu,\nu)$ completely determains the Wigner function $W(q,p)$ the classical disribution of componets completely determains the quantum system. In fact we can say that the quantum system states is known if the distribution

$w(X,\mu,\nu)$ is known.

Remark 1.6.12. An general quantum mechanical formalism based on the probability representation of states was also proposed by A.Yu.Khrennikov [37].In [37] the main structures of quantum theory (interference of probabilities, Born's rule, complex probabilistic amplitudes, Hilbert state space, representation of observables by operators) are present in a latent form in the Kolmogorov model. In particular, "interference of probabilities" is obtained without appealing to the Hilbert space formalism.

However in this paper we dealing by using axiomatical approach, see section I.7 bellow.We have to combined canonical QM approach based on a Hilbert state space, representation of observables by operators, etc. and Kolmogorov probabilistic model which completed canonical QM postulates.

I.7. A new quantum mechanical formalism based on the probability representation of quantum states.

I.7.1.Generalized Postulates for Continuous Valued Observables.

Suppose we have an n-dimensional physical quantum system.

I.Then we claim the following:

Q.I.1. Any given n-dimensional quantum system is identified by a set $\mathbf{Q}$:

$$\mathbf{Q} \triangleq \langle \mathbf{H}, \Im, \Re, \mathcal{L}_{2,1}, \Im^*(\mathbf{H}), \mathbf{G}, |\psi_t\rangle\rangle$$

where:

(i) $\mathbf{H}$ that is some infinite-dimensional complex Hilbert space,

(ii) $\Im = (\Omega, \mathcal{F}, \mathbf{P})$ that is complete probability space,

(iii) $\Re = (\mathbb{R}^n, \Sigma)$ that is measurable space,

(iv) $\mathcal{L}_{2,1}(\Omega)$ that is complete space of random variables $X : \Omega \to \mathbb{R}^n$ such that

$$\int_\Omega \|X(\omega)\| d\mathbf{P} < \infty, \int_\Omega \|X(\omega)\|^2 d\mathbf{P} < \infty \tag{1.7.1}$$

(v) $\mathbf{G} : \mathbf{C}^*(\mathbf{H}) \times \mathbf{H} \to \mathcal{L}_{2,1}(\Omega)$ that is one to one correspondence such that

$$\left|\langle\psi|\widehat{Q}|\psi\rangle\right| = \int_\Omega \left(\mathbf{G}\left[\widehat{Q}, |\psi\rangle\right](\omega)\right) d\mathbf{P} = \mathbf{E}_\Omega\left(\mathbf{G}\left[\widehat{Q}|\psi\rangle\right](\omega)\right),$$
$$\mathbf{G}\left[\widehat{\mathbf{1}}, |\psi\rangle\right](\omega) = 1 \tag{1.7.2}$$

for any $|\psi\rangle \in \mathbf{H}$ and for any Hermitian adjoint operator $\widehat{Q} : \mathbf{H} \to \mathbf{H}, \widehat{Q} \in C^*(\mathbf{H})$,where $C^*(\mathbf{H})$ is C^*- algebra of the Hermitian adjoint operators in $\mathbf{H}$ and $\Im^*(\mathbf{H})$ an commutative subalgebra of $C^*(\mathbf{H})$.

(vi) $|\psi_t\rangle$ is an continuous vector function $|\psi_t\rangle : \mathbb{R}_+ \to \mathbf{H}$ which representedthe evolution of the quantum system $\mathbf{Q}$.

Q.I.2. For any $|\psi_1\rangle, |\psi_2\rangle \in \mathbf{H}$ and for any Hermitian operator $\widehat{Q} : \mathbf{H} \to \mathbf{H}$ such that

$$\left\langle \psi_1 \middle| \widehat{Q} \middle| \psi_2 \right\rangle = \left\langle \psi_2 \middle| \widehat{Q} \middle| \psi_1 \right\rangle = 0 \tag{1.7.3}$$

valid the equality

$$\mathbf{G}\left[\widehat{Q}(|\psi_1\rangle + |\psi_2\rangle) \right](\omega) = \mathbf{G}\left[\widehat{Q}|\psi_1\rangle \right](\omega) + \mathbf{G}\left[\widehat{Q}|\psi_2\rangle \right](\omega). \tag{1.7.4}$$

Definition 1.7.1. A random variable $X : \Omega \to E$ is a measurable function from the set of
possible outcomes Ω to some set E.

Definition 1.7.2. Given a probability space $\Im = (\Omega, \mathcal{F}, \mathbf{P})$ and a measurable space $\Re = (\mathbb{R}^n, \Sigma)$, any $\mathbb{R}^n$-valued stochastic process that is a collection of $\mathbb{R}^n$-valued random variables on Ω, indexed by a totally ordered set T ("time"). That is, a stochastic process $X_t(\omega)$ is a collection $\{X_t(\omega)|t \in T\}$, where each $X_t(\omega)$ is an $\mathbb{R}^n$-valued random variable on Ω. The space $\mathbb{R}^n$ is then called the state space of the process.

Q.I.3. Suppose that the evolution of the quantum system is represented by continuous vector function $|\psi_t\rangle : \mathbb{R}_+ \to \mathbf{H}$. Then any process of continuous measurements on measuring observable $\widehat{Q}$ for the system in state $|\psi_t\rangle$ one can to describe by an continuous $\mathbb{R}^n$-valued stochastic process

$$X_t(\omega) = X_t\left(\omega; \left|\widehat{Q}\psi_t\right\rangle\right) \triangleq X_{\left|\widehat{Q}\psi_t\right\rangle}(\omega)$$

given on probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a measurable space $(\mathbb{R}^n, \Sigma)$.

Remark 1.7.1.We assume now for short but without loss of generality that $n = 1$.

Remark 1.7.2. Let $X(\omega)$ be random variable $X(\omega) \in \mathcal{L}_{2,1}(\Omega)$ such that $X(\omega) = \mathbf{G}[|\psi\rangle](\omega)$, then we denote such random variable by $X_{|\psi\rangle}(\omega)$. The probability density of random variable $X_{|\psi\rangle}(\omega)$ we denote by $p_{|\psi\rangle}(q), q \in \mathbb{R}$.

Definition 1.7.3. The *classical pure states* correspond to vectors $\mathbf{v} \in \mathbf{H}$ of norm $\|\mathbf{v}\| \equiv 1$. Thus the set of all classical pure states corresponds to the unit sphere $\mathbf{S}^\infty \subset \mathbf{H}$ in a Hilbert space $\mathbf{H}$.

Definition 1.7.4. The projective Hilbert space $P(\mathbf{H})$ of a complex Hilbert space $\mathbf{H}$ is the set of equivalence classes $[\mathbf{v}]$ of vectors $\mathbf{v}$ in $\mathbf{H}$, with $\mathbf{v} \neq \mathbf{0}$, for the equivalence relation given by $\mathbf{v} \sim_P \mathbf{w} \Leftrightarrow \mathbf{v} = \lambda \mathbf{w}$ for some non-zero complex number $\lambda \in \mathbb{C}$. The equivalence classes for the relation $\sim_P$ are also called rays or projective rays.

Remark 1.7.3.The physical significance of the projective Hilbert space $P(\mathbf{H})$ is that in canonical quantum theory, the states $|\psi\rangle$ and $\lambda|\psi\rangle$ represent the same physical state of the quantum system, for any $\lambda \neq 0$. It is conventional to choose a state $|\psi\rangle$ from the ray $[|\psi\rangle]$ so that it has unit norm $\sqrt{\langle\psi|\psi\rangle} = 1$.

Remark 1.7.4. In contrast with canonical quantum theory we have used instead contrary to $\sim_P$ equivalence relation $\sim_Q$, a Hilbert space $\mathbf{H}$, see Definition 1.7.7.

Definition 1.7.5.The *non-classical pure states* correspond to the vectors $\mathbf{v} \in \mathbf{H}$ of a norm $\|\mathbf{v}\| \neq 1$. Thus the set of all non-classical pure states corresponds to the set $\mathbf{H} \backslash \mathbf{S}^\infty \subset \mathbf{H}$ in the Hilbert space $\mathbf{H}$.

Suppose we have an observable Q of a quantum system that is found through an exhaustive series of measurements, to have a set $\Im$ of values $q \in \Im$ such that $\Im = \bigcup_{i=1}^{m} (\theta_1^i, \theta_2^i), m \geq 2,\ (\theta_1^i, \theta_2^i) \cap \left(\theta_1^j, \theta_2^j\right) = \varnothing, i \neq j.$ Note that in practice any observable Q is measured to an accuracy δq determined by the measuring device. We represent now by $|q\rangle$ the idealized state of the system in the limit $\delta q \to 0,$ for which the observable definitely has the value q.

II.Then we claim the following:

Q.II.1.The states $\{|q\rangle : q \in \Im\}$ form a complete set of δ-function normalized basis states

for the state space $\mathbf{H}_\Im$ of the system.That the states $\{|q\rangle : q \in \Im\}$ form a complete set of

basis states means that any state $|\psi[\Im]\rangle \in \mathbf{H}_\Im$ of the system can be expressed as:

$$|\psi[\Im]\rangle = \int_\Im c_{\psi[\Im]}(q) dq, \tag{1.7.5}$$

where supp$(c_{\psi[\Im]}(q)) \subseteq \Im$ and while δ-function normalized means that $\langle q|q'\rangle = \delta(q-q')$

from which follows $c_{\psi[\Im]}(q) = \langle q|\psi[\Im]\rangle$ so that

$$|\psi[\Im]\rangle = \int_\Im |q\rangle\langle q|\psi[\Im]\rangle dq. \tag{1.7.6}$$

The completeness condition can then be written as

$$\int_\Im |q\rangle\langle q| dq = \widehat{\mathbf{1}}_{\mathbf{H}_\Im}. \tag{1.7.7}$$

Q.II.2.For the system in state $|\psi[\Im]\rangle$ the probability $P(q, q+dq; |\psi[\Im]\rangle)$ of obtaining the result

$q \in \Im$ lying in the range $(q, q+dq) \subset \Im$ on measuring observable Q is given by

$$P(q, q+dq; |\psi[\Im]\rangle) = p_{|\psi[\Im]\rangle}(q) dq \tag{1.7.8}$$

for any $|\psi[\Im]\rangle \in \mathbf{H}_\Im$.

Remark 1.7.5. Note that in general case $p_{|\psi[\Im]\rangle}(q) \neq |c_{\psi[\Im]}(q)|^2$.

Q.II.3.The observable $Q_\Im$ is represented by a Hermitian operator $\widehat{Q}_\Im : \mathbf{H}_\Im \to \mathbf{H}_\Im$ whose eigenvalues are the possible results $\{q : q \in \Im\}$, of a measurement of $Q_\Im$, and the associated eigenstates are the states $\{|q\rangle : q \in \Im\}$, i.e. $\widehat{Q}_\Im|q\rangle = q|q\rangle, q \in \Im$.

Remark 1.7.6. Note that the spectral decomposition of the operator $\widehat{Q}_\Im$ is then

$$\widehat{Q}_\Im = \int_\Im q|q\rangle\langle q| dq. \tag{1.7.9}$$

Definition 1.7.6. A connected set in $\mathbb{R}$ is a set $X \subset \mathbb{R}$ that cannot be partitioned into two nonempty subsets which are open in the relative topology induced on the set. Equivalently, it is a set which cannot be partitioned into two nonempty subsets such that each subset has no points in common with the set closure of the other.

Definition 1.7.7. The *well localized pure states* $|\psi[\Theta]\rangle$ *with a support* $\Theta = (\theta_1, \theta_2)$ correspond to vectors of the norm 1 and such that: supp$(c_{\psi[\Theta]}(q)) = \Theta$ is a connected set in $\mathbb{R}$ Thus the set of all well localized pure states corresponds to the unit sphere $\mathbf{S}_\Theta^\infty \subsetneqq \mathbf{S}^\infty \subset \mathbf{H}$ in the Hilbert space $\mathbf{H}_\Theta \subsetneqq \mathbf{H}$.

Suppose we have an observable Q_Θ of a system that is found through an exhaustive series of measurements, to have a continuous range of values q : $\theta_1 < q < \theta_2$.

III.Then we claim the following:

Q.III.1.For the system in well localized pure statestate $|\psi[\Theta]\rangle$ such that:

(i) $|\psi[\Theta]\rangle \in \mathbf{S}_\Theta^\infty$ and

(ii) $\mathrm{supp}(c_{\psi[\Theta]}(q)) \triangleq \{q|c_{\psi[\Theta]}(q) \neq 0\}$ is a connected set in $\mathbb{R}$, then the probability $P(q,q+dq;|\psi[\Theta]\rangle)$ of obtaining the result q lying in the range $(q,q+dq)$ on measuring

observable Q_Θ is given by

$$P(q,q+dq;|\psi[\Theta]\rangle) = |\langle q|\psi[\Theta]\rangle|^2 dq = |c_{\psi[\Theta]}(q)|^2 dq. \tag{1.7.10}$$

Q.III.2. $p_{|\psi[\Theta]\rangle}(q)dq = |\langle q|\psi[\Theta]\rangle|^2 dq = |c_{\psi[\Theta]}(q)|^2 dq.$

Q.III.3.Let $|\psi[\Theta_1]\rangle$ and $|\psi[\Theta_2]\rangle$ be well localized pure states with $\Theta_1 = (\theta_1^1,\theta_2^1)$ and

$\Theta_2 = (\theta_1^2,\theta_2^2)$ correspondingly. Let $X_1(\omega) = X_{|\psi[\Theta_1]\rangle}(\omega)$ and $X_2(\omega) = X_{|\psi[\Theta_2]\rangle}(\omega)$

correspondingly. Assume that $\overline{\Theta}_1 \cap \overline{\Theta}_2 = \varnothing$ (here the closure of $\Theta_i, i = 1,2$ is denoted by

$\overline{\Theta}_i, i = 1,2$) then random variables $X_1(\omega)$ and $X_2(\omega)$ are independent.

Q.III.4. If the system is in well localized pure state $|\psi[\Theta]\rangle$ the state $|\psi[\Theta]\rangle$ described by a

wave function $\psi(q,\Theta) = \langle q||\psi[\Theta]\rangle$ and the value of observable Q_Θ is measured once each

on many identically prepared system, the average value of all the measurements will be

$$\langle Q_\Theta\rangle = \frac{\int_\Theta q|\psi(q,\Theta)|^2 dq}{\int_\Theta |\psi(q,\Theta)|^2 dq}. \tag{1.7.11}$$

The completeness condition can then be written as $\int_\Theta |q\rangle\langle q|dq = \widehat{\mathbf{1}}_{\mathbf{H}_\Theta}$.
Completeness means that for any state $|\psi[\Theta]\rangle \in \mathbf{S}_\Theta^\infty$ it must be the case that $\int_\Theta |\langle q|\psi[\Theta]\rangle|^2 dq \neq 0$, i.e. there must be a non-zero probability to get some result on measuring observable Q_Θ.

Q.III.5.(**von Neumann measurement postulate**) Assume that

(i) $|\psi\rangle \in \mathbf{S}_\Theta^\infty$ and (ii) $\mathrm{supp}(c_\psi(q)) = \Theta$ is a connected set in $\mathbb{R}$. Then if on performing a measurement of Q_Θ with an accuracy δq, the result is obtained in the range $(q - \frac{1}{2}\delta q, q + \frac{1}{2}\delta q)$, then the system will end up in the state

$$\frac{\widehat{P}(q,\delta q)|\psi[\Theta]\rangle}{\sqrt{\langle\psi|\widehat{P}(q,\delta q)|\psi[\Theta]\rangle}} = \frac{\int_{|q-q'|\leq\delta q/2}|q'\rangle\langle q'||\psi[\Theta]\rangle dq'}{\sqrt{\int_{|q-q'|\leq\delta q/2}|\langle q'||\psi[\Theta]\rangle|^2 dq'}}. \tag{1.7.12}$$

IV.**We claim the following**:

Q.**IV**.**1** For the system in state $|\psi^a[\Theta]\rangle = a|\psi[\Theta]\rangle \in \mathbf{H}_\Theta$, where: (i) $|\psi[\Theta]\rangle \in \mathbf{S}_\Theta^\infty, |a| \neq 1$,

(ii) supp$(c_{\psi[\Theta]}(q))$ is a connected set in $\mathbb{R}$ and (iii) $|\psi[\Theta]\rangle = \int_{\theta_1}^{\theta_2} c_{\psi[\Theta]}(q)|q\rangle dq$

$$\mathbf{G}\left[\widehat{Q}_\Theta|\psi^a[\Theta]\rangle\right] = |a|^2\mathbf{G}\left[\widehat{Q}_\Theta|\psi[\Theta]\rangle\right]. \tag{1.7.13}$$

Q.**IV**.**2**. Assume that the system in state $|\psi^a[\Theta]\rangle = a|\psi[\Theta]\rangle \in \mathbf{H}_\Theta$, where (i) $|\psi[\Theta]\rangle \in \mathbf{S}_\Theta^\infty$,

$|a| \neq 1$, (ii) supp$(c_{\psi[\Theta]}(q))$ is a connected set in $\mathbb{R}$ and (iii) $|\psi[\Theta]\rangle = \int_{\theta_1}^{\theta_2} c_{\psi[\Theta]}(q)|q\rangle dq$.

Then if the system is in state $|\psi^a[\Theta]\rangle$ described by a wave function $\psi^a(q;\Theta) = \langle q||\psi^a[\Theta]\rangle$ and the value of observable Q_Θ is measured once each on many identically prepared system, the average value of all the measurements will be

$$\langle Q_\Theta\rangle = \int_\Theta q|\psi^a(q;\Theta)|^2 dq. \tag{1.7.14}$$

Q.**IV**.**3**. The probability $P(q,q+dq;|\psi^a[\Theta]\rangle)dq$ of obtaining the result q lying in the range

$(q,q+dq)$ on measuring Q_Θ is

$$P(q,q+dq;|\psi^a[\Theta]\rangle)dq = |a|^{-2}|c_{\psi[\Theta]}(q|a|^{-2})|^2 dq. \tag{1.7.15}$$

Remark 1.**7**.**7**.Note that Q.IV.3 immediately folows from Q.IV.1 and Q.III.2.

Q.**IV**.**4**. (**Generalized von Neumann measurement postulate**) If on performing a measurement of observable Q_Θ with an accuracy δq, the result is obtained in the range $(q-\frac{1}{2}\delta q, q+\frac{1}{2}\delta q)$, then the system immediately after measurement will end up in the state

$$\frac{\widehat{P}(q,\delta q)|\psi^a[\Theta]\rangle}{\sqrt{\langle\psi|\widehat{P}(q,\delta q)|\psi[\Theta]\rangle}} = \frac{\int_{|q-q'|\leq\delta q/2}|q'\rangle\langle q'||\psi^a[\Theta]\rangle dq'}{\sqrt{\int_{|q-q'|\leq\delta q/2}|\langle q'||\psi[\Theta]\rangle|^2 dq'}} = \tag{1.7.16}$$

$$\frac{a\int_{|q-q'|\leq\delta q/2}|q'\rangle\langle q'||\psi[\Theta]\rangle dq'}{\sqrt{\int_{|q-q'|\leq\delta q/2}|\langle q'||\psi[\Theta]\rangle|^2 dq'}} \in \mathbf{H}_\Theta.$$

Q.V.1. Let $|\Psi^{a_1,a_2}[\Theta_1,\Theta_2]\rangle = |\psi_1^{a_1}[\Theta_1]\rangle + |\psi_2^{a_2}[\Theta_2]\rangle \in \mathbf{H}_{1,2} \triangleq \mathbf{H}_{\Theta_1} \oplus \mathbf{H}_{\Theta_2} \subsetneqq \mathbf{H}$, where

(i) $|\psi_i^{a_i}[\Theta_i]\rangle = a_i|\psi_i[\Theta_i]\rangle \in \mathbf{H}_{\Theta_i}, |\psi_i\rangle = |\psi_i[\Theta_i]\rangle \in \mathbf{S}^\infty_{\Theta_i}, |a_i| \neq 1, i = 1,2;$

(ii) supp$(c_{\psi_i[\Theta_i]}(q)), i = 1,2$ is a connected sets in $\mathbb{R}$;

(iii) $\left(\text{supp}(c_{\psi_1[\Theta_1]}(q))\right) \cap \left(\text{supp}(c_{\psi_2[\Theta_2]}(q))\right) = \varnothing$ and

(iv) $|\psi_i[\Theta_i]\rangle = \int_{\theta_1}^{\theta_2} c_{\psi_i[\Theta_i]}(q)|q\rangle dq, i = 1,2.$

Then if the system is in a state $|\Psi^{a_1,a_2}[\Theta_1,\Theta_2]\rangle$ described by a wave function $\Psi^{a_1,a_2}(q;\Theta_1,\Theta_2) = \langle q||\Psi^{a_1,a_2}[\Theta_1,\Theta_2]\rangle, q \in \Theta_1 \cup \Theta_2$ and the value of observable Q_{Θ_1,Θ_2} is measured once each on many identically prepared system, the average value of all the measurements will be

$$\langle Q_{\Theta_1,\Theta_2}\rangle = \int\limits_{\Theta_1\cup\Theta_2} q|\Psi^{a_1,a_2}(q;\Theta_1,\Theta_2)|^2 dq. \tag{1.7.17}$$

Q.V. 2. The probability of getting a result q with an accuracy δq such that $(q - \frac{1}{2}\delta q, q + \frac{1}{2}\delta q) \in supp(c_{\psi_1}(q))$ or $(q - \frac{1}{2}\delta q, q + \frac{1}{2}\delta q) \in supp(c_{\psi_2}(q))$ given by

$$\int_{|q-q'|\leq\delta q/2}\left[\left(|\langle q'||\psi_1^{a_1}[\Theta_1]\rangle|^2\right) * \left(|\langle q'||\psi_2^{a_2}[\Theta_2]\rangle|^2\right)\right]dq'. \tag{1.7.18}$$

Remark 1.7.8.Note that Q.IV.3 immediately folows from Q.III.3.

Q.V. 3. Assume that the system is initially in the state $|\Psi^{a_1,a_2}[\Theta_1,\Theta_2]\rangle$. If on performing a measurement of Q_{Θ_1,Θ_2} with an accuracy δq, the result is obtained in the range $(q - \frac{1}{2}\delta q, q + \frac{1}{2}\delta q)$, then the state of the system immediately after measurement given by

$$\frac{\widehat{P}(q_i,\delta q)|\Psi^{a_1,a_2}[\Theta_1,\Theta_2]\rangle}{\sqrt{\langle\psi|\widehat{P}(q_i,\delta q)|\psi\rangle}} =$$

$$\frac{\int_{|q_i-q'|\leq\delta q/2}(|q'\rangle\langle q'||\psi_1^{a_1}[\Theta_1]\rangle + |q'\rangle\langle q'||\psi_2^{a_2}[\Theta_2]\rangle)dq'}{\sqrt{\int_{|q_i-q'|\leq\delta q/2}\left[|\langle q'||\psi_1[\Theta_1]\rangle|^2 + |\langle q'||\psi_2[\Theta_2]\rangle|^2\right]dq'}} = \tag{1.7.19}$$

$$\frac{\int_{|q_i-q'|\leq\delta q/2}(a_1|q'\rangle\langle q'||\psi_1[\Theta_1]\rangle + a_2|q'\rangle\langle q'||\psi_2[\Theta_2]\rangle)dq'}{\sqrt{\int_{|q_i-q'|\leq\delta q/2}\left[|\langle q'||\psi_1[\Theta_1]\rangle|^2 + |\langle q'||\psi_2[\Theta_2]\rangle|^2\right]dq'}} \in \mathbf{H}_{\Theta_i},$$

$$q_i \in \Theta_i, i = 1,2.$$

Definition 1.7.8. Let $\mathbf{H}_{1,2}$ be $\mathbf{H}_{1,2} \triangleq \mathbf{H}_{\Theta_1} \oplus \mathbf{H}_{\Theta_2}$.

Definition 1.7.9. Let $|\psi^a\rangle$ be a state $|\psi^a\rangle = a|\psi\rangle$, where $|\psi\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi\rangle = \int_{\theta_1}^{\theta_2} c_\psi(q)|q\rangle dq$. Let $|\psi_a\rangle$ be an state such that $|\psi_a\rangle \in \mathbf{S}^\infty$. States $|\psi^a\rangle$ and $|\psi_a\rangle$ is a Q-equivalent: $|\psi^a\rangle \sim_Q |\psi_a\rangle$ iff

$$P(q,q+dq;|\psi^a\rangle) = |a|^{-2}|c_\psi(q|a|^{-2})|^2dq = P(qq+dq;|\psi_a\rangle)dq \tag{1.7.20}$$

Q.V.For any state $|\psi^a\rangle = a|\psi\rangle$,where $|\psi\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi\rangle = \int_{\theta_1}^{\theta_2} c_\psi(q)|q\rangle dq$ there exist an state $|\psi_a\rangle \in \mathbf{S}^\infty$ such that: $|\psi^a\rangle \sim_Q |\psi_a\rangle$.

Definition 1.7.10. Let $|\psi^a\rangle$ be a state $|\psi^a\rangle = a|\psi\rangle$, where $|\psi\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi\rangle = \int_{\theta_1}^{\theta_2} c_\psi(q)|q\rangle dq$. Let $|\psi_a\rangle$ be an state such that $|\psi_a\rangle \in \mathbf{S}^\infty$. States $|\psi^a\rangle$ and $|\psi_a\rangle$ is a $\widehat{Q}$-equivalent ($|\psi^a\rangle \sim_{\widehat{Q}} |\psi_a\rangle$) iff: $\langle\psi^a|\widehat{Q}|\psi^a\rangle = \langle\psi_a|\widehat{Q}|\psi_a\rangle$.

Q.VI.For any state $|\psi^a\rangle = a|\psi\rangle$,where $|\psi\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi\rangle = \int_{\theta_1}^{\theta_2} c_\psi(q)|q\rangle dq$ there exist an state $|\psi_a\rangle \in \mathbf{S}^\infty$ such that: $|\psi^a\rangle \sim_{\widehat{Q}} |\psi_a\rangle$.

I.7.2. The Position Representation.Position observable of a particle in one dimension.

The position representation is used in quantum mechanical problems where it is the position of the particle in space that is of primary interest. For this reason, the position representation, or the wave function, is the preferred choice of representation.

P.1. In one dimension, the position x of a particle can range over the values $-\infty < x < +\infty$. Thus the Hermitean operator $\widehat{x}$ corresponding to this observable will have eigenstates $|x\rangle$ and associated eigenvalues x such that:
$\widehat{x}|x\rangle = x|x\rangle, -\infty < x < +\infty$.

P.2. As the eigenvalues cover a continuous range of values, the completeness relation will be expressed as an integral: $|\psi_t\rangle = \int_{-\infty}^{+\infty}|x\rangle\langle x|\psi_t\rangle dx,$ where $\langle x|\psi_t\rangle = \psi(x,t)$ is the wave function associated with the particle at each instant t. Since there is a continuously infinite number of basis states $|x\rangle$, these states are δ-function normalized: $\langle x|x'\rangle = \delta(x-x')$.

P.3. The operator $\widehat{x}$ itself can be expressed as: $\widehat{x} = \int_{-\infty}^{+\infty} x|x\rangle\langle x|dx.$

Definition 1.7.11. A connected set is a set $X \subset \mathbb{R}$ that cannot be partitioned into two nonempty subsets which are open in the relative topology induced on the set. Equivalently, it is a set which cannot be partitioned into two nonempty subsets such that each subset has no points in common with the set closure of the other.

P.4.The wave function is, of course, just the components of the state vector $|\psi_t\rangle \in \mathbf{S}^\infty$ with respect to the position eigenstates as basis vectors. Hence, the wave function is often referred to as being the state of the system in the position representation. The probability amplitude $\langle x|\psi_t\rangle$ is just the wave function, written $\langle x|\psi_t\rangle \triangleq \psi(x,t)$ and is such that $|\psi(x,t)|^2dx$ is the probability $P(x,t;|\psi_t\rangle)$ of the particle being observed to have a coordinate in the range x to $x+dx$

Definition 1.7.12. Let $|\psi_t^a\rangle, t \in [0,+\infty)$ be a state $|\psi_t^a\rangle = a|\psi_t\rangle$, where $|\psi_t\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and

$|\psi_t\rangle = \int_{-\infty}^{+\infty}\psi(x,t)|x\rangle dx.$ Let $|\psi_{t,a}\rangle, t \in [0,+\infty)$ be an state such that $|\psi_{t,a}\rangle \in \mathbf{S}^\infty$, $t \in [0,+\infty)$.

States $|\psi_t^a\rangle$ and $|\psi_{t,a}\rangle$ is x-equivalent ($|\psi_t^a\rangle \sim_x |\psi_{t,a}\rangle$) iff

$$P(x,t;|\psi_t^a\rangle)dx = |a|^{-2}|\psi(x|a|^{-2},t)|^2dx = P(x,t;|\psi_{t,a}\rangle)dx \quad (1.7.21)$$

P.5.From postulate Q.V above (see subsection 1.7.1) follows: for any state $|\psi_t^a\rangle = a|\psi_t\rangle,$where $|\psi_t\rangle \in \mathbf{S}^\infty, |a| \neq 1, t \in [0,+\infty)$ and $|\psi_t\rangle = \int_{-\infty}^{+\infty}\psi(x,t)|x\rangle dx$ there exist an state $|\psi_{t,a}\rangle \in \mathbf{S}^\infty$, $t \in [0,+\infty)$ such that: $|\psi_t^a\rangle \sim_x |\psi_{t,a}\rangle$.

Definition 1.7.13.Let $|\psi_t^a\rangle, t \in [0,+\infty)$ be a state $|\psi_t^a\rangle = a|\psi_t\rangle$, where $|\psi_t\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi_t\rangle = \int_{-\infty}^{+\infty}\psi(x,t)|x\rangle dx.$ Let $|\psi_{t,a}\rangle, t \in [0,+\infty)$ be an state such that $|\psi_{t,a}\rangle \in \mathbf{S}^\infty$, $t \in [0,+\infty)$. States $|\psi_t^a\rangle$ and $|\psi_{t,a}\rangle$ is $\widehat{x}$-equivalent ($|\psi_t^a\rangle \sim_{\widehat{x}} |\psi_{t,a}\rangle$) iff: $\langle\psi_t^a|\,\widehat{x}|\psi_t^a\rangle = \langle\psi_{t,a}|\,\widehat{x}|\psi_{t,a}\rangle$.

D.6.From postulate C.7 (see Appendix C) follows: for any state $|\psi_t^a\rangle = a|\psi_t\rangle,$where $|\psi_t\rangle \in \mathbf{S}^\infty, |a| \neq 1, t \in [0,+\infty)$ and $|\psi_t\rangle = \int_{-\infty}^{+\infty}\psi(x,t)|x\rangle dx$ there exist an state $|\psi_{t,a}\rangle \in \mathbf{S}^\infty$, $t \in [0,+\infty)$ such that: $|\psi_t^a\rangle \sim_{\widehat{x}} |\psi_{t,a}\rangle$.

Definition1.7.14.The pure state $|\psi_t\rangle \in \mathbf{S}^\infty, t \in [0,+\infty), |\psi_t\rangle = \int_{-\infty}^{+\infty}\psi(x,t)|x\rangle dx$ is a weakly

Gaussian in the position representation iff

$$|\psi(x,t)|^2dx = \frac{1}{\sigma_t\sqrt{2\pi}}\exp\left[-\frac{(x-\bar{x}_t)^2}{\sigma_t^2}\right]dx, \quad (1.7.22)$$

where $\bar{x}_t$ and σ_t an given functions which depend only on variable t.

P.7.From statement P.5 follows: for any state $|\psi_t^a\rangle = a|\psi_t\rangle$, where $|\psi_t\rangle \in \mathbf{S}^\infty, |a| \neq 1,\ t \in [0,+\infty)$ and $|\psi_t\rangle = \int_{-\infty}^{+\infty} \psi(x,t)|x\rangle dx$ is a weakly Gaussian state there exist an weakly Gaussian state $|\psi_{t,a}\rangle \in \mathbf{S}^\infty,\ t \in [0,+\infty)$ such that:

$$P(x,t;|\psi_t^a\rangle)dx = |a|^{-1}|\psi(x|a|^{-1},t)|^2 dx = \\ = \frac{1}{|a|\sigma_t\sqrt{2\pi}} \exp\left[-\frac{(x-|a|\bar{x}_t)^2}{|a|^2\sigma_t^2}\right]dx. \tag{1.7.23}$$

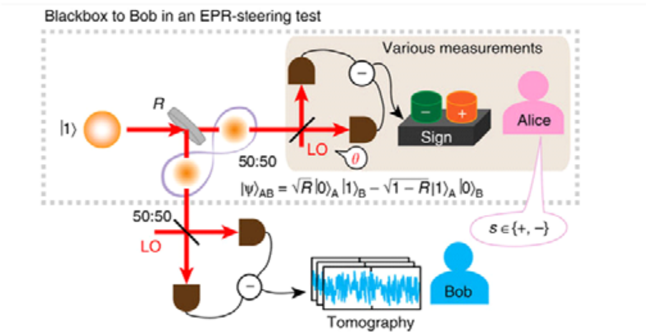
I.8. The EPR paradox

In 1935, Einstein, Podolsky and Rosen (EPR) originated the famous “EPR paradox” [38]. This argument concerns two spatially separated particles which have both perfectly correlated positions and momenta, as is predicted possible by quantum mechanics. The EPR paper spurred investigations into the nonlocality of quantum mechanics, leading to a direct challenge of the philosophies taken for granted by most physicists.The EPR conclusion was based on the assumption of local realism, and thus the EPR argument pinpoints a contradiction between local realism and the completeness of quantum mechanics.

I.8.1. Einstein’s 1927 gedanken experiment

Einstein never accepted orthodox quantum mechanics because he did not believe that its nonlocal collapse of the wavefunction could be real. When he first made this argument in 1927 [39],he considered just a single particle. The particle’s wavefunction was diffracted through a tiny hole so that it ‘dispersed’ over a large hemispherical area before encountering a screen of that shape covered in photographic film. Since the film only ever registers the particle at one point on the screen, orthodox quantum mechanics must postulate a ‘peculiar mechanism of action at a distance, which prevents the wave... from producing an action in two places on the screen. That is, according to the theory, the detection at one point must instantaneously collapse the wavefunction to nothing at all other points.

Remark 1.8.1. It was only in 2010, nearly a century after Einstein’s original proposal, that a scheme to rigorously test Einstein’s ‘spooky action at a distance [39],[40] using a single particle (a photon), as in his original conception, was conceived [41]. In this scheme, Einstein’s 1927 gedankenexperiment is simplified so that the single photon is split into just two wavepackets, one sent to a laboratory supervised by Alice and the other to a distant laboratory supervised by Bob. However, there is a key difference, which enables demonstration of the nonlocal collapse experimentally: rather than simply detecting the presence or absence of the photon, homodyne detection is used. This gives Alice the power to make different measurements, and enables Bob to test (using tomography) whether Alice’s measurement choice affects the way his conditioned state collapses, without having to trust anything outside his own laboratory.



Pic. Pic. 1.8.1.Simplified version of Einstein's original gedankenexperiment Adapted from [42].

Simplified version of Einstein's original gedankenexperiment [42]. A single photon is incident on a beam splitter of reflectivity R and then subjected to homodyne measurements at two spatially separated locations. Alice is trying to convince Bob that she can steer his portion of the single photon to different types of local quantum states by performing various measurements on her side. She does this by using different values of her LO phase θ, and extracting only the sign $s \in \{+,-\}$ of the quadrature she measures. Meanwhile, Bob scans his LO and performs full quantum-state tomography to reconstruct his local quantum state. He reconstructs unconditional and conditional local quantum states to test if his portion of the single photon has collapsed to different states according to Alice's LO setting θ, and result s see Pic. 1.8.1.

The key role of measurement choice by Alice in demonstrating 'spooky action at a distance' was introduced in the famous Einstein–Podolsky–Rosen (EPR) paper [38] of 1935. In its most general form, this phenomenon has been called EPR-steering, to acknowledge the contribution and terminology of Schrödinger [43], who talked of Alice 'steering' the state of Bob's quantum system. From a quantum information perspective, EPR-steering is equivalent to the task of entanglement verification when Bob (and his detectors) can be trusted but Alice (or her detectors) cannot. This is strictly harder than verifying entanglement with both parties trusted [44], but strictly easier than violating a Bell inequality [45], where neither party is trusted [44].

Remark 1.8.2. A recent experimental test of entanglement for a single photon via an entanglement witness has no efficiency loophole [46] however, it demonstrates a weaker form on nonlocality than EPR-steering. In [42], it was demonstrated experimentally that there exist Einstein's elusive 'spooky action at a distance' for a single particle without opening the efficiency loophole without claim to have closed the separation loophole. That is the one-sided device-independent verification of spatial-mode entanglement for a single photon.

I.8.2.The continuous variable EPR

paradox.EPR-Reid's criteria

We remind that EPR treated the case of a non-factorizable pure state $|\psi\rangle$ which describes the results for measurements performed on two spatially separated systems at A and B (Fig.1.8.2). "Non- factorizable" means "entangled", that is, we cannot express $|\psi\rangle$ as a simple product $|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B$, where $|\psi\rangle_A$ and $|\psi\rangle_B$ are quantum states for the results of measurements at A and B, respectively.

Fig.1.8.2.The original EPR gedanken experiment.Two particles move from the source into spatially separated regions A and B, and yet continue to have maximaly correlated positions: $x_A + x_0 = x_B$ and anti-correlated momenta: $p_A = -p_B$. Adapted from [47].

In the first part of their paper, EPR point out in a general way the problematic aspects of such entangled states. The key issue is that one can expand $|\psi\rangle$ in terms of more than one basis, that correspond to different experimental settings, which we parametrize by ϕ. Let us consider the state

$$|\psi\rangle = \int dx |\psi_x\rangle_{\phi,A} \otimes |u_x\rangle_{\phi,B}, \tag{1.8.1}$$

where the eigenvalue x could be continuous or discrete. The parameter setting ϕ at the detector B is used to define a particular orthogonal measurement basis $|u_x\rangle_{\phi,B}$. On measurement at B, this projects out a wave-function $|\psi_x\rangle_{\phi,A}$ at A, the process called "reduction of the wave packet".

Remark 1.8.3. The locality assumption postulates no action-at-a-distance, so that measurements at a location B cannot immediately "disturb" the system at a spatially separated location A .

Remark 1.8.4. The problematic issue is that different choices of measurements ϕ at B will cause reduction of the wave packet at A in more than one possible way. EPR state that, "as a consequence of two different measurements" at B, the "second system may be left in states with two different wavefunctions". Yet, "no real change can take place in the second system in consequence of anything that may be done to the first system".

The problem was established by EPR by a specific example, shown in Fig.1.8.2. EPR considered two spatially separated subsystems, at A and B, each with two observables $\hat{x}$ and $\hat{p}$ where $\hat{x}$ and $\hat{p}$ are non-commuting quantum operators, with commutator

$$[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = 2C \neq 0. \tag{1.8.2}$$

The results of the measurements $\hat{x}$ and $\hat{p}$ are denoted x and p respectively, and this convention we follow throughout the paper. We note that EPR assumed a continuous variable spectrum and considered wavefunction ψ defined in a position representation by

$$\psi(x,x^B) = \int e^{(ip/\hbar)(x-x^B-x_0)}dp, \tag{1.8.3}$$

where x_0 is an constant implying space-like separation. Here the pairs x and p refer to the results for position and momentum measurements at A, while x^B and p^B denote the position and momentum measurements at B. We leave off the superscript for system A, to emphasize the inherent asymmetry that exists in the EPR argument, where one system A is steered by the other, B.

Remark 1.8.5. According to canonical quantum mechanics, one can "predict with certainty" that a measurement $\hat{x}$ will give result $x^B + x_0$, if a measurement $\widehat{x^B}$, with result x^B, was already performed at B. One may also "predict with certainty" the result of measurement $\hat{p}$, for a different choice of measurement at B. If the momentum at B is measured to be p, then the result for $\hat{p}$ is $-p$. These predictions are made "without disturbing the second system" at A, based on the assumption, implicit in the original EPR paper, of "locality".

Remark 1.8.6.The locality assumption can be strengthened if the measurement events at A and B are causally separated (such that no signal can travel from one event to the other, unless faster than the speed of light)

Remark 1.8.7.The remainder of the EPR argument may be summarized as follows. Assuming local realism, one deduces that both the measurement outcomes, for x and p at A, are predetermined. The perfect correlation of x with $x^B + x_0$ implies the existence of an "element of reality" for the measurement $\hat{x}$. Similarly, the correlation of p with $-p^B$ implies an "element of reality" for $\hat{p}$. Although not mentioned by EPR, it will prove useful to mathematically represent the "elements of reality" for $\hat{x}$ and $\hat{p}$ by the respective variables μ_x^A and μ_p^A , whose "possible values are the predicted results of the measurement"

Remark 1.8.8.To continue the argument, *local realism* implies the existence of two elements of reality, μ_x^A and μ_p^A, that *simultaneously* predetermine, with absolute definiteness, the results for measurement x or p at A. These "elements of reality" for the localized subsystem A are not themselves consistent with quantum mechanics. Simultaneous determinacy for both the position and momentum is not possible for any quantum state. Hence, assuming the validity of local realism, one concludes quantum mechanics to be incomplete or even inconsistent!

Remark 1.8.9.We claim that any assumption of local realism is completely wrong.

Such claim meant as minimum the weak postulate of nonlocality.

The weak postulate of nonlocality for continuous variables.

The Heisenberg uncertainty relations

$$\Delta x^{\mathbf{A}} \Delta p^{\mathbf{A}} \geq 1 \tag{1.8.4}$$

cannot be violated in any cases:

(i) of course according to quantum mechanics, the Heisenberg uncertainty relations (1.8.4) cannot be violated if the coordinate $x^{\mathbf{A}}$ and momentum $p^{\mathbf{A}}$ of the particle **A** are measured directly by measurements performed on the particle **A**,

(ii) the Heisenberg uncertainty relations (1.8.4) cannot be violated even if the coordinate $x^{\mathbf{A}}$ and momentum $p^{\mathbf{A}}$ of the particle **A** are measured indirectly, i.e. by using measurement on particle **B**, as required in EPR gedanken experiment,

(iii) in any cases true coordinate $x^{\mathbf{A}}$ and momentum $p^{\mathbf{A}}$ of the particle **A** cannot be predicted samultaneously with a sufficiently small uncertainty $\Delta x^{\mathbf{A}}$ and $\Delta p^{\mathbf{A}}$ such that the Reid's inequality [50]:

$$\Delta x^{\mathbf{A}} \Delta p^{\mathbf{A}} < 1 \tag{1.8.5}$$

based on local realism would be satisfied, i.e., alwais

$$\Delta x^{\mathbf{A}} \Delta p^{\mathbf{A}} \nless 1. \tag{1.8.6}$$

We claim strictly stronger assumptions of nonlocality than mentioned above.

The strong postulate of nonlocality for continuous variables.

Let $|\psi_t^x\rangle_{\mathbf{A}}$ and $|\psi_t^x\rangle_{\mathbf{B}}$ be a state vector in x-representation at instant t of the particle **A** and particle **B** correspondingly.

Let $|\psi_t^p\rangle_{\mathbf{A}}$ and $|\psi_t^p\rangle_{\mathbf{B}}$ be a state vector in p-representation at instant t of the particle **A** and particle **B** correspondingly.

Let $\psi_t^{\mathbf{A}}(x) = \langle x|\psi_t^x\rangle_{\mathbf{A}}, \psi_t^{\mathbf{B}}(x) = \langle x|\psi_t^x\rangle_{\mathbf{B}}$ be a wave functions in x-representation of the particle **A** and particle **B** correspondingly.

Let $\psi_t^{\mathbf{A}}(p) = \langle p|\psi_t^p\rangle_{\mathbf{A}}, \psi_t^{\mathbf{B}}(p) = \langle p|\psi_t^p\rangle_{\mathbf{B}}$ be a wave functions in p-representation of the particle **A** and particle **B** correspondingly.

Let $\psi_t^{\mathbf{A/B}}(x_{\mathbf{A}}, x_{\mathbf{B}})$ be corresponding two-particle wave function in x-representation and let $\psi_t^{\mathbf{A/B}}(p_{\mathbf{A}}, p_{\mathbf{B}})$ be corresponding two-particle wave function in p-representation.

We claim that:

(i) whenever a measurement of the coordinate x of a particle **B** is performed at instant t with result $\bar{x}^{\mathbf{B}} \in [x^{\mathbf{B}} - \varepsilon, x^{\mathbf{B}} + \varepsilon], \varepsilon \ll 1,$ then:

(a) according to quantum mechanics a state vector $|\psi_t^x\rangle_{\mathbf{B}}$ collapses at instant t to

the state
vector

$$\left|\psi^{x}_{t,\delta,\varepsilon,x^{\mathbf{B}}}\right\rangle_{\mathbf{B}} \sim \widehat{L}^{\mathbf{B}}_{x^{\mathbf{B}}}(\delta,\varepsilon)|\psi^{x}_{t}\rangle_{\mathbf{B}} \quad (1.8.7)$$

given by law (1.2.20), where $\widehat{L}^{\mathbf{B}}_{\mathbf{x}^{\mathbf{B}}}(\delta,\varepsilon)$ is a norm-reducing, positive, self-adjoint, linear
operator in the 2-particle non projective Hilbert space $\mathbf{H}$, representing the localization of
particle $\mathbf{B}$ around the point $x^{\mathbf{B}}$, (see subsection I.2.4),
(b) according postulate of nonlocality a state vector $|\psi^{x}_{t}\rangle_{\mathbf{A}}$ immediately collapses at instant t
to the state vector

$$\left|\psi^{x}_{t,\delta,\varepsilon,x^{\mathbf{A}}}\right\rangle_{\mathbf{A}} \sim \widehat{L}^{\mathbf{A}}_{x^{\mathbf{B}}+x_0}(\delta,\varepsilon)|\psi^{x}_{t}\rangle_{\mathbf{A}} \quad (1.8.8)$$

given by law (1.2.20) and this is true independent of the distance in Minkovski spacetime
$M_4 = \mathbb{R}^{1,3}$ that separates the particles. Thus

$$|\psi^{x}_{t}\rangle_{\mathbf{B}} \overset{\text{collapse}}{\longrightarrow} \left|\psi^{x}_{t,\delta,\varepsilon,x^{\mathbf{B}}}\right\rangle_{\mathbf{B}} \Rightarrow |\psi^{x}_{t}\rangle_{\mathbf{A}} \overset{\text{collapse}}{\longrightarrow} \left|\psi^{x}_{t,\delta,\varepsilon,x^{\mathbf{B}}+x_0}\right\rangle_{\mathbf{A}} \quad (1.8.9)$$

(ii) under conditions given by Eq.(1.8.7)-Eq.(1.8.9) two-particle wave function
$\psi^{\mathbf{A/B}}_{t}(x_{\mathbf{A}},x_{\mathbf{B}})$ collapses at instant t by law

$$\psi^{\mathbf{A/B}}_{t}(x_{\mathbf{A}},x_{\mathbf{B}}) \overset{\text{collapse}}{\longrightarrow} \widehat{L}^{\mathbf{A}}_{x^{\mathbf{B}}+x_0}\widehat{L}^{\mathbf{B}}_{x^{\mathbf{B}}}(\delta,\varepsilon)\psi^{\mathbf{A/B}}_{t}(x_{\mathbf{A}},x_{\mathbf{B}}) \quad (1.8.10)$$

(iii) whenever a measurement of the momentum $p^{\mathbf{B}}$ of a particle **B** is performed at instant
t with result $\bar{p}^{\mathbf{B}} \in [p^{\mathbf{B}} - \varepsilon, p^{\mathbf{B}} + \varepsilon], \varepsilon \ll 1,$ then:
(a) according to quantum mechanics a state vector $|\psi^{p}_{t}\rangle_{\mathbf{B}}$ collapses at instant t to the state
vector

$$\left|\psi^{p}_{t,\delta,\varepsilon,p^{\mathbf{B}}}\right\rangle_{\mathbf{B}} \sim \widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon)|\psi^{p}_{t}\rangle_{\mathbf{B}}, \quad (1.8.11)$$

where $\widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon)$ is a norm-reducing, positive, self-adjoint, linear operator in the 2-particle
non projective Hilbert space $\mathbf{H}$, representing the localization of momentum of the particle $\mathbf{B}$
around the value $p^{\mathbf{B}}$. The localization operators $\widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon)$ have been chosen to have the
following form:

$$\widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon) = \left(\frac{1}{\delta\pi_{\delta}}\right)^{3/4} \exp\left[-\frac{1}{2\delta}(\widehat{p} - p^{\mathbf{B}})^2\right] \quad (1.8.12)$$

where $\delta \in (0,1]$ and $\lim_{\delta\to 0}\pi_{\delta} = \pi$.
(b) according postulate of nonlocality a state vector $|\psi^{p}_{t}\rangle_{\mathbf{A}}$ immediately collapses at instant t

to the state vector

$$\left|\psi^{p}_{t,\delta,\varepsilon,x^{\mathbf{A}}}\right\rangle_{\mathbf{A}} \sim \widehat{L}^{\mathbf{A}}_{-p^{\mathbf{B}}}(\delta,\varepsilon)|\psi^{p}_{t}\rangle_{\mathbf{A}} \quad (1.8.13)$$

and this is true independent of the distance in Minkovski spacetime $M_4 = \mathbb{R}^{1,3}$ that

separates the particles. Thus

$$|\psi^{p}_{t}\rangle_{\mathbf{B}} \overset{\text{collapse}}{\to} \left|\psi^{p}_{t,\delta,\varepsilon,p^{\mathbf{B}}}\right\rangle_{\mathbf{B}} \Rightarrow |\psi^{p}_{t}\rangle_{\mathbf{A}} \overset{\text{collapse}}{\to} \left|\psi^{p}_{t,\delta,\varepsilon,-p^{\mathbf{B}}}\right\rangle_{\mathbf{A}} \quad (1.8.14)$$

(iv) under conditions given by Eq.(1.8.11)-Eq.(1.8.13) two-particle wave function $\psi^{\mathbf{A/B}}_{t}(p_{\mathbf{A}},p_{\mathbf{B}})$ collapses at instant t by law

$$\psi^{\mathbf{A/B}}_{t}(p_{\mathbf{A}},p_{\mathbf{B}}) \overset{\text{collapse}}{\to} \widehat{L}^{\mathbf{A}}_{-p^{\mathbf{B}}}\widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon)\psi^{\mathbf{A/B}}_{t}(p_{\mathbf{A}},p_{\mathbf{B}}). \quad (1.8.15)$$

Remark 1.8.10. Let $p^{\mathbf{A}}_{t}$ and $p^{\mathbf{B}}_{t}$ be the momentum at instant t of the particle **A** and particle

B correspondingly. Note that whenever a measurement of the coordinate x of a particle **B** is

performed at instant t with an accuracy $\varepsilon_{x^{\mathbf{B}}} \ll 1$ then:

(i) immediately after this measurement the momentum $p^{\mathbf{B}}_{t}$ at instant t changed

according

to quantum mechanics by the Heisenberg uncertainty relations (1.8.4);

(ii) immediately after this measurement the momentum $p^{\mathbf{A}}_{t}$ at instant t changed

according

to postulate of nonlocality by the Heisenberg uncertainty relations (1.8.4)

Remark 1.8.11.Let $x^{\mathbf{A}}_{t}$ and $x^{\mathbf{B}}_{t}$ be the coordinate at instant t of the particle **A** and particle

B correspondingly. Note that whenever a measurement of the momentum p of a particle **B**

is performed at instant t with an accuracy $\varepsilon_{p^{\mathbf{B}}} \ll 1$ then:

(i) immediately after this measurement the coordinate $x^{\mathbf{B}}_{t}$ at instant t changed

according

to quantum mechanics by the Heisenberg uncertainty relations (1.8.4);

Remark 1.8.12.Schrödinger [43] pointed out that the EPR two-particle wavefunction in Eq.(1.8.3) was verschränkten - which he later translated as entangled - i.e., not of the separable form $\psi_A\psi_B$. Schrödinger considered as a possible resolution of the paradox that this “entanglement” degrades as the particles separate spatially, so that EPR correlations would not be physically realizable.

Definition 1.8.1.Quantum inseparability (entanglement) for a general mixed quantum state is defined as the failure of

$$\widehat{\rho} = \int d\lambda P(\lambda)\widehat{\rho}^{\mathbf{A}}_{\lambda} \otimes \widehat{\rho}^{\mathbf{B}}_{\lambda}\,, \quad (1.8.16)$$

where $\int d\lambda P(\lambda) = 1$ and $\widehat{\rho}$ is the density operator. Here λ is a discrete or continuous label for component states, and $\widehat{\rho}_\lambda^{\mathbf{A}}$ and $\widehat{\rho}_\lambda^{\mathbf{B}}$ correspond to density operators that are restricted to the Hilbert spaces $\mathbf{A}$ and B respectively.

Remark 1.8.13.The definition of inseparability extends beyond that of the EPR situation, in that one considers a whole spectrum of measurement choices, parametrized by θ for those performed on system A, and by ϕ for those performed on B. We use canonical notation $\hat{x}_\theta^A$ and $\hat{x}_\phi^B$ to describe all measurements at A and B. Denoting the eigenstates of $\hat{x}_\theta^A$ by $|x_\theta^A\rangle$, we define $P_Q(x_\theta^A|\theta,\lambda) = \langle x_\theta^A|\widehat{\rho}_\lambda^A|x_\theta^A\rangle$ and $P_Q(x_\phi^B|\phi,\lambda) = \langle x_\phi^B|\widehat{\rho}_\lambda^B|x_\phi^B\rangle$, which are the localized probabilities for observing results x_θ^A and x_ϕ^B respectively. The separability condition (1.8.9) then implies that joint probabilities $P(x_\theta^A,x_\phi^B)$ are given as [50]:

$$P(x_\theta^A,x_\phi^B) = \int d\lambda P(\lambda)P_Q(x_\theta^A|\lambda)P_Q(x_\phi^B|\lambda)\ . \tag{1.8.17}$$

Remark 1.8.14.We note the canonical restriction

$$\Delta^2(x^A|\lambda)\Delta^2(p^A|\lambda) \geq 1 \tag{1.8.18}$$

where $\Delta^2(x^{\mathbf{A}}|\lambda)$ and $\Delta^2(p^{\mathbf{A}}|\lambda)$ are the variances of $P_Q(x_\theta^{\mathbf{A}}|\theta,\lambda)$ for the choices θ corresponding to position x and momentum p, respectively. Thus

$$\Delta^2(x^{\mathbf{A}}|\lambda)\Delta^2(p^{\mathbf{A}}|\lambda) < 1 \tag{1.8.19}$$

is an EPR criterion, meaning that this would imply an EPR "paradox".

Remark 1.8.13.Note that the original EPR state of Eq. (1.8.3) is not separable.

Suppose that, based on a result $x^{\mathbf{B}}$ for the measurement at $\mathbf{B}$, an estimate $x_{\mathbf{est}}(x_{\mathbf{B}})$ is made of the result x at $\mathbf{A}$. We may define the average error $\Delta_{\mathbf{inf}}x$ of this inference as the root mean square (RMS) of the deviation of the estimate from the actual value, so that [50]:

$$\Delta_{\mathbf{inf}}^2 x = \int dxdx^{\mathbf{B}}P_t(x,x^{\mathbf{B}})(x - x_{\mathbf{est}}(x^B))^2\ . \tag{1.8.20}$$

An inference variance $\Delta_{\mathbf{inf}}^2 p$ is defined similarly,i.e.

$$\Delta_{\mathbf{inf}}^2 p = \int dpdp^{\mathbf{B}}P_t(p,p^{\mathbf{B}})(p - p_{\mathbf{est}}(p^B))^2\ . \tag{1.8.21}$$

Remark 1.8.14.Let $\psi_t^{\mathbf{A/B}}(x_{\mathbf{A}},x_{\mathbf{B}})$ be corresponding two-particle wave function in x-representation and let $\psi_t^{\mathbf{A/B}}(p_{\mathbf{A}},p_{\mathbf{B}})$ be corresponding two-particle wave function in p-representation.Note that:

(i) $P_t(x,x^{\mathbf{B}})$ is the joint probability of obtaining an outcome x at $\mathbf{A}$ and $x^{\mathbf{B}}$ at $\mathbf{B}$ at instant t is of the form

$$P_t(x,x^{\mathbf{B}}) \sim \left|\psi_t^{\mathbf{A/B}}(x_{\mathbf{A}},x_{\mathbf{B}})\right|^2, \tag{1.8.22}$$

(ii) $P_t(p,p^{\mathbf{B}})$ is the joint probability of obtaining an outcome p at $\mathbf{A}$ and $p^{\mathbf{B}}$ at $\mathbf{B}$ at instant t is of the form

$$P_t(p,p^{\mathbf{B}}) \sim \left|\psi_t^{\mathbf{A/B}}(p_{\mathbf{A}},p_{\mathbf{B}})\right|^2. \tag{1.8.23}$$

The best estimate, which minimizes $\Delta_{\mathbf{inf}}x$, is given by choosing $x_{\mathbf{est}}$ for each $x^{\mathbf{B}}$

to be the mean $\langle x|x^{\mathbf{B}}\rangle$ of the conditional distribution $P_t(x|x^{\mathbf{B}})$. This is seen upon noting that for each result $x^{\mathbf{B}}$, we can define the RMS error in each estimate as

$$\Delta^2_{\mathbf{inf}}(t,x|x^{\mathbf{B}}) = \int dx P_t(x|x^{\mathbf{B}})(x - x_{\mathbf{est}}(x^{\mathbf{B}}))^2 . \quad (1.8.24)$$

The average error in each inference is minimized for $x_{est} = \langle x|x^{\mathbf{B}}\rangle$, when each $\Delta^2_{\mathbf{inf}}(t,x|x^{\mathbf{B}})$ becomes the variance $\Delta^2(t,x|x^B)$ of $P_t(x|x^B)$. We thus define the minimum inference error $\Delta_{\mathbf{inf}}x$ for position, averaged over all possible values of $x^{\mathbf{B}}$, as

$$V^x_{\mathbf{A|B}} = \min(\Delta^2_{\mathbf{inf}}x) = \int dx^B P_t(x^B)\Delta^2(t,x|x^B) , \quad (1.8.25)$$

where $P(x^{\mathbf{B}})$ is the probability density for a result $x^{\mathbf{B}}$ upon measurement of $\hat{x}^{\mathbf{B}}$. This minimized inference variance is the average of the individual variances for each outcome at $\mathbf{B}$. Similarly, we can define a minimum inference variance, $V^p_{A|B}$, for momentum,i.e.

$$V^p_{\mathbf{A|B}} = \min(\Delta^2_{\mathbf{inf}}p) = \int dp^B P_t(p^B)\Delta^2(t,p|p^B) . \quad (1.8.26)$$

Remark 1.8.14.Let $\psi^{\mathbf{A/B}}_t(x_{\mathbf{A}},x_{\mathbf{B}})$ be corresponding two-particle wave function in x-representation and let $\psi^{\mathbf{A/B}}_t(p_{\mathbf{A}},p_{\mathbf{B}})$ be corresponding two-particle wave function in p-representation.Note that:

(i) according to local realism the conditional distributions densities $P_{\mathbf{loc}}(x|x^B)$ and $P_{\mathbf{loc}}(p|p^B)$ are given by formulae

$$P_{\mathbf{loc}}(x|x^B) \sim \widehat{L}^{\mathbf{B}}_{x^{\mathbf{B}}}(\delta,\varepsilon)\psi^{\mathbf{A/B}}_t(x,x_{\mathbf{B}}) \quad (1.8.27)$$

and

$$P_{\mathbf{loc}}(p|p^B) \sim \widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon)\psi^{\mathbf{A/B}}_t(p_{\mathbf{A}},p_{\mathbf{B}}). \quad (1.8.28)$$

(ii) distributions densities $P_{\mathbf{loc}}(t,x^{\mathbf{B}})$ and $P_{\mathbf{loc}}(t,p^B)$ are given by formulae

$$P_{\mathbf{loc}}(t,x^B) = \int dx P_{\mathbf{loc}}(t,x|x^B) \quad (1.8.29)$$

and

$$P_{\mathbf{loc}}(t,p^B) = \int dp P_{\mathbf{loc}}(t,p|p^B). \quad (1.8.30)$$

Remark 1.8.15.Let $\psi^{\mathbf{A/B}}_t(x_{\mathbf{A}},x_{\mathbf{B}})$ be corresponding two-particle wave function in x-representation and let $\psi^{\mathbf{A/B}}_t(p_{\mathbf{A}},p_{\mathbf{B}})$ be corresponding two-particle wave function in p-representation.Note that:

(i) according to postulates of nonlocality the conditional distributions densities $P_{\mathbf{n.loc}}(t,x|x^B)$ and $P_{\mathbf{n.loc}}(t,p|p^B)$ are given by formulae

$$P_{\mathbf{n.loc}}(t,x|x^B) = \widehat{L}^{\mathbf{A}}_{x^{\mathbf{B}}+x_0}\widehat{L}^{\mathbf{B}}_{x^{\mathbf{B}}}(\delta,\varepsilon)\psi^{\mathbf{A/B}}_t(x,x_{\mathbf{B}}) \quad (1.8.31)$$

and

$$P_{\mathbf{n.loc}}(t,p|p^{B}) \sim \widehat{L}^{\mathbf{A}}_{-p^{\mathbf{B}}}\widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon)\psi_t^{\mathbf{A/B}}(p,p_{\mathbf{B}}), \tag{1.8.32}$$

see Eq.(1.8.10) and Eq.(1.8.15) respectively.

(ii) distributions $P_{\mathbf{n.loc}}(t,x^{\mathbf{B}})$ and $P_{\mathbf{n.loc}}(t,p^{B})$ are given by formulae

$$P_{\mathbf{n.loc}}(t,x^{B}) = \int dx P_{\mathbf{n.loc}}(t,x|x^{B}) \tag{1.8.33}$$

and

$$P_{\mathbf{n.loc}}(t,p|^{B}) = \int dp P_{\mathbf{n.loc}}(t,p|p^{B}) \tag{1.8.34}$$

Thus we can define corresponding RMS errors as

$$\begin{aligned}\Delta^2_{\mathbf{loc.inf}}(t,x|x^{\mathbf{B}}) &= \int dx P_{\mathbf{loc}}(t,x|x^{\mathbf{B}})(x - x_{\mathbf{est}}(x^{\mathbf{B}}))^2\\ \Delta^2_{\mathbf{loc.inf}}(t,p|p^{\mathbf{B}}) &= \int dx P_{\mathbf{loc}}(t,p|p^{\mathbf{B}})(p - x_{\mathbf{est}}(p^{\mathbf{B}}))^2\end{aligned} \tag{1.8.35}$$

and

$$\begin{aligned}\Delta^2_{\mathbf{n.loc.inf}}(t,x|x^{\mathbf{B}}) &= \int dx P_{\mathbf{loc}}(t,x|x^{\mathbf{B}})(x - x_{\mathbf{est}}(x^{\mathbf{B}}))^2,\\ \Delta^2_{\mathbf{n.loc.inf}}(t,p|p^{\mathbf{B}}) &= \int dx P_{\mathbf{loc}}(t,p|p^{\mathbf{B}})(p - x_{\mathbf{est}}(p^{\mathbf{B}}))^2\end{aligned} \tag{1.8.36}$$

respectively.We thus define the minimum inference error $\Delta_{\mathbf{inf}}x$ for position, averaged over all possible values of $x^{\mathbf{B}}$ and $p^{\mathbf{B}}$ as

$$\begin{aligned}\min(\Delta^2_{\mathbf{loc.inf}}x) &= \int dx^B P_{\mathbf{loc}}(t,x^B)\Delta^2_{\mathbf{loc.}}(t,x|x^B)\ ,\\ \min(\Delta^2_{\mathbf{loc.inf}}p) &= \int dp^B P_{\mathbf{loc}}(t,p^B)\Delta^2_{\mathbf{loc.}}(t,p|p^B)\end{aligned} \tag{1.8.37}$$

and

$$\begin{aligned}\min(\Delta^2_{\mathbf{n.loc.inf}}x) &= \int dx^B P_{\mathbf{n.loc}}(t,x^B)\Delta^2_{\mathbf{n.loc.}}(t,x|x^B)\ ,\\ \min(\Delta^2_{\mathbf{n.loc.inf}}p) &= \int dp^B P_{\mathbf{n.loc}}(t,p^B)\Delta^2_{\mathbf{n.loc.}}(t,p|p^B).\end{aligned} \tag{1.8.38}$$

respectively.From Eq.(1.8.37) and Eq.(1.8.38) we obtain the EPR-nonlocality criteria

$$\begin{aligned}&\left|\ \min\Delta^2_{\mathbf{loc.inf.}}\ x - \min\ \Delta^2_{\mathbf{n.loc.inf.}}\ x\right| =\\ &\left|\int dx^B[P_{\mathbf{loc}}(t,x^B)\Delta^2_{\mathbf{loc.}}(t,x|x^B) - P_{\mathbf{n.loc}}(t,x^B)\Delta^2_{\mathbf{n.loc.}}(t,x|x^B)]\right| > 0,\\ &\left|\min\Delta^2_{\mathbf{loc.inf.}}\ p - \min\Delta^2_{\mathbf{n.loc.inf.}}\ p\right| =\\ &\left|\int dp^B[P_{\mathbf{loc}}(t,p^B)\Delta^2_{\mathbf{loc.}}(t,p|p^B) - P_{\mathbf{n.loc}}(t,p^B)\Delta^2_{\mathbf{n.loc.}}(t,p|p^B)]\right| > 0\end{aligned} \tag{1.8.39}$$

and

$$\left|\left(\min\Delta^2_{\mathbf{loc.inf.}}\ x\right)\left(\min\Delta^2_{\mathbf{loc.inf.}}\ p\right) - \left(\min\ \Delta^2_{\mathbf{n.loc.inf.}}\ x\right)\left(\left(\Delta^2_{\mathbf{n.loc.inf}}p\right)\right)\right| > 0. \tag{1.8.40}$$

I.9.The EPR-Bohm paradox. Reid’s criteria for EPR-Bohm paradox.

Bohm [49] considered two spatially-separated spin-$1/2$ particles at A and B produced in an entangled singlet state (often referred to as the "EPR-Bohm state" or the "Bell-state"):

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\right\rangle_A\left|-\frac{1}{2}\right\rangle_B - \left|-\frac{1}{2}\right\rangle_A\left|\frac{1}{2}\right\rangle_B\right) \quad (1.9.1)$$

Here $|\pm\frac{1}{2}\rangle_A$ are eigenstates of the spin operator $\hat{J}_z^A$, and we use $\hat{J}_z^A$, $\hat{J}_x^A$, $\hat{J}_y^A$ to define the spin-components measured at location A. The spin-eigenstates and measurements at B are defined similarly. By considering different quantization axes, one obtains different but equivalent expansions of $|\psi\rangle$ in Eq. (1.8.1), just as EPR suggested.

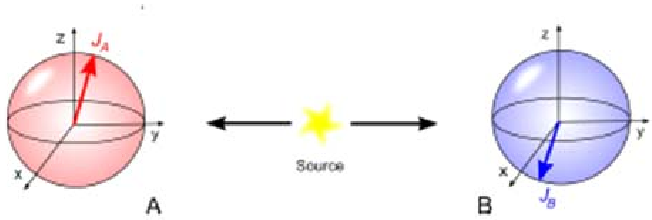


Fig.1.9.1.The Bohm gedanken EPR experiment.Two spin-$1/2$ partiles prepared in a singlet state from the source into spatially separated regions A and B,and give anti-correlated outomes for J_θ^A and J^B,where θ is x,y or z.Adapted from [50].

Bohm's paradox is based on the existence, for Eq. (1.9.1), of a maximum anti-correlation between not only $\hat{J}_z^A$ and $\hat{J}_z^B$, but $\hat{J}_y^A$ and $\hat{J}_y^B$, and also $\hat{J}_x^A$ and $\hat{J}_x^B$. An assumption of local realism would lead to the conclusion that the three spin components of particle A were simultaneously predetermined, with absolute definiteness. Since no such quantum description exists, this is the situation of an EPR paradox.

Remark 1.9.1. Bohm's paradox is based on the existence, for Eq. (1.9.1), of a maximum
anti-correlation between not only $\hat{J}_z^A$ and $\hat{J}_z^B$, but $\hat{J}_y^A$ and $\hat{J}_y^B$, and also $\hat{J}_x^A$ and $\hat{J}_x^B$.

Remark 1.9.2. Note that an assumption of local realism would lead to the conclusion that
the three spin components of particle A were simultaneously predetermined, with absolute
definiteness. Since no such quantum description exists, this is the situation of an EPR
paradox.

Remark 1.9.3. Criteria sufficient to demonstrate Bohm's EPR paradox can be derived using

Reid's canonical inferred uncertainty approach [41]. Using the Heisenberg spin uncertainty relation

$$\Delta J_x^{\mathbf{A}} \Delta J_y^{\mathbf{A}} \geq |\langle J_z^{\mathbf{A}} \rangle|/2, \tag{1.9.2}$$

one obtains the following canonical spin-EPR criterion that is useful for the Bell state given by Eq. (1.9.1)

$$\Delta_{\mathbf{inf}} J_x^{\mathbf{A}} \Delta_{\mathbf{inf}} J_y^{\mathbf{A}} < \frac{1}{2} \sum_{J_z^B} P(J_z^{\mathbf{B}}) \left| \langle J_z^{\mathbf{A}} \rangle_{J_z^{\mathbf{B}}} \right|. \tag{1.9.3}$$

Here $\langle J_z^{\mathbf{A}} \rangle_{J_z^{\mathbf{B}}}$ is the mean of the conditional distribution $P(J_z^{\mathbf{A}} | J_z^{\mathbf{B}})$.Calculations for Eq.(1.9.1) including the effect of detection efficiency η reveals this EPR criterion to be satisfied for $\eta > 0.62$.The concept of spin-EPR has been experementally tested in the continuum limit with purely optical systems for states with $\langle J_z^A \rangle \neq 0$.In this case the EPR criterion linked closely to definition of spin squeesing

$$\Delta_{\mathbf{inf}} J_x^A \Delta_{\mathbf{inf}} J_y^A < |\langle J_z^A \rangle|. \tag{1.9.4}$$

Remark 1.9.4. We claim that any assumption of local realism is completely wrong.The three spin components of particle A were simultaneously predetermined, does not with absolute definiteness but only with uncertainties which requred by Heisenberg spin uncertainty relations (1.9.5). Such claim meant as minimum the weak postulate of nonlocality.

The weak postulate of nonlocality.

The Heisenberg spin uncertainty relations

$$\begin{aligned} \Delta J_x^{\mathbf{A}} \Delta J_y^{\mathbf{A}} &\geq |\langle J_z^{\mathbf{A}} \rangle|/2, \\ \Delta J_x^{\mathbf{A}} \Delta J_z^{\mathbf{A}} &\geq |\langle J_y^{\mathbf{A}} \rangle|/2, \\ \Delta J_z^{\mathbf{A}} \Delta J_y^{\mathbf{A}} &\geq |\langle J_x^{\mathbf{A}} \rangle|/2 \end{aligned} \tag{1.9.5}$$

does not violated in any cases:

(i) if the three spin components of the particle **A** are measured directly by measurements performed on the particle **A**

(ii) and even if some spin components of the particle **A** are measured indirectly as required in Bohm gedanken EPR experiment.

Think of the following situation: a particle with zero spin decays into two particles (**A** and **B**), each with $1/2$-spin. Due to the fact that spin angular momentum must be conserved during the decay, if initially the total spin angular momentum was zero, then after the decaying process it must still be zero. Therefore, particles **A** and **B** have opposite spin.Take as an example the dissociation of an exited hydrogen molecule into two hydrogen atoms. If the decaying mechanism does not change total angular momentum, then the spins on the hydrogen atoms will be anti-correlated.

Remark 1.9.5.Whenever a measurement of the spin of **A** is found to be positive with respect of the z-axis (we shall note this state as $|\uparrow\rangle_z$, then,under local realism,we could infer that the spin of the **B** particle must be negative $|\downarrow\rangle_z$, and this is true independent of the distance that separates the particles. The spin of these particles are then entangled.

Remark 1.9.6. We claim again that any assumption of local realism is completely wrong.

The strong postulate of nonlocality.

Let $|\psi_t\rangle_{\mathbf{A}}$ and $|\psi_t\rangle_{\mathbf{B}}$ a state at instant t of the particle **A** and particle **B** correspondingly.

Let $|\uparrow\rangle_{z,\mathbf{A/B}}$ be eigenstates of the spin operator $\mathbf{S}^z_{\mathbf{A/B}}$:

$$\mathbf{S}^z_{\mathbf{A/B}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.9.6}$$

We claim that:

(i) whenever a measurement of the spin of a particle **A** is performed at instant $t_1 \geq t$ and

particle **A** is found in the state $|\uparrow\rangle_z$, i.e., a state $|\psi_{t_1}\rangle_A$ collapses at instant t_1 to the state

$|\uparrow\rangle_{z,A}$ with respect of the Heisenberg spin uncertainty relations (1.9.5), then a state $|\psi_{t_1}\rangle_B$

immediately collapses at instant t_1 to the state $|\downarrow\rangle_{z,B}$ with respect of the Heisenberg spin

uncertainty relations (1.9.5), and this is true independent of the distance in Minkovski

spacetime that separates the particles:

$$|\psi_{t_1}\rangle_A \overset{\text{col}\,\mathbf{lapse}}{\rightarrow} |\uparrow\rangle_{z,A} \Rightarrow |\psi_{t_1}\rangle_B \overset{\text{col}\,\mathbf{lapse}}{\rightarrow} |\downarrow\rangle_{z,B} \tag{1.9.7}$$

(ii) whenever a measurement of the spin of a particle **A** is performed at instant $t_1 \geq t$ and

particle **A** is found in the state $|\downarrow\rangle_z$, i.e., a state $|\psi_{t_1}\rangle_A$ collapses at instant t_1 to the state

$|\downarrow\rangle_{z,A}$ with respect of the Heisenberg spin uncertainty relations (1.9.5), then a state $|\psi_{t_1}\rangle_B$

immediately $|\psi_t\rangle_B$ collapses at instant t_1 to the state $|\uparrow\rangle_{z,B}$ with respect of the Heisenberg

spin uncertainty relations (1.9.5), and this is true independent of the distance in Minkovski spacetime that separates the particles:

$$|\psi_{t_1}\rangle_A \overset{\text{col}\,\mathbf{lapse}}{\longrightarrow} |\downarrow\rangle_{z,A} \Rightarrow |\psi_{t_1}\rangle_B \overset{\text{col}\,\mathbf{lapse}}{\longrightarrow} |\uparrow\rangle_{z,B} \tag{1.9.8}$$

Note that,we can not predict which spin will be positive (or negative) with respect of the

z-axis, so the state that describes the spins of the particles could be for instance the spin

singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \tag{1.9.9}$$

We have a probability of 50% for the spin of particle **A** to be positive (and the spin of **B** negative) and a probability of 50% of it being the other way around.

Remark 1.9.7. So far we have assumed that we are performing a measurement along the z-axis, but measurements are not restricted to this particular election, we could measure for instance the spin of particle **A** along the $\mathbf{a}$-axis and the spin of **B** along the $\mathbf{b}$-axis. Let's see what happens if we decide to measure the spin along the x-axis: $\mathbf{a} = \mathbf{b} = x$. As it known for $1/2$-spins, the spin operator $\mathbf{S}^x_{\mathbf{A/B}}$ can be represented by the 2×2 hermitian matrix

$$\mathbf{S}^x_{\mathbf{A/B}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.9.10}$$

By performing a change of basis we can rewrite the state $|\psi\rangle$ in terms of the eigenstates of the spin operator $\mathbf{S}^x_{\mathbf{A/B}}$:

$$|u\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle + |\uparrow\rangle), |v\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle - |\uparrow\rangle), \tag{1.9.11}$$

and using Eq. (1.9.10), we can rewrite the state $|\psi\rangle$ as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|vu\rangle - |uv\rangle). \tag{1.9.12}$$

The strong postulate of nonlocality in this case takes the form similarly mentioned above.Just like before, by choosing to measure the spin of **A** along the x-axis we can determine it's value and infer the value of the spin of particle $\widetilde{\mathbf{B}}$ $\left(\widetilde{\mathbf{B}} \neq \mathbf{B}\right)$ in the state $|\psi\rangle_{x,\widetilde{\mathbf{B}}} = |u\rangle_{x,\mathbf{B}} \neq |\psi\rangle_{x,\mathbf{B}}$ without the need to measure it (and vice versa).

Furthermore, it turns out that this is the case independent of the election of the axis we choose to measure! (Provided that $\mathbf{a} = \mathbf{b} = \nu$).

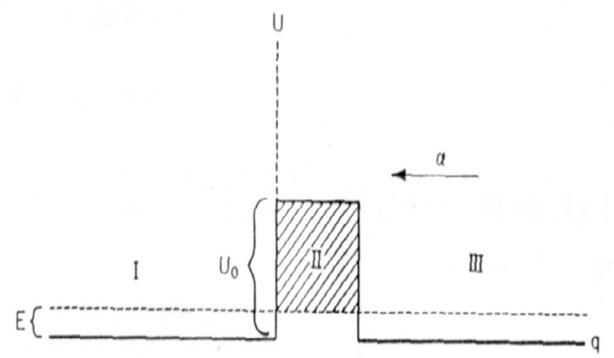
This is exactly the same situation such that a simple choice of the axis along which to measure the spin **A** allow us to establish the value of the spin of **B** along this same axis without the need to measure it. And this is also the case (as we already saw) for physical properties described by non-commuting operators ($\mathbf{S}^x$ and $\mathbf{S}^z$ do not commute).

II.Generalized Gamow theory of the alpha decay via tunneling using GRW collapse model.Nonlocal Schrödinger equation corresponding to alpha decay.

II.1.Generalized Gamow theory of the alpha decay via tunneling using GRW collapse model.

By 1928, George Gamow had solved the theory of the alpha decay via tunneling [42]. The alpha particle is trapped in a potential well by the nucleus. Classically, it is forbidden to escape, but according to the (then) newly discovered principles of quantum mechanics, it has a tiny (but non-zero) probability of "tunneling" through the barrier and appearing on the other side to escape the nucleus. Gamow solved a model potential for the nucleus and derived, from first principles, a relationship between the half-life of the decay, and the energy of the emission.

The α-particle has total energy E and is incident on the barrier from the right to left.



Pic. 2.1.1.The particle has total energy E and is incident on the barrier $V(x)$ from right to left. Adapted from [42].

The Schrödinger equation in each of regions $\mathbf{I} = \{x|x < 0\}, \mathbf{II} = \{x|0 \leq x \leq l\}$ and $\mathbf{III} = \{x|x > l\}$ takes the folloving form

$$\frac{\partial^2 \Psi(x)}{\partial x^2} + \frac{2m}{\hbar^2}[E - U(x)]\Psi(x) = 0, \tag{2.1.1}$$

where

$$U(x) = \begin{cases} 0 \text{ for } x < 0 \\ U_0 \text{ for } 0 \leq x \leq l \\ 0 \text{ for } x > l \end{cases} \tag{2.1.2}$$

The solutions reads [8]:

$$\begin{aligned} \Psi_{\mathbf{III}}(x) &= C_+ \exp(ikx) + C_- \exp(-ikx), \\ \Psi_{\mathbf{II}}(x) &= B_+ \exp(k'x) + B_- \exp(-k'x), \\ \Psi_{\mathbf{I}}(x) &= A\cos(kx) = \frac{A}{2}[\exp(ikx) + \exp(-ikx)], \end{aligned} \tag{2.1.3}$$

where

$$\begin{aligned} k &= \frac{2\pi}{\hbar}\sqrt{2mE}, \\ k' &= \frac{2\pi}{\hbar}\sqrt{2m(U_0 - E)}. \end{aligned} \tag{2.1.4}$$

At the boundary $x = 0$ we have the following boundary conditions:

$$\Psi_{\mathbf{I}}(0)|_{x=0} = \Psi_{\mathbf{II}}(0)|_{x=0}, \frac{\partial \Psi_{\mathbf{I}}(x)}{\partial x}\bigg|_{x=0} = \frac{\partial \Psi_{\mathbf{II}}(x)}{\partial x}\bigg|_{x=0}. \tag{2.1.5}$$

At the boundary $x = l$ we have the following boundary conditions

$$\Psi_{\mathbf{II}}(l)|_{x=l} = \Psi_{\mathbf{III}}(l)|_{x=l}, \frac{\partial \Psi_{\mathbf{II}}(x)}{\partial x}\bigg|_{x=l} = \frac{\partial \Psi_{\mathbf{III}}(x)}{\partial x}\bigg|_{x=l}. \tag{2.1.6}$$

From the boundary conditions (2.1.5)-(2.1.6) one obtains [42]:

$$\begin{aligned} B_+ &= \frac{A}{2}\left(1 + i\frac{k}{k'}\right), B_- = \frac{A}{2}\left(1 - i\frac{k}{k'}\right), \\ C_+ &= A[\mathbf{ch}(k'l) + iD\mathbf{sh}(k'l)], C_- = i(AS\mathbf{sh}(k'l)\exp(ikl)), \\ D &= \frac{1}{2}\left(\frac{k}{k'} - \frac{k'}{k}\right), S = \frac{1}{2}\left(\frac{k}{k'} + \frac{k'}{k}\right). \end{aligned} \tag{2.1.7}$$

From Eqs.(2.1.7) one obtains the conservation law

$$|A|^2 = |C_+|^2 - |C_-|^2.$$

Let us introduce now a function $E_{\mathbf{II}}(x,l) = \theta_2(x,l)E_2(x,l)$ where

$$E_2(x,l) = \begin{cases} (\pi r_c^2)^{-1/4}\exp\left(-\frac{x^2}{2r_c^2}\right) \text{ for } -\infty < x < \frac{l}{2} \\ (\pi r_c^2)^{-1/4}\exp\left(-\frac{(x-l)^2}{2r_c^2}\right) \text{ for } \frac{l}{2} \leq x < \infty \end{cases} \tag{2.1.8}$$

$$\theta_2(x,l) = \begin{cases} 1 \text{ for } x \in [0,l] \\ 0 \text{ for } x \notin [0,l] \end{cases}$$

Assumption 2.1.1. We assume now that:

(i) at instant $t = 0$ the wave function $\Psi_{\mathbf{I}}(x)$ experiences a sudden jump of the form

$$\Psi_{\mathbf{I}}(x) \to \Psi_{\mathbf{I}}^{\#}(x) = \frac{\Re_{\mathbf{I}}(\widehat{x})\Psi_{\mathbf{I}}(x)}{\|\Re_{\mathbf{I}}(\widehat{x})\Psi_{\mathbf{I}}(x)\|_2}, \tag{2.1.9}$$

where $\Re_{\mathbf{I}}(\widehat{x})$ is a linear operator which is chosen equal to:

$$\Re_{\mathbf{I}}(\widehat{x}) = (\pi r_c^2)^{-1/4}\theta_1(\widehat{x},l)\exp\left[-\frac{\widehat{x}^2}{2r_c^2}\right]; \tag{2.1.10}$$

where

$$\theta_1(x,l) = \begin{cases} 1 \text{ for } x \in [-l,0], \\ 0 \text{ for } x \notin [-l,0]. \end{cases}$$

Remark 2.1.1. Note that: $\operatorname{supp}(\Psi_{\mathbf{I}}^{\#}(x)) \subseteq [-l,0]$

(ii) at instant $t = 0$ the wave function $\Psi_{\mathbf{II}}(x)$ experiences a sudden jump of the form

$$\Psi_{\mathbf{II}}(x) \to \Psi_{\mathbf{II}}^{\#}(x) = \frac{\Re_{\mathbf{II}}(\widehat{x})\Psi_{\mathbf{II}}(x)}{\|\Re_{\mathbf{II}}(\widehat{x})\Psi_{\mathbf{II}}(x)\|_2}, \tag{2.1.11}$$

where $\Re_{\mathbf{II}}(\widehat{x})$ is a linear operator which is chosen equal to:

$$\Re_{\mathbf{II}}(\widehat{x}) = E_{\mathbf{II}}(\widehat{x},l); \tag{2.1.12}$$

Remark 2.1.2. Note that: $\operatorname{supp}(\Psi_{\mathbf{II}}^{\#}(x)) \subseteq [0,l]$.

(iii) at instant $t = 0$ the wave function $\Psi_{\mathbf{III}}(x)$ experiences a sudden jump of the form

$$\Psi_{\mathbf{III}}(x) \to \Psi^{\#}_{\mathbf{III}}(x) = \frac{\Re_{\mathbf{III}}(\hat{x})\Psi_{\mathbf{III}}(x)}{\|\Re_{\mathbf{III}}(\hat{x})\Psi_{\mathbf{III}}(x)\|_2}, \tag{2.1.13}$$

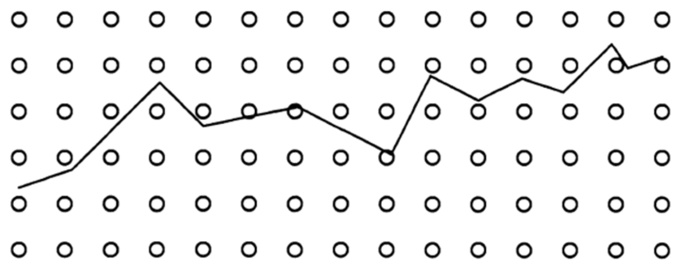
where $\Re_{\mathbf{III}}(\hat{x})$ is a linear operator which is chosen equal to:

$$\Re_{\mathbf{III}}(\hat{x}) = (\pi r_c^2)^{-1/4} \exp\left[-\frac{(\hat{x}-l)^2}{2r_c^2}\right]. \tag{2.1.14}$$

Remark 2.1.3. Note that. We have choose operators (2.1.10),(2.1.12) and (2.1.14) such that the boundary conditions (2.1.5),(2.1.6) is satisfied.

II.2.The nonlocal Schrödinger equation corresponding to alpha decay.

Note that α-particle moves through some medium and therefore the states of the atoms of the medium are changed as a result of the interaction with it. As a result, information about positions of the particle at different instants of time is recorded in the state of the medium. This means that the position is monitored. Simultaneously the motion of the particle is influenced by the medium. The resulting influence of the medium leads to the decoherence of the α-particle.



Pic. 2.2.1.Quantum diffusion:Decoherence by the structure of atoms.Adopted from [9].

The following master equation was derived for the density matrix of the such α-particle [9]:

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] - \frac{k}{2}[\mathbf{r},[\mathbf{r},\rho]], \quad k = \frac{2}{l^2\tau}, \tag{2.2.1}$$

where l is the distance at which the particle excites an atom and τ a characteristic time of interaction between the atom and the α-particle. In this case decoherence is caused by the internal structure of atoms. This illustrated by the Pic. 2.2.1.The medium to be considered is a system of an infinite number of non-interacting quantum oscillators located at the nodes of a cubic lattice.The oscillators are labelled by the index $k(k = 1,2,\ldots)$. The interaction of the k-th oscillator with the measured particle given by the following simple Hamiltonian [9]:

$$H_{\mathbf{int}} = \sum_k H_k = \gamma \sum_k q_k \exp\left[-\frac{(\mathbf{r}-\mathbf{c}_k)^2}{l^2}\right], \tag{2.2.2}$$

where q_k is a canonical coordinate of the oscillator while $\mathbf{c}_k$ is its location, γ is the interaction constant, l characterizes its range, and $\mathbf{r}$ is the coordinate of the α-particle.The complete Hamiltonian of the problem has the form

$$\widehat{\mathbf{H}} = \frac{\mathbf{P}^2}{2M} + V(\mathbf{r}) + \sum_k \frac{p_k^2}{2m} + \sum_k \frac{m\omega^2 q_k^2}{2} + H_{\mathbf{int}}, \tag{2.2.3}$$

where $\mathbf{P}, \mathbf{M}$ are the linear momentum and the mass of the α-particle.

Remark 2.2.1. (i) In classical handbooks it assumed that collapse of α-particle is caused

only by the interection with atoms of the medium, see for example [9].

(ii) In contrast with a classical case we assume that α-particle immediately collapses after

decay in accordance with nonclassical collapse models, see subsection **I**.2.4-**I**.2.5.

Remark 2.2.2. Schrödinger equation (2.1.1) can be solved by canonical Feynman propagator [9],[26].

Definition 2.2.1. Let $\Psi(x)$ be an solution of the Schrödinger equation (2.1.1). The stationary Schrödinger equation (2.1.1) is a weakly well preserved in region $\Gamma \subseteq \mathbb{R}$ by collapsed wave function $\Psi^{\#}(x)$ if the estimate

$$\int_{\Gamma} \left\{ \frac{\partial^2 \Psi^{\#}(x)}{\partial x^2} + \frac{2m}{\hbar^2}[E - U(x)]\Psi^{\#}(x) \right\} dx = O(\hbar^{\alpha}), \tag{2.2.4}$$

where $1/4 < \alpha \leq 1/2$, is satisfied.

Definition 2.2.2.The stationary integral equation (2.2.4) is called nonlocal Schrödinger equation of the order $\hbar^{\alpha}$ (corresponding to alpha decay).

Proposition 2.2.1.The Schrödinger equation (2.1.1) in each of regions $\mathbf{I}, \mathbf{II}, \mathbf{III}$

is a weakly well preserved by collapsed wave function $\Psi^{\#}_{\mathbf{I}}(x), \Psi^{\#}_{\mathbf{II}}(x)$ and $\Psi^{\#}_{\mathbf{III}}(x)$ correspondingly.

Proof. See Appendix B.

Definition 2.2.3.Let us consider the time-dependent Schrödinger equation:

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \widehat{\mathbf{H}}\Psi(x,t), \widehat{\mathbf{H}} = \frac{\partial^2}{\partial x^2} + \frac{2m}{\hbar^2}[E - U(x)], \tag{2.2.5}$$

$$t \in [0,T], x \in \mathbb{R}.$$

The time-dependent Schrödinger equation (2.2.5) is a weakly well preserved by corresponding to $\Psi(x,t)$ collapsed wave function $\Psi^{\#}(x,t)$

$$\Psi^{\#}(x,t) = \Psi(x,t;\widetilde{x}) = \frac{\Re(\widetilde{x})\Psi(x,t)}{\|\Re(\widetilde{x})\Psi(x,t)\|_2},$$

$$\Re(\widetilde{x}) = (\pi r_c^2)^{-3/4}\exp\left[-\frac{(\widehat{x}-\widetilde{x})^2}{2r_c^2}\right],$$

in region $\Gamma \subseteq \mathbb{R}$ if the estimate

$$\int_{\Gamma}\left\{i\hbar\frac{\partial\Psi^{\#}(x,t)}{\partial t} - \widehat{\mathbf{H}}\Psi^{\#}(x,t)\right\}dx = O(\hbar^{\alpha}), 1/4 < \alpha \le 1/2, \quad (2.2.6)$$

$$t \in [0,T], x\in\mathbb{R},$$

is satisfied.

Definition 2.2.4.The time-dependent integral equation (2.2.6) is colled the time-dependent nonlocal Schrödinger equation of the order $\hbar^{\alpha}$.

Definition 2.2.5. Let $\Psi^{\#}(\mathbf{x},t)$=$\Psi^{\#}(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_d,t)$ be a function $\Psi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_d,t;\widetilde{\mathbf{x}}_1,\ldots,\widetilde{\mathbf{x}}_d)$. Let us consider the Probability Current Law

$$\frac{\partial}{\partial t}P(\Gamma,t) + \int_{\partial\Gamma}\mathbf{J}(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_d,t)\cdot\mathbf{n}d^{2d}x = 0,$$

$$\mathbf{J}(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_d,t) = \Psi(\mathbf{x},t)\nabla\overline{\Psi(\mathbf{x},t)} - \overline{\Psi(\mathbf{x},t)}\nabla\Psi(\mathbf{x},t), \quad (2.2.7)$$

$$t \in [0,T], \mathbf{x} \in\mathbb{R}^{3d},$$

corresponding to Schrödinger equation (2.2.16). Probability Current Law (2.2.7) is a

weakly well preserved by corresponding to $\Psi(\mathbf{x},t)$ collapsed wave function $\Psi^{\#}(\mathbf{x},t)$ in region $\Gamma \subseteq \mathbb{R}^{3d}$ if there exist an wave function $\Psi(\mathbf{x},t)$ such that the estimate

$$\frac{\partial}{\partial t}P(\Gamma,t) + \int_{\partial\Gamma}\mathbf{J}^{\#}(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_d,t)\cdot\mathbf{n}d^{2d}x = O(\hbar^{\alpha}),$$

$$\mathbf{J}^{\#}(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_d,t) = \Psi^{\#}(\mathbf{x},t)\nabla\overline{\Psi^{\#}(\mathbf{x},t)} - \overline{\Psi^{\#}(\mathbf{x},t)}\nabla\Psi^{\#}(\mathbf{x},t) \quad (2.2.8)$$

$$= O(\hbar^{\alpha}), 1/4 < \alpha \le 1/2,$$

$$t \in [0,T], \mathbf{x} \in\mathbb{R}^{3d},$$

is satisfied.

Proposition 2.2.2. Assume that there exist an wave function $\Psi(\mathbf{x},t)$ such that

the estimate

(2.2.7) is satisfied. Then Probability Current Law (2.2.4) is a weakly well preserved by corresponding to $\Psi(\mathbf{x},t)$ collapsed wave function $\Psi^{\#}(\mathbf{x},t)$ in region $\Gamma \subseteq \mathbb{R}^{3d}$, i.e. the estimate (2.2.8) is satisfied on the wave function $\Psi^{\#}(\mathbf{x},t)$.

Definition 2.2.6.The time-dependent integral equation (2.2.9)

$$\begin{gathered}\int\limits_{\Gamma\subset\mathbb{R}^{3d}} \left\{ i\hbar \frac{\partial \Psi(\mathbf{x},t)}{\partial t} - \widehat{\mathbf{H}}\Psi(x,t) \right\} d^{3d}x = 0, \\ \widehat{\mathbf{H}} = \frac{\partial^2}{\partial x^2} + \frac{2m}{\hbar^2}[E - U(x)] \\ t \in [0,T], x \in \mathbb{R},\end{gathered} \tag{2.2.9}$$

is colled the time-dependent nonlocal Schrödinger equation corresponding to alpha decay.

II.3.The nonlocal stochastic nonlinear Schrödinger equation corresponding to alpha decay

Let us consider the nonlocal stochastic nonlinear Colombeau-Schrödinger equation describing the (one-dimensional) evolution of a free quantum particle subject to spontaneous localizations in space. The stochastic dynamics is governed by a standard Colombeau-Wiener process $(W_{\epsilon,t})_\epsilon$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $\{\mathcal{F}_t : t \geq 0\}$ defined on i

$$\begin{gathered}\left(\int\limits_{\mathbb{R}} dx [\psi_{\epsilon,T}(x) - \psi_{\epsilon,0}(x)] \right)_\epsilon = \\ \int\limits_{\mathbb{R}} dx \times \\ \left(\left[\int\limits_0^T \left(-\frac{i}{\hbar} \frac{\widehat{p}^2}{2m} dt + \sqrt{\lambda}\, (\widehat{q} - \langle \widehat{q} \rangle_{\epsilon,t}) \right) \psi_{\epsilon,t}(x) dW_{\epsilon,t} - \frac{\lambda}{2} \int\limits_0^T (\widehat{q} - \langle \widehat{q} \rangle_{\epsilon,t})^2 \psi_{\epsilon,t}(x) dt \right] \right)_\epsilon,\end{gathered} \tag{2.3.1}$$

where $\widehat{q}$ and $\widehat{p}$ are the position and momentum operators, respectively, and $\langle \widehat{q} \rangle_{\epsilon,t} \equiv \langle \psi_{\epsilon,t} | \widehat{q} | \psi_{\epsilon,t} \rangle$.

In order to obtain let us consider corresponding stochastic nonlinear Colombeau- Schrödinger equation

$$\begin{gathered}(\psi_{\epsilon,T}(x) - \psi_{\epsilon,0}(x))_\epsilon = \left(\int\limits_0^T \left(-\frac{i}{\hbar} \frac{\widehat{p}^2}{2m} \psi_{\epsilon,t}(x) - \frac{\lambda}{2} (q - \langle q \rangle_{\epsilon,t})^2 \psi_{\epsilon,t}(x) \right) dt \right)_\epsilon + \\ + \left(\int\limits_0^T \sqrt{\lambda}\, (\widehat{q} - \langle \widehat{q} \rangle_{\epsilon,t})\, \psi_{\epsilon,t}(x) dW_{\epsilon,t} \right)_\epsilon .\end{gathered}, \tag{2.3.2}$$

In order to obtain an solution of the Eq.(2.3.2) Let us consider the following Colombeau linear stochastic differential equation

$$(d\phi_{\epsilon,t}(x))_\epsilon = \left(\left[-\frac{i}{\hbar}\frac{p^2}{2m}dt + \sqrt{\lambda}\,\widehat{q}\,d\xi_{\epsilon,t} - \frac{\lambda}{2}\widehat{q}^2 dt\right]\phi_{\epsilon,t}(x)\right)_\epsilon, \tag{2.3.3}$$

where $\xi_{\epsilon,t}$ is a standard Colombeau-Wiener process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, where $\mathbb{Q}$ is a new probability measure, whose connection with $\mathbb{P}$ will be clear in what follows. Contrary to Eq.(2.3.2), the above equation does not preserve the norm of statevectors, so let us define the normalized vectors:

$$(\psi_{\epsilon,t})_\epsilon = \begin{cases} (\phi_{\epsilon,t}/\|\phi_{\epsilon,t}\|)_\epsilon & \text{if } (\|\phi_{\epsilon,t}\|)_\epsilon \neq 0 \\ \text{a fixed unit vector} & \text{if } (\|\phi_{\epsilon,t}\|)_\epsilon = 0 \end{cases} \tag{2.3.4}$$

By using Colombeau-Itô calculus, it is not difficult to show that $(\psi_{\epsilon,t})_\epsilon$ defined by (2.3.4) is a solution of Eq. (2.3.2), whenever $(\phi_{\epsilon,t})_\epsilon$ solves Eq. (2.3.3). We now briefly explain the relations between the two probability measures $\mathbb{Q}$ and $\mathbb{P}$, and between the two Colombeau-Wiener processes $(\xi_{\epsilon,t})_\epsilon$ and $(W_{\epsilon,t})_\epsilon$.

The key property of Eq.(2.3.3) is that $(p_{\epsilon,t})_\epsilon \equiv (\|\phi_{\epsilon,t}\|^2)_\epsilon$ is a martingale satisfying the following Colombeau stochastic differential equation

$$(dp_{\epsilon,t}) = 2\sqrt{\lambda}\,(\langle q\rangle_{\epsilon,t}p_{\epsilon,t}\,d\xi_{\epsilon,t})_\epsilon, \tag{2.3.5}$$

with $(\langle q\rangle_{\epsilon,t})_\epsilon = (\langle\psi_{\epsilon,t}|q|\psi_{\epsilon,t}\rangle)_\epsilon$. As a consequence of the martingale property (and assuming, as we shall always do, that $\|\phi_{\epsilon,0}\| = 1$) $(p_{\epsilon,t})_\epsilon$ can be used to generate a new probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ We choose now $\mathbb{Q}$ in such a way that the new measure $\tilde{\mathbb{P}}$ coincides with $\mathbb{P}$.Given Girsanov's theorem provides a simple relation between the Colombeau-Wiener process $(\xi_{\epsilon,t})_\epsilon$ defined on $(\Omega, \mathcal{F}, \mathbb{Q})$, and the Colombeau-Wiener process $(W_{\epsilon,t})_\epsilon$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$

$$(W_{\epsilon,t})_\epsilon = (\xi_{\epsilon,t})_\epsilon - 2\left(\sqrt{\lambda}\int_0^t \langle q\rangle_{\epsilon,s}\,ds\right)_\epsilon. \tag{2.3.6}$$

The above results imply that one can find the solution $(\psi_{\epsilon,t})_\epsilon$ of Eq.(2.3.2), given the initial condition $(\psi_{\epsilon,0})_\epsilon$, by using the following procedure:(i) find the solution $(\phi_{\epsilon,t})_\epsilon$ of Eq. (2.3.3), with the initial condition $(\phi_{\epsilon,0})_\epsilon = (\psi_{\epsilon,0})_\epsilon$, (ii) normalize the solution:$(\phi_{\epsilon,t})_\epsilon \to (\psi_{\epsilon,t})_\epsilon = (\phi_{\epsilon,t})_\epsilon/(\|\phi_{\epsilon,t}\|)_\epsilon$,(iii) make the substitution: $(d\xi_{\epsilon,t})_\epsilon \to (dW_{\epsilon,t})_\epsilon + 2\sqrt{\lambda}\,(\langle q\rangle_{\epsilon,t})_\epsilon$.We choose now, as a solution, a single-Gaussian wavefunction:

$$(\phi_{\epsilon,t}(x))_\epsilon = ([-a_t(x-\bar{x}_{\epsilon,t})^2 + i\bar{k}_{\epsilon,t}x + \gamma_{\epsilon,t}])_\epsilon, \tag{2.3.7}$$

where a_t and $(\gamma_{\epsilon,t})_\epsilon$ are supposed to be complex functions of time, while $(\bar{x}_{\epsilon,t})_\epsilon$ and $(\bar{k}_{\epsilon,t})_\epsilon$ are taken to be real. By inserting (2.3.7) into Eq.(2.3.3), one finds the following sets of Colombeau stochastic differential equations for the relevant parameters:

$$
\begin{aligned}
da_t &= \left[\lambda - \frac{2i\hbar}{m}(a_t)^2\right]dt,\\
(d\bar{x}_{\epsilon,t})_\epsilon &= \frac{\hbar}{m}(\bar{k}_{\epsilon,t}\,dt)_\epsilon + \frac{\sqrt{\lambda}}{2\operatorname{Re}(a_t)}\left[\left(d\xi_{\epsilon,t}\right)_\epsilon - 2\sqrt{\lambda}\,(\bar{x}_{\epsilon,t}\,dt)_\epsilon\right],\\
(d\bar{k}_{\epsilon,t}) &= -\sqrt{\lambda}\,\frac{\operatorname{Im}(a_t)}{\operatorname{Re}(a_t)}\left[\left(d\xi_{\epsilon,t}\right)_\epsilon - 2\sqrt{\lambda}\,(\bar{x}_{\epsilon,t}\,dt)_\epsilon\right],\\
(d\operatorname{Re}(\gamma_{\epsilon,t}))_\epsilon &= \left[\lambda(\bar{x}^2_{\epsilon,t})_\epsilon + \frac{\hbar}{m}\operatorname{Im}(a_t)\right]dt + \sqrt{\lambda}\left(\bar{x}_{\epsilon,t}\left[d\xi_{\epsilon,t} - 2\sqrt{\lambda}\,\bar{x}_{\epsilon,t}\,dt\right]\right)_\epsilon\\
(d\operatorname{Im}(\gamma_{\epsilon,t}))_\epsilon &= \left[-\frac{\hbar}{m}\operatorname{Re}(a_t) - \frac{\hbar}{2m}\left(\bar{k}^2_{\epsilon,t}\right)_\epsilon\right]dt + \sqrt{\lambda}\,\frac{\operatorname{Im}(a_t)}{\operatorname{Re}(a_t)}\left(\bar{x}_{\epsilon,t}\left[d\xi_{\epsilon,t} - 2\sqrt{\lambda}\,\bar{x}_{\epsilon,t}\,dt\right]\right)_\epsilon.
\end{aligned}
\quad (2
$$

For a single–Gaussian wavefunction, the two equations for $(\gamma_{\epsilon,t})_\epsilon$ can be omitted since the real part of $(\gamma_{\epsilon,t})_\epsilon$ is absorbed into the normalization factor, while the imaginary part gives an irrelevant global fase.The normalization procedure is trivial, and also the Girsanov transformation (2.3.6) is easy since, for a Gaussian wavefunction like (2.3.7), one simply has $(\langle q\rangle_{\epsilon,t})_\epsilon = (\bar{x}_{\epsilon,t})_\epsilon$. We then have the following set of stochastic differential equations for the relevant parameters:

$$
\begin{aligned}
da_t &= \left[\lambda - \frac{2i\hbar}{m}(a_t)^2\right]dt, a_t|_{t=0} = a_0,\\
d\bar{x}_t &= \frac{\hbar}{m}\bar{k}_t\,dt + \frac{\sqrt{\lambda}}{2\operatorname{Re}(a_t)}\,dW_t, \bar{x}_t|_{t=0} = \bar{x}_0,\\
d\bar{k}_t &= -\sqrt{\lambda}\,\frac{\operatorname{Im}(a_t)}{\operatorname{Re}a_t}\,dW_t, \bar{k}_t\big|_{t=0} = \bar{k}_0.
\end{aligned}
\tag{2.3.9}
$$

Eq. (2.3.9) for a_t can be easily solved

$$
a_t = c\tanh[bt + k], \tag{2.3.10}
$$

with:

$$
c = (1-i)\frac{1}{2}\sqrt{\frac{m\lambda}{\hbar}}, \quad b = (1+i)\sqrt{\frac{\hbar\lambda}{m}}, k = \tanh^{-1}\left[\frac{a_0}{c}\right]. \tag{2.3.11}
$$

After some algebra, one obtains the following analytical expressions for the real and the imaginary parts of a_t:

$$
\begin{aligned}
\operatorname{Re}(a_t) &= \frac{\lambda}{\omega}\,\frac{\sinh(\omega t + \varphi_1) + \sin(\omega t + \varphi_2)}{\cosh(\omega t + \varphi_1) + \cos(\omega t + \varphi_2)},\\
\operatorname{Im}(a_t) &= -\frac{\lambda}{\omega}\,\frac{\sinh(\omega t + \varphi_1) - \sin(\omega t + \varphi_2)}{\cosh(\omega t + \varphi_1) + \cos(\omega t + \varphi_2)}.
\end{aligned}
\tag{2.3.12}
$$

An important property of $\operatorname{Re}(a_t)$ t is positivity: $a_0 > 0 \Rightarrow a_t > 0$.

The quantities $(\langle q\rangle_{\epsilon,t})_\epsilon = (\bar{x}_{\epsilon,t})_\epsilon$ and $(\langle p\rangle_{\epsilon,t})_\epsilon = \hbar(\bar{k}_{\epsilon,t})_\epsilon$, corresponding to the peak of the Gaussian wavefunction in the position and momentum spaces, respectively, satisfy the following stochastic differential equations:

$$
\begin{aligned}
(d\langle q\rangle_{\epsilon,t})_\epsilon &= \frac{1}{m}\,(\langle p\rangle_{\epsilon,t}dt)_\epsilon \;+\; \sqrt{\lambda}\,\frac{1}{2\,\mathrm{Re}(a_t)}\,(dW_{\epsilon,t})_\epsilon, \\
(d\langle p\rangle_{\epsilon,t})_\epsilon &= -\sqrt{\lambda}\,\hbar\,\frac{\mathrm{Im}(a_t)}{\mathrm{Re}(a_t)}\,(dW_{\epsilon,t})_\epsilon.
\end{aligned}
\tag{2.3.13}
$$

We now study the time evolution of the superposition of two Gaussian wavefunctions; such an analysis is interesting since it allows to understand in a quite simple and clear way how the reduction mechanism works, i.e. how the superposition of two different position states is reduced into one of them. To this purpose, let us consider the following wavefunction:

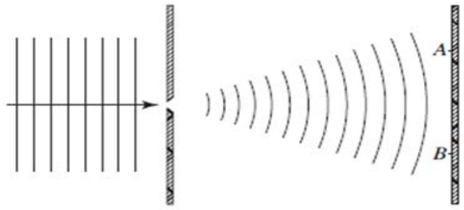
$$
\begin{aligned}
\phi_t^D(x) &= \phi_{1t}(x) + \phi_{2t}(x) = \\
&= [-a_{1t}(x-\bar{x}_{1t})^2 + i\bar{k}_{1t}x + \gamma_{1t}] \\
&+ [-a_{2t}(x-\bar{x}_{2t})^2 + i\bar{k}_{2t}x + \gamma_{2t}];
\end{aligned}
\tag{2.3.14}
$$

we follow the strategy outlined in Sec. II, by first finding the solution of the linear equation.

Because of linearity, $\phi_t^D(x)$ is automatically a solution of Eq. (ref: le), provided that its

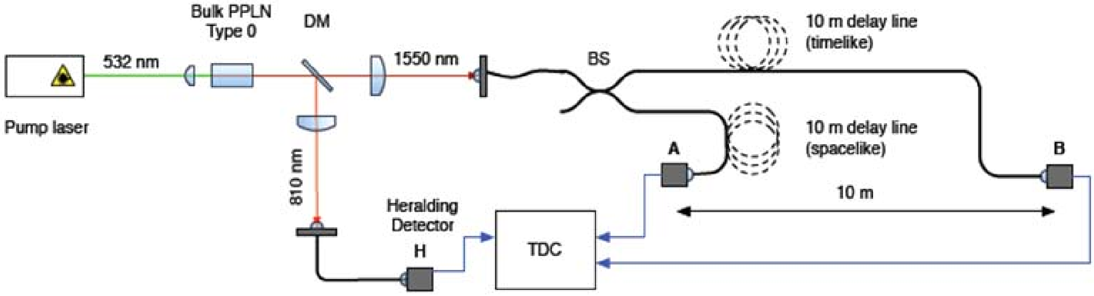
II.4. Einstein's 1927 gedanken experiment revisited.

During the famous 5th Solvay conference in 1927, Einstein [39] considered a single particle which, after diffraction in a pin-hole encounters a “detection plate” (e.g. in the case of photons, a photographic plate), see Fig 2.2.1. We simplify this thought experiment, though keeping the essence, by replacing the “detection plate” by two detectors. Einstein noted that there is no question that only one of them can detect the particle, otherwise energy would not be conserved. However, he was deeply concerned about the situation in which the two detectors are space-like separated, as this prevents - according to relativity - any possible coordination among the detectors: “It seems to me,” Einstein continued, “that this difficulty cannot be overcome unless the description of the process in terms of the Schr¨odinger wave is supplemented by some detailed specification of the localization of the particle during its propagation. I think M. de Broglie is right in searching in this direction.”



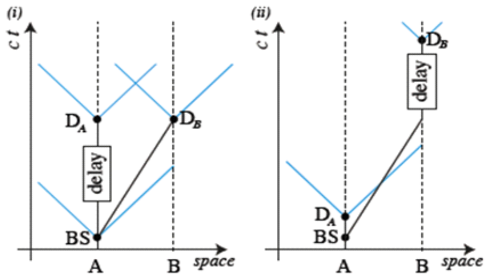
Pic. 2.4.1.Einstein’s 1927 gedanken experiment.
A and B are points on the photographic plate,
for which the events of detection can be space-
like separated from each other.Adapted from [39].

But what happened to Einstein’s original “Gedanken experiment”? This simple - with today’s technology - experiment had been done originally by T. Guerreiro, B. Sanguinetti, H. Zbinden N. Gisin, and A. Suarez, see [53].This experiment consists in verifying that when a single photon is thrown at a beam splitter, it is detected in only one arm, i.e. the probability $\mathbf{P}_{A\wedge B}$ of getting a coincidence between the two detectors A and B is much smaller than the product of the probabilities of detection on each side $\mathbf{P}_A \times \mathbf{P}_B$, as would be expected in the case of uncorrelated events.The experimental setup is shown in Fig. 2.2.2 and consists of a source of heralded single photons which is coupled into a single mode fiber and injected into a fiber beamsplitter (BS). Each of the two outputs of the beamsplitter goes to a single photon detector (IDQ ID200), detector A being close to the source and detector B being separated by a distance of approximately 10 meters.



Pic. 2.2.2.Experimental setup: photon pairs a regenerated by Spontaneous Parametric Down Conversion at the wavelengths of 1550 nm and 810 nm. These pairs are split by a dichroic mirror (DM), and the 810 nm photon is sent to detector D,used to herald the presence of the1550 nm photon which follows to the beam splitter (BS). Arbitrary electronic delays were applied beforeTDC to ensure the coincidence peak swould remain on scale. Adapted from [53].

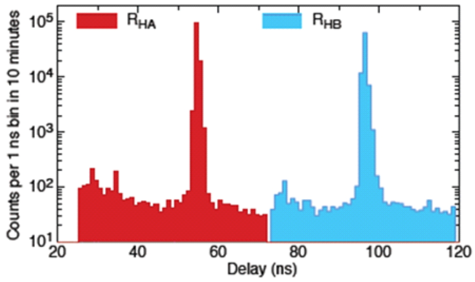
If we ensure that the fiber lengths before each detector are equal by inserting a 10 m (50ns) fiber delay loop before detector A, the detections will happen simultaneously in some reference frame, thus being space-like separated (a signal would take 33 ns to travel between the two detectors at the speed of light; simultaneity of detection is guaranteed to within 1ns by the matched length of fiber both before and inside the detectors). It is also possible to make the detections time-like separated by removing the 10m delay line from detector A and adding it to detector B.



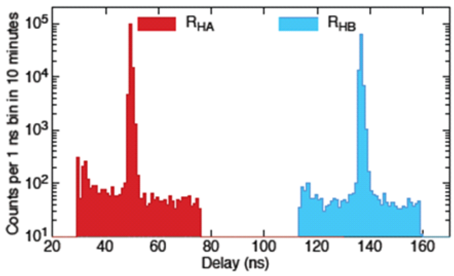
Pic. 2.2.4.[53]. Spacetime diagrams for spacelike (i) and timelike (ii) configurations. A and B represent the locations of the detectors. Adapted from [53].

First one measure the probabilities of detecting a photon at detector A or at detector B given that a heralding photon has been detected at H. We denote R_{HA} the total number of coincident counts at detector H and detector A during the time of measurement, and $R_{H(A)}$ the total number of counts at detector H alone during the same measurement; R_{HB} and $R_{H(B)}$ denote similar quantities for the measurement with H and B. Next we measure the probability of detectors A and B clicking at the same time, again given a heralding signal. R_{HAB} denotes the number of triple coincident counts at the detectors H, A and B, and $R_{H(AB)}$ the total number of counts at detector H alone during the same measurement. All these quantities are measured directly for both a space-like configuration and a time-like configuration.

Next one measure the probability of detectors A and B clicking at the same time, again given a heralding signal. R_{HAB} denotes the number of triple coincident counts at the detectors H, A and B, and $R_{H(AB)}$ the total number of counts at detector H alone during the same measurement. All these quantities are measured directly for both a space-like configuration and a time-like configuration.



Pic. 2.2.5.Coincidences between the heralding detector and each of the detectors A (red) and B (blue) with spacelike separation,measured in a window of 1ns during a time period of 10 minutes. $R_{HA} = 9.49 \times 104/10$ min, $R_{HB} = 6.39 \times 104/10$ min.The noise is on average:$R_N = 50/10$ min. Adapted from [53].



Pic. 2.2.6.Coincidences between the heralding detector and each of the detectors A (red) and B (blue) with timelike separation,measured in a window of 1ns during a time period of 10 minutes. R_{HA} = 9.90 × 104/10 min, R_{HB} = 6.22 × 104/10 min. Adapted from [53].

The raw TDC data is shown in Figures 5-6 and the results are summarized in Table I.

Spacelike separation

R_{HA}	$R_{H(A)}$	$P_A^{SL} = R_{HA}/R_{H(A)}$
$(94.8 \pm 0.3)\cdot 10^3$	$(5570 \pm 2)\ \cdot 10^3$	$(1.703 \pm 0.006)\cdot 10^{-2}$
R_{HB}	$R_{H(B)}$	$P_B^{SL} = R_{HB}/R_{H(B)}$
$(63.8 \pm 0.2)\cdot 10^3$	$(5860 \pm 2)\ \cdot 10^3$	$(1.090 \pm 0.004)\cdot 10^{-2}$
R_{HAB}	$R_{H(AB)}$	$P^{SL}(1,1) = R_{HAB}/R_{H(AB)}$
4 ± 2	$(17145 \pm 4)\cdot 10^3$	$(2.3 \pm 1.2)\cdot 10^{-7}$
R_{HN}	$R_{H(N)}$	$P_N^{SL} = R_{HN}/R_{H(N)}$
50 ± 7	$(5500 \pm 2)\cdot 10^3$	$(9.0 \pm 1.3)\cdot 10^{-6}$

Timelike separation

R_{HA}	$R_{H(A)}$	$P_A^{TL} = R_{HA}/R_{H(A)}$
$(99.0 \pm 0.3)\cdot 10^3$	$(6130 \pm 2)\cdot 10^3$	$(1.616 \pm 0.005)\cdot 10^{-2}$
R_{HB}	$R_{H(B)}$	$P_B^{TL} = R_{HB}/R_{H(B)}$
$(62.2 \pm 0.2)\cdot 10^3$	$(6100 \pm 2)\cdot 10^3$	$(1.019 \pm 0.004)\cdot 10^{-2}$
R_{HAB}	$R_{H(AB)}$	$P^{TL}(1,1) = R_{HAB}/R_{H(AB)}$
4 ± 2	$(18345 \pm 4)\cdot 10^3$	$(2.2 \pm 1.1)\cdot 10^{-7}$

Table I.Summary of results.Values obtained for the different counting rates and corresponding probabilities defined inthetext,measured with spacelike and timelike separation. Adapted from [53].

The number of counts given by detector noise and twophoton events can be estimated by looking at the counts away from the peak. As an example, for the space-like configuration (Figure 2.2.5.) in a window of 1ns the noise rate is on average R_{HN} = 50 (7) for a 10 minutes integration time. This corresponds to a noise probability $\mathbf{P}_N$ = 9· 10^{-6}(1.3·10^{-6}). From the values in Table 1 one derives the following probability values for spacelike separation:

$$\mathbf{P}_A^{SL} \bullet P_B^{SL} = 1.86 \pm 0.01 \cdot 10^{-4},$$
$$\mathbf{P}_{A \wedge B}^{SL} = 0.002 \pm 0.001 \cdot 10^{-4}. \tag{2.2.1}$$

For timelike separation one derives the values:

$$\mathbf{P}_A^{TL} \cdot P_B^{TL} = 1.65 \pm 0.01 \cdot 10^{-4},$$
$$\mathbf{P}_{A \wedge B}^{TL} = 0.002 \pm 0.001 \cdot 10^{-4}. \tag{2.2.2}$$

For the probability $\mathbf{P}_N^{SL}$ that A and B detect photons coming from different pairs (noise) one derives the value:

$$\mathbf{P}_N^{SL}(1,1) = \mathbf{P}_N^{SL} \cdot \mathbf{P}_A^{SL} + \mathbf{P}_N^{SL} \cdot \mathbf{P}_B^{SL} \approx$$
$$0.0025 \pm 0.0026 \cdot 10^{-4} \tag{2.2.3}$$

Definition 2.2.1.[54].A measure algebra $\mathcal{F} = (\mathbf{B}, \mathbf{P})$ with a probability measure $\mathbf{P}$, is a

Boolean algebra $\mathbf{B}$ with a countably additive probability measure.

Definition 2.2.2.(**i**) A measure algebra of physical events $\mathcal{F}^{ph} = (\mathbf{B}, \mathbf{P})$ with a

probability measure $\mathbf{P}$, is an Boolean algebra of physical events $\mathbf{B}$ with an countably

additive probability measure.

(**ii**) A Boolean algebra of physical events can be formally defined as a set $\mathbf{B}$ of elements

$a, b, \ldots$ with the following properties:

1. $\mathbf{B}$ has two binary operations, $\wedge$ (logical AND, or "wedge") and $\vee$ (logical OR, or "vee"),

which satisfy:

the idempotent laws:

(1) $a \wedge a = a \vee a = a$,

the commutative laws:

(2) $a \wedge b = b \wedge a$,

(3) $a \vee b = b \vee a$,

and the associative laws:

(4) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,

(5) $a \vee (b \vee c) = (a \vee b) \vee c$.

2. The operations satisfy the absorption law:

(6) $a \wedge (a \vee b) = a \vee (a \wedge b) = a$.

3. The operations are mutually distributive

(7) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

(8) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

4. $\mathbf{B}$ contains universal bounds $\mathbf{0}$ and $\mathbf{1}$ which satisfy

(9) $\mathbf{0} \wedge a = \mathbf{0}$

(10) $\mathbf{0} \vee a = a$

(11) $\mathbf{1} \wedge a = a$

(12) $\mathbf{1} \vee a = \mathbf{1}$.

5. $\mathbf{B}$ has a unary operation $\neg a$ (or a') of complementation (logical negation), which

obeys the laws:

(13) $a \wedge \neg a = \mathbf{0}$

(14) $a \vee \neg a = \mathbf{1}$

All properties of negation including the laws below follow from the above two laws alone.

6. Double negation law: $\neg(\neg a) = a$

7.De Morgan's laws: (i) $\neg a \wedge \neg b = \neg(a \vee b)$, (ii) $\neg a \vee \neg b = \neg(a \wedge b)$.

8.Operations composed from the basic operations include the following importent

examples:

The first operation, $a \to b$ (logical material implication):

(i) $a \to b \triangleq \neg a \vee b$.

The second operation, $a \oplus b$, is called exclusive. It excludes the possibility of both a and b

(ii) $a \oplus b \triangleq (a \vee b) \wedge \neg(a \wedge b)$.

The third operation, the complement of exclusive or, is equivalence or Boolean equality:

(iii) $a \equiv b \triangleq \neg(a \oplus b)$

9. $\mathbf{B}$ has a unary predicate $\mathbf{Occ}(a)$, which meant that event a has occurred, and which

obeys the laws:

(i) $\mathbf{Occ}(a \wedge b) \Leftrightarrow \mathbf{Occ}(a) \wedge \mathbf{Occ}(b)$,

(ii) $\mathbf{Occ}(a \vee b) \Leftrightarrow \mathbf{Occ}(a) \vee \mathbf{Occ}(b)$,

(iii) $\mathbf{Occ}(\neg a) \Leftrightarrow \neg\mathbf{Occ}(a)$.

Remark 2.2.4. A probability measure $\mathbf{P}$ on a measure space (Ω, Σ) gives a probability

measure algebra $\mathcal{F} = (\Omega, \Sigma, \mathbf{P})$ on the Boolean algebra of measurable sets modulo null

sets.

Definition 2.2.3.(i) Let $\mathbf{B}$ be a Boolean algebra of physical events. A Boolean algebra $\mathbf{B}_{M_4}$

of physical events in Minkowski spacetime $M_4 = \mathbb{R}^{1,3}$ that is cartesian product

$\mathbf{B}_{M_4} = \mathbf{B} \times M_4$.

(ii) Let $\mathbf{B}_{M_4}$ be a Boolean algebra of physical events in Minkowski spacetime. A measure

algebra of physical events $\mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ in Minkowski spacetime that is a Boolean

algebra $\mathbf{B}_{M_4}$ with a probability measure $\mathbf{P}$.

(iii) Let $\mathbf{B}_{M_4}$ be Boolean algebra of the all physical events in Minkowski spacetime and let

$\mathcal{F}^{ph}_{M_4}$ be an measure algebra $\mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ with a probability measure $\mathbf{P}$. We denote such physical events by $A(\mathbf{x}),B(\mathbf{x}),\ldots$ etc.,where $\mathbf{x} = (t,x_1,x_2,x_3) \in M_4$ or $A,B,\ldots$etc.

(iv) We will be write for a short $A^{Oc}(\mathbf{x}),B^{Oc}(\mathbf{x}),\ldots$ etc., instead $\mathbf{Occ}(A(\mathbf{x})),\mathbf{Occ}(B(\mathbf{x})),\ldots$ etc.

Definition 2.2.4. Let $\mathbf{Aut_P}(\mathbf{B}_{M_4})$ be a set of the all measure-preserving automorphism of

$\mathbf{B}_{M_4}$. This is a group,being a subgroup of the group $\mathbf{Aut}(\mathbf{B}_{M_4})$ of all Boolean automorphism

of $\mathbf{B}_{M_4}$. Let $P_{+}^{\uparrow}$ be Poincaré group.

Remark 2.2.5. We assume now that: any element $\Theta = (\Lambda,a) \in P_{+}^{\uparrow}$ induced an element

$\widetilde{\Theta} \in \mathbf{Aut_P}(\mathbf{B}_{M_4})$ by formula $\widetilde{\Theta} = \Theta[A(\mathbf{x})] = A(\Lambda\mathbf{x}+\mathbf{a}) \in \mathbf{B}_{M_4}$.

Definition 2.2.5. Given two events A and B from the algebra $\mathcal{F}_{M_4}^{ph} = (\mathbf{B}_{M_4},\mathbf{P})$ the conditional probability of A given B is defined as the quotient of the probability of the joint of events A and B, and the probability of B :

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \wedge B)}{\mathbf{P}(B)} = \frac{\mathbf{P}_{A\wedge B}}{\mathbf{P}_B} = \mathbf{P}_{A|B}, \tag{2.2.4}$$

where $\mathbf{P}(B) \neq 0$.

Definition 2.2.6. (i) Events A and B from the algebra $\mathcal{F}_{M_4}^{ph} = (\mathbf{B}_{M_4},\mathbf{P})$ are defined to be statistically independent or uncorrelated iff

$$\mathbf{P}_{A\wedge B} = \mathbf{P}_A \cdot \mathbf{P}_B, \tag{2.2.5}$$

where $\mathbf{P}_B \neq 0$, then this is equivalent to the statement that $\mathbf{P}_{A|B} = \mathbf{P}_A$. Similarly, if $\mathbf{P}_A$ is not zero, then $\mathbf{P}_{B|A} = \mathbf{P}_B$ is also equivalent.

(ii) Events A and B from the algebra $\mathcal{F} = (\mathbf{B}_{M_4},\mathbf{P})$ are defined to be statistically almost independent or almost uncorrelated iff

$$\begin{gathered}\mathbf{P}_{A\wedge B} \approx \mathbf{P}_A \cdot \mathbf{P}_B,\\ \mathbf{P}_{A\wedge B} = \mathbf{P}_A \cdot \mathbf{P}_B - \delta(A,B), 0 < \delta(A,B) \ll \mathbf{P}_A \cdot \mathbf{P}_B.\end{gathered} \tag{2.2.6}$$

Remark 2.2.6. Note that

$$\mathbf{P}_{A\vee B} = \mathbf{P}_A + \mathbf{P}_B - \mathbf{P}_{A\wedge B}. \tag{2.2.7}$$

Although mathematically equivalent, this may be preferred philosophically; under major probability interpretations such as the subjective theory, conditional probability is considered a primitive entity. Further, this "multiplication axiom" introduces a symmetry with the summation axiom for mutually exclusive events, i.e.

$$\mathbf{P}_{A\vee B} = \mathbf{P}_A + \mathbf{P}_B - \mathbf{P}_{A\wedge B}. \tag{2.2.8}$$

Definition 2.2.7. (i) Events $A_1,A_2,\ldots,A_n \in \mathcal{F}_{M_4}^{ph} = (\mathbf{B}_{M_4},\mathbf{P})$ are said to be exactly mutually exclusive if the occurrence of any one of them implies the non-occurrence of the remaining $n-1$ events. Therefore, two mutually exclusive events cannot both occur. Formally said, the conjunction of each two of them is $\mathbf{0}$ (the null event): $A \wedge B = \mathbf{0}$. In consequence, exactly mutually exclusive events A and B have the property:

$$\mathbf{P}(A \wedge B) = 0. \tag{2.2.9}$$

(ii) Events $A_1, A_2, \ldots, A_n \in \mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ are said to be almost mutually exclusive if

$A_1, A_2, \ldots, A_n$ have the property:

$$\begin{gathered} \mathbf{P}(A_1 \wedge A_2 \wedge \ldots \wedge A_n) \approx 0, \\ \mathbf{P}(A_1 \wedge A_2 \wedge \ldots \wedge A_n) \ll \mathbf{P}(A_1) \cdot \mathbf{P}(A_2) \cdots \mathbf{P}(A_n). \end{gathered} \tag{2.2.10}$$

In consequence, almost mutually exclusive events A and B have the property:

$$\begin{gathered} \mathbf{P}(A \wedge B) \approx 0, \\ \mathbf{P}(A \wedge B) \ll \mathbf{P}(A) \cdot \mathbf{P}(B). \end{gathered} \tag{2.2.11}$$

Remark 2.2.7. Let A^{ph}, B^{ph} be events such that detectors A,B detect photon at an instants t_1 and t_2 correspondingly. Note that (2.2.1) and (2.2.2) show that whether the separation between the detectors is timelike or spacelike, the number of coincidences is three orders of magnitude smaller than what would be expected had the events been statistically almost uncorrelated, i.e., $\mathbf{P}_{A \wedge B} \approx \mathbf{P}_A \cdot \mathbf{P}_B$, see Def.2.2.5(ii).

Remark 2.2.8. Let A^{ph}, B^{ph} be events such that detectors **A**,**B** detect photon at an instants t_1 and t_2 correspondingly. Note that:

(i) from Eq.(2.2.1) follows probability value for spacelike separation:

$$\mathbf{P}^{SL}_{A^{ph} \wedge B^{ph}} = 0.002 \pm 0.001 \cdot 10^{-4} \neq 0, \tag{2.2.12}$$

(ii) from Eq.(2.2.2) follows probability value for timelike separation:

$$\mathbf{P}^{TL}_{A^{ph} \wedge B^{ph}} = 0.002 \pm 0.001 \cdot 10^{-4} \neq 0. \tag{2.2.13}$$

Therefore in both cases the property (2.2.9) are violated, i.e. $\mathbf{P}_{A^{ph} \wedge B^{ph}} \neq 0$ but however in both cases the property (2.2.11) is satisfied

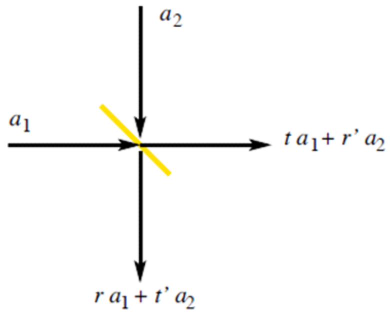
$$\begin{gathered} 0.002 \pm 0.001 \cdot 10^{-4} = \mathbf{P}^{SL}_{A^{ph} \wedge B^{ph}} \ll \mathbf{P}^{SL}_{A^{ph}} \cdot \mathbf{P}^{SL}_{B^{ph}} = 1.86 \pm 0.01 \cdot 10^{-4}, \\ 0.002 \pm 0.001 \cdot 10^{-4} = \mathbf{P}^{TL}_{A^{ph} \wedge B^{ph}} \ll \mathbf{P}^{TL}_{A^{ph}} \cdot \mathbf{P}^{TL}_{B^{ph}} = 1.65 \pm 0.01 \cdot 10^{-4} \end{gathered} \tag{2.2.12}$$

and therefore in both cases the events A^{ph}, B^{ph} are almost mutually exclusive events.

Beamsplitter transformation.

A beamsplitter is the most simple way to mix two modes, see Figure 2.2.7. From classical electrodynamics, one gets the following amplitudes for the outgoing modes:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}_{\mathbf{in}} \mapsto \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}_{\mathbf{out}} = \begin{pmatrix} t & r \\ r' & t' \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}_{\mathbf{in}}. \tag{2.2.13}$$



Pic. 2.2.7.Mixing of two modes by a beam splitter.

The recipe for quantization is now: ‘replace the classical amplitudes by annihilation operators’. If the outgoing modes are still to be useful for the quantum theory, they have to satisfy the commutation relations:

$$[A_i(out), A_j(out)] = \delta_{ij} \tag{2.2.14}$$

These conditions give constraints on the reflection and transmission amplitudes, for example $|t'|^2 + |r'|^2 = 1.$ We are now looking for an unitary operator $\mathbf{S}$ [the $\mathbf{S}$-matrix] that implements this beamsplitter transformation in the following sense:

$$A_i = \mathbf{S}^\dagger a_i \mathbf{S}, i = 1,2. \tag{2.2.15}$$

Let us start from the general transformation (summation over double indices)

$$a_i \mapsto A_i = B_{ij} a_i, \vec{a} \mapsto \vec{A} = \mathbf{B}\vec{a} \tag{2.2.16}$$

where we have introduced matrix and vector notation. Using this $\mathbf{S}$-matrix one can also compute the transformation of the states: $|out\rangle = \mathbf{S}|in\rangle$. For the unitary transformation, we make the ansatz

$$\mathbf{S}(\theta) = \exp(i\theta J_{ki} a_k^\dagger a_i) \tag{2.2.17}$$

with J_{kl} a hermitean matrix (ensuring unitarity). The action of this unitary on the photon mode operators is now required to reduce to

$$a_i \mapsto A_i(\theta) = \mathbf{S}^\dagger(\theta) a_i \mathbf{S}(\boldsymbol{\theta}) = B_{ij} a_j. \tag{2.2.18}$$

We compute this ‘operator conjugation’ by using a differential equation:

$$\frac{dA_i(\theta)}{d\theta} = iJ_{ki} A_i(\theta). \tag{2.2.19}$$

This is a system of linear differential equations with constant coefficients, so that one obtains a solution

$$\vec{A}(\theta) = \exp(i\theta \mathbf{J}). \tag{2.2.20}$$

We thus conclude that the so-called generator $\mathbf{J}$ of the beam splitter matrix is fixed by equatuon

$$\mathbf{B} = \exp(i\theta \mathbf{J}). \tag{2.2.21}$$

If the transformation $\mathbf{B}$ is part of a continuous group and depends on θ as a parameter, we can expand it around unity. Doing the same for the matrix exponential, we get

$$\mathbf{B} = 1 + i\theta\mathbf{J} + \ldots. \tag{2.2.22}$$

Equation (2.2.22) explains the name generator for the matrix $\mathbf{J}$: it actually generates a subgroup of matrices $\mathbf{B} = \mathbf{B}(\theta)$ parametrized by the angle θ. The unitary transformation we are looking for is thus determined via the same generator $\mathbf{J}$. For the two-mode beam splitter, an admissible transformation is given by

$$\mathbf{B}(\theta) = \begin{pmatrix} t & r \\ r' & t' \end{pmatrix} = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} \tag{2.2.23}$$

The factor i is just put for convenience so that the reflection amplitudes are the same for both sides, $r = r'$, as expected by symmetry. Expanding for small θ, the generator is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \tag{2.2.24}$$

so that the unitary operator for this beamsplitter is

$$S(\theta) = \exp[i\theta(a_1^\dagger a_2 + a_2^\dagger a_1)]. \tag{2.2.25}$$

Therefore, the effective Hamiltonian of the beam splitter is given by

$$\widehat{H}_{\mathbf{eff}} = a_1^\dagger a_2 + a_2^\dagger a_1. \tag{2.2.26}$$

Splitting a two-photon state

Let us consider two single photon states $|in\rangle = |1,1\rangle$ incident on the beam splitter such that mentioned above.Then

$$\begin{gathered} |\psi(\theta)\rangle = |out\rangle = \mathbf{S}|in\rangle = \mathbf{S}a_1^\dagger S^\dagger S a_2^\dagger S^\dagger S|0,0\rangle = \\ (a_1^\dagger\cos\theta + ia_2^\dagger\sin\theta)(a_2^\dagger\cos\theta + ia_1^\dagger\sin\theta)|0,0\rangle = \\ (|2,0\rangle - |0,2\rangle)\frac{\sin\theta}{\sqrt{2}} + |1,1\rangle\cos\theta. \end{gathered} \tag{2.2.27}$$

Let $\mathbf{H}$ be a complex Hilbert space such that

$$\begin{gathered} \forall\theta[|\psi(\theta)\rangle_{cl} \in \mathbf{H}], \\ \forall\theta\forall\delta(\delta \in (0,1])\forall\varepsilon(\varepsilon \in (0,1])\left[|\psi^i_{\delta,\varepsilon,\mathbf{x}}(\theta)\rangle_{cl} \in \mathbf{H}\right], \\ |\psi^i_{\delta,\varepsilon,\mathbf{x}}(\theta)\rangle_{cl} = L^i_{\mathbf{x}}(\delta,\varepsilon)|\psi(\theta)\rangle_{cl}. \end{gathered} \tag{2.2.28}$$

By postulate C.I (see Appendix C) quantum system wih Hamiltonian given by Eq.(2.2.26) is identified by a set $\Xi \triangleq \left\langle \mathbf{H}, \widehat{H}_{\mathbf{eff}}, \Im, \Re, \mathcal{L}_{2,1}(\Omega), \mathbf{G}, |\psi_t\rangle \right\rangle$, where

(i) $\mathbf{H}$ that is a complex Hilbert space defined above,
(ii) $\Im = (\Omega, \mathcal{F}, \mathbf{P})$ that is complete probability space,
(iii) $\Re = (\mathbb{R}^n, \Sigma)$ that is measurable space ,

(iv) $\mathcal{L}_{2,1}(\Omega)$ that is complete space of random variables $X : \Omega \to \mathbb{R}^n$ such that

$$\int_\Omega \|X(\omega)\| d\mathbf{P} < \infty, \int_\Omega \|X(\omega)\|^2 d\mathbf{P} < \infty, \tag{2.2.29}$$

(v) $\mathbf{G} : \mathbf{H} \to \mathcal{L}_{2,1}(\Omega)$ that is one to one correspondence such that

$$\left|\langle\psi|\widehat{Q}|\psi\rangle\right| = \int_\Omega \left(\mathbf{G}\left[\widehat{Q}|\psi\rangle\right](\omega)\right) d\mathbf{P} = \mathbf{E}_\Omega\left[\mathbf{G}\left[\widehat{Q}|\psi\rangle\right](\omega)\right] \tag{2.2.30}$$

for any $|\psi\rangle \in \mathbf{H}$ and for any Hermitian operator $\widehat{Q} : \mathbf{H} \to \mathbf{H}$,

(vi) $|\psi_t\rangle$ is an continuous vector function $|\psi_t\rangle : \mathbb{R}_+ \to \mathbf{H}$ which representedthe canonical evolution of the quantum system Ξ.

Remark 2.2.9. Note that $\Im^{ph}_{M_4} = \mathcal{F} \times M_4 = (\Omega, \Sigma, \mathbf{P}) \times M_4$,where $\mathcal{F}$ is a probability measure

algebra $\mathcal{F} = (\Omega, \Sigma, \mathbf{P})$ on the Boolean algebra of measurable sets modulo null sets, see

Remark 2.2.4.

Let $\mathbf{B}_{M_4}$ be Boolean algebra of the all physical events in Minkowski spacetime M_4 and let

$\widetilde{\mathcal{F}}_{M_4}$ be an measure algebra $\widetilde{\mathcal{F}}_{M_4} = \left(\mathbf{B}_{M_4}, \widetilde{\mathbf{P}}\right)$ with a probability measure $\widetilde{\mathbf{P}}$, see

Definition 2.2.2 (vii).

We assume now that there exist subalgebra $\mathcal{F}^{\#}_{M_4} \subsetneq \widetilde{\mathcal{F}}_{M_4}$ and isomorphism

$\lambda[\bullet] : \mathcal{F}^{\#}_{M_4} \mapsto \Im^{ph}_{M_4}$ such that for any event $A(\mathbf{x}) \in \mathcal{F}^{\#}_{M_4}, \mathbf{x} = (t, x_1, x_2, x_3) \in M_4$ (see Definition

2.2.2):

$$\begin{gathered} \lambda[A(\mathbf{x})] = \lambda[A](\mathbf{x}), \\ \widetilde{\mathbf{P}}(A(\mathbf{x})) = \mathbf{P}(\lambda[A](\mathbf{x})) \triangleq \mathbf{P}(A_\lambda(\mathbf{x})). \end{gathered} \tag{2.2.31}$$

Proposition 2.2.1.Suppose that A and B are events in measure algebra

$\widetilde{\mathcal{F}}_{M_4} = \left(\mathbf{B}_{M_4}, \widetilde{\mathbf{P}}\right)$. Then following properties is satisfied:

$$\begin{aligned} &1.\, \mathbf{P}(A|B) > \mathbf{P}(A) \Leftrightarrow \mathbf{P}(B|A) > \mathbf{P}(B) \Leftrightarrow \mathbf{P}(A \wedge B) > \mathbf{P}(A)\mathbf{P}(B) \\ &2.\, \mathbf{P}(A|B) < \mathbf{P}(A) \Leftrightarrow \mathbf{P}(B|A) < \mathbf{P}(B) \Leftrightarrow \mathbf{P}(A \wedge B) < \mathbf{P}(A)\mathbf{P}(B) \\ &3.\, \mathbf{P}(A|B) = \mathbf{P}(A) \Leftrightarrow \mathbf{P}(B|A) = \mathbf{P}(B) \Leftrightarrow \mathbf{P}(A \wedge B) = \mathbf{P}(A)\mathbf{P}(B) \end{aligned} \tag{2.2.32}$$

Proposition 2.2.2.Suppose that A and B are events in measure algebra $\Im_{M_4} = (\Omega, \mathcal{F}, \mathbf{P})$.

Then following properties is satisfied:

$$\begin{aligned} &1.\, \mathbf{P}(A|B) > \mathbf{P}(A) \Leftrightarrow \mathbf{P}(B|A) > \mathbf{P}(B) \Leftrightarrow \mathbf{P}(A \cap B) > \mathbf{P}(A)\mathbf{P}(B) \\ &2.\, \mathbf{P}(A|B) < \mathbf{P}(A) \Leftrightarrow \mathbf{P}(B|A) < \mathbf{P}(B) \Leftrightarrow \mathbf{P}(A \cap B) < \mathbf{P}(A)\mathbf{P}(B) \\ &3.\, \mathbf{P}(A|B) = \mathbf{P}(A) \Leftrightarrow \mathbf{P}(B|A) = \mathbf{P}(B) \Leftrightarrow \mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B) \end{aligned} \tag{2.2.33}$$

Definition 2.2.8.In case (1), A and B are said to be positively correlated.

Intuitively, the occurrence of either event means that the other event is more likely.

In case (2), A and B are said to be negatively correlated.

Intuitively, the occurrence of either event means that the other event is less likely.

In case (3), A and B are said to be uncorrelated or independent.

Intuitively, the occurrence of either event does not change the probability of the other event.

Remark 2.2.10. Suppose that A and B are events in measure algebra $\Im_{M_4} = (\Omega, \mathcal{F}, \mathbf{P})$.

Note from the result above that if $A \subseteq B$ or $B \subseteq A$ then A and B are positively correlated. If

A and B are disjoint then A and B are negatively correlated.

Proposition 2.2.3.Suppose that A and B are events in measure algebra $\widetilde{\mathcal{F}}_{M_4} = \left(\mathbf{B}_{M_4}, \widetilde{\mathbf{P}}\right)$. Then:

(i) A and B have the same correlation (positive, negative, or zero) as $\neg A$ and $\neg B$.

(ii) A and B have the opposite correlation as A and $\neg B$ (that is, positive-negative, negative-positive, or zero-zero).

Proposition 2.2.4.Suppose that A and B are events in measure algebra $\Im_{M_4} = (\Omega, \mathcal{F}, \mathbf{P})$.

Then:

(i) A and B have the same correlation (positive, negative, or zero) as A^c and B^c.

(ii) A and B have the opposite correlation as A and B^c (that is, positive-negative, negative-positive, or zero-zero).

Definition 2.2.9.Let $A(\mathbf{x}_1) = A(t_1, \mathbf{r}_1)$ and $B(\mathbf{x}_2) = B(t_2, \mathbf{r}_2)$ be an events $A(\mathbf{x}_1) \in \mathcal{F}^{\#}_{M_4}$ which occurs at instant t_1 and $B(\mathbf{x}_2) \in \mathcal{F}^{\#}_{M_4}$ at instant t_2 correspondingly.

Let $\mathbf{x}_{1,2}$ be a vector: $\mathbf{x}_{1,2} = \{c(t_1 - t_2), \mathbf{r}_1 - \mathbf{r}_2\} = (ct_{1,2}, \mathbf{r}_{1,2}), t_{1,2} = t_1 - t_2, \mathbf{r}_{\mathbf{1,2}} = \mathbf{r}_1 - \mathbf{r}_2$.

Vectors $\mathbf{x}_{1,2} = (ct_{1,2}, \mathbf{r}_{1,2})$ are classified according to the sign of $c^2 t_{1,2}^2 - \mathbf{r}_{1,2}^2$. A vector is

(i) timelike if $c^2 t_{1,2}^2 > \mathbf{r}_{\mathbf{1,2}}^{\mathbf{2}}$, (ii) spacelike if $c^2 t_{1,2}^2 < \mathbf{r}_{\mathbf{1,2}}^{\mathbf{2}}$, and null or lightlike if (iii) $c^2 t_{1,2}^2 = \mathbf{r}_{\mathbf{1,2}}^{\mathbf{2}}$.

Pairs of events $\{A(t_1, \mathbf{r}_1), B(t_2, \mathbf{r}_2)\} \in \mathcal{F}^{\#}_{M_4} \times \mathcal{F}^{\#}_{M_4}$ are classified according to the sign of

$c^2 t_{1,2}^2 - \mathbf{r}_{1,2}^2$:

(i) a pair $\{A(t_1, \mathbf{r}_1), B(t_2, \mathbf{r}_2)\}$ is timelike separated if $c^2 t_{1,2}^2 > \mathbf{r}_{\mathbf{1,2}}^{\mathbf{2}}$,

and we denoted such pairs by $\{A(t_1, \mathbf{r}_1), B(t_2, \mathbf{r}_2)\}_{\mathbf{t.l.s.}}$

(ii) a pair $\{A(t_1, \mathbf{r}_1), B(t_2, \mathbf{r}_2)\}$ is spacelike separated if $c^2 t_{1,2}^2 < \mathbf{r}_{\mathbf{1,2}}^{\mathbf{2}}$,

and we denoted such pairs by $\{A(t_1, \mathbf{r}_1), B(t_2, \mathbf{r}_2)\}_{\mathbf{s.l.s.}}$

(iii) a pair $\{A(t_1, \mathbf{r}_1), B(t_2, \mathbf{r}_2)\}$ is null or lightlike separated if $c^2 t_{1,2}^2 = \mathbf{r}_{\mathbf{1,2}}^{\mathbf{2}}$.

and we denoted such pairs by $\{A(t_1, \mathbf{r}_1), B(t_2, \mathbf{r}_2)\}_{\mathbf{l.l.s.}}$

Definition 2.2.10.(i) Let $[\mathcal{F}^{\#}_{M_4}, \{(t_1, \mathbf{r}_1), (t_2, \mathbf{r}_2)\}]_{\mathbf{t.l.s.}}$ be a set of the all timelike separated pairs $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\}_{\mathbf{t.l.s.}}$ which are corresponding to a given vector $\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\} \in M_4 \times M_4$, i.e.,

$$[\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}} = \{\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\} \in \mathcal{F}^{\#}_{M_4} \times \mathcal{F}^{\#}_{M_4} | c^2 t^2_{1,2} > \mathbf{r}^2_{1,2}\}. \quad (2.2.34.a)$$

(ii) Let $[\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}}$ be a set of the all spacelike separated pairs $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\}_{\mathbf{s.l.s.}}$ which is corresponding to a given vector $\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\} \in M_4 \times M_4$, i.e.,

$$[\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}} = \{\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\} \in \mathcal{F}^{\#}_{M_4} \times \mathcal{F}^{\#}_{M_4} | c^2 t^2_{1,2} < \mathbf{r}^2_{1,2}\}. \quad (2.2.34.b)$$

Remark 2.2.11. Let $[\mathcal{F}^{\#}_{M_4},\{(t,\mathbf{r}_1),(t,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}}$ be a set of the all pairs $\{A(t,\mathbf{r}_1),B(t,\mathbf{r}_2)\}$
which is corresponding to a given vector $\{(t,\mathbf{r}_1),(t,\mathbf{r}_2)\} \in M_4 \times M_4,\ \mathbf{r}_1 \neq \mathbf{r}_2$, i.e.,

$$[\mathcal{F}^{\#}_{M_4},\{(t,\mathbf{r}_1),(t,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}} = \{\{A(t,\mathbf{r}_1),B(t,\mathbf{r}_2)\} \in \mathcal{F}^{\#}_{M_4} \times \mathcal{F}^{\#}_{M_4} | 0 < \mathbf{r}^2_{1,2}\}, \quad (2.2.35)$$
$$\mathbf{r}_{1,2} = \mathbf{r}_1 - \mathbf{r}_2.$$

Such pairs obviously is spacelike separated. Note that

$$\forall t \forall \mathbf{r}_1 \forall \mathbf{r}_2 (\mathbf{r}_1 \neq \mathbf{r}_2) \left\{[\mathcal{F}^{\#}_{M_4},\{(t,\mathbf{r}_1),(t,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}} \neq \varnothing\right\}. \quad (2.2.36)$$

Definition 2.2.11. Let $A^{t_1} \triangleq A(\mathbf{x}_1) = A(t_1,x_A)$ and $B^{t_2} \triangleq B(\mathbf{x}_2) = B(t_2,x_B)$ be a symbols such that A^{t_1} and B^{t_2} represent there is detection events $A(\mathbf{x}_1) \in \mathcal{F}^{\#}_{M_4}$ at instant t_1 and $B(\mathbf{x}_2) \in \mathcal{F}^{\#}_{M_4}$ at instant t_2 correspondingly, where symbols x_A and x_B represent the locations of the detectors A and B correspondingly (see Pic. 2.2.4). We assume that

$$\{A^{t_1},B^{t_2}\} \in [\mathcal{F}^{\#}_{M_4},\{(t_1,x_A),(t_2,x_B)\}]_{\mathbf{s.l.s.}}. \quad (2.2.37)$$

Remark 2.2.12. We assume now without loss of generality that $t_1 = t_2 = t$, note that such assuption valid by properties: $A(\Lambda\mathbf{x}_1) \in \mathcal{F}^{\#}_{M_4}$ and $B(\Lambda\mathbf{x}_2) \in \mathcal{F}^{\#}_{M_4}$, required above, see Remark 2.2.5.

Einstein's 1927 gedanken experiment resolution

In classical case considered by A. Einstein in his 1927 gedanken experiment, by postulates of canonical QM, both events $A^t \in \mathcal{F}^{\#}_{M_4}$ and $B^t \in \mathcal{F}^{\#}_{M_4}$ cannot occur simultaneously, i.e. that is mutually exclusive events with a probability $= 1$, and therefore $A^t \wedge B^t = \mathbf{0}$. Such exactly mutually exclusive events have the property:

$$\widetilde{\mathbf{P}}(A^t \wedge B^t) = 0, \quad (2.2.38)$$

see Definition 2.2.6.

We remind that the probability density $p^{\mathbf{ph}}(x,\delta,\varepsilon)$ for the occurrence of a photon localization at point x is assumed to be

$$p^{\mathbf{ph}}(x,\delta,\varepsilon) = \left\| \left|\psi^{\mathbf{ph}}_{\delta,\varepsilon,x}(\theta)\right\rangle_{cl} \right\|^2, \quad \delta \in (0,1], \varepsilon \in (0,1], \tag{2.2.39}$$

where

$$\left|\psi^{\mathbf{ph}}_{\delta,\varepsilon,x}\right\rangle_{cl} = L_x(\delta,\varepsilon)|\psi^{\mathbf{ph}}(\theta)\rangle_{cl}. \tag{2.2.40}$$

and where the localization operators $L_{\mathbf{x}}(\delta,\varepsilon)$ have been chosen to have the form:

$$\widehat{L}_x(\widehat{q},\delta,\varepsilon) = \begin{cases} \left(\frac{1}{\delta\pi_\delta}\right)^{1/4} \exp\left[-\frac{1}{2\delta}(\widehat{q}-x)^2\right] \text{ iff } |\widehat{q}-x| \leq \varepsilon \ll 1, \\ 0 \text{ iff } |\widehat{q}-x| > \varepsilon. \end{cases} \tag{2.2.41}$$

see subsection **I**.2.4.

Remark 2.2.13. Note that: (i) from (2.2.28) follows that $\left|\psi^{\mathbf{ph}}_{\delta,\varepsilon,\mathbf{x}}(\theta)\right\rangle_{cl} \in \mathbf{H}$,
(ii) from (2.2.39) and (2.2.41) where $\delta \ll 1$ follows that

$$\begin{gathered} p^{\mathbf{ph}}(x,\delta,\varepsilon) = \left\| \left|\psi^{\mathbf{ph}}_{\delta,\varepsilon,x}(\theta)\right\rangle_{cl} \right\|^2 = \int dq \left\langle \psi^{\mathbf{ph}}_{\delta,\varepsilon,x}(\theta) \middle| |q\rangle\langle q| \middle| \psi^{\mathbf{ph}}_{\delta,\varepsilon,x}(\theta) \right\rangle = \\ \int dq \left\langle \widehat{L}_x(\widehat{q},\delta,\varepsilon)\psi^{\mathbf{ph}}(\theta) \middle| |q\rangle\langle q| \middle| \widehat{L}_x(\widehat{q},\delta,\varepsilon)\psi^{\mathbf{ph}}(\theta) \right\rangle = \\ \int dq L^2_x(q,\delta,\varepsilon)\langle\psi^{\mathbf{ph}}(\theta)||q\rangle\langle q||\psi^{\mathbf{ph}}(\theta)\rangle = \|\langle x||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\|^2 + O(\delta) \asymp \\ \asymp \|\langle x||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\|^2, \\ \delta \ll 1, \varepsilon \in (0,1], \end{gathered} \tag{2.2.42}$$

From postulate C.I.3 follows that there exist unique random variable $X\left(\omega; \left||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right\rangle\right)$ given on a probability space $(\Omega,\mathcal{F},\mathbf{P})$ and a measurable space $(\mathbb{R}^n,\Sigma)$ by formula

$$X\left(\omega; \left||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right\rangle\right) \triangleq X_{|\psi^{\mathbf{ph}}(\theta)\rangle}(\omega) = \mathbf{G}\left[\left||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right\rangle\right] \tag{1.2.43}$$

The probability density of random variable $X_{|\psi^{\mathbf{ph}}(\theta)\rangle}(\omega)$ we denote by $p_{|\psi^{\mathbf{ph}}(\theta)\rangle}(q), q \in \mathbb{R}$.

Remark 2.2.14. From postulate Q.II.2 (see subsection I.7.1) follows that for the system in state $\left||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right\rangle$ the probability $P\left(q,q+dq; \left||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right\rangle\right)$ of obtaining the result q lying in the range $(q,q+dq)$ on measuring observable $\widehat{q}$ given by

$$P\left(q,q+dq; \left||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right\rangle\right) = p_{|\psi^{\mathbf{ph}}(\theta)\rangle_{cl}}(q)dq = \left|c_{|\psi^{\mathbf{ph}}(\theta)_{cl}\rangle}(q)\right|^2 = \left|\langle q|\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right|^2 \tag{2.2.44}$$

Now we go to explain Einstein's 1927 gedanken experiment. Let $A^{ph}(t,x_A)$ and $B^{ph}(t,x_A)$ be events such that detectors A,B detect photon at an instant t correspondingly. By properties (2.2.31) we obtain

$$\mathbf{P}\left(A_\lambda^{ph}(t,x_A)\right) \triangleq \mathbf{P}(\lambda[A^{ph}](t,x_A)) = \widetilde{\mathbf{P}}(A^{ph}(t,x_A)),$$
$$\mathbf{P}\left(B_\lambda^{ph}(t,x_B)\right) \triangleq \mathbf{P}(\lambda[B^{ph}](t,x_B)) = \widetilde{\mathbf{P}}(B^{ph}(t,x_B)). \quad (2.2.45)$$

Note that

$$A^t \triangleq A_\lambda^{ph}(t,x_A) = \left\{\omega | x_A - \epsilon \leq X_{|\psi^{\mathbf{ph}}(\theta)\rangle}(\omega) \leq x_A - \epsilon\right\},$$
$$B^t \triangleq B_\lambda^{ph}(t,x_B) = \left\{\omega | x_B - \epsilon \leq X_{|\psi^{\mathbf{ph}}(\theta)\rangle}(\omega) \leq x_B - \epsilon\right\}, \quad (2.2.46)$$
$$\epsilon \in (0,\gamma], \gamma \ll 1,$$

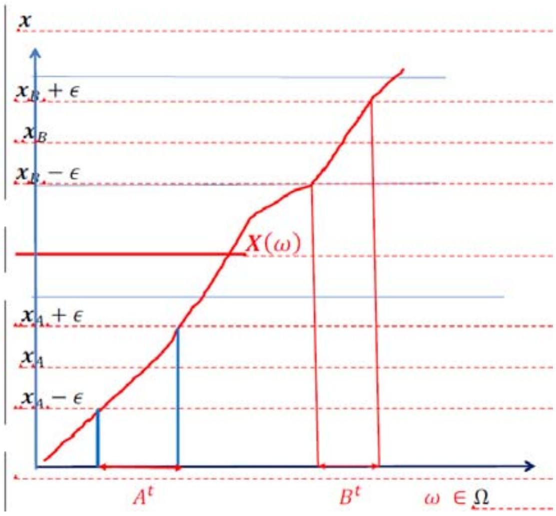
where a small parameter $\epsilon \ll |x_A - x_B|$ dependent on measuring device. Thus by general definition of random variable one obtains directly

$$A_\lambda^{ph}(t,x_A) \cap B_\lambda^{ph}(t,x_B) = \varnothing \quad (2.2.47)$$

and therefore

$$\mathbf{P}\left(A_\lambda^{ph}(t,x_A) \cap B_\lambda^{ph}(t,x_B)\right) = 0 \quad (2.2.48)$$

The property (2.2.48) follows directly from (2.2.45).



Pic. 2.2.8.The plot of the random variable $X_{|\psi^{\mathbf{ph}}(\theta)\rangle}(\omega)$.

$A^t = A_\lambda^{ph}(t,x_A), B^t = B_\lambda^{ph}(t,x_B), A^t \cap B^t = \varnothing$.

Remark 2.2.15. Let $[\mathcal{F}_{M_4}^{\#}, \{(t,x_A),(t,x_B)\}]_{\mathbf{s.l.s.}}$ be a set of the all pairs $\{A(t,x_A), B(t,x_B)\}$

which is corresponding to a given vector $\{(t,x_A,0,0),(t,x_B,0,0)\} \in M_4 \times M_4$, $x_A \neq x_B$, i.e.,

$$[\mathcal{F}_{M_4}^{\#}, \{(t,x_A),(t,x_B)\}]_{\mathbf{s.l.s.}} =$$
$$\{\{A(t,x_A), B(t,x_B)\} \in \mathcal{F}_{M_4}^{\#} \times \mathcal{F}_{M_4}^{\#} | 0 < (x_A - x_B)^2\}. \quad (2.2.49)$$

Such pairs obviously is spacelike separated. Note that

$$\forall t \forall x_A \forall x_B (x_A \neq x_B)\left\{[\mathcal{F}_{M_4}^{\#}, \{(t,x_A),(t,x_B)\}]_{\mathbf{s.l.s.}} \neq \varnothing\right\}. \quad (2.2.50)$$

Now we go to explain non zero result $\widetilde{\mathbf{P}}(A^t \wedge B^t) \neq 0$ given above by (2.2.1) and (2.2.2):

$$\begin{aligned} &\widetilde{\mathbf{P}}^{TL}_{A^t \wedge B^t} = 0.002 \pm 0.001 \cdot 10^{-4}, \mathbf{P}^{TL}_{A^t} \cdot P^{TL}_{B^t} = 1.65 \pm 0.01 \cdot 10^{-4}, \\ &\widetilde{\mathbf{P}}^{SL}_{A^t \wedge B^t} = 0.002 \pm 0.001 \cdot 10^{-4}. \mathbf{P}^{SL}_{A^t \wedge B^t} = 0.002 \pm 0.001 \cdot 10^{-4}. \end{aligned} \tag{2.2.51}$$

We consider this problem in general case.

Remark 2.2.16. Note that: (i) a probability density $p(x, \delta_A, \epsilon,)$ for the occurrence of a localization inside interval $[x - \epsilon, x + \epsilon]$ in arm with detector **A** (see Pic. 2.2.2) is given by formula (see Remark 1.2.4)

$$p(x, \delta_A, \epsilon) = \frac{\| |\psi_{\delta_A, \epsilon, x}\rangle_{cl} \|^2}{\Delta(\delta_A, \epsilon)}, \tag{2.2.52}$$

where

$$\begin{aligned} &\| |\psi_{\delta_A, \epsilon, x}\rangle_{cl} \|^2 = \left(\frac{1}{\delta_A \pi_{\delta_A}} \right)^{1/2} \int\limits_{|q - x| \leq \epsilon} dq |\psi(q)|^2 \exp\left[-\frac{1}{\delta_A} (q - x)^2 \right], \\ &\psi(q) = \langle q \| \psi \rangle, \\ &\Delta(\delta_A, \epsilon) = \int\limits_{-\infty}^{\infty} \| |\psi_{\delta_A, \epsilon, x}\rangle_{cl} \|^2 dx, \end{aligned} \tag{2.2.53}$$

and where parametr δ_A depend on arm with detector **A**.

(ii) a probability density $p(x, \delta_B, \epsilon,)$ for the occurrence of a localization inside interval $[x - \epsilon, x + \epsilon]$ in arm with detector **B** (see Pic. 2.2.2) is given by formula (see Remark 1.2.4)

$$p(x, \delta_B, \epsilon) = \frac{\| |\psi_{\delta_B, \epsilon, x}\rangle_{cl} \|^2}{\Delta(\delta_B, \epsilon)}, \tag{2.2.54}$$

where

$$\begin{aligned} &\| |\psi_{\delta_B, \epsilon, x}\rangle_{cl} \|^2 = \left(\frac{1}{\delta_B \pi_{\delta_B}} \right)^{1/2} \int\limits_{|q - x| \leq \epsilon} dq |\psi(q)|^2 \exp\left[-\frac{1}{\delta_B} (q - x)^2 \right], \\ &\psi(q) = \langle q \| \psi \rangle, \\ &\Delta(\delta_B, \epsilon) = \int\limits_{-\infty}^{\infty} \| |\psi_{\delta_B, \epsilon, x}\rangle_{cl} \|^2 dx, \end{aligned} \tag{2.2.55}$$

and where parametr δ_B depend on arm with detector **B**.

Remark 2.2.17.Note that parametr δ in formula (1.2.27) (see Remark 1.2.4) of course depend on measurement device and there no exist two equivalet devices such that $\delta_A = \delta_B$.

We assume now that

$$\begin{gathered}\delta_A \simeq \delta_B \ll 1,\\ 0 < |\delta_A - \delta_B|,\\ \int_{-\infty}^{\infty}\left[|\psi(x)|^2\right]'' dx < \infty,\end{gathered} \tag{2.2.56}$$

From Eq.(2.2.53) and (2.2.56) by using laplace method [24], we obtain:

$$\begin{gathered}\| |\psi_{\delta_A,\epsilon,x}\rangle_{cl}\|^2 = \left(\frac{1}{\delta_A \pi_{\delta_A}}\right)^{1/2} \int\limits_{|q-x|\leq\epsilon} dq |\psi(q)|^2 \exp\left[-\frac{1}{\delta_A}(q-x)^2\right] \approx\\ \approx |\psi(x)|^2 + \delta_A O\left(\left[|\psi(x)|^2\right]''\right) = |\psi(x)|^2 + \delta_A c_1^A \left[|\psi(x)|^2\right]'',\\ \Delta(\delta_A,\epsilon) = \int\limits_{-\infty}^{\infty} \| |\psi_{\delta_A,\epsilon,x}\rangle_{cl}\|^2 dx = 1 + c_2^A \delta_A, c_2^A = O\left(\int_{-\infty}^{\infty}\left[|\psi(x)|^2\right]'' dx\right).\end{gathered} \tag{2.2.57}$$

From Eq.(2.2.55) and (2.2.56) by using laplace method [24], we obtain:

$$\begin{gathered}\| |\psi_{\delta_B,\epsilon,x}\rangle_{cl}\|^2 = \left(\frac{1}{\delta_B \pi_{\delta_B}}\right)^{1/2} \int\limits_{|q-x|\leq\epsilon} dq |\psi(q)|^2 \exp\left[-\frac{1}{\delta_B}(q-x)^2\right] \approx\\ |\psi(x)|^2 + \delta_B O\left(\left[|\psi(x)|^2\right]''\right) = |\psi(x)|^2 + \delta_B c_1^B \left[|\psi(x)|^2\right]'',\\ \Delta(\delta_B,\epsilon) = \int\limits_{-\infty}^{\infty} \| |\psi_{\delta_B,\epsilon,x}\rangle_{cl}\|^2 dx = 1 + c_2^B \delta_B, c_2^B = O\left(\int_{-\infty}^{\infty}\left[|\psi(x)|^2\right]'' dx\right).\end{gathered} \tag{2.2.58}$$

From Eq.(2.2.52) and Eq.(2.2.56) we obtain

$$p(x,\delta_A,\epsilon) = \frac{\| |\psi_{\delta_A,\epsilon,x}\rangle_{cl}\|^2}{\Delta(\delta_A,\epsilon)} = \frac{|\psi(x)|^2 + \delta_A c_1^A \left[|\psi(x)|^2\right]''}{1 + c_2^A \delta_A,}. \tag{2.2.59}$$

From Eq.(2.2.54) and Eq.(2.2.57) we obtain

$$p(x,\delta_B,\epsilon) = \frac{\| |\psi_{\delta_B,\epsilon,x}\rangle_{cl}\|^2}{\Delta(\delta_B,\epsilon)} = \frac{|\psi(x)|^2 + \delta_B c_1^B \left[|\psi(x)|^2\right]''}{1 + c_2^B \delta_B,}. \tag{2.2.60}$$

Definition 2.2.12.We define now the probability measures $\mathbf{P}_{|\psi_{\delta_A,\epsilon,x}\rangle}(A^t)$ and $\mathbf{P}_{|\psi_{\delta_B,\epsilon,x}\rangle}(A^t)$ by formulae

$$\begin{gathered}\mathbf{P}_{|\psi_{\delta_A,\epsilon,x}\rangle}(A^t) = \int\limits_{A^t} p(x,\delta_A,\epsilon)d\mu(x),\\ \mathbf{P}_{|\psi_{\delta_B,\epsilon,x}\rangle}(A^t) = \int\limits_{A^t} p(x,\delta_B,\epsilon)d\mu(x),\end{gathered} \tag{2.2.61}$$

where $A^t \in \Sigma_{a,b}$ and $d\mu(x)$ is the Lebesgue measure and $\Sigma_{a,b} = B([a,b])$ is the Borel algebra on a set $[a,b]$,(see subsection **I.6** Definition 1.6.3).

Definition 2.2.13.We assume now that $\mathbf{P}_{|\psi_{\delta_A,\epsilon,x}\rangle} \ll \mathbf{P}$ and $\mathbf{P}_{|\psi_{\delta_B,\epsilon,x}\rangle} \ll \mathbf{P}$, i.e. $\mathbf{P}_{|\psi\rangle}$ is absolutely continuous with respect to $\mathbf{P}$ (see subsection **I.6**,Eq.(1.6.5)).By

Radon-Nicodym theorem we obtain for any $A^t \in \Sigma_{a,b}$:

$$\mathbf{P}_{\left|\psi_{\delta_A,\epsilon,x}\right\rangle}(A^t) = \int_{A^t} X_{\left|\psi_{\delta_A,\epsilon,x}\right\rangle}(\omega) d\mathbf{P},$$
$$X_{\left|\psi_{\delta_A,\epsilon,x}\right\rangle}(\omega) = \frac{d\mathbf{P}_{\left|\psi_{\delta_A,\epsilon,x}\right\rangle}}{d\mathbf{P}},$$
$$\mathbf{P}_{\left|\psi_{\delta_A,\epsilon,x}\right\rangle}(A^t) = \int_{A^t} X_{\left|\psi_{\delta_A,\epsilon,x}\right\rangle}(\omega) d\mathbf{P},$$
$$X_{\left|\psi_{\delta_A,\epsilon,x}\right\rangle}(\omega) = \frac{d\mathbf{P}_{\left|\psi_{\delta_A,\epsilon,x}\right\rangle}}{d\mathbf{P}}. \quad (2.2.62)$$

We write below for a short

$$X_1(\omega) \triangleq X_{\left|\psi_{\delta_A,\epsilon,x}\right\rangle}(\omega), X_2(\omega) \triangleq X_{\left|\psi_{\delta_B,\epsilon,x}\right\rangle}(\omega). \quad (2.2.63)$$

Remark 2.2.18.We assume now without loss of generality that

$$X_2(\omega) - X_1(\omega) \geq 0 \text{ a.s.} \quad (2.2.64)$$

see Pic. 2.2.9.

Let us consider now the quantity

$$\eta_{1,2} = \int_{\Omega} |X_1(\omega) - X_2(\omega)| d\mathbf{P} = \int_{\Omega} [X_2(\omega) - X_1(\omega)] d\mathbf{P}. \quad (2.2.65)$$

We assume now that

$$\int_{-\infty}^{\infty} x|\psi(x)|^2 dx < \infty, \int_{-\infty}^{\infty} x\left[|\psi(x)|^2\right]'' dx < \infty, \quad (2.2.66)$$

From Eq.(2.2.65) by using Eq.(2.2.59) and Eq.(2.2.60) we obtain

$$\eta_{1,2} =$$

$$\int_{\mathbb{R}} xp(x,\delta_B,\epsilon)dx - \int_{\mathbb{R}} xp(x,\delta_A,\epsilon)dx = \frac{1}{1+c_2^B\delta_B}\int_{\mathbb{R}} x\Big[|\psi(x)|^2 + \delta_B c_1^B\big[|\psi(x)|^2\big]''\Big]dx -$$

$$-\frac{1}{1+c_2^A\delta_A}\int_{\mathbb{R}} x\Big[|\psi(x)|^2 + \delta_A c_1^A\big[|\psi(x)|^2\big]''\Big]dx \simeq$$

$$(1-c_2^B\delta_B)\int_{\mathbb{R}} x\Big[|\psi(x)|^2 + \delta_B c_1^B\big[|\psi(x)|^2\big]''\Big]dx -$$

$$-(1-c_2^A\delta_A)\int_{\mathbb{R}} x\Big[|\psi(x)|^2 + \delta_A c_1^A\big[|\psi(x)|^2\big]''\Big]dx = \tag{2.2.67}$$

$$\delta_B c_1^B\int_{\mathbb{R}} x\big[|\psi(x)|^2\big]''dx - c_2^B\delta_B\int_{\mathbb{R}} x|\psi(x)|^2dx - \delta_B^2 c_1^B c_2^B\int_{\mathbb{R}} x\big[|\psi(x)|^2\big]''dx -$$

$$-\delta_A c_1^A\int_{\mathbb{R}} x\big[|\psi(x)|^2\big]''dx + c_2^A\delta_A\int_{\mathbb{R}} x|\psi(x)|^2dx + \delta_A^2 c_1^A c_2^A\int_{\mathbb{R}} x\big[|\psi(x)|^2\big]''dx =$$

$$(\delta_B c_1^B - \delta_A c_1^A - \delta_B^2 c_1^B c_2^B + \delta_A^2 c_1^A c_2^A)\int_{\mathbb{R}} x\big[|\psi(x)|^2\big]''dx + (c_2^A\delta_A - c_2^B\delta_B)\int_{\mathbb{R}} x|\psi(x)|^2dx \simeq$$

$$\alpha_1(c_2^A\delta_A - c_2^B\delta_B) + \alpha_2(\delta_B c_1^B - \delta_A c_1^A),$$

where

$$\alpha_1 = \int_{\mathbb{R}} x|\psi(x)|^2dx, \alpha_2 = \int_{\mathbb{R}} x\big[|\psi(x)|^2\big]''dx. \tag{2.2.68}$$

Lemma 2.2.1. Let $(\Omega,\Sigma,\mathbf{P})$ be a measure space, and let f be an real-valued measurable function defined on Ω. Then for any real number $t > 0$:

$$\mathbf{P}\{\omega\in\Omega \| f(\omega)| \geq t\} \leq \frac{1}{t}\int_{|f(\omega)|\geq t}\big|f(\omega)\big|d\mathbf{P}(\omega). \tag{2.2.69}$$

From inequality (2.2.69) and Eq.(2.2.67) we obtain

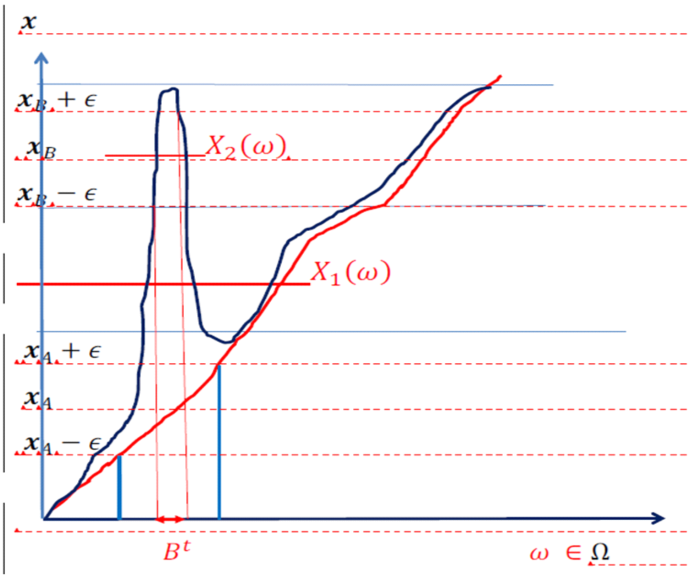
$$\mathbf{P}\big\{\omega\in\Omega: \ |X_1(\omega) - X_2(\omega)| \geq t\big\} \leq \frac{1}{t}\int_{|X_1(\omega)-X_2(\omega)|\geq t}[|X_1(\omega) - X_2(\omega)|]d\mathbf{P}(\omega)$$

$$< \frac{1}{t^2}\int_{\Omega}[X_1(\omega) - X_2(\omega)] = \frac{\eta_{1,2}}{t} \simeq \frac{\alpha_1(c_2^A\delta_A - c_2^B\delta_B) + \alpha_2(\delta_B c_1^B - \delta_A c_1^A)}{t}. \tag{2.2.70}$$

We define now

$$A^t = A_\lambda^{ph}(t,x_A) = \{\omega | x_A - \epsilon \leq X_1(\omega) \leq x_A - \epsilon\},$$
$$B^t = B_\lambda^{ph}(t,x_B) = \{\omega | x_B - \epsilon \leq X_2(\omega) \leq x_B - \epsilon\}, \tag{2.2.71}$$

and chose in (2.2.68) number $t = x_B - x_A \gg 1$.



Pic. 2.2.9.The plot of the random variables $X_1(\omega)$ and $X_2(\omega)$.

$A^t = A_\lambda^{ph}(t, x_A), B^t = B_\lambda^{ph}(t, x_B), A^t \cap B^t = B^t.$

Note that $\mathbf{P}(A^t \cap B^t) \leq \mathbf{P}(B^t)$, see Pic. 2.2.9. From (2.2.68) follows that

$$\mathbf{P}(A^t \cap B^t) < \frac{\alpha_1(c_2^A\delta_A - c_2^B\delta_B) + \alpha_2(\delta_B c_1^B - \delta_A c_1^A)}{(x_B - x_A)^2} \ll 1. \qquad (2.2.70)$$

III. Schrödinger's Cat paradox resolution

In this section we shall consider the problem of the collapse of the cat state vector on the basis of two different hypotheses:

(**A**) The canonical postulate of QM is correct in all cases.

(**B**) The canonical interpretation of the wave function $\psi = c_1\psi_1 + c_2\psi_2$ is correct only when the supports the wave functions ψ_1 and ψ_2 essentially overlap. When the wave functions ψ_1 and ψ_2 have separated supports (as in the case of the experiment that we are considering in section II) we claim that canonical interpretation of the wave function $\psi = c_1\psi_1 + c_2\psi_2$ is no longer valid for a such cat state, for details see Appendix **C**.

III.1. Considerationtion of the Schrödinger's cat paradox using canonical von Neumann postulate

Let $|s_1(t)\rangle$ and $|s_2(t)\rangle$ be

$$|s_1(t)\rangle = \left|\text{undecayed nucleus at instant } t\right\rangle,$$
$$|s_2(t)\rangle = \left|\text{decayed nucleus at instant } t\right\rangle. \tag{3.1.1}$$

In a good approximation we assume now that

$$|s_1(0)\rangle = \int_{-\infty}^{+\infty} \Psi^{\#}_{\mathbf{II}}(x)|x\rangle dx \tag{3.1.2}$$

and

$$|s_2(0)\rangle = \int_{-\infty}^{+\infty} \Psi^{\#}_{\mathbf{I}}(x)|x\rangle dx. \tag{3.1.3}$$

Remark 3.1.1. Note that: (i) $|s_2(0)\rangle = \left|\text{decayed nucleus at instant } 0\right\rangle =$ $=\left|\text{free } \alpha\text{-particle at instant } 0\right\rangle$. (ii) Feynman propagator of a free α-particle are [52]:

$$K_2(x,t,x_0) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp\left\{\frac{i}{\hbar}\left[\frac{m(x-x_0)^2}{2t}\right]\right\}. \tag{3.1.4}$$

Therefore from Eq.(3.1.3),Eq.(2.1.9) and Eq.(3.1.4) we obtain

$$|s_2(t)\rangle = \int_{-\infty}^{+\infty} \Psi^{\#}_{\mathbf{I}}(x,t)|x\rangle dx,$$

$$\Psi^{\#}_{\mathbf{I}}(x,t) = \int_{-\infty}^{0} \Psi^{\#}_{\mathbf{I}}(x_0)K_2(x,t,x_0)dx_0 =$$

$$(\pi r_c^2)^{-1/4} \times \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \times \int_{-\infty}^{0} \theta_1(x_0,l)\exp\left(-\frac{x_0^2}{2r_c^2}\right)\exp\left(-i\frac{2\pi}{\hbar}\sqrt{2mE}\,x_0\right) \times$$

$$\times \exp\left\{\frac{i}{\hbar}\left[\frac{m(x-x_0)^2}{2t}\right]\right\}dx_0 =$$

$$(3.1.5)$$

$$(\pi r_c^2)^{-1/4} \times \left(\frac{m}{2\pi i\hbar_\varepsilon t}\right)^{1/2} \times \int_{-l}^{0} \theta_1(x_0,l)\exp\left(-\frac{x_0^2}{2r_c^2}\right) \times$$

$$\times \exp\left\{\frac{i}{\hbar}\left[\frac{m(x-x_0)^2}{2t} - \pi\sqrt{4mE}\,x_0\right]\right\}dx_0 =$$

$$(\pi r_c^2)^{-1/4} \times \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \times \int_{-l}^{0} \theta_1(x_0,l)\exp\left(-\frac{x_0^2}{2r_c^2}\right) \times \exp\left\{\frac{i}{\hbar}[S(t,x,x_0)]\right\}dx_0,$$

where

$$S(t,x,x_0) = \frac{m(x-x_0)^2}{2t} - \pi\sqrt{8mE}\,x_0. \quad (3.1.6)$$

We assume now that

$$\hbar \ll 2r_c^2 \ll l^2 < 1. \quad (3.1.7)$$

Oscillatory integral in RHS of Eq.(3.1.5) is calculated now directly using stationary phase approximation. The phase term $S(x,x_0)$ given by Eq.(3.1.6) is stationary when

$$\frac{\partial S(t,x,x_0)}{\partial x_0} = -\frac{m(x-x_0)}{t} - \pi\sqrt{8mE} = 0. \quad (3.1.8)$$

Therefore

$$-\frac{m(x-x_0)}{t} - \pi\sqrt{8mE} = 0,$$
$$-(x-x_0) = \pi t\sqrt{8E/m}, \quad (3.1.9)$$

and thus stationary point $x_0(t,x)$ are

$$x_0(t,x) = \pi t\sqrt{8E/m} + x. \qquad (3.1.10)$$

Thus from Eq.(3.1.5) and Eq.(3.1.10) using stationary phase approximation we obtain

$$|s_2(t)\rangle = \int_{-\infty}^{+\infty} \Psi_{\mathbf{I}}^{\#}(x,t)|x\rangle dx,$$
$$\Psi_{\mathbf{I}}^{\#}(x,t) = \qquad (3.1.11)$$
$$(\pi r_c^2)^{-1/4} \times \theta_1(x_0(t,x),l)\exp\left[-\frac{x_0^2(t,x)}{2r_c^2}\right] \times \exp\left\{\frac{i}{\hbar}[S(t,x,x_0(t,x))]\right\} + O(\hbar),$$

where

$$S(x,x_0(t,x)) = \frac{m(x - x_0(t,x))^2}{2t} - \pi\sqrt{8mE}\,x_0(t,x). \qquad (3.12)$$

From Eq.(3.10) we obtain

$$\overline{\Psi_{\mathbf{I}}^{\#}(x,t)}\Psi_{\mathbf{I}}^{\#}(x,t) \simeq (\pi r_c^2)^{-1/2} \times \theta_1\left(x + \pi t\sqrt{8E/m}\,,l\right)\exp\left[-\frac{\left(x + \pi t\sqrt{8E/m}\right)^2}{r_c^2}\right]. \qquad (3.1.13)$$

Remark 3.1.2. From the inequality (3.1.7) and Eq.(3.1.13) follows that α-particle at each instant $t \geq 0$ moves quasiclassically from right to left by the law

$$x(t) = -\pi t\sqrt{8E/m}\,, \qquad (3.1.14)$$

i.e. i.e.,estimating the position $x(t,x_0,t_0;\hbar)$ at each instant $t \geq 0$ with final error r_c gives $|\langle x\rangle(t) - x(t)| \leq r_c, i = 1,\ldots,d$ with a probability

$$\mathbf{P}\{|\langle x\rangle(t,0,0;\hbar) - x(t)| \leq r_c\} = 1.$$

Remark 3.1.3. We assume now that a distance between radioactive source and internal monitor which detects a single atom decaying (see Pic.1.1.1) is equal to L.

Proposition 3.1.1. After α-decay at instant $t = 0$ the collaps: $|\text{live cat}\rangle \to |\text{death cat}\rangle$ arises at instant

$$T = \frac{L}{\pi\sqrt{8E/m}} \qquad (3.1.15)$$

with a probability $\mathbf{P}_T\left(|\text{death cat}\rangle\right)$ to observe a state $|\text{death cat}\rangle$ at instant T is $\mathbf{P}_T\left(|\text{death cat}\rangle\right) = 1.$

Proof. Note that. In this case Schrödinger's cat in fact permorm the single measurement of α-particle position with accuracy of $\delta x = l$ at instant $t = T$ (given by

Eq.(3.1.15)) by internal monitor (see Pic.1.1.1.). The probability of getting a result L with accuracy of $\delta x = l$ given by

$$\int_{|L-x|\leq l/2} |\langle x||s_2(T)\rangle|^2 dx = 1. \quad (3.1.16)$$

Therefore at instant T the α-particle kills Schrödinger's cat with a probability $\mathbf{P}_T\big(\big|\text{death cat}\big\rangle\big) = 1.$

Remark 3.1.4. Note that. When Schrödinger's cat has permormed this measurement the immediate post measurement state of α-particle (by von Neumann postulate Q.III.5, see subsection **I.7.1**) will end up in the state

$$|\Psi_T\rangle = \frac{\int_{|L-x|\leq l/2} |x\rangle\langle x||s_2(T)\rangle dx}{\sqrt{\int_{|L-x|\leq l/2} |\langle x||s_2(T)\rangle|^2 dx}} = \int_{|L-x|\leq l/2} |x\rangle\langle x||s_2(T)\rangle dx \in \mathbf{S}_\Theta, \Theta = \{x||L-x| \leq l/2\} \quad (3.1.17)$$

From Eq.(3.1.17) one obtains

$$\langle x'||\Psi_T\rangle = \int_{|L-x|\leq l/2} \langle x'||x\rangle\langle x||s_2(T)\rangle dx = \int_{|L-x|\leq l/2} \delta(x'-x)\langle x||s_2(T)\rangle dx = \Psi^{\#}_{\mathbf{I}}(x',t). \quad (3.1.18)$$

Therefore the state $|\Psi_T\rangle$ again kills Schrödinger's cat with a probability $\mathbf{P}_T\big(\big|\text{death cat}\big\rangle\big) = 1.$

Suppose now that a nucleus $\mathbf{n}$, whose Hilbert space is spanned by orthonormal states $|s_i(t)\rangle,\ i = 1,2,$where $|s_1(t)\rangle = \big|\text{undecayed nucleus at instant } t\big\rangle$ and $|s_2(t)\rangle = \big|\text{decayed nucleus at instant } t\big\rangle$ is in the superposition state

$$|\Psi_t\rangle_{\mathbf{n}} = c_1|s_1(t)\rangle + c_2|s_2(t)\rangle, |c_1|^2 + |c_2|^2 = 1. \quad (3.1.19)$$

Remark 3.1.5. Note that: (i) $|s_1(0)\rangle = \big|\text{undecayed nucleus at instant } t = 0\big\rangle =$ $= \big|\ \alpha\text{-particle iside region } (0,l] \text{ at instant } t = 0\big\rangle$. (ii) Feynman propagator of α-particle inside region $(0,l]$ are [52]:

$$K_2(x,t,x_0) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp\left\{\frac{i}{\hbar}[S(t,x,x_0)]\right\}, \quad (3.1.20)$$

where

$$S(t,x,x_0) = \frac{m(x-x_0)^2}{2t} + mt(U_0 - E). \quad (3.1.21)$$

Therefore from Eq.(2.1.11)-Eq.(2.1.12) and Eq.(3.1.20)-Eq.(3.1.21) we obtain

$$|s_1(t)\rangle = \int_{-\infty}^{+\infty} \Psi^{\#}_{\mathbf{II}}(x,t)|x\rangle dx,$$

$$\Psi^{\#}_{\mathbf{II}}(x,t) = \int_0^l \Psi^{\#}_{\mathbf{II}}(x_0)K_2(x,t,x_0)dx_0 = \tag{3.1.22}$$

$$\left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \int_0^l E(x_0,l)\Psi_{\mathbf{II}}(x_0)\theta_l(x_0)\exp\left\{\frac{i}{\hbar}[S(t,x,x_0)]\right\}dx_0,$$

where

$$\theta_l(x) = \begin{cases} 1 \text{ for } x \in [0,l] \\ 0 \text{ for } x \notin [0,l] \end{cases}$$

Remark 3.1.6. We assume for simplification now that

$$k' \leq 1. \tag{3.1.23}$$

Therefore oscillatory integral in RHS of Eq.(3.1.22) is calculated now directly using stationary phase approximation. The phase term $S(x,x_0)$ given by Eq.(3.21) is stationary when

$$\frac{\partial S(t,x,x_0)}{\partial x_0} = -\frac{m(x-x_0)}{t} = 0. \tag{3.1.24}$$

and thus stationary point $x_0(t,x)$ are

$$\begin{aligned} -x + x_0 &= 0 \\ x_0(t,x) &= x. \end{aligned} \tag{3.1.25}$$

Thus from Eq.(3.1.22) and Eq.(3.1.25) using stationary phase approximation we obtain

$$\begin{aligned} &\Psi^{\#}_{\mathbf{II}}(x,t) = \\ &E(x_0(t,x),l)\Psi_{\mathbf{II}}(x_0(t,x))\theta_l(x_0(t,x))\exp\left\{\frac{i}{\hbar}[S(t,x,x_0(t,x))]\right\} + O(\hbar) = \\ &= E(x,l)\Psi_{\mathbf{II}}(x)\theta_l(x)\exp\left\{\frac{i}{\hbar}[mt(U_0 - E)]\right\} + O(\hbar) = \\ &E(x,l)\theta_l(x)O(1)\exp\left\{\frac{i}{\hbar}[mt(U_0 - E)]\right\} + O(\hbar). \end{aligned} \tag{3.1.26}$$

Therefore from Eq.(3.22) and Eq.(3.26) we obtain

$$|\Psi^{\#}_{\mathbf{II}}(x,t)|^2 = E^2(x,l)\theta_l(x)O(1) + O(\hbar). \tag{3.1.27}$$

Remark 3.1.7. Note that for each instant $t > 0$:

$\mathbf{supp}(\Psi^{\#}_{\mathbf{II}}(x,t)) \cap \mathbf{supp}(\Psi^{\#}_{\mathbf{I}}(x,t)) = \varnothing.$

Remark 3.1.8. Note that. From Eq.(3.1.11),Eq.(3.1.13), Eq.(3.1.19), Eq.(3.1.22)-Eq.(3.1.27) and Eq.(A.13) by Remark 3.1.7 we obtain

$$\begin{gathered}
{}_{\mathbf{n}}\langle\Psi_t|\widehat{x}|\Psi_t\rangle_{\mathbf{n}} = |c_1|^2\langle s_1(t)|\widehat{x}|s_1(t)\rangle + |c_2|^2\langle s_2(t)|\widehat{x}|s_2(t)\rangle + \\
c_1c_2^*\langle s_2(t)|\widehat{x}|s_1(t)\rangle + c_1^*c_2\langle s_2(t)|\widehat{x}|s_1(t)\rangle^* = \\
|c_1|^2\langle s_1(t)|\widehat{x}|s_1(t)\rangle + |c_2|^2\langle s_2(t)|\widehat{x}|s_2(t)\rangle = |c_1|^2 l + |c_2|^2 T\pi\sqrt{8E/m}\,.
\end{gathered} \tag{3.1.28}$$

Proposition 3.1.2. (**i**) Suppose that a nucleus $\mathbf{n}$ is in the superposition state $|\Psi_t\rangle_{\mathbf{n}}$ ($|\Psi_t\rangle_{\mathbf{n}}$-particle) given by Eq.(3.19). Then the collaps: $\left|\text{live cat}\right\rangle \to \left|\text{death cat}\right\rangle$ arises at instant

$$T_{\mathrm{col}} \approx \frac{L \pm l}{|c_2|^2\sqrt{8\pi^2 E/m}}. \tag{3.1.29}$$

with a probability $\mathbf{P}_{T_{\mathrm{col}}}\left(\left|\text{death cat}\right\rangle\right)$ to observe a state $\left|\text{death cat}\right\rangle$ at instant T_{col} is $\mathbf{P}_{T_{\mathrm{col}}}\left(\left|\text{death cat}\right\rangle\right) = |c_2|^2$.

(**ii**) Assume now a Schrödinger's cat has performed the single measurement of $|\Psi_t\rangle_{\mathbf{n}}$-particle position with accuracy of $\delta x = l$ at instant $T = T_{\mathrm{col}}$ (given by Eq.(3.1.29)) by internal monitor (see Pic.1.1.1) and the result $x \approx L \pm l$ is not observed by Schrödinger's cat. Then the collaps: $\left|\text{live cat}\right\rangle \to \left|\text{death cat}\right\rangle$ never arises at any instant $T > T_{\mathrm{col}}$ and a probability $\mathbf{P}_{T>T_{\mathrm{col}}}\left(\left|\text{death cat}\right\rangle\right)$ to observe a state $\left|\text{death cat}\right\rangle$ at instant $T > T_{\mathrm{col}}$ is $\mathbf{P}_{T>T_{\mathrm{col}}}\left(\left|\text{death cat}\right\rangle\right) = 0$.

Proof. (**i**) Note that for $t > 0$ the marginal density matrix $\rho(t)$ is diagonal

$$\rho(t) = \begin{pmatrix} |c_1|^2\int|\Psi^{\#}_{\mathbf{II}}(x,t)|^2dx & 0 \\ 0 & |c_2|^2\int|\Psi^{\#}_{\mathbf{I}}(x,t)|^2dx \end{pmatrix}$$

In this case a Schrödinger's cat in fact perform the single measurement of $|\Psi_t\rangle_{\mathbf{n}}$-particle position with accuracy of $\delta x = l$ at instant $t = T_{\mathrm{col}}$ (given by Eq.(3.1.29)) by internal monitor (see Pic.1.1.1). The probability of getting a result L at instant $T \approx T_{\mathrm{col}}$ with accuracy of $\delta x = l$ given by

$$\begin{gathered}
\int_{|L-x|\leq l/2}|\langle x||\Psi_T\rangle_{\mathbf{n}}|^2dx = \int_{|L-x|\leq l/2}|\langle x|c_1|s_1(T)\rangle + c_2|s_2(T)\rangle|^2dx = \\
\int_{|L-x|\leq l/2}|c_1\langle x||s_1(T)\rangle + c_2\langle x||s_2(T)\rangle|^2dx = \\
\int_{|L-x|\leq l/2}|c_1^2\Psi^{\#2}_{\mathbf{II}}(x,T) + c_2^2\Psi^{\#2}_{\mathbf{I}}(x,T) + 2c_1c_2\Psi^{\#}_{\mathbf{I}}(x,T)\Psi^{\#}_{\mathbf{II}}(x,T)|dx.
\end{gathered} \tag{3.1.30}$$

From Eq.(3.1.30) by Remark 3.1.7 and Eq.(3.1.13) one obtains

$$\int_{|L-x|\leq l/2} |\langle x||\Psi_T\rangle_{\mathbf{n}}|^2 dx = \int_{|L-x|\leq l/2} |c_2^2 \Psi_{\mathbf{I}}^{\#2}(x,T)| dx = |c_2|^2 \int_{|L-x|\leq l/2} |\Psi_{\mathbf{I}}^{\#}(x,T)|^2 dx = |c_2|^2. \qquad (3.1.31)$$

Note that. When Schrödinger's cat has permormed this measurement and the result $x \approx L \pm l$ is observed, then the immediate post measurement state of α-particle (by von Neumann measurement postulate **Q.III.5**, see subsection **I.7.1**.) will end up in the state

$$|\Psi_{T_{\mathrm{col}}}\rangle_{\mathbf{n}} = \frac{\int_{|L-x|\leq l/2} |x\rangle\langle x||\Psi_{T_{\mathrm{col}}}\rangle_{\mathbf{n}} dx}{\sqrt{\int_{|L-x|\leq l/2} |\langle x||\Psi_{T_{\mathrm{col}}}\rangle_{\mathbf{n}}|^2 dx}} = \frac{\int_{|L-x|\leq l/2} |x\rangle\langle x|(c_1|s_1(T_{\mathrm{col}})\rangle + c_2|s_2(T_{\mathrm{col}})\rangle) dx}{\sqrt{\int_{|L-x|\leq l/2} |\langle x||\Psi_{T_{\mathrm{col}}}\rangle_{\mathbf{n}}|^2 dx}} =$$
$$\frac{c_1 \int_{|L-x|\leq l/2} |x\rangle\langle x||s_1(T_{\mathrm{col}})\rangle + c_2 \int_{|L-x|\leq l/2} |x\rangle\langle x||s_2(T_{\mathrm{col}})\rangle dx}{\sqrt{\int_{|L-x|\leq l/2} |\langle x||\Psi_{T_{\mathrm{col}}}\rangle_{\mathbf{n}}|^2 dx}} \in \mathbf{S}_{\Theta}, \Theta = \{x||L-x| \leq l/2\}. \qquad (3.1.32)$$

From Eq.(3.1.32) by Eq.(3.1.31) and by Remark 3.1.7 one obtains

$$|\Psi_{T_{\mathrm{col}}}\rangle_{\mathbf{n}} =$$
$$\frac{\int_{|L-x|\leq l/2} |x\rangle\langle x||\Psi_{T_{\mathrm{col}}}\rangle_{\mathbf{n}} dx}{\sqrt{\int_{|L-x|\leq l/2} |\langle x||\Psi_{T_{\mathrm{col}}}\rangle_{\mathbf{n}}|^2 dx}} = \frac{\int_{|L-x|\leq l/2} |x\rangle\langle x|(c_1|s_1(T_{\mathrm{col}})\rangle + c_2|s_2(T_{\mathrm{col}})\rangle) dx}{\sqrt{\int_{|L-x|\leq l/2} |\langle x||\Psi_{T_{\mathrm{col}}}\rangle_{\mathbf{n}}|^2 dx}} = \qquad (3.1.32)$$
$$= \frac{c_2}{|c_2|} \int_{|L-x|\leq l/2} |x\rangle\langle x||s_2(T_{\mathrm{col}})\rangle dx.$$

Obviously by Remark 3.1.4 the staite $|\Psi_{T_{\mathrm{col}}}\rangle_{\mathbf{n}}$ kills Schrödinger's cat with a probability $\mathbf{P}_{T_{\mathrm{col}}}\left(|\text{death cat}\rangle\right) = 1$.

Proof. (**ii**) The probability of getting a result L at any instant $T > T_{\mathrm{col}}$ with

accuracy of $\delta x = l$ by Eq.(3.1.31) and Eq.(3.1.13) given by

$$\int_{|L-x|\leq l/2} |\langle x||\Psi_T\rangle_{\mathbf{n}}|^2 dx = \int_{|L-x|\leq l/2} |c_2^2 \Psi_{\mathbf{I}}^{\#2}(x,T)| dx = |c_2|^2 \int_{|L-x|\leq l/2} |\Psi_{\mathbf{I}}^{\#}(x,T)|^2 dx =$$
$$= (\pi r_c^2)^{-1/2} \int_{|L-x|\leq l/2} \theta_1\left(x + \pi T\sqrt{8E/m}, l\right) \exp\left[-\frac{\left(x + \pi T\sqrt{8E/m}\right)^2}{r_c^2}\right] \asymp 0. \qquad (3.1.33)$$

Thus standard formalism of continuous quantum measurements [2],[3],[4],[5] leads to a definite but unpredictable measurement outcomes, and therefore $|\Psi_t\rangle_{\mathbf{n}}$ suddenly "collapses" at unpredictable instant t' into one of the states $|s_i(t')\rangle, i = 1,2$.

III.2.Resolution of the Schrödinger's cat paradox using generalized von Neumann postulate.

Proposition 3.2.1. Suppose that a nucleus $\mathbf{n}$ is in the superposition state given by Eq.(3.1.19). Then the collaps: $|\text{live cat}\rangle \to |\text{death cat}\rangle$ arises at instant

$$T = \frac{L}{|c_2|^2\sqrt{8\pi^2 E/m}}. \tag{3.2.1}$$

with a probability $\mathbf{P}_T\big(|\text{death cat}\rangle\big)$ to observe a state $|\text{death cat}\rangle$ at instant T is $\mathbf{P}_T\big(|\text{death cat}\rangle\big) = 1.$

Proof. Let us consider now a state $|\Psi_t\rangle_{\mathbf{n}}$ given by Eq.(3.1.19). This state consists of a summ of two wave packets $c_1\Psi^{\#}_{\mathbf{II}}(x,t)$ and $c_2\Psi^{\#}_{\mathbf{I}}(x,t)$. Wave packet $c_1\Psi^{\#}_{\mathbf{II}}(x,t)$ present an $\alpha_{\mathbf{II}}$-particle which lives in region $\mathbf{II}$ with a probability $|c_1|^2$ (see Pic. 2.1.1). Wave packet $c_2\Psi^{\#}_{\mathbf{I}}(x,t)$ present an $\alpha_{\mathbf{I}}$-particle which lives in region $\mathbf{I}$ with a probability $|c_2|^2$ (see Pic. 2.1.1) and moves from the right to the left. Note that $\mathbf{I} \cap \mathbf{II} = \varnothing$. From Eq.(3.1.28) follows that $\alpha_{\mathbf{I}}$-particle at each instant $t \geq 0$ moves quasiclassically from right to left by the law

$$x(t) = -|c_2|^2\pi t\sqrt{8E/m}\,, \tag{3.2.2}$$

From Eq.(3.2.2) one obtains

$$T = T_{\text{col}} \simeq \frac{L}{|c_2|\sqrt{8\pi^2 E/m}}. \tag{3.2.3}$$

Note that. In this case Schrödinger's cat in fact permorm a single measurement of $|\Psi_t\rangle_{\mathbf{n}}$-particle position with accuracy of $\delta x = l$ at instant $t = T = T_{\text{col}}$ given by Eq.(3.2.3), by internal monitor (see Pic.1.1.1). The probability $\mathbf{P}(L,l,T_{\text{col}})$ of getting

the result L at instant $t = T_{\text{col}}$ with accuracy of $\delta x = l$ [by Remark 3.1.7 and by Generalized von Neumann measurement postulate Q.IV.4 and by postulate Q.IV.3 (see subsection 1.7.1) and by reconcile Bohr rule, see section 1.6, Eq.(1.6.19)] is given by formula

$$\begin{aligned}
&\mathbf{P}(L,l,T_{\text{col}}) = \int_{|L-x|\leq l/2}\Big[|\langle x|c_1|s_1(T_{\text{col}})\rangle|^2 * |\langle x|c_2|s_2(T_{\text{col}})\rangle|^2\Big]dx = \\
&\int_{|L-x|\leq l/2}|c_2|^{-2}|c_1|^{-2}\Big[|\langle x|c_1|^{-2}||s_1(T_{\text{col}})\rangle|^2 * |\langle x|c_2|^{-2}||s_2(T_{\text{col}})\rangle|^2\Big]dx = \\
&\int_{|L-x|\leq l/2}|c_2|^{-2}|c_1|^{-2}\Big[|\Psi^{\#}_{\mathbf{I}}(x|c_2|^{-2},T_{\text{col}})|^2 * |\Psi^{\#}_{\mathbf{II}}(x|c_1|^{-2},T_{\text{col}})|^2\Big]dx = 1.
\end{aligned} \tag{3.2.4}$$

Note that. When Schrödinger's cat has permormed this measurement and the result $x \approx L \pm l$ is observed, then the immediate post measurement state of α-particle (by generalized von Neumann postulate Q.IV.4, see subsection 1.7.1) will end up in the state

$$\begin{aligned}
&|\Psi_{T_{\text{col}}}\rangle_{\mathbf{n}} = \\
&\frac{\int_{|L-x|\leq l/2}|x\rangle\langle x||\Psi_{T_{\text{col}}}\rangle_{\mathbf{n}}dx}{\sqrt{\int_{|L-x|\leq l/2}\Big[|\langle x||s_1(T_{\text{col}})\rangle|^2 + |\langle x||s_2(T_{\text{col}})\rangle|^2\Big]dx}} = \frac{|c_2|^{-2}\int_{|L-x|\leq l/2}|x\rangle\langle x||s_2(T_{\text{col}})\rangle dx}{\sqrt{\int_{|L-x|\leq l/2}\Big[|\langle x||s_2(T_{\text{col}})\rangle|^2\Big]dx}} \in \mathbf{H}_{\Theta}, \\
&\Theta = \{x||L-x| \leq l/2\}.
\end{aligned} \tag{3.2.5}$$

The staite $|\Psi_{T_{\text{col}}}\rangle_{\mathbf{n}}$ again kills Schrödinger's cat with a probability

$\mathbf{P}_{T\text{col}}\big(|\text{death cat}\rangle\big) = 1.$

Thus is the collapsed state of the cat always shows definite and predictable

outcomes even if cat also consists of a superposition:

$$|\text{cat}\rangle = c_1|\text{live cat}\rangle + c_2|\text{death cat}\rangle.$$

Contrary to van Kampen's [10] and some others' opinions, "looking" at the outcome changes nothing, beyond informing the observer of what has already happened.

We remain: there are widespread claims that Schrödinger's cat is not in a definite alive or dead state but is, instead, in a superposition of the two. van Kampen, for example, writes "The whole system is in a superposition of two states: one in which no decay has occurred and one in which it has occurred. Hence, the state of the cat also consists of a superposition: $|\text{cat}\rangle = c_1|\text{live cat}\rangle + c_2|\text{death cat}\rangle$. The state remains a superposition until an observer looks at the cat" [10].

III.3.Schrödinger's cat does not a live cat and dead cat samultaniously.

In this subsection we prove that Schrödinger's cat does not a live cat and dead cat samultaniously.Being interested only in distinction in changes of macroscopical and microscopic variables, it is easy to show that the macroscopical variable under suitable conditions can be described own wave function and the own Shrödinger equation i.e. if $X, \mathbf{Y} = (y_1,\ldots,y_n)$ designates an macroscopic coordinates, and $\mathbf{x} = (x_1,\ldots,x_m)$ is a set of microscopic variables then own functions of system corresponding to an value of energy, contain functions of the form [6]

$$\Phi(X,\mathbf{Y},\mathbf{x},t) = \Psi(X,\mathbf{Y},t)\psi(\mathbf{x},t), \tag{3.3.1}$$

Here $\Psi(X,\mathbf{Y},t)$ is a "macroscopic" wave function such that

$$i\hbar\frac{\partial\Psi(X,\mathbf{Y},t)}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\Psi(X,\mathbf{Y},t) + V(X,\mathbf{Y})\Psi(X,\mathbf{Y},t), \tag{3.3.2}$$

Where the variable X unequivocally corresponds to live or dead cat, $V(X,\mathbf{Y})$ is a conservative potential. Further, for the purpose of simplifications, we will assume that macroscopic wave function $\Psi(X,\mathbf{Y},t)$ depends only on one variable X. By variable X it is possible to choose body temperature of a cat, i.e. $X = \theta$ if it is a question of the real cat. Then we obtain Shrödinger equation

$$\begin{aligned} &i\hbar\frac{\partial\Psi(\theta,\mathbf{Y},t)}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\Psi(\theta,\mathbf{Y},t) + V(\theta,\mathbf{Y})\Psi(\theta,\mathbf{Y},t),\\ &\Psi(\theta,\mathbf{Y},0) = \Psi_0(\theta,\mathbf{Y}). \end{aligned} \tag{3.3.3}$$

Definition 3.3.1. Let θ-supp$[\Psi(\theta,\mathbf{Y},t)]$ be a set such that

$$\forall\theta\big\{\theta \in \theta\text{-supp}[\Psi(\theta,\mathbf{Y},t)] \text{ iff } \Psi(\theta,\mathbf{Y},t) \neq 0\big\}. \tag{3.3.4}$$

Assumption 3.3.1.We assume now that:

(i) for a live cat any observer which measured body temperature of a cat always obtains result $\widehat{\theta}$ such that $\widehat{\theta} \in [\theta_1^{\mathbf{live}}, \theta_2^{\mathbf{live}}]$, where $0 < \theta_1^{\mathbf{live}} < \theta_2^{\mathbf{live}}$,

(ii) for a dead cat any observer which measured body temperature of a cat always obtains result $\widehat{\theta}$ such that $\widehat{\theta} \in [\theta_1^{\mathbf{dead}}, \theta_2^{\mathbf{dead}}]$, where $\theta_2^{\mathbf{dead}} \leq 0$ and

(iii) for a sick but a live cat any observer which measured body temperature of a cat always obtains result $\widehat{\theta}$ such that $\widehat{\theta} \in (\theta_2^{\mathbf{dead}}, \theta_1^{\mathbf{live}})$.

Let $\Psi^{\mathbf{live}}(\theta, \mathbf{Y}, 0) = \Psi_0^{\mathbf{live}}(\theta, \mathbf{Y})$ be a wave function of a live cat at instant $t = 0$. Then obviously

$$\int_{\theta_1^{\mathbf{live}}}^{\theta_2^{\mathbf{live}}} \int_{\mathbb{R}^n} \theta \|\Psi_0^{\mathbf{live}}(\theta, \mathbf{Y})\|^2 d\theta d^n Y = \overline{\theta}_1 \in [\theta_1^{\mathbf{live}}, \theta_2^{\mathbf{live}}] \tag{3.3.5}$$

and θ-supp$[\Psi(\theta, \mathbf{Y}, t)] = [\theta_1^{\mathbf{live}}, \theta_2^{\mathbf{live}}]$.

Let $\Psi^{\mathbf{dead}}(\theta, \mathbf{Y}, 0) = \Psi_0^{\mathbf{dead}}(\theta, \mathbf{Y})$ be a wave function of a dead cat at instant $t = 0$. Then obviously

$$\int_{\theta_1^{\mathbf{dead}}}^{\theta_2^{\mathbf{dead}}} \int_{\mathbb{R}^n} \theta \|\Psi_0^{\mathbf{dead}}(\theta, \mathbf{Y})\|^2 d\theta d^n Y = \overline{\theta}_2 \in [\theta_1^{\mathbf{dead}}, \theta_2^{\mathbf{dead}}] \tag{3.3.6}$$

We assume now that

$$\begin{gathered} \int_{\mathbb{R}^n} \|\Psi_0^{\mathbf{live}}(\theta, \mathbf{Y})\|^2 d^n Y = \psi_1(\theta), \\ \int_{\mathbb{R}^n} \|\Psi_0^{\mathbf{dead}}(\theta, \mathbf{Y})\|^2 d^n Y = \psi_2(\theta), \end{gathered} \tag{3.3.7}$$

where

$$\psi_1(\theta) = \begin{cases} \left(\dfrac{1}{2\delta\pi_\delta}\right)^{1/4} \exp\left[-\dfrac{1}{4\delta}(\theta - \overline{\theta}_1)^2\right] \text{ iff } |\theta - \overline{\theta}_1| \leq \varepsilon \ll 1, \\ 0 \text{ iff } |\theta - \overline{\theta}_1| > \varepsilon \end{cases} \tag{3.3.8}$$

and

$$\psi_2(\theta) = \begin{cases} \left(\dfrac{1}{2\delta\pi_\delta}\right)^{1/4} \exp\left[-\dfrac{1}{4\delta}(\theta - \overline{\theta}_2)^2\right] \text{ iff } |\theta - \overline{\theta}_2| \leq \varepsilon \ll 1, \\ 0 \text{ iff } |\theta - \overline{\theta}_2| > \varepsilon \end{cases} \tag{3.3.9}$$

We assume now that at instant $t = 0$ cat consist of superposition

$$\begin{gathered} \Psi_0(\theta, \mathbf{Y}) = c_1 \Psi_0^{\mathbf{live}}(\theta, \mathbf{Y}) + c_2 \Psi_0^{\mathbf{dead}}(\theta, \mathbf{Y}), \\ |c_1|^2 + |c_2|^2 = 1. \end{gathered} \tag{3.3.10}$$

Let $\psi(\theta)$ be a function

$$\psi(\theta) = \int_{\mathbb{R}^n} \Psi_0(\theta, \mathbf{Y}) d^n Y. \tag{3.3.11}$$

Thus

$$\psi(\theta) = c_1 \psi_1(\theta) + c_2 \psi_2(\theta). \tag{3.3.12}$$

Let $p(\theta,\alpha_1,\alpha_2),\ \alpha_1 = |c_1|^2,\alpha_2 = |c_2|^2$ be a probability density to observe at instant $t = 0$ a body temperature of a cat with result θ. By reconcile Bohr rule, [see section 1.6, Eq.(1.6.19)]

$$\begin{gathered} p(\theta,\alpha_1,\alpha_2) = [p_1(\theta,\alpha_1)] * [p_2(\theta,\alpha_2)], \\ p(\theta,\alpha_1) = \alpha_1^{-1}\left|\psi_1\left(\frac{\theta}{\alpha_1}\right)\right|^2, p_2(\theta,\alpha_2) = \alpha_2^{-1}\left|\psi_2\left(\frac{\theta}{\alpha_2}\right)\right|^2. \end{gathered} \tag{3.3.13}$$

Thus from Eq.(3.3.8)-Eq.(3.3.9) we obtain

$$p_1(\theta,\alpha_1) = \begin{cases} \alpha_1^{-1}\left(\frac{1}{2\delta\pi_\delta}\right)^{1/2}\exp\left[-\frac{1}{2\alpha_1^2\delta}(\theta-\alpha_1\bar{\theta}_1)^2\right] \text{ iff } \left|\frac{\theta}{\alpha_1}-\bar{\theta}_1\right| \leq \varepsilon \ll 1, \\ 0 \text{ iff } \left|\frac{\theta}{\alpha_1}-\bar{\theta}_1\right| > \varepsilon \end{cases} \tag{3.3.14}$$

and

$$p_2(\theta,\alpha_2) = \begin{cases} \alpha_2^{-1}\left(\frac{1}{2\delta\pi_\delta}\right)^{1/2}\exp\left[-\frac{1}{2\alpha_2^2\delta}(\theta-\alpha_2\bar{\theta}_2)^2\right] \text{ iff } \left|\frac{\theta}{\alpha_2}-\bar{\theta}_2\right| \leq \varepsilon \ll 1, \\ 0 \text{ iff } \left|\frac{\theta}{\alpha_2}-\bar{\theta}_2\right| > \varepsilon. \end{cases} \tag{3.3.15}$$

From Eq.(3.3.13)-Eq.(3.3.15) by using formulae

$$\begin{gathered} f_1(x) = \frac{1}{\sigma_1\sqrt{2\pi}}\exp\left[-\frac{(x-m_1)}{2\sigma_1^2}\right], f_2(x) = \frac{1}{\sigma_2\sqrt{2\pi}}\exp\left[-\frac{(x-m_2)^2}{2\sigma_2^2}\right], \\ f_1(x) * f_2(x) = \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}\exp\left\{-\frac{[x-(m_1+m_2)]^2}{2(\sigma_1^2+\sigma_2^2)}\right\} \end{gathered} \tag{3.3.16}$$

we obtain

$$\begin{gathered} p(\theta,\alpha_1,\alpha_2) \asymp \\ \frac{1}{\delta\sqrt{2\pi_\delta(\alpha_1^2+\alpha_2^2)}}\exp\left\{-\frac{[\theta-(\alpha_1\bar{\theta}_1+\alpha_2\bar{\theta}_2)]^2}{2\delta^2(\alpha_1^2+\alpha_2^2)}\right\} \text{ iff } |\theta-(\alpha_1\bar{\theta}_1+\alpha_2\bar{\theta}_2)| \leq \varepsilon \ll 1, \\ p(\theta,\alpha_1,\alpha_2) = 0 \text{ iff } |\theta-(\alpha_1\bar{\theta}_1+\alpha_2\bar{\theta}_2)| > \varepsilon \end{gathered} \tag{3.3.17}$$

We assume now that potential $V(\theta,\mathbf{Y})$ is a polynomial function of variables $z = (z_1,\ldots,z_{n+1}) = (\theta,\mathbf{Y}) = (\theta,y_1,\ldots,y_n)$. Using replacement

$$\begin{gathered} z_i \to \frac{y_i}{(1+\varepsilon^{2k}|x|^2k)}, i = 1,\ldots,n+1, \\ \varepsilon \in (0,1], k \geq 1, \end{gathered} \tag{3.3.18}$$

we obtain from potential $V(z) = V(\theta,\mathbf{Y})$ regularized potential $V_\epsilon(z), \epsilon \in (0,1]$, such that $V_{\varepsilon=0}(z) = V(z)$ and

$$\sup_{z\in\mathbb{R}^{n+1}}(|V_\varepsilon(z)|) < +\infty, \varepsilon \in (0,1]. \tag{3.3.19}$$

Finally we obtain from Schrödinger equation (3.3.3) regularized Schrödinger equation of the Colombeau form

$$\begin{aligned} &i\hbar\left(\frac{\partial\Psi_\varepsilon(z,t)}{\partial t}\right)_\epsilon = -\frac{\hbar^2}{2M}(\nabla^2\Psi_\varepsilon(z,t))_\varepsilon + (V_\varepsilon(z)\Psi_\varepsilon(z,t))_\varepsilon, \\ &(\Psi_\varepsilon(z,0))_\varepsilon = \Psi_0(z). \end{aligned} \tag{3.3.20}$$

Theorem 3.3.1.Let us consider Cauchy problem (3.3.20) with $\Psi_0(z)$ given by formula

$$\Psi_0(z) = \begin{cases} \left(\frac{1}{2\delta\pi_\delta}\right)^{1/4}\exp\left[-\frac{1}{4\delta}(z-z_0)^2\right] \text{ iff } \|z-z_0\| \leq \varepsilon_1 \ll 1, \\ 0 \text{ iff } \|z-z_0\| > \varepsilon_1, \end{cases} \tag{3.3.21}$$

1.We assume that:

(i) $(V_\varepsilon(z))_\varepsilon \in G(\mathbb{R}^{n+1})$,

(ii) function V(x) is a polynomial on variable $z = (z_1,\ldots,z_{n+1})$, i.e. $V(z) = \sum_{\|\alpha\|\leq m} g_\alpha z^\alpha, \alpha = (i_1,\ldots,i_{n+1}), z^\alpha = z_1^{i_1}\times\ldots\times z_{n+1}^{i_{n+1}}, \|\alpha\| = \sum_{r=1}^{n+1} i_r$

2.Let $u(\tau,t,\lambda,z,y) = (u_1(\tau,t,\lambda,z,y),\ldots,u_{n+1}(\tau,t,\lambda,z,y))$ be the solution of the boundary
problem:

$$\begin{aligned} &\frac{\partial^2 u^T(\tau,t,\boldsymbol{\lambda},z,y)}{\partial\tau^2} = Hess[V(\lambda,\tau)]u^T(\tau,t,\boldsymbol{\lambda},x,y) + [V'(\boldsymbol{\lambda},\tau)]^T, \\ &u(0,t,\boldsymbol{\lambda},z,y) = y, u(t,t,\boldsymbol{\lambda},z,y) = z. \end{aligned} \tag{3.3.22}$$

Here

$$\begin{aligned} &\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_{n+1}) \in \mathbb{R}^{n+1}, u^T(\tau,t,\boldsymbol{\lambda},z,y) = (u_1(\tau,t,\boldsymbol{\lambda},z,y),\ldots,u_{n+1}(\tau,t,\boldsymbol{\lambda},z,y))^T, \\ &V'(\boldsymbol{\lambda},\tau) = ([\partial V(z,t)/\partial z_1]_{z=\boldsymbol{\lambda}},\ldots,[\partial V(z,t)/\partial z_{n+1}]_{z=\boldsymbol{\lambda}}), \\ &Hess[V(\boldsymbol{\lambda},\tau)] = [(\partial^2 V(z,t))/\partial x_i\partial x_j]_{x=\lambda} \end{aligned} \tag{3.3.23}$$

3.Let $S(t,\boldsymbol{\lambda},z,y)$ be the function given by formula

$$S(t,\boldsymbol{\lambda},z,y) = \int_0^t \mathcal{L}(\dot{u}(\tau,t,\boldsymbol{\lambda},z,y), u(\tau,t,\boldsymbol{\lambda},z,y),\tau)d\tau, \tag{3.3.24}$$

where master Lagrangian $\mathcal{L}(\dot{u},u,\tau)$ is

$$\begin{aligned} &\mathcal{L}(\dot{u},u,\tau) = m/2\dot{u}^2(\tau,t,\boldsymbol{\lambda},z,y) - \widehat{V}(u(\tau,t,\boldsymbol{\lambda},z,y),\tau), \dot{u} = \\ &((\partial u_1)/\partial\tau,\ldots,(\partial u_{n+1})/\partial\tau), \dot{u}^2 = \langle\dot{u},\dot{u}\rangle, \\ &\widehat{V}(u(\tau,t,\boldsymbol{\lambda},z,y),\tau) = u(\tau,t,\lambda,z,y)Hess[V(\lambda,\tau)]u^T(\tau,t,\lambda,z,y) + V'(\lambda,\tau)u^T(\tau,t,\lambda,z,y). \end{aligned} \tag{3.3.25}$$

4.Let $y_{cr} = y_{cr}(t,\boldsymbol{\lambda},z) \in \mathbb{R}^{n+1}$ be solution of the linear system of the algebraic equations

$$[\partial S(t,\boldsymbol{\lambda},z,y)/(\partial y_i)]_{y=y_{cr}} = 0, i = 1,\ldots,n+1. \tag{3.3.26}$$

5. Let $\widehat{z} = \widehat{z}(t,\boldsymbol{\lambda},z_0) \in \mathbb{R}^{n+1}$ be solution of the linear system of the algebraic equations

$$y_{cr}(t,\boldsymbol{\lambda},\widehat{z}) + \boldsymbol{\lambda} - z_0 = 0. \tag{3.3.27}$$

6.Assume that: for a given values of the parameters $t,\boldsymbol{\lambda},z_0$ the point $\widehat{z} = \widehat{z}(t,\boldsymbol{\lambda},z_0)$ is not a focal point on a corresponding trajectory is given by corresponding solution of the boundary problem (3.3.22). Then:

(i) for the limiting quantum average given by formula

$$\begin{gathered}(\lim_{\hbar\to 0}\langle i,t,z_0,\varepsilon;\hbar\rangle)_\varepsilon = \\ (\lim_{\hbar\to 0}\int z_i|\Psi_\varepsilon(z,t;\hbar)|^2 dz)_\varepsilon, \varepsilon \in (0,1], z \in \mathbb{R}^{n+1}, i = 1,\dots,n+1.\end{gathered} \tag{3.3.28}$$

the following inequalities are satisfies

$$\begin{gathered}\lim_{\substack{\hbar\to 0\\ \varepsilon\to 0}}|\langle i,t,z_0,\varepsilon;\hbar\rangle - \lambda_i(t,z_0)| \leq \\ 2[|\det(S_{y_{cr}y_{cr}}(t,\boldsymbol{\lambda},\widehat{z}(t,\boldsymbol{\lambda},z_0),y_{cr}(t,\boldsymbol{\lambda},\widehat{z}(t,\boldsymbol{\lambda},z_0))))|]^{-1}|\widehat{z}_i(t,\boldsymbol{\lambda},z_0)|, \\ (S_{y_{i,cr}y_{j,cr}}) = [\partial S(t,\boldsymbol{\lambda},z,y)/\partial y_i\partial y_j)]_{y_i=y_{i,cr},y_j=y_{j,cr}} \\ i,j = 1,\dots,n+1.\end{gathered} \tag{3.3.29}$$

(ii) therefore one can to calculate the limiting quantum trajectories $z_i^*(t,z_0,t_0)$

$$z_i^*(t,z_0,t_0) \triangleq \lambda_i(t,z_0), i = 1,\dots,n+1 \tag{3.3.30}$$

corresponding to potential $V(z,t)$ by using following sistem of the transcendental master equations

$$\widehat{z}_i(t,\boldsymbol{\lambda},z_0) = 0, i = 1,\dots,n+1. \tag{3.3.31}$$

(iii) In the limit $\hbar \to 0, \varepsilon \to 0$ Schrödinger equation Eq.(3.3.20) completely evolve quasiclassically i.e. for expectation value of the position $\{\langle z_i\rangle(t,x_0,t_0,\varepsilon;\hbar)\}_{i=1}^{n+1}$ at each instant t the inequality

$$\lim_{\substack{\hbar\to 0\\ \varepsilon\to 0}} |\langle z_i\rangle(t,z_0,t_0,\varepsilon;\hbar) - z_i^*(t,z_0,t_0)| \leq \delta \ll 1, i = 1,\dots,n+1, \tag{3.3.32}$$

is satisfied with a probability 1, e.g.,

$$\lim_{\substack{\hbar\to 0\\ \varepsilon\to 0}} \mathbf{P}\{|\langle z_i\rangle(t,z_0,t_0,\varepsilon;\hbar) - z_i^*(t,z_0,t_0)| \leq \delta\} = 1. \tag{3.3.33}$$

Proof. The proof completely similarly as the proof of the theorem 3.1 from paper [19].

Theorem 3.3.2. Let us consider Cauchy problem (3.3.20) with $\Psi_0(z)$ given by formula

$$\begin{gathered}\Psi_0(z) = c_1\Psi_{1,0}(z) + c_2\Psi_{2,0}(z), \\ |c_1|^2 + |c_2|^2 = 1,\end{gathered} \tag{3.3.34}$$

where

$$\Psi_{1,0}(z) = \begin{cases} \left(\frac{1}{2\delta\pi_\delta}\right)^{1/4} \exp\left[-\frac{1}{4\delta}(z - z_{1,0})^2\right] \text{ iff } \|z - z_0\| \leq \varepsilon_1 \ll 1, \\ 0 \text{ iff } \|z - z_0\| > \varepsilon_1, \end{cases} \tag{3.3.35}$$

and

$$\Psi_{2,0}(z) = \begin{cases} \left(\frac{1}{2\delta\pi_\delta}\right)^{1/4} \exp\left[-\frac{1}{4\delta}(z - z_{2,0})^2\right] \text{ iff } \|z - z_{2,0}\| \leq \varepsilon_1 \ll 1, \\ 0 \text{ iff } \|z - z_0\| > \varepsilon_1, \end{cases} \tag{3.3.36}$$

Then

(i) in the limit $\hbar \to 0, \varepsilon \to 0$ Schrödinger equation Eq.(3.3.33)-Eq.(3.3.35) completely evolve quasiclassically i.e. for expectation value of the observable $\{\langle z_i \rangle(t, x_0, t_0, \varepsilon; \hbar)\}_{i=1}^{n+1}$ at each instant t, the inequality

$$\begin{gathered} \lim_{\substack{\hbar\to 0 \\ \varepsilon\to 0}} |\langle z_i \rangle(t, z_0, t_0, \varepsilon; \hbar) - z_i^*(t, z_0, t_0)| \leq \delta \ll 1, i = 1, \ldots, n+1, \\ z_0 = \alpha_1 z_{1,0} + \alpha_2 z_{2,0}, \\ \alpha_1 = |c_1|^2, \alpha_2 = |c_2|^2 \end{gathered} \tag{3.3.37}$$

is satisfied with a probability 1, e.g.,

$$\lim_{\substack{\hbar\to 0 \\ \varepsilon\to 0}} \mathbf{P}\{|\langle z_i \rangle(t, z_0, t_0, \varepsilon; \hbar) - z_i^*(t, z_0, t_0)| \leq \delta\} = 1. \tag{3.3.38}$$

(ii) here $z_i^*(t, z_0, t_0), i = 1, \ldots, n+1$ given by Eq.(3.3.30)-Eq.(3.3.31) with $z_0 = \alpha_1 z_{1,0} + \alpha_2 z_{2,0}$,
$\alpha_1 = |c_1|^2, \alpha_2 = |c_2|^2$.

Proof. Let $\Psi_1(z,t)$ be a solution of the Cauchy problem (3.3.20) with $\Psi_0(z) = \Psi_{1,0}(z)$ given by Eq.(3.3.35) and let $\Psi_2(z,t)$ be a solution of the Cauchy problem (3.3.20) with $\Psi_0(z) = \Psi_{2,0}(z)$ given by Eq.(3.3.36). From Theorem 3.3.2 follows that

$$\forall t(t \geq 0)\left[\left(\text{supp}(\Psi_1(z,t))\right) \cap \left(\text{supp}(\Psi_2(z,t))\right) = \varnothing\right] \tag{3.3.39}$$

Let $p(z, t, z_{1,0}, z_{2,0}, \alpha_1, \alpha_2)$, $\alpha_1 = |c_1|^2, \alpha_2 = |c_2|^2$ be a probability density to observe at instant t vector z with result z. By reconcile Bohr rule, [see section 1.6, Eq.(1.6.19)] we obtain

$$\begin{gathered} p(z, t, z_{1,0}, z_{2,0}, \alpha_1, \alpha_2) = [p_1(z, t, z_{1,0}, \alpha_1)] * [p_2(z, t, z_{2,0}, \alpha_2)], \\ p(z, t, z_{1,0}, \alpha_1) = \alpha_1^{-(n+1)} |\Psi_1(\alpha_1^{-1} z, t, z_{1,0})|^2, \\ p_2(z, t, z_{2,0}, \alpha_2) = \alpha_2^{-(n+1)} |\Psi_2(\alpha_2^{-1} z, t, z_{2,0},)|^2. \end{gathered} \tag{3.3.40}$$

In order to obtain the inequality (3.3.37) we need to estimate the quantities

$$\langle z_i \rangle(t, z_{1,0}, z_{2,0}, t_0, \alpha_1, \alpha_2, \varepsilon; \hbar) = \int_{\mathbb{R}^{n+1}} |z_i - \lambda_i| p(z, t, z_{1,0}, z_{2,0}, \alpha_1, \alpha_2) dz^{n+1}. \tag{3.3.41}$$

In order to estimate the quantities we dealing completely similarly as in the proof of the theorem 3.1 from paper [19].

Quasiclassical quantum "cat" with a cubic potential supplemented by additive sinusoidal driving.

As an example we calculate now exact quasi-classical asymptotic for quantum cat with a cubic potential supplemented by additive sinusoidal driving. Using Theorem3.3.2 we obtain limiting quantum trajectories given given by Eq.(3.3.30)-Eq.(3.3.31) with $z_0 = \alpha_1 z_{1,0} + \alpha_2 z_{2,0},\ \alpha_1 = |c_1|^2, \alpha_2 = |c_2|^2$. Let us consider quantum cat with a cubic potential

$$V(\theta) = (m\omega^2)/2\theta^2 - a\theta^3 + b\theta, \theta \in [\theta_1, \theta_2], a, b > 0 \tag{3.3.42}$$

supplemented by additive sinusoidal driving

$$V(\theta) = \frac{m\omega^2}{2}\theta^2 - a\theta^3 + b\theta - [A\sin(\Omega t)]\theta. \tag{3.3.43}$$

The corresponding master Lagrangian given by Eq.(3.3.25),is

$$L(u, \dot{u}, \tau) = (m/2)\dot{u}^2 - m((\omega^2/2) + (3a\lambda/m))u^2 - (m\omega^2\lambda + 3a\lambda^2 - b - A\sin(\Omega t))u. \tag{3.3.44}$$

We assume now that: $\omega^2/2 + 3a\lambda/m \geq 0$ and rewrite (3.3.44) in the following form

$$L(\dot{u}, u, \tau) = (m/2)\dot{u}^2 - (m\varpi^2\lambda/2)u^2 + g(\lambda, t)u, \tag{3.3.45}$$

where $\varpi(\lambda) = \sqrt{2|\omega^2/2 + 3a\lambda/m|}$ and $g(\lambda, t) = -[m\omega^2\lambda + 3a\lambda^2 - b - A\sin(\Omega t)]$. The corresponding master action $S(t, \lambda, x, y)$ given by Eq.(3.3.24), is

$$\begin{gathered} S(t,\lambda,x,y) = \frac{m\varpi}{2\sin\varpi t}\Big[(\cos\varpi t)(y^2 + x^2) - 2xy + \frac{2x}{m\varpi}\int_0^t g(\lambda,\tau)\sin(\varpi\tau)d\tau + \\ + \frac{2y}{m\varpi}\int_0^t g(\lambda,\tau)\sin(\varpi(t-\tau))d\tau - \frac{2}{m^2\varpi^2}\int_0^t\int_0^\tau g(\lambda,\tau)g(\lambda,s)\sin\varpi(t-\tau)\sin(\varpi s)dsd\tau]\Big]. \end{gathered} \tag{3.3.46}$$

Therefore a linear system of the algebraic equations(3.3.) is

$$\partial S(t,\lambda,x,y)/\partial y = 2y\cos\varpi t - 2x + \frac{2}{m\varpi}\int_0^t g(\lambda,t)sin((\varpi(t-\tau))d\tau = 0. \tag{3.3.47}$$

Therefore

$$y_{cr}(t,\lambda,x) = \frac{x}{\cos\varpi t} - \frac{1}{m\varpi\cos\varpi t}\int_0^t g(\lambda,t)\sin((\varpi(t-\tau))d\tau. \tag{3.3.48}$$

The linear system of the algebraic equations (3.3.26) is

$$\frac{x}{\cos\varpi t} - \frac{1}{m\varpi\cos\varpi t}\int_0^t g(\lambda,t)\sin((\varpi(t-\tau))d\tau + \lambda - \theta_0 = 0. \tag{3.3.49}$$

Therefore solution of the linear system of the algebraic equations (3.3.49) is

$$x(t,\lambda,\theta_0) = \frac{1}{m\varpi}\int_0^t g(\lambda,t)\sin((\varpi(t-\tau))d\tau + (\lambda - \theta_0)\cos\varpi t. \tag{3.3.50}$$

Transcendental master equation (3.3.31) is

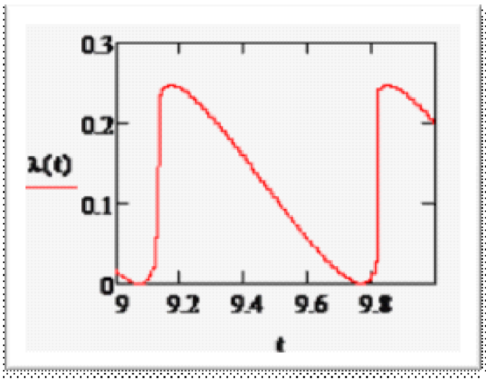
$$\int_0^t g(\lambda,t)\sin((\varpi(t-\tau))d\tau + m\varpi(\lambda-\theta_0)\cos\varpi t = 0 \tag{3.3.51}$$

Finally from Eq.(3.3.51) we obtain

$$\begin{gathered} d(\lambda(t))(cos(\varpi t)/\varpi - 1/\varpi) + A(\varpi sin(\Omega t) - \Omega sin(\varpi t))/(\varpi^2 - \Omega^2) - \\ (\lambda(t) - x\_0)m\varpi cos(\varpi t) = 0, \end{gathered} \tag{3.3.52}$$

where $d(\lambda) = m\omega^2\lambda + 3a\lambda^2 - b$.

Numerical Example. $\theta_0 = 0, m = 1, \Omega = 0, \omega = 9, a = 3, b = 10, A = 0$.



$Pic.1.$ Limiting quantum trajectory $\theta(t) = \lambda(t)$.

III.4. Stern-Gerlach experiment revisited and Schrödinger's cat paradox resolution.

III.4.1. Stern-Gerlach experiment revisited

In 1922, by studying the deflection of a beam of silver atoms in a strongly inhomogeneous magnetic field (cf. Fig. 3.4.1) Otto Stern and Walter Gerlach obtained an experimental result that contradicts the common sense prediction: the beam, instead of expanding, splits into two separate beams giving two spots of equal intensity N^{+} and N^{-} on a detector, at equal distances from the axis of the original beam. Historically, this is the experiment which helped establish spin quantization. Theoretically, it is the seminal experiment posing the problem of measurement in quantum mechanics.

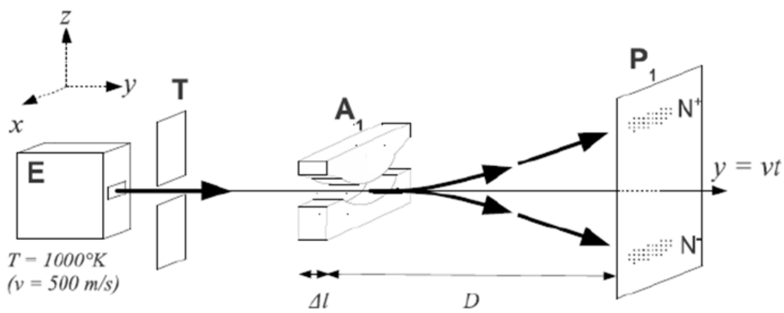


Fig.3.4.1.Schematic configuration of the Stern-Gerlach experiment. Adapted from [55].

$$\Psi(z,y,t)|_{t=0} = \Psi^0(z,y) = \Psi_1^0(z)\Psi_2^0(y). \quad (3.4.1)$$

We assume now that both density $\Psi^0(z)$ and $\Psi^0(y)$ is very narrow, in fact constrained such that

$$\begin{aligned} \Psi_1^0(z) &= \Psi_1^0(z,\delta) = 0 \quad \text{iff} \quad |x| > \delta, \\ \Psi_2^0(y) &= \Psi_2^0(y,\delta) = 0 \quad \text{iff} \quad |y| > \delta, \end{aligned} \tag{3.4.2}$$

and

$$\begin{aligned} &\Psi_1^0(z) = \Psi_1^0(z,\delta) = (2\pi_{\sigma_0}\sigma_0^2)^{-\frac{1}{4}} e^{-\frac{z^2}{4\sigma_0^2}} \begin{pmatrix} \cos\frac{\theta_0}{2} e^{-i\frac{\varphi_0}{2}} \\ \sin\frac{\theta_0}{2} e^{i\frac{\varphi_0}{2}} \end{pmatrix} \text{ iff } |z| \leq \delta, \\ &\| \Psi_1^0(z,\delta) \|_2^2 = 1; \\ &\Psi_2^0(y) = \Psi_2^0(y,\delta) = (2\pi_{\sigma_0}\sigma_0^2)^{-\frac{1}{4}} e^{-\frac{y^2}{4\sigma_0^2}} \text{ iff } |y| \leq \delta, \\ &\| \Psi_2^0(y,\delta) \|_2^2 = 1 \\ &\sigma_0 \ll 1. \end{aligned} \tag{3.4.3}$$

Silver atoms contained in the oven E (Fig.3.4.1.) are heated to a high temperature and escape through a narrow opening. A second aperture, T, selects those atoms whose velocity, $\mathbf{v}_0$, is parallel to the y-axis. The atomic beam crosses the gap of the electromagnet A_1 before condensing on the detector, P_1.Before crossing the electromagnet, the magnetic moment of each silver atom is oriented randomly (isotropically). In the beam, we represent each atom by its wave function; one can assume that at the entrance to the electromagnet, A_1, and at the initial time $t = 0$, each atom can be approximatively described by a quasi-Gaussian spinor in plain (z,y) given by Eqs.(3.4.1-3.4.3) corresponding to a pure state. As will be it is proved later the variable y will be treated strictly quasiclassically, i.e. almost classically, with

$$\begin{aligned} \mathbf{P}\left\{ \left| y - \mathbf{v}_y^{\pm}(\mathbf{v}_0,\theta_0)t \right| < \delta \right\} &= 1, \\ \mathbf{P}\left\{ \left| y - \mathbf{v}_y^{\pm}(\mathbf{v}_0,\theta_0)t \right| \geq \delta \right\} &= 0 \end{aligned} \tag{3.4.4}$$

and $\sigma_0 \leq \sigma_0' = 10^{-4}m$, where σ_0' corresponds to the size of the slot T along the z-axis and where the expression of the functions $\mathbf{v}_y^{+}(\mathbf{v}_0,\theta_0)$ and $\mathbf{v}_y^{-}(\mathbf{v}_0,\theta_0)$will be given later.

The approximation by a quasi-Gaussian initial spinor will allow explicit calculations. Because the slot is much wider along the x-axis, the variable z will be also treated strictly quasiclassically with

$$\begin{aligned} \mathbf{P}\left\{ \left| z - z_\Delta^{\pm} - \mathbf{v}_z^{\pm}(u,\theta_0)t \right| \leq \delta \right\} &= 1, \\ \mathbf{P}\left\{ \left| z - \mathbf{v}_z^{\pm}(u,\theta_0)t \right| > \delta \right\} &= 0, \end{aligned} \tag{3.4.5}$$

where the expression of the functions $\mathbf{v}_z^{\pm}(u,\theta_0), u = \dfrac{\mu_B B_0'(\Delta t)}{m}$ will be given later. In order to obtain an explicit solution of the Stern-Gerlach experiment, we take for the silver atom, we have $m = 1.8 \times 10^{-25} kg, \mathbf{v}_0 = 500\ m/s$ (corresponding to the temperature of $T = 1000°K$). In equation (3.4.3.) and in figure 3.4.2., θ_0 and φ_0 are the polar angles characterizing the initial orientation of the magnetic moment, θ_0 corresponds to the angle with the z-axis. The experiment is a statistical mixture of pure states where the θ_0 and the φ_0 are randomly chosen: θ_0 is drawn in a uniform way from $[0,\pi]$ and that φ_0 is drawn in a uniform way from $[0,2\pi]$.

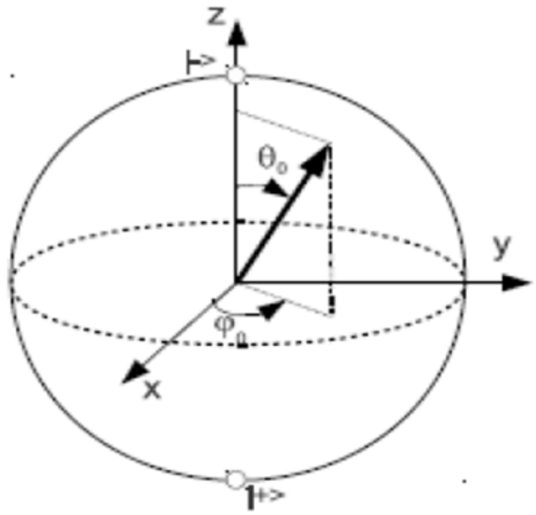


Fig.3.4.2.Orientation of the magnetic moment θ_0 and ϕ_0 are the polar angles characterizing the spin vector in the de Broglie-Bohm interpretation. Adapted from [55].

Assumption 3.4.1.We assume that a particle collapses in a magnetic field $\mathbf{B}$ at some instant t' by two particle,i.e.the spinor $\Psi(z,y,t)$ collapses in a magnetic field $\mathbf{B}$
at some instant t' by two spinors $\Psi_+(z,y,t,t',\delta)$ and $\Psi_-(z,y,t,t',\delta)$ given by Eq.(3.4.9.a)-Eq.(3.4.9.b). Note that such collapse obviously occurs except spinors such
that:$\sqrt{2^{-1}}\left(\psi_+^{(z)} + \psi_-^{(z)}\right) = \psi_+^{(x)}$,etc.

Remark 3.4.1.Note that standard assumption consist that spinor collapses on detector P1
with respect of the Born rule.

Thus the evolution of the spinor

$$\Psi(z,y,t,t') = \begin{pmatrix} \Psi_+(z,y,t,t') \\ \Psi_-(z,y,t,t') \end{pmatrix}$$

in a magnetic field $\mathbf{B}$ is then given by the nonlocal Pauli equation:

$$i\hbar\begin{pmatrix} \int dzdy \int dt \frac{\partial\Psi_+(z,y,t,t')}{\partial t} \\ \int dzdy \int dt \frac{\partial\Psi_-(z,y,t,t')}{\partial t} \end{pmatrix} = \tag{3.4.6}$$

$$= -\frac{\hbar^2}{2m}\int dt \int dzdy \Delta\begin{pmatrix} \Psi_+(z,y,t,t') \\ \Psi_-(z,y,t,t') \end{pmatrix} + \mu_B \int dt \int dzdy \mathbf{B}\sigma\begin{pmatrix} \Psi_+(z,y,t,t') \\ \Psi_-(z,y,t,t') \end{pmatrix}$$

where $\mu_B = \frac{e\hbar}{2m_e}$ is the Bohr magneton and where $\sigma = (\sigma_x,\sigma_y,\sigma_z)$ corresponds to the three Pauli matrices.

Remark 3.4.2. The particle first enters an electromagnetic field $\mathbf{B}$ directed along the z-axis, $B_x = B_0'x, B_y = 0, B_z = B_0 - B_0'z$, with $B_0 = 5$ Tesla, $B_0' = \left|\frac{\partial B}{\partial z}\right| = 10^3\ Tesla/m$ over a length $\Delta l = 1\ cm$.

Remark 3.4.3. On exiting the magnetic field, the both particles is free until it reaches the detector P_1 placed at a $D = 20\ cm$ distance.

The particles stays within the magnetic field for a time Δt with

$$\Delta t = \frac{\Delta l}{\mathbf{v}_0}. \tag{3.4.7}$$

Assumption 3.4.2. We assume now for simplification that

$$t' \approx \Delta t. \tag{3.4.8}$$

Thus during this time $t \in [0,t') \approx [0,\Delta t)$, the spinor $\Psi(z,y,t,t',\delta)$ is:

$$\Psi(z,y,t,t',\delta) = \begin{pmatrix} \Psi_+(z,y,t,t',\delta) \\ \Psi_-(z,y,t,t',\delta) \end{pmatrix} = \begin{pmatrix} \Psi_+(z,t,t',\delta)\Psi(y,t,t',\delta) \\ \Psi_-(z,t,t',\delta)\Psi(y,t,t',\delta) \end{pmatrix}, \tag{3.4.9.a}$$

where

$$\Psi_+(z,t,t',\delta) = \cos\frac{\theta_0}{2}e^{-i\frac{\varphi_0}{2}}(2\pi\sigma_0^2)^{-\frac{1}{2}}\exp\left[-\frac{\left(z-\frac{\mu_B B_0'}{2m}t^2\right)^2}{4\sigma_0^2}\right]\times$$

$$\exp\left[i\frac{\mu_B B_0' tz - \frac{\mu_B^2 B_0'^2}{6m}t^3 + \mu_B B_0 t + 0.5\hbar\varphi_0}{\hbar}\right] \text{ iff } \left|z-\frac{\mu_B B_0'}{2m}t^2\right| \leq \delta,$$

$$\Psi_+(z,t,\delta) = 0 \text{ iff } \left|z-\frac{\mu_B B_0'}{2m}t^2\right| > \delta,$$

$$\Psi_-(z,t,t',\delta) = i\sin\frac{\theta_0}{2}e^{i\frac{\varphi_0}{2}}(2\pi\sigma_0^2)^{-\frac{1}{2}}\exp\left[-\frac{\left(z+\frac{\mu_B B_0'}{2m}t^2\right)^2}{4\sigma_0^2}\right]\times \qquad (3.4.9.b)$$

$$\exp\left[i\frac{-\mu_B B_0' tz - \frac{\mu_B^2 B_0'^2}{6m}t^3 - \mu_B B_0 t - 0.5\hbar\varphi_0}{\hbar}\right] \text{ iff } \left|z+\frac{\mu_B B_0'}{2m}t^2\right| \leq \delta,$$

$$\Psi_-(z,t,t',\delta) = 0 \text{ iff } \left|z+\frac{\mu_B B_0'}{2m}t^2\right| > \delta;$$

$$\Psi(y,t,t',\delta) = (2\pi\sigma_0^2)^{-\frac{1}{2}}\exp\left[-\frac{(y-\mathbf{v}_0 t)^2}{4\sigma_0^2}\right] \text{ iff } |y-\mathbf{v}_0 t| \leq \delta,$$

$$\Psi(y,t,t',\delta) = 0 \text{ iff } |y-\mathbf{v}_0 t| > \delta.$$

After the magnetic field, at time $t+\Delta t$ $(t>0)$ in the free space, the both spinors becomes:

$$\Psi_+(z,y,t+t',\delta) \simeq \Psi_+(z,y,t+\Delta t,\delta) = \Psi_+(z,t+\Delta t,\delta)\Psi(y,t+\Delta t,\delta) \qquad (3.4.10)$$

and

$$\Psi_-(z,y,t+t',\delta) \simeq \Psi_-(z,y,t+\Delta t,\delta) = \Psi_-(z,t+\Delta t,\delta)\Psi(y,t+\Delta t,\delta). \qquad (3.4.11)$$

Here

$$\Psi_+(z,t+\Delta t,\delta) \simeq$$

$$\begin{cases} \cos\frac{\theta_0}{2}(2\pi\sigma_0^2)^{-\frac{1}{2}}\exp\left[-\frac{(z-z_\Delta-ut)^2}{4\sigma_0^2}\right]e^{i\frac{muz+\hbar\varphi_+}{\hbar}} \text{ iff } |z-z_\Delta-ut| \leq \delta, \\ 0 \text{ iff } |z-z_\Delta-ut| > \delta \end{cases} \qquad (3.4.12)$$

and

$$\Psi_{-}(z,t+\Delta t,\delta) \simeq$$

$$\begin{cases} \sin\dfrac{\theta_0}{2}(2\pi\sigma_0^2)^{-\frac{1}{4}}\exp\left[-\dfrac{(z+z_\Delta+ut)^2}{4\sigma_0^2}\right]e^{i\frac{-muz+h\varphi_-}{h}} \text{ iff } |z+z_\Delta+ut| \leq \delta, \\ 0 \text{ iff } |z+z_\Delta+ut| > \delta, \end{cases} \quad (3.4.13)$$

and

$$\Psi(y,t,\delta) = (2\pi\sigma_0^2)^{-\frac{1}{4}}\exp\left[-\frac{(y-\mathbf{v}_0(t+\Delta t))^2}{4\sigma_0^2}\right] \text{ iff } |y-\mathbf{v}_0(t+\Delta_{\theta_0}t)| \leq \delta,$$
$$\Psi(y,t,\delta) = 0 \text{ iff } |y-\mathbf{v}_0(t+\Delta t)| > \delta. \quad (3.4.14)$$

where

$$z_\Delta = \frac{\mu_B B_0'([\Delta t]^2}{2m}, \qquad u = \frac{\mu_B B_0'(\Delta t)}{m}. \quad (3.4.15)$$

From Eq.(3.4.10),Eq.(3.4.12) and Eq.(3.4.14) we obtain

$$\Psi_{+}(z,y,t+\Delta t,\delta) =$$

$$\begin{cases} \cos\frac{\theta_0}{2}(2\pi\sigma_0^2)^{-\frac{1}{2}}\exp\left[-\dfrac{(z-z_\Delta-ut)^2}{4\sigma_0^2}\right]e^{i\frac{muz+h\varphi_+}{h}} \times & \text{iff } |z-z_\Delta-ut| \leq \delta \\ \exp\left[-\dfrac{(y-\mathbf{v}_0(t+\Delta t))^2}{4\sigma_0^2}\right] & \text{and } |y-\mathbf{v}_0(t+\Delta t)| \leq \delta \\ 0 & \text{otherwise} \end{cases} \quad (3.4.16)$$

From Eq.(3.4.16) by postulate Q.IV.3 for the probability density with respect to observable z we obtain the expression

$$c_{|\Psi_+\rangle}(z,t) =$$

$$\begin{cases} (2\pi\sigma_0^2)^{-\frac{1}{2}}\exp\left[-\dfrac{\left(\dfrac{z}{\eta_{\theta_0}^+}-z_\Delta-ut\right)^2}{2\sigma_0^2}\right] & \text{iff } \left|\dfrac{z}{\eta_{\theta_0}^+}-z_\Delta-ut\right| \leq \delta \\ 0 & \text{otherwise} \end{cases} \quad (3.4.17)$$

$$\eta_{\theta_0}^+ = \cos^2\frac{\theta_0}{2}$$

and with respect to observable y we obtain the expression

$$c_{|\Psi_+\rangle}(y,t) =$$

$$\begin{cases} (2\pi\sigma_0^2)^{-\frac{1}{2}} \exp\left[-\dfrac{\left(\dfrac{y}{\eta_{\theta_0}^+} - \mathbf{v}_0(t+\Delta t)\right)^2}{4\sigma_0^2} \right] & \text{iff } \left| \dfrac{y}{\eta_{\theta_0}^+} - \mathbf{v}_0(t+\Delta t) \right| \leq \delta \\ 0 & \text{otherwise} \end{cases} \quad (3.4.18)$$

$$\eta_{\theta_0}^+ = \cos^2 \frac{\theta_0}{2}$$

and therefore corresponding particle movin by strictly quasiclassical law

$$\begin{aligned} &\mathbf{P}\left\{ \left| z^+(t) - z_\Delta^+ - \mathbf{v}_z^+(u,\theta_0)t \right| \leq \delta \right\} = 1, \\ &\mathbf{P}\left\{ \left| z^+(t) - z_\Delta^+ - \mathbf{v}_z^+(u,\theta_0)t \right| > \delta \right\} = 0, \\ &\mathbf{P}\{ |y^+(t) - \mathbf{v}_y^+(\mathbf{v}_0,\theta_0)(t+\Delta t)| \leq \delta \} = 1, \\ &\mathbf{P}\{ |y^+(t) - \mathbf{v}_y^+(\mathbf{v}_0,\theta_0)(t+\Delta t)| > \delta \} = 0, \\ &z_\Delta^+ = \eta_{\theta_0}^+ z_\Delta, \\ &\mathbf{v}_z^+(\theta_0) = \eta_{\theta_0}^+ u, \mathbf{v}_y^+(\theta_0) = \eta_{\theta_0}^+ \mathbf{v}_0. \end{aligned} \quad (3.4.19)$$

From Eq.(3.4.11),Eq.(3.4.13) and Eq.(3.4.14) we obtain

$$\Psi_-(z,y,t+\Delta t,\delta) =$$

$$\begin{cases} \sin\dfrac{\theta_0}{2}(2\pi\sigma_0^2)^{-\frac{1}{4}} \exp\left[-\dfrac{(z+z_\Delta+ut)^2}{4\sigma_0^2} \right] e^{i\frac{-muz+h\varphi_-}{h}} \times & \text{iff} \\ & |z+z_\Delta+ut| \leq \delta \\ \exp\left[-\dfrac{(y-\mathbf{v}_0(t+\Delta t))^2}{4\sigma_0^2} \right] & \text{and} \\ & |y-\mathbf{v}_0(t+\Delta_{\theta_0} t)| \leq \delta \\ 0 & \text{otherwise} \end{cases} \quad (3.4.20)$$

From Eq.(3.4.20) by postulate Q.IV.3 for the probability density with respect to observable z we obtain the expression

$$c_{|\Psi_-\rangle}(z,t) =$$

$$\begin{cases} (2\pi\sigma_0^2)^{-\frac{1}{2}} \exp\left[-\dfrac{\left(\dfrac{z}{\eta_{\theta_0}^-} + z_\Delta + ut\right)^2}{2\sigma_0^2} \right] & \text{iff } \left| \dfrac{z}{\eta_{\theta_0}^-} + z_\Delta + ut \right| \leq \delta \\ 0 & \text{otherwise} \end{cases} \quad (3.4.21)$$

$$\eta_{\theta_0}^- = \sin^2 \frac{\theta_0}{2}$$

and with respect to observable y we obtain the expression

$$c_{|\Psi_-\rangle}(y,t) =$$

$$\begin{cases} (2\pi\sigma_0^2)^{-\frac{1}{2}} \exp\left[-\frac{\left(\frac{y}{\eta_{\theta_0}^-} - \mathbf{v}_0(t+\Delta t)\right)^2}{4\sigma_0^2} \right] & \text{iff } \left| \frac{y}{\eta_{\theta_0}^-} - \mathbf{v}_0(t+\Delta_{\theta_0} t) \right| \leq \delta \\ 0 & \text{othervice} \end{cases} \quad (3.4.22)$$

$$\eta_{\theta_0}^- = \sin^2 \frac{\theta_0}{2}$$

and therefore corresponding particle movin by strictly quasiclassical law

$$\begin{aligned} &\mathbf{P}\left\{ \left| z^-(t) + z_\Delta^- + \mathbf{v}_z^-(u,\theta_0)t \right| \leq \delta \right\} = 1, \\ &\mathbf{P}\left\{ \left| z^-(t) + z_\Delta^- + \mathbf{v}_z^-(u,\theta_0)t \right| > \delta \right\} = 0, \\ &\mathbf{P}\{|y^-(t) - \mathbf{v}_y^-(\mathbf{v}_0,\theta_0)(t+\Delta t)| \leq \delta\} = 1, \\ &\mathbf{P}\{|y^-(t) - \mathbf{v}_y^-(\mathbf{v}_0,\theta_0)(t+\Delta t)| > \delta\} = 0, \\ &z_\Delta^- = \eta_{\theta_0}^- z_\Delta, \\ &\mathbf{v}_z^-(u,\theta_0) = \eta_{\theta_0}^- u, \mathbf{v}_y^-(\mathbf{v}_0,\theta_0) = \eta_{\theta_0}^- \mathbf{v}_0. \end{aligned} \quad (3.4.23)$$

All interpretations are based on the equations (3.4.18)-(3.4.21).One deduce from Eq.(3.4.18)-Eq.(3.4.21) the probability density of a pure state in the free space after the electromagnet:

$$\begin{aligned} &\rho_{\theta_0}(z,y,t+\Delta t) = (2\pi\sigma_0^2)^{-\frac{1}{2}} \rho_{\theta_0}(z,t+\Delta_{\theta_0} t) \sum_{\pm} \exp\left[-\frac{\left(\frac{y}{\eta_{\theta_0}^{\pm}} - \mathbf{v}_0(t+\Delta t)\right)^2}{4\sigma_0^2} \right]; \\ &\rho_{\theta_0}(z,t+\Delta t) = (2\pi\sigma_0^2)^{-\frac{1}{2}} \left\{ \cos^{-2}\frac{\theta_0}{2}(2\pi\sigma_0^2)^{-\frac{1}{2}} \exp\left[-\frac{(\frac{z}{\eta_{\theta_0}^+} - z_\Delta - ut)^2}{2\sigma_0^2} \right] + \right. \\ &\left. \sin^{-2}\frac{\theta_0}{2}(2\pi\sigma_0^2)^{-\frac{1}{2}} \exp\left[-\frac{(\frac{z}{\eta_{\theta_0}^-} + z_\Delta + ut)^2}{4\sigma_0^2} \right] \right\}. \end{aligned} \quad (3.4.24)$$

The decoherence time $t_{\mathbf{dec}}$, where the two spots N^+ and $N^-$are separated, is then given by the equation:

$$t_{\mathbf{dec}} \asymp \frac{3\sigma_0 - z_\Delta}{u(\eta_{\theta_0}^+ + \eta_{\theta_0}^-)} = \frac{3\sigma_0 - z_\Delta}{u}. \quad (3.4.25)$$

This decoherence time is usually the time required to diagonalize the marginal density matrix $\rho_{\theta_0}^S(t,\delta)$ of spin variables associated with a pure state

$$\rho^{S}_{\theta_0}(t,\delta) = \begin{pmatrix} \int |\Psi_+(z,y,t+\Delta t,\delta)|^2 dzdy & \int \Psi^*_-(z,y,t+\Delta t,\delta)\Psi_+(z,y,t+\Delta t,\delta)dzdy \\ \int \Psi_-(z,y,t+\Delta t,\delta)\Psi^*_+(z,y,t+\Delta t,\delta)dzdy & \int |\Psi_+(z,y,t+\Delta t,\delta)|^2 dzdy \end{pmatrix} \quad (3.4.26)$$

For $t \geq t_{\mathbf{dec}}$, the product $\Psi_-(z,y,t+\Delta t,\delta)\Psi^*_+(z,y,t+\Delta t,\delta)$ is null and the density matrix $\rho^{S}_{\theta_0}(t,\delta)$ is diagonal. We then obtain atoms with a spin oriented only along the z-axis (positively or negatively). Let us consider the spinor $\Psi(z,y,t+\Delta t,\delta)$ given by equations (3.4.10)-(3.4.15).

Remark 3.4.4. Experimentally, we do not measure the spin directly, but the z position of the particle impact on the detector P1 (Fig.3.4.3.).

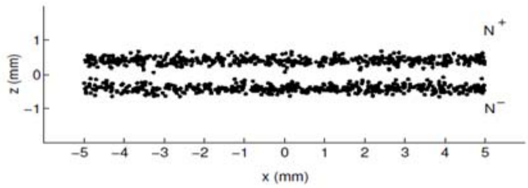


Fig.3.4.3.Silver atom impacts on the detector P1.

Adapted from [55].

Remark 3.4.5. Note that if we measure the z-position of the particle at instant t, we also measure the y-position of the particle at the same instant t.

Remark 3.4.6. Let $\mathbf{P}_t(D-\delta,D,y^+_t)$ be the probability of obtaining the result y^+_t at instant t, lying in the range $(D-\delta,D)$ on measuring observable y in respect to spinor $\Psi_+(z,y,t+\Delta t,\delta)$. From Eq.(3.4.19) we obtain

$$\begin{gathered} \mathbf{P}_t(D-\delta,D+\delta,y^+_t) = 1 \text{ iff} \\ y^+(t) = D \text{ and } |y^+(t) - \mathbf{v}^+_y(\mathbf{v}_0,\theta_0)(t+\Delta t)| \leq \delta. \end{gathered} \quad (3.4.27)$$

From Eq.(3.4.27) follows that:

$$\begin{gathered} \mathbf{P}_t(D-\delta,D+\delta,y^+_t) = 1 \\ \text{if} \\ t \triangleq t_+(D) \approx \frac{D}{\mathbf{v}^+_y(\mathbf{v}_0,\theta_0)} = \frac{D}{\mathbf{v}_0 \cos^2 \frac{\theta_0}{2}}. \end{gathered} \quad (3.4.28)$$

Remark 3.4.7. Let $\mathbf{P}_t(\widetilde{z}_+ - \delta,\widetilde{z}_+ + \delta,z^+_t)$ be the probability of obtaining the result z^+_t at instant t, lying in the range $(\widetilde{z}_+ - \delta,\widetilde{z}_+ + \delta), \widetilde{z}_+ \in N^+$ on measuring observable z in respect to spinor $\Psi_+(z,y,t+\Delta t,\delta)$. From Eq.(3.4.19) we obtain

$$\begin{gathered} \mathbf{P}_t(\widetilde{z}_+ - \delta,\widetilde{z}_+ + \delta,z^+_t) = 1 \text{ iff} \\ z^+(t) = \widetilde{z}_+ \text{ and } |z^+(t) - z^+_\Delta - \mathbf{v}^+_z(u,\theta_0)t| \leq \delta. \end{gathered} \quad (3.4.29)$$

From Eq.(3.4.29) follows that:

$$\mathbf{P}_t(\widetilde{z}_+ - \delta, \widetilde{z}_+ + \delta, z_t^+) = 1$$
if
$$t \triangleq t(\widetilde{z}_+) \approx \frac{\widetilde{z}_+}{\mathbf{v}_z^+(u,\theta_0)} = \frac{\widetilde{z}_+}{u\cos^2\frac{\theta_0}{2}}. \tag{3.4.30}$$

Remark 3.4.8.Note that from Remark 3.4.5 it follows that $t(\widetilde{z}_+) \approx t(D)$ and therefore
from Eq.(3.4.28) and Eq.(3.4.30) one obtains

$$\frac{\widetilde{z}_+}{u\cos^2\frac{\theta_0}{2}} \approx \frac{D}{\mathbf{v}_0\cos^2\frac{\theta_0}{2}} \Rightarrow \frac{\widetilde{z}_+}{u} \approx \frac{D}{\mathbf{v}_0} \tag{3.4.31}$$

as it should be, because the equality $\frac{\widetilde{z}_+}{u} \approx \frac{D}{\mathbf{v}_0}$ is required by the condition of the

Stern-Gerlach experiment.

Remark 3.4.9. Let $\mathbf{P}_t(D-\delta, D, y_t^-)$ be the probability of obtaining the result y_t^- at instant t, lying in the range $(D-\delta, D)$ on measuring observable y in respect to spinor $\Psi_-(z,y,t+\Delta t,\delta)$. From Eq.(3.4.23) we obtain

$$\mathbf{P}_t(D-\delta, D+\delta, y_t^-) = 1 \text{ iff}$$
$$y^-(t) = D \text{ and } |y^-(t) - \mathbf{v}_y^-(\mathbf{v}_0,\theta_0)(t+\Delta t)| \le \delta. \tag{3.4.32}$$

From Eq.(3.4.32) follows that:

$$\mathbf{P}_t(D-\delta, D+\delta, y_t^-) = 1$$
if
$$t \triangleq t_-(D) \approx \frac{D}{\mathbf{v}_y^-(\mathbf{v}_0,\theta_0)} = \frac{D}{\mathbf{v}_0\sin^2\frac{\theta_0}{2}}. \tag{3.4.33}$$

Remark 3.4.10. Let $\mathbf{P}_t(\widetilde{z}_- - \delta, \widetilde{z}_- + \delta, z_t^-)$ be the probability of obtaining the result z_t^- at instant t, lying in the range $(\widetilde{z}_- - \delta, \widetilde{z}_- + \delta), \widetilde{z}_- \in N^-$ on measuring observable z in respect to spinor $\Psi_-(z,y,t+\Delta t,\delta)$.From Eq.(3.4.32) we obtain

$$\mathbf{P}_t(\widetilde{z}_- - \delta, \widetilde{z}_- + \delta, z_t^-) = 1 \text{ iff}$$
$$z^-(t) = \widetilde{z}_- \text{ and } |z^-(t) - z_\Delta^- - \mathbf{v}_z^-(u,\theta_0)t| \le \delta. \tag{3.4.34}$$

From Eq.(3.4.29) follows that:

$$\mathbf{P}_t(\widetilde{z}_- - \delta, \widetilde{z}_- + \delta, z_t^-) = 1$$
if
$$t \triangleq t(\widetilde{z}_-) \approx \frac{|\widetilde{z}_-|}{\mathbf{v}_z^-(u,\theta_0)} = \frac{|\widetilde{z}_-|}{u\sin^2\frac{\theta_0}{2}}. \tag{3.4.35}$$

Remark 3.4.11.Note that from Remark 3.4.5 it follows that $t(\tilde{z}_+) \approx t(D)$ and therefore

from Eq.(3.4.33) and Eq.(3.4.35) one obtains

$$\frac{|\tilde{z}_-|}{u\sin^2\frac{\theta_0}{2}} \approx \frac{D}{\mathbf{v}_0\sin^2\frac{\theta_0}{2}} \Rightarrow \frac{\tilde{z}_-}{u} \approx \frac{D}{\mathbf{v}_0} \tag{3.4.36}$$

as it should be, because the equality $\frac{|\tilde{z}_-|}{u} \approx \frac{D}{\mathbf{v}_0}$ is required by the condition of the

Stern-Gerlach experiment.

III.4.2.Schrödinger’s cat which measure spin.Schrödinger’s cat paradox resolution.

Let us consider again the Schrödinger’s cat which measure spin by using the Stern-Gerlach apparatus, see subsection 1.6, (Fig.1.6.2). When a measurement is made, with the “up” outcome Schrödinger’s cat is dead. When a measurement is made, with a “down” outcome Schrödinger’s cat is alive. It known many years that conventional QM with canonical explanation of the Stern-Gerlach experiment cannot give predicable and [27] As pointed out in subsection 1.6

Theorem 3.4.1. Any spinor

$$\begin{pmatrix} \Psi_+(z,y,t,t') \\ \Psi_-(z,y,t,t') \end{pmatrix} \tag{3.4.37}$$

given by Eq.(3.4.9.a)-Eq.(3.4.9.b) with θ_0 such that $\cos\frac{\theta_0}{2} \neq 0$ always kills the Schrödinger’s cat at instant t :

$$t \approx \frac{D}{\mathbf{v}_0\cos^2\frac{\theta_0}{2}}. \tag{3.4.38}$$

IV.EPR Paradox Resolution

IV.1.The relaxed locality principle.

The Special Theory of Relativity limits the speed at which any physical influences and any

real information can travel to the speed of light, c.

The Einstein’s principle of locality (**EPL**): any effects do not propogate faster than the

speed of light, i.e. speed of light is a limiting factor.

The principle of locality claimed that:

(**i**) Any physical event $\mathbf{A}(t_1,\mathbf{r}_1)$ which has occured at point $A(t_1,\mathbf{r}_1) \in M_4$ (see Definition
2.2.8) cannot cause (by physical interection) a physical event $\mathbf{B}(t_2,\mathbf{r}_2)$ (result) which has
occured at point $B(t_2,\mathbf{r}_2) \in M_4$ in a time less than $T = D/c,$ where $D,$ is the distance
between the points.

(**ii**) An physical event $\mathbf{A}(t,\mathbf{r}_1)$ which has occured at point $A(t,\mathbf{r}_1) \in M_4$ cannot cause a
simultaneous physical event $\mathbf{B}(t,\mathbf{r}_2)$ (result) which has occured at another point
$B(t,\mathbf{r}_2) \in M_4$.

(**iii**) Any real physical information about physical event $\mathbf{A}(t_1,\mathbf{r}_1)$ at point $A(t_1,\mathbf{r}_1)$ cannot be
obtained by observer at point $B(t_2,\mathbf{r}_2)$ in a time less than $T = D/c,$ where $D,$ is the distance
between the points.

Definition 4.1.1. Let $[\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]^{\Rightarrow}_{\mathbf{t.l.s.}}$ be a set of the all timelike separated
pairs of events $\{\mathbf{A}(t_1,\mathbf{r}_1),\mathbf{B}(t_2,\mathbf{r}_2)\}_{\mathbf{t.l.s.}} \in [\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{t.l.s.}}$,(see Definition 2.2.10.a)
such that $t_2 > t_1$ and $\mathbf{A}^{Oc}(t_1,\mathbf{r}_1) \Rightarrow \mathbf{B}^{Oc}(t_2,\mathbf{r}_2)$.

Note that $[\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]^{\Rightarrow}_{\mathbf{t.l.s.}} \subsetneq [\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{t.l.s.}}$.

Remark 4.1.1. Note that the claim (i) obviously meant that

$$\begin{gathered}\forall(t_1 > t_2)\forall\mathbf{A}(t_1,\mathbf{r}_1)\forall\mathbf{B}(t_2,\mathbf{r}_2)\Big\{[\mathbf{A}^{Oc}(t_1,\mathbf{r}_1) \Rightarrow \mathbf{B}^{Oc}(t_2,\mathbf{r}_2)] \Leftrightarrow \\ \{\mathbf{A}(t_1,\mathbf{r}_1),\mathbf{B}(t_2,\mathbf{r}_2)\} \in [\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{t.l.s.}}\Big\}.\end{gathered} \tag{4.1.1}$$

Remark 4.1.2. In spacetime diagram, see FIG.4.1.1, the interval s^2_{AB} is "time-like" i.e., there is a frame of reference in which events **A** and **B** occur at the same location in space, separated only by occurring at different times. If **A** precedes **B** in that frame, then **A** precedes **B** in all frames. It is hypothetically possible for matter (or information) to travel from A to B, so there can be a causal relationship (with **A** the cause and **B** the effect).

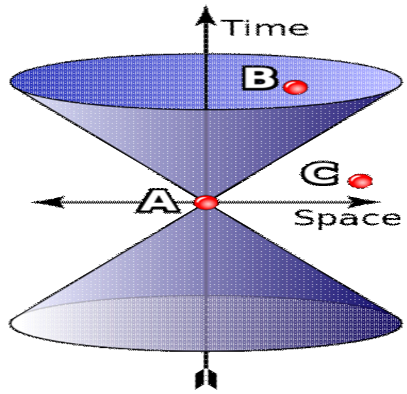


FIG.4.1.1.Spacetime diagram.

Remark 4.1.3. Note that:

(i) the interval s_{AC}^2 in the diagram, see FIG.4.1.1, is "space-like"; i.e., there is a frame $\mathcal{F}_t$ of reference in which events $\mathbf{A}(t,\mathbf{r}_1)$ and $\mathbf{C}(t,\mathbf{r}_2)$ occur simultaneously at instant t, separated only in space. There are also frames in which **A** precedes **C** and frames in which **C** precedes **A**.

(ii) If it were possible for a cause-and-effect relationship to exist between events **A** and **C**, then paradoxes of causality would result. For example, if **A** was the cause, and **C** the effect, then there would be frames of reference in which the effect preceded the cause. Although this in itself won't give rise to a paradox, one can show that faster than light signals can be sent back into one's own past. A causal paradox can then be constructed by sending the signal if and only if no signal was received previously.

(iii) Obviously there exist space-like separated pairs of physical events $\{\mathbf{A}(t,\mathbf{r}_1),\mathbf{B}(t,\mathbf{r}_2)\}_{\mathbf{s.l.s.}}$ such that the events $\mathbf{A}(t,\mathbf{r}_1)$ and $\mathbf{C}(t,\mathbf{r}_2)$ alwais occur only simultaneously at any instant t i.e.,

$$\mathbf{A}^{Oc}(t,\mathbf{r}_1) \Leftrightarrow \mathbf{C}^{Oc}(t,\mathbf{r}_2). \tag{4.1.2}$$

Example 4.1.1. Let us consider two synchronized clock A and B which at rest on given inertial frame $\mathcal{F}_I$. Assume that clock A at rest in point $\mathbf{r}_1$ and clock B at rest in point $\mathbf{r}_2$ correspondingly.

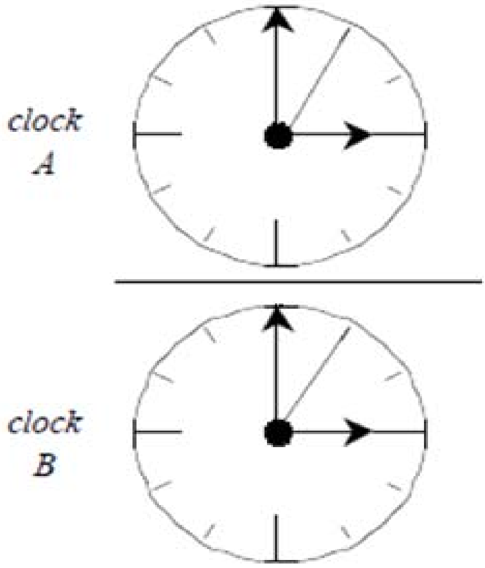


FIG.4.1.2.Clock A and clock B which at rest on
given inertial frame $\mathcal{F}_I$.

Let $\mathbf{A}(t,\mathbf{r}_1)$ be event which consist that time on clock A is t at time t according to clock A and let $\mathbf{B}(t,\mathbf{r}_1)$ be event which consist that time on clock B is t at time t according to clock B.It is clear that $\mathbf{A}^{Oc}(t,\mathbf{r}_1) \Leftrightarrow \mathbf{B}^{Oc}(t,\mathbf{r}_2)$.

Definition 4.1.2. Let $[\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]^{\Leftrightarrow}_{\mathbf{s.l.s.}}$ be a set of the all spacelike separated

pairs of events $\{\mathbf{A}(t_1,\mathbf{r}_1),\mathbf{B}(t_2,\mathbf{r}_2)\}_{\mathbf{s.l.s.}} \in [\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}}$,(see Definition

2.2.10.b) such that

$$\mathbf{A}^{Oc}(t_1,\mathbf{r}_1) \Leftrightarrow \mathbf{B}^{Oc}(t_2,\mathbf{r}_2). \tag{4.1.3}$$

Remark 4.1.3. Note that the conditions (4.1.3) does not violeted the Einstein's principle of locality and gives only an additional properties of the algebra $\mathcal{F}^{\#}_{M_4}$.

Remark 4.1.4. Note that from (4.1.3) folows that

$[\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]^{\Leftrightarrow}_{\mathbf{s.l.s.}} \subsetneq [\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}}$.

On the basis of this Gedankenexperiment, which is also realized by photons, the EPR-paradox can be derived if the following two principles are taken as postulates.

1. The principle of reality R :

If the value A_i of an observable A can be determined without altering the

quantum system S, then any property $P(A_i)$ which corresponds to this value of A pertains

to the sustem S.

2. The principle of locality L :

2.**1**.If two quantum systems S_1 and S_2 cannot interact with each other, then a measurement with respect to one system cannot alter the other system and therefore we

can assume the existance of state vectors $|S_1\rangle$ and $|S_2\rangle$.

2.**2**.Let $\widehat{x}_1$ and $\widehat{x}_2$ be two observables measured with respect to systems S_1 and S_2

mentioned above. Then by result of measurement of the quantity $\bar{x}_2 = \langle S_2|\widehat{x}_2|S_2\rangle$

at instant t, impossible to get any information on result of measurement of the quantity

$\bar{x}_1 = \langle S_1|\widehat{x}_1|S_1\rangle$ at the same instant t.

We assume now the relaxed principle of locality. Intuitively this principle says that for even spacelike separated entangled quantum systems S_1 and S_2 any measurement at instant t with respect to system S_1 alwais immediately alter the other system S_2 at the same instant t. But no additional information about the system S_1 can be found out upon measurement on the system S_2 except the canonical information which can be predicted by using correlation relations which follows from concrete type of entanglement.

3. **The relaxed principle of locality** L_{rel} :

3.**1**. Any spacelike separated quantum systems $\mathbf{S}_1$ and $\mathbf{S}_2$ cannot interact with each other

and therefore we can assume the existance of state vectors $|\mathbf{S}_1\rangle$ and $|\mathbf{S}_2\rangle$ correspondingly.

3.**2**. Let $\mathbf{S}_{1(2)}(t,\mathbf{r}_1)$ and $\mathbf{S}_{2(1)}(t,\mathbf{r}_2)$ be two spacelike separated entangled quantum systems

located in points $(t,\mathbf{r}_1)$ and $(t,\mathbf{r}_2)$ correspondingly.

(i) Assume that a state vector $|\mathbf{S}_{1(2)}(t,\mathbf{r}_1)\rangle$ suddenly collapses at instant t to state vector

$\left|\mathbf{S}^{\mathbf{s\text{-}col}}_{1(2)}(t,\mathbf{r}_1)\right\rangle$:

$$|\mathbf{S}_{1(2)}(t,\mathbf{r}_1)\rangle \overset{\mathbf{s\text{-}collapse}}{\rightarrow} \left|\mathbf{S}^{\mathbf{s\text{-}col}}_{1(2)}(t,\mathbf{r}_1)\right\rangle, \tag{4.1.4}$$

then a state vector $|\mathbf{S}_{2(1)}(t,\mathbf{r}_2)\rangle$ immediately collapses to state vector $|\mathbf{S}^{\mathbf{col}}_{2(1)}(t,\mathbf{r}_2)\rangle$:

$$|\mathbf{S}_{2(1)}(t,\mathbf{r}_2)\rangle \overset{\mathbf{collapse}}{\rightarrow} |\mathbf{S}^{\mathbf{col}}_{2(1)}(t,\mathbf{r}_2)\rangle \tag{4.1.5}$$

(ii) Assume that a state vector $|\mathbf{S}_{1(2)}(t,\mathbf{r}_1)\rangle$ after measurement immediately collapses at

instant t to state vector $\left|\mathbf{S}^{\mathbf{m\text{-}col}}_{1(2)}(t,\mathbf{r}_1)\right\rangle$:

$$|\mathbf{S}_{1(2)}(t,\mathbf{r}_1)\rangle \overset{\mathbf{m\text{-}collapse}}{\rightarrow} \left|\mathbf{S}^{\mathbf{m\text{-}col}}_{1(2)}(t,\mathbf{r}_1)\right\rangle, \tag{4.1.6}$$

then a state vector $|\mathbf{S}_{2(1)}(t,\mathbf{r}_2)\rangle$ immediately collapses to state vector $|\mathbf{S}^{\mathbf{col}}_{2(1)}(t,\mathbf{r}_2)\rangle$:

$$|\mathbf{S}_{2(1)}(t,\mathbf{r}_2)\rangle \overset{\mathbf{collapse}}{\rightarrow} |\mathbf{S}^{\mathbf{col}}_{2(1)}(t,\mathbf{r}_2)\rangle \tag{4.1.7}$$

(iii) Let $\mathbf{S}^{\text{s-col}}_{1(2)}(t,\mathbf{r}_1)$ and $\mathbf{S}^{\text{col}}_{2(1)}(t,\mathbf{r}_2)$ be a physical events defined by formulae (4.1.4) and
(4.1.5) correspondingly, then

$$\mathbf{Occ}\left[\mathbf{S}^{\text{s-col}}_{1(2)}(t,\mathbf{r}_1)\right] \Leftrightarrow \mathbf{Occ}[\mathbf{S}^{\text{col}}_{2(1)}(t,\mathbf{r}_2)], \tag{4.1.8}$$

see Definition 2.2.8 (ii.9).

(iv) Let $\mathbf{S}^{\text{m-col}}_{1(2)}(t,\mathbf{r}_1)$ and $\mathbf{S}^{\text{col}}_{2(1)}(t,\mathbf{r}_2)$ be a physical events defined by formulae (4.1.6) and
(4.1.7) correspondingly,then

$$\mathbf{Occ}\left[\mathbf{S}^{\text{m-col}}_{1(2)}(t,\mathbf{r}_1)\right] \Leftrightarrow \mathbf{Occ}[\mathbf{S}^{\text{col}}_{2(1)}(t,\mathbf{r}_2)], \tag{4.1.9}$$

3.**3**.No any additional information about the system S_1 upon measurement at instant t can
be found out upon measurement on the system S_2 upon measurement at instant t except
the canonical information which can be predicted by using correlation relations which
follows from concrete type of entanglement.

Remark 4.**1**.**5**. Note that conditions (4.1.8)-(4.1.9) very similarly to the condition (4.1.3)
and gives only an additional properties of the algebra $\mathcal{F}^{\#}_{M_4}$.

Remark 4.**1**.**6**. Note that from (4.1.8) follows that

$$\{\mathbf{S}^{\text{s-col}}_{1(2)}(t,\mathbf{r}_1),\mathbf{S}^{\text{col}}_{2(1)}(t,\mathbf{r}_2)\} \in [\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]^{\Leftrightarrow}_{\text{s.l.s.}} \tag{4.1.10}$$

from (4.1.9) follows that

$$\{\mathbf{S}^{\text{m-col}}_{1(2)}(t,\mathbf{r}_1),\mathbf{S}^{\text{col}}_{2(1)}(t,\mathbf{r}_2)\} \in [\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]^{\Leftrightarrow}_{\text{s.l.s.}} \tag{4.1.11}$$

Remark 4.**1**.**7**. Note that:

(i) collapse of a state vector $|\mathbf{S}_{2(1)}(t,\mathbf{r}_2)\rangle$ given by (4.1.5) occurs without any interaction
between quantum systems $S_{1(2)}$ and $S_{2(1)}$ but only by property given by formulae (4.1.8);

(ii) collapse of a state vector $|\mathbf{S}_{2(1)}(t,\mathbf{r}_2)\rangle$ given by (4.1.7) occurs without any interaction
between quantum systems $S_{1(2)}$ and $S_{2(1)}$ but only by property given by formulae (4.1.9);

Remark.**4**.**1**.**8**.We find that the EPR-paradox can be resolved by nonprincipal and convenient relaxing of the Einstein's locality principle . Hoever it follows also, that the nonlocalities which are introduced above cannot be explained within the conventional quantum theory.

IV.2.Generalized EPR argument and Postulate of Nonlocality

Entanglement is one of the most interesting properties of quantum mechanics, and is an important ingredient of quantum information protocols such as quantum densecoding and quantum computation. In the Schrödinger picture, a necessary and sufficient criterion for the emergence of entanglement is that the state describing the entire system is inseparable, i.e. the wavefunction of the total system cannot be factored into a product of separate contributions from each sub-system. Using the Heisenberg approach, a sufficient criterion for the presence of entanglement is that correlations between conjugate observables of two subsystems allow the statistical inference of either observable in one sub-system, upon a measurement in the other, to be smaller than the standard quantum limit, i.e. the presence of non-classical correlations. The latter approach was originally proposed in the paper of Einstein, Podolsky and Rosen [21]. These two different pictures result in two distinct methods of characterizing entanglement. One is to identify an observable signature of the mathematical criterion for wave-function entanglement, i.e. inseparability of the state. The second looks directly for the onset of non-classical correlations. For pure states these two approaches return the same result suggesting consistency of the two methods. However, when decoherence is present, causing the state to be mixed, difference scan occur.

IV.2.1.The EPR-Reid criterion

We remind now EPR-Reid criterion [22]. EPR originally argued as follows. Consider two spatially separated subsystems at A and B. EPR considered two observables $\hat{x}$ (the "position") and $\hat{p}$ ("momentum") for subsystem A, where $\hat{x}$ and $\hat{p}$ do not commute, so that (C is nonzero)

$$[\hat{x},\hat{p}] = 2C. \tag{4.2.1}$$

Suppose now that one may predict with certainty the result of measurement $\hat{x}$, based on the result of a measurement performed at B. Also, for a different choice of measurement at B, suppose one may predict the result of measurement $\hat{p}$ at A. Such correlated systems are predicted by quantum theory. Assuming "local realism" EPR deduce the existence of an "element of reality", $\tilde{x}$, for the physical quantity $\hat{x}$ and also an element of reality, $\tilde{p}$, for $\hat{p}$. Local realism implies the existence of two hidden variables $\tilde{x}$ and $\tilde{p}$ that simultaneously predetermine, with no uncertainty, the values for the result of an $\hat{x}$ or $\hat{p}$ measurement on subsystem A, should it be performed. This hidden variable state for the subsystem A alone is not describable within quantum mechanics, since simultaneous eigenstates of $\hat{x}$ and $\hat{p}$ do not exist. Hence, EPR argued, if quantum mechanics is to be compatible with local realism, we must regard quantum mechanics to be incomplete.

We remind that in original publication [21],Einstein, Podolsky and Rosen describe two particles A and B with correlated position

$$x_B = x_A + x_0 \tag{4.2.2}$$

and anti-correlated momentum

$$p_B = -p_A, \tag{4.2.3}$$

(see Fig.4.2.1).

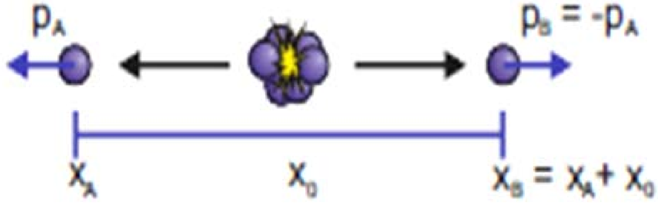


Fig.4.2.1.

In the idealized entangled state proposed by EPR,

$$|EPR\rangle = \int_{-\infty}^{\infty} |x,x\rangle dx = \int_{-\infty}^{\infty} |p,p\rangle dp$$

the positions and momenta of the two particles are perfectly correlated. Note that: this state is non-normalizable and cannot be realized in the laboratory. When coordinates x^A and p^A are measured in independent realizations of the same state, the correlations allow for an exact prediction of x^B and p^B. EPR assumed that such exact predictions necessitate an ”element of reality” which predetermines the outcome of the measurement. Quantum mechanics however prohibits the exact knowledge of two noncommuting variables like x^B and p^B, since their measurement uncertainties are subject to the Heisenberg relation

$$\Delta x^B \Delta p^B \geq \hbar/2. \tag{4.2.4}$$

Classsical notion of EPR correlations was generalized to a more realistic scenario, yielding a Reid criterion [22] for the uncertainties $\Delta x^B_{\mathbf{inf}}$ and $\Delta p^B_{\mathbf{inf}}$ of the inferred predictions for x^B and p^B. The EPR criterion is met if these uncertainties violate the Heisenberg inequality for the inferred uncertainties $\Delta x^B_{\mathbf{inf}} \Delta p^B_{\mathbf{inf}} \geq \hbar/2$.

Reid extended classical EPR argument to situations where the result of measurement $\hat{x}$ at A cannot be predicted with absolute certainty [22]. The assumption of local realism allows us to deduce the existence of an “element of reality” of some type for $\hat{x}$ at A, since we can make a prediction of the result at A, without disturbing the subsystem at A, under the locality assumption. Let $\Psi\left(\hat{x}^A, \hat{x}^B\right)$ be a wave function of composite system $A \cup B$. Let x_i^B be the result of a measurement, $\hat{x}^B$ say, performed at B, where i is used to label the possible results, discrete or otherwise, of the measurement $\hat{x}^B$. As a result of the measurement of the coordinate, we have a new wave function of composite system $A \cup B$ which is given by Eq.(4.2.3) (see Remark.4.2.1)

$$\Psi_{x_i^B}\left(x, \hat{x}^B\right) = \Psi_{x_i^B}\left(\hat{x}^A, \hat{x}^B\right) = R(\hat{x}^B - x_i^B)\Psi\left(\hat{x}^A, \hat{x}^B\right) = R(\hat{x}^B - x_i^B)\Psi\left(x, \hat{x}^B\right) \tag{4.2.5}$$

and therefore adjoint probability density $p_{x_i^B}\left(x, \hat{x}^B\right) = p\left(x, \hat{x}^B \,|x_i^B\right)$ at instant at once after measurement is given by

$$p_{x_i^B}\left(x, \hat{x}^B\right) = p\left(x, \hat{x}^B \,|x_i^B\right) = \left\| R(\hat{x}^B - x_i^B)\Psi\left(x, \hat{x}^B\right) \right\|^2 \tag{4.2.6}$$

Then the conditional probability density $p_{x_i^B}(x) = p(x|x_i^B)$ conditional on a result x_i^B for QM measurement at B is given by

$$p_{x_i^B}(x) = p(x|x_i^B) = \int_{-\infty}^{\infty} p_{x_i^B}\left(x, \hat{x}^B\right) d\hat{x}^B = \int_{-\infty}^{\infty} d\hat{x}^B \left\| \Psi_{x_i^B}\left(x, \hat{x}^B\right) \right\|^2 = \int_{-\infty}^{\infty} d\hat{x}^B \left\| R(\hat{x}^B - x_i^B)\Psi\left(x, \hat{x}^B\right) \right\|^2. \quad (4.2.7)$$

The predicted results for the measurement at A, based on the measurement at B, are however no longer a set of definite numbers with zero uncertainty, but become fuzzy, being described by a set of distributions $P(x|x_i^B)$ giving the probability of a result for the measurement at A, conditional on a result x_i^B for measurement at B. We define $\Delta_i^2 x$ to be the variance of the conditional distribution $P(x|x_i^B)$. Similarly we may infer the result of measurement $\hat{p}$ at A, based on a (different) measurement, $\hat{p}^B$ say, at B. Denoting the results of the measurement $\hat{p}^B$ at B by p_j^B, we then define the probability distribution, $P(p|p_j^B)$ which is the predicted result of the measurement for $\hat{p}$ at A conditional on the result p_j^B for the measurement $\hat{p}^B$ at B. The variance of the conditional distribution $P(p|p_j^B)$ is denoted by $\Delta_j^2 p$.

Remark.4.2.1. We remind now that the QM-measurement is represented by the canonical scheme [5]

$$|\psi\rangle \overset{a'}{\to} |\psi_{a'}\rangle = \Re_{a'}|\psi\rangle, \int da' \Re_{a'}^{\dagger}\Re_{a'} = \mathbf{1}, p_{a'} = \|\psi_{a'}\|^2 = \langle\psi|\Re_{a'}^{\dagger}\Re_{a'}|\psi\rangle, \quad (4.2.8)$$

where $p_{a'}$ is a corresponding probability density. To obtain the probability that the parameter a' turns out to belong to the set Δ one has to integrate over this set:

$$\mathbf{P}[a' \in \Delta] = \int_{\Delta} da' p_{a'}. \quad (4.2.9)$$

If the state $|\psi\rangle$ is represented by the wave function $\psi(a)$ the operator $\Re_{a'}$ describing the measurement giving the result a' will be taken in the following form

$$\Re_{a'}\psi(a) = R(a - a')\psi(a), \quad (4.2.10)$$

where $R(a)$ is a function with a support concentrated in some vicinity of zero and representing the 'fuzziness' of the measurement. It is a characteristic function of the measurement and may, for example, be (and typically is) a Gaussian function. The width of this function corresponds to the resolution of the measurement. Normalization $\int da' \Re_{a'}^{\dagger}\Re_{a'} = \mathbf{1}$ of the operators $\Re_{a'}$ is provided by the corresponding normalization of the function $R(a)$ as follows:

$$\int da|R^2(a)| = 1. \quad (4.2.11)$$

If the measurement is described by the Gaussian function

$$R(a) = \exp\left[-\frac{(a - a')^2}{4\Delta^2}\right] \quad (4.2.12)$$

it is a minimally disturbing measurement of the coordinate a' with resolution Δ [5].

Remark.4.2.2. Consider the momentum representation $\widetilde{\psi}(p)$ of the initial wave function $\psi(q)$

$$\widetilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int dq \psi(q) \exp\left[-\frac{i}{\hbar} pq\right]. \tag{4.2.13}$$

As a result of the measurement of the coordinate,

$$\Re_{q'}\psi(q) = R(q-q')\psi(q) = \psi_{q'}(q), \tag{4.2.14}$$

we have a new wave function and its momentum representation has the form [5]

$$\widetilde{\psi}_{q'}(p) = \int dp' \widetilde{R}_{q'}(p-p')\widetilde{\psi}(p'), \tag{4.2.15}$$

where $\widetilde{R}_{q'}(p)$ is a momentum representation of the function $R_{q'}(q)$. Note that

$$\begin{aligned} \widetilde{R}_{q'}(p) &= \widetilde{R}(p) \exp\left[-\frac{i}{\hbar} pq'\right], \\ \widetilde{R}(p) &= \frac{1}{2\pi\hbar} \int dq R(q) \exp\left[-\frac{i}{\hbar} pq\right]. \end{aligned} \tag{4.2.16}$$

Remark.4.2.3. Consider now a coordinate measurement having a Gaussian characteristic function of width of the order of Δ

$$R(q) \simeq \exp\left[-\frac{q^2}{4\Delta^2}\right]. \tag{4.2.17}$$

Then the momentum representation of this function (characterizing the structure of the momentum uncertainty $\hbar/\Delta$ acquired in the measurement) is also Gaussian with width of the order of $\hbar/\Delta$:

$$\widetilde{R}(p) \simeq \exp\left[\frac{-p^2}{\left(\frac{\hbar}{\Delta}\right)^2}\right]. \tag{4.2.18}$$

For a given experiment one could in principle measure the individual variances $\Delta_i^2 x$ of the conditional distributions $P(x|x_i^B)$ (and also $\Delta_j^2 p$ for the $P(p|p_j^B)$). Obviously if each of the variances $\Delta_i^2 x$ and $\Delta_j^2 p$ satisfy $\Delta_i^2 x = 0$ and $\Delta_j^2 p = 0$ this would imply the demonstration of the original EPR paradox. This situation however is not practical for continuous variable measurements [5]. Instead of considering the problem of simultaneous eigenstates as originally proposed by EPR, one can suggest an different and experimentally realizable criterion based on the Heisenberg Uncertainty Principle: $\Delta\hat{x}\Delta\hat{p} \geq C$. For the sake of notational convenience we now consider in the remainder of this subsection that appropriate scaling enables $\hat{x}$ and $\hat{p}$ to be dimensionless and $C = 1$.

EPR correlations however would be demonstrated in a convincing manner if the experimentalist could measure each of the conditional distributions $P(x|x_i^B)$ and establish that each of the distributions is very narrow, in fact constrained such that [5]:

$$P(x|x_i^B) = 0 \quad \text{iff} \quad |x - \mu_i| > \delta,$$
$$P(p|p_j^B) = 0 \quad \text{iff} \quad |p - \nu_j| > \delta. \tag{4.2.19}$$

Here μ_i is the mean value of the conditional distribution $P(x|x_i^B)$ and ν_j is the mean value of the conditional distribution $P(p|p_j^B)$. In this case the assumption of local realism would imply, since the measurement $\hat{x}^B$ at B will always imply the result of $\hat{x}$ at A to be within the range $\mu_i \pm \delta_x$, that the result of the measurement at A is predetermined to be within a bounded range of width 2δ. In a straightforward extension of EPR's argument, we replace the words "predict with certainty" with "predict with certainty that the result is constrained to be within the range $\mu_i \pm \delta$", and then define an "element of reality" with this intrinsic bounded by fuzziness δ. We now consider the situation where an experimenter has demonstrated that for *every outcome* x_i^B (and p_j^B) for the measurement $\hat{x}^B$ (and $\hat{p}^B$) performed at B, the variance $\Delta_i x$ (and $\Delta_j p$) of the appropriate conditional distribution satisfies

$$\Delta_i x < 1, \Delta_j p < 1 \tag{4.2.20}$$

for any $i,j \in \mathbb{N}$. The measurement at B always allows an inference of the result at A to a precision better than given by the uncertainty bound 1.

Remark.4.2.4. In this case we do not predict a result at A "with certainty", as in EPR's original paradox. The measurement $\hat{x}^B$ at B however does predict by Eq.(4.2.3) [or by Eq.(4.2.9) in general case] with a certain probability constraints on the result for $\hat{x}$ at A.

Remark 4.2.5. Following the EPR argument, which assumes

no action-at-a-distance,

so that the measurement at B does not cause any instantaneous influence to the system at A, one can attribute a probabilistic predetermined "element of reality" to the system at A.

Remark.4.2.6. There is a similar predicted result for the measurement $\hat{p}$ at A based on a result of measurement at B, and a corresponding predetermined description based again on the

no-action-at-a-distance

assumption.

Remark.4.2.7. The important point in establishing the EPR paradox for this more general yet practical situation is that under the EPR premises the predetermined statistics (or generalised "elements of reality") for the physical quantities

$\hat{x}$ **and** $\hat{p}$ **are attributed simultaneously to the subsystem at** A.

Assuming no action-at-a-distance, the choice of the experimenter (Bob) at B to infer

information about either $\hat{x}$ or $\hat{p}$ cannot actually induce the result of the measurement at A.

As there is no disturbance created by Bob's measurement, the (appropriately extended)

EPR definition of realism is that the prediction for x is something (a probabilistic "element of

reality") that can be attributed to the subsystem at A, whether or not Bob makes his

measurement.

Remark 4.2.8. This is also true of the prediction for $\hat{p}$, and therefore the two "elements of

reality" representing the physical quantities $\hat{x}$ and $\hat{p}$ exist to describe the predictions for $\hat{x}$

and $\hat{p}$ simultaneously.

The paradox can then be established by proving the impossibility of such a simultaneous level of prediction for both $\hat{x}$ and $\hat{p}$ for any quantum description of the subsystem A alone. By this we mean explicitly that there can be no procedure allowed, within the predictions of quantum mechanics, to make *simultaneous inferences* by measurements performed at B or any other location, of both the result $\hat{x}$ and $\hat{p}$ at A, to the precision indicated by $\Delta_i x < 1$, $\Delta_j p < 1$.

Remark 4.2.9. Recall that the inference of the result at A by measurement at B is actually a measurement of $\hat{x}$ performed with the accuracy determined by the $\Delta_i x$. However simultaneous measurements of $\hat{x}$ and $\hat{p}$ to the accuracy (4.2.9) are not possible (predicted by quantum mechanics). The reduced density matrix describing the state at A after such measurements would violate the H.U.P.

A simpler quantitative, experimentally testable criterion for EPR was proposed by Reid in 1989 see for example [22]. The 1989 inferred H.U.P. criterion is based on the average variance of the conditional distributions for inferring the result of measurement $\hat{x}$ (and also for $\hat{p}$). The EPR paradox is demonstrated when the product of the average errors in the inferred results for $\hat{x}$ and $\hat{p}$ violate the corresponding Heisenberg Uncertainty Principle. The spirit of the original EPR paradox is present, in that one can perform a measurement on B to enable an estimate of the result x at A (and similarly for $\hat{p}$).

Abbreviation 4.2.1. For the sake of notational convenience we now abbreviate in the remainder of the paper: $\Delta_{\mathbf{loc}.i} x$ and $\Delta_{\mathbf{loc}.i} p$ instead $\Delta_i x$ and $\Delta_j p$ for the variance $\Delta_i x$ and $\Delta_j p$ which were calculated under assumption no action-at-a-distance, see Remark 4.2.5-4.2.6.

We define now [22]:

$$\Delta^2_{\mathbf{loc.inf.}}\, x = \sum_i P(x_i^B)\Delta^2_{\mathbf{loc.}i}x,$$
$$\Delta^2_{\mathbf{loc.inf.}}\, p = \sum_j P(p_j^B)\Delta^2_{\mathbf{loc.}j}p. \tag{4.2.21}$$

Here $\Delta^2_{\mathbf{loc.inf.}}\hat{x}$ is the average variance for the prediction (inference) under assumption no action-at-a-distance of the result x for $\hat{x}$ at A, conditional on a measurement $\hat{x}^B$ at B. Here $i \in \mathbb{N}$ labels all outcomes of the measurement $\hat{x}$ at A, and μ_i and $\Delta_i x$ are the mean and standard deviation, respectively, of the conditional distribution $P(x|x_i^B)$, where x_i^B is the result of the measurement $\hat{x}^B$ at B. We define a $\Delta^2_{\mathbf{loc.inf}}\hat{p}$ similarly to represent the weighted variance for the prediction (inference) under assumption no action-at-a-distance of the result $\hat{p}$ at A, based on the result of the measurement at B. Here $P(x_i^B)$ is the probability for a result x_i^B upon measurement of $\hat{x}^B$, and $P(p_j)$ is defined similarly.

The Reid's criterion to demonstrate the EPR "paradox", the Reid's local signature of the EPR paradox, is

$$\left(\Delta^2_{\mathbf{loc.inf.}}\, x\right)\left(\Delta^2_{\mathbf{loc.inf.}}\, p\right) < 1. \tag{4.2.22}$$

This criterion is a clear criterion for the demonstration of the EPR "paradox", by way of the argument presented above. Such a prediction (4.2.21) for $\hat{x}$ and $\hat{p}$ with the average inference variances given, cannot be achieved by any quantum description of the subsystem alone. This EPR criterion has been achieved experimentally.

IV.2.2.The Postulate of Nonlocality and signature of the EPR "paradox"

Remark.4.2.10. A most critical component of the EPR argument was the principle of locality. Indeed, one may regard the EPR paradox as a statement of the mutual incompatibility of locality, entanglement, and completeness. Experimental tests of Bell's inequalities have indicated that quantum mechanics is complete by ruling out the possibility of hidden variables. Therefore it is generally agreed that the assumption of locality is invalid for entangled states: measurement of either particle of an entangled system projects both particles onto a state consistent with the result of measurement, regardless of how far apart the particles are. In the situation proposed by EPR, the position or momentum of the unmeasured particle becomes a reality when, and only when, the corresponding quantity of the other particle is measured.

Remark.4.2.11. The assumption of nonlocality allows us to deduce the existence of an fuzzy "element of reality" of some type for $\widehat{x}$ at A, since we can make a prediction of the result at A, but with some disturbing of the subsystem at A, under the measurement, $\widehat{x}^B$ say, performed at B. This prediction is subject to the result x_i^B of a measurement, $\widehat{x}^B$ say, performed at B, where i is used to label the possible results, discrete or otherwise, of the measurement $\widehat{x}^B$.

We accept now the following postulate:

Postulate of Nonlocality

(**i**) Let A and B two entangled particles. Let $\Psi(\hat{x}^A, \hat{x}^B)$ be a wave function of composite system $A \cup B$. Let x_i^B be the result of a measurement, $\hat{x}^B$ say, performed at B, where i is used to label the possible results, discrete or otherwise, of the measurement $\hat{x}^B$. As a result of the measurement of the coordinate, we have a new wave function of composite system $A \cup B$ which [in contrast with Eq.(4.2.5)] is given by

$$\begin{gathered}\Psi_{x_i^B}(\hat{x}^A, \hat{x}^B) = R_2(\hat{x}^A - x_i^A(x_i^B))R_1(\hat{x}^B - x_i^B)\Psi(\hat{x}^A, \hat{x}^B),\\ x_i^A(x_i^B) + x_0 \simeq x_i^B.\end{gathered} \tag{4.2.23}$$

(**ii**) Let A and B two entangled particles. Let $\widetilde{\Psi}(\hat{p}^A, \hat{p}^B)$ be a wave function of composite system $A \cup B$. Let p_j^B be the result of a measurement, $\hat{p}^B$ say, performed at B, where j is used to label the possible results, discrete or otherwise, of the measurement $\hat{p}^B$. As a result of the measurement of the coordinate, we have a new wave function of composite system $A \cup B$ which is given by

$$\begin{gathered}\widetilde{\Psi}_{p_j^B}(\hat{p}^A, \hat{p}^B) = \widehat{R}_2(\hat{p}^A - p_j^A(p_j^B))\widehat{R}_1(\hat{p}^B - p_j^B)\widetilde{\Psi}(\hat{p}^A, \hat{p}^B),\\ p_j^A(p_j^B) \simeq -p_j^B.\end{gathered} \tag{4.2.24}$$

Remark 4.2.12.The spirit of the original EPR paradox now is present, in that the canonical EPR correlations (4.2.2) and (4.2.3) well preserved.

Remark 4.2.13.Note that EPR correlations $x_i^A(x_i^B) + x_0 \simeq x_i^B$ and $p_j^A(p_j^B) \simeq -p_j^B$ however would be demonstrated in a convincing manner if the experimentalist could measure each of the conditional distributions $P(x|x_i^B)$ and establish that each of the distributions is very narrow, in fact constrained so that [5]

$$\begin{gathered}p(x|x_i^B) \simeq 0 \quad \text{iff} \quad |x - \mu_i| > \delta,\\ p(p|p_j^B) \simeq 0 \quad \text{iff} \quad |p - \nu_j| > \delta,\\ P(x|x_i^B) \simeq 0 \quad \text{iff} \quad |x - \mu_i| > \delta,\\ P(p|p_j^B) \simeq 0 \quad \text{iff} \quad |p - \nu_j| > \delta.\end{gathered} \tag{4.2.25}$$

Here μ_i is the mean of the conditional distribution $P(x|x_i^B)$ and ν_j is the mean of the conditional distribution $P(p|p_j^B)$.

Remark 4.2.14.We assume now that a coordinate and momentum measurements have a Gaussian characteristic function of width of the order of 2δ

$$\begin{gathered}R_1(x) = R_2(x) = R(x) \simeq \exp\left[-\frac{x^2}{4\delta^2}\right]\\ \widehat{R}_1(p) = \widehat{R}_2(p) = \widehat{R}(p) \simeq \exp\left[-\frac{p^2}{4\delta^2}\right]\end{gathered} \tag{4.2.26}$$

In this case the Postulate of Nonlocality would imply, since the measurement $\hat{x}^B$ at B will always imply the result of $\hat{x}$ at A to be within the range $\mu_i \pm \delta_x$, that the result of the measurement at A is predetermined to be within a bounded range of width 2δ. In a straightforward extension of EPR's argument, we replace the words “predict with certainty” with “predict with certainty that the result is constrained to be within the range $\mu_i \pm \delta$. We now consider the situation where an experimenter has demonstrated that for every outcome x_i^B (and p_j^B) for the measurement $\hat{x}^B$ (and $\hat{p}^B$) performed at B, the variance $\Delta_i x$ (and $\Delta_j p$) of the appropriate conditional distribution satisfies

$$\Delta_i x < 1, \Delta_j p < 1 \qquad (4.2.27)$$

for all i,j. The measurement at B always allows an inference of the result at A to a precision better than given by the uncertainty bound 1.

In this case we do not predict a result at A “with certainty”, as in EPR’s original paradox. The measurement $\widehat{x}_B$ at B however does predict with a certain probability constraints on the result for $\widehat{x}$ at A.

Remark 4.2.15. Note that adjoint probability density $p\left(\widehat{x}^A, \widehat{x}^B \middle| x_i^B\right)$ at instant at once after measurement [in contrast with Eq.(4.1.6)] is given by

$$\begin{gathered} p\left(\widehat{x}^A, \widehat{x}^B \middle| x_i^B\right) = \left\| \Psi_{x_i^B}\left(\widehat{x}^A, \widehat{x}^B\right) \right\|^2 = \\ \left\| R(\widehat{x}^A - x_i^A(x_i^B)) R(\widehat{x}^B - x_i^B) \Psi\left(\widehat{x}^A, \widehat{x}^B\right) \right\|^2, \\ x_i^A(x_i^B) + x_0 \simeq x^B \pm \mu_i. \end{gathered} \qquad (4.2.27)$$

Then the conditional probability density $p_{x_i^B}(x) = p(x|x_i^B)$ conditional on a result x_i^B for QM measurement at B is given by

$$\begin{gathered} p_{x_i^B}(x) = p(x|x_i^B) = \int\limits_{-\infty}^{\infty} p_{x_i^B}\left(x, \widehat{x}^B\right) d\widehat{x}^B = \int\limits_{-\infty}^{\infty} d\widehat{x}^B \left\| \Psi_{x_i^B}\left(x, \widehat{x}^B\right) \right\|^2 = \\ \int\limits_{-\infty}^{\infty} d\widehat{x}^B \left\| R(\widehat{x}^A - x_i^A(x_i^B)) R(\widehat{x}^B - x_i^B) \Psi\left(x, \widehat{x}^B\right) \right\|^2. \end{gathered} \qquad (4.2.28)$$

There is a similar predicted result for the measurement $\widehat{p}$ at A based on a result of measurement at B, and a corresponding predetermined description based on the QM constraints

$$\begin{gathered} \widetilde{\Psi}_{p_j^B}\left(\widehat{p}^A, \widehat{p}^B\right) = \widehat{R}(\widehat{p}^A - p_j^A(p_j^B)) \widehat{R}(\widehat{p}^B - p_j^B) \widetilde{\Psi}\left(\widehat{p}^A, \widehat{p}^B\right), \\ p_j^A(p_j^B) \simeq -p^B. \end{gathered} \qquad (4.2.29)$$

The spirit of the original EPR "paradox" is present, in that one can perform a measurement on B to enable an estimate of the result x at A (and similarly for $\widehat{p}$).

Abbreviation 4.2.2. For the sake of notational convenience we now abbreviate in the remainder of the paper: $\Delta_{\mathbf{nonloc}.i} x$ and $\Delta_{\mathbf{nonloc}.j} p$ instead $\Delta_i x$ and $\Delta_j p$ for the variance $\Delta_i x$ and $\Delta_j p$ which were calculated under nonlocality assumption (postulate) by conditional probability density given by Eq.(4.1.28).

We define now

$$\Delta^2_{\mathbf{nonloc.inf.}}\, x = \sum_i P(x_i^B)\Delta^2_{\mathbf{nonloc.}i}x,$$
$$\Delta^2_{\mathbf{nonloc.inf.}}\, p = \sum_j P(p_j^B)\Delta^2_{\mathbf{nonloc.}j}p. \quad (4.2.30)$$

Here $\Delta^2_{\mathbf{nonloc.inf.}}\, x$ is the average variance for the prediction (inference) of the result x for $\hat{x}$ at A, conditional on a measurement $\hat{x}^B$ at B. Here i labels all outcomes of the measurement $\hat{x}$ at A, and μ_i and $\Delta_i x$ are the mean and standard deviation, respectively, of the conditional distribution $P(x|x_i^B)$, where x_i^B is the result of the measurement $\hat{x}^B$ at B. We define a $\Delta^2_{\mathbf{nonloc.inf.}}\, p$ similarly to represent the weighted variance for the prediction (inference) of the result $\hat{p}$ at A, based on the result of the measurement at B. Here $P(x_i^B)$ is the probability for a result x_i^B upon measurement of $\hat{x}^B$, and $P(p_j^B)$ is defined similarly. The criterion to demonstrate the EPR paradox, the signature of the EPR paradox, is The criterion to demonstrate the EPR "paradox", the nonlocal signature of the EPR paradox, is given by

$$\left(\Delta_{\mathbf{nonloc.inf.}}\, x\right)\left(\Delta_{\mathbf{nonloc.inf.}}\, p\right) < 1. \quad (4.2.31)$$

This criterion is a clear criterion for the demonstration of the EPR "paradox", by way of the argument presented above. Such a prediction for $\hat{x}$ and $\hat{p}$ with the average inference variances given, cannot be achieved by any quantum description of the subsystem alone.

IV.2.3.The EPR-nonlocality criteria

Remark 4.2.16.A critical component of the EPR argument was the principle of locality. Indeed, one may regard the EPR paradox as a statement of the mutual incompatibility of locality, entanglement, and completeness. Experimental tests of Bell's inequalities have indicated that quantum mechanics is complete by ruling out the possibility of hidden variables. Therefore it is generally agreed that the assumption of locality is invalid for entangled states: measurement of either particle of an entangled system projects both particles onto a state consistent with the result of measurement, regardless of how far apart the particles are. In the situation proposed by EPR, the position or momentum of the unmeasured particle becomes a reality when, and only when, the corresponding quantity of the other particle is measured. Since only one quantity or the other is measured, the position and the momentum of the unmeasured particle need not be simultaneous realities. In this way the EPR "paradox" also is resolved. From Eq.(4.2.21) and Eq.(4.2.30) we obtain the EPR-nonlocality criteria

$$\left|\Delta^2_{\mathbf{loc.inf.}}\, x - \Delta^2_{\mathbf{nonloc.inf.}}\, x\right| = \left|\sum_i P(x_i^B)[\Delta^2_{\mathbf{loc.}i}x - \Delta^2_{\mathbf{nonloc.}i}x]\right| > 0,$$
$$\left|\Delta^2_{\mathbf{loc.inf.}}\, p - \Delta^2_{\mathbf{nonloc.inf.}}\, p\right| = \left|\sum_j P(p_j^B)\left[\Delta^2_{\mathbf{loc.}j}p - \Delta^2_{\mathbf{nonloc.}j}p\right]\right| > 0, \quad (4.2.32)$$

and

$$\left|\left(\Delta_{\mathbf{nonloc.inf.}}\, x\right)\left(\Delta_{\mathbf{nonloc.inf.}}\, p\right) - \left(\Delta_{\mathbf{loc.inf.}}\, x\right)\left(\Delta_{\mathbf{loc.inf.}}\, p\right)\right| > 0. \quad (4.2.33)$$

These EPR-nonlocality criteria has been achieved experimentally [66],[67], (see subsection IV.5, Remark 4.5.3-Remark 4.5.4).

IV.3.Nonlocal Schrödinger equation implies the Postulate of Nonlocality

In this subsection we obtain nonlocal Schrödinger equation (**NSE**) which corresponding to position-momentum entangled pairs A and B (see Fig.4.2.1) with well correlated position

$$\langle x_B \rangle \simeq \langle x_A \rangle + x_0 \tag{4.3.1}$$

and anti-correlated momentum

$$\langle p_B \rangle \simeq -\langle p_A \rangle. \tag{4.3.2}$$

Remark 4.3.1. As pointed out in subsection IV.2 it is generally agreed that the assumption of locality is invalid for entangled states: measurement of either particle of an entangled system projects both particles onto a state consistent with the result of measurement, regardless of how far apart the particles are. It allow us to use special nonlocal generalization of the canonical Schrödinger equation.

Remark 4.3.2. As pointed out in subsection II.2 from nonlocal Schrödinger equation (2.1.17) one obtains collapsed wave function corresponding to GRW collapse model.

It allow us to use similar nonlocal Schrödinger equation also for entangled states.

Remark 4.3.3. The spirit of the original EPR paradox is present, in that the canonical EPR correlations (4.3.1) and (4.3.2) gives an boundary conditions for the solutions of the nonlocal Schrödinger equation.

Remark 4.3.4. In this subsection we denote (i) $x_A = x_1, x_B = x_2$, (ii) $x^A = \widetilde{x}_1$, $x^B = \widetilde{x}_2 = \widetilde{x}_1 + x_0$.

Definition 4.3.1. Let us consider the time-dependent canonical Schrödinger equation

$$i\hbar \frac{\partial \Psi(x_1,x_2,t)}{\partial t} = H\Psi(x_1,x_2,t), \tag{4.3.3}$$
$$t \in [0,T], (x_1,x_2) \in \mathbb{R}^2.$$

Let $\Psi(x_1,x_2,t)$ be a classical solution of the time-dependent Schrödinger equation (4.3.3). The time-dependent Schrödinger equation (4.3.3) is a weakly well preserved (in sense of Colombeau generalized functions) by corresponding to $\Psi(x_1,x_2,t)$ collapsed Colombeau generalized wave function $(\Psi^{\#}_{\varepsilon}(x_1,x_2,t))_{\varepsilon}, \varepsilon \in (0,1]$, where

$$(\Psi_{\varepsilon}^{\#}(x_1,x_2,t))_{\varepsilon} = (\Psi_{\varepsilon}(x_1,x_2,t;\widetilde{x}_1(t),\widetilde{x}_2(t)))_{\varepsilon} =$$

$$= \left(\frac{\Re_{1,2}(x_1,\widetilde{x}_1(t),x_2,\widetilde{x}_2(t);\delta,\varepsilon)\Psi(x_1,x_2,t)}{\|\Re_{1,2}(\widetilde{x}_1(t),\widetilde{x}_2(t);\delta,\varepsilon)\Psi(x_1,x_2,t)\|_2}\right)_{\varepsilon},$$

$$(\Re_{1,2}(x_1,\widetilde{x}_1,x_2,\widetilde{x}_2;\delta,\varepsilon))_{\varepsilon} = \prod_{i=1}^{2}(\Re_i(x_i,\widetilde{x}_i;\delta,\varepsilon))_{\varepsilon}, \tag{4.3.4}$$

$$\Re_i(x,\widetilde{x}_i(t);\delta,\varepsilon) = \begin{cases} (\pi_\delta\delta^2)^{-1/4}\exp\left[-\frac{(x_i-\widetilde{x}_i(t))^2}{2\delta^2}\right] \text{ iff } \|x_i-\widetilde{x}_i\| \leq \varepsilon, \\ 0 \text{ iff } \|x_i-\widetilde{x}_i\| > \varepsilon. \end{cases}$$

$$i = 1,2.$$

in region $\Gamma \subseteq \mathbb{R}^2$ if there exist an solution $\Psi(x_1,x_2,t)$ of Schrödinger equation (4.2.1) such that the estimate

$$\left(\int_{\Gamma}\left\{i\hbar\frac{\partial\Psi_{\varepsilon}^{\#}(x_1,x_2,t)}{\partial t} - \widehat{H}\Psi_{\varepsilon}^{\#}(x_1,x_2,t)\right\}dx_1dx_2\right)_{\varepsilon} = (O(\hbar^{\alpha}))_{\varepsilon}, \tag{4.3.5}$$

$$t \in [0,T], x_1,x_2 \in \Gamma^2,$$

with $1/2 \leq \alpha,$ is satisfied.

Definition 4.3.2. Equation (4.3.5) with a following boundary conditions

$$\begin{aligned} \langle x_B^t \rangle &\simeq \langle x_A^t \rangle + x_0, \\ \langle x_A^t \rangle &= \left(\int x_A |\Psi_{\varepsilon}^{\#}(x_A,x_B,t)|^2 dx_A dx_B\right)_{\varepsilon}, \\ \langle x_B^t \rangle &= \left(\int x_B |\Psi_{\varepsilon}^{\#}(x_A,x_B,t)|^2 dx_A dx_B\right)_{\varepsilon}, \end{aligned} \tag{4.3.6}$$

that is time-dependent nonlocal Schrödinger equation corresponding to EPR entangled state.

Definition 4.3.3.(i) The time-dependent integral equation (4.3.5) with a boundary conditions (4.3.6) is colled the time-dependent nonlocal Schrödinger equation of the order $\hbar^{\alpha}$ corresponding to EPR entangled state.

(ii) Such collapsed wave function $\Psi^{\#}(x_1,x_2,t,)$ as mentioned in Definition 4.3.2 is colled the

$\hbar^{\alpha}$- solution of the nonlocal Schrödinger equation (4.3.5)-(4.3.6) of the order α.

Definition 4.3.4.Let us consider the time-independent canonical Schrödinger equation

$$\widehat{H}\Psi(x_1,x_2) = 0, (x_1,x_2) \in \mathbb{R}^2. \tag{4.3.7}$$

Let $\Psi(x_1,x_2)$ be a classical solution of the time-independent Schrödinger equation (4.3.7).The time-independent Schrödinger equation (4.3.7) is a weakly well preserved (in sense of Colombeau generalized functions) by corresponding to $\Psi(x_1,x_2)$ Colombeau generalized collapsed wave function $(\Psi^{\#}_{\varepsilon}(x_1,x_2))_{\varepsilon}, \varepsilon \in (0,1]$, where

$$(\Psi^{\#}_{\varepsilon}(x_1,x_2,\delta))_{\varepsilon} = (\Psi_{\varepsilon}(x_1,x_2;\widetilde{x}_1,\widetilde{x}_2,\delta))_{\varepsilon} =$$

$$= \left(\frac{\Re_{1,2}(x_1,\widetilde{x}_1,x_2,\widetilde{x}_2;\delta,\varepsilon)\Psi(x_1,x_2)}{\|\Re_{1,2}(\widetilde{x}_1,\widetilde{x}_2;\delta,\varepsilon)\Psi(x_1,x_2)\|_2}\right)_{\varepsilon},$$

$$(\Re_{1,2}(x_1,\widetilde{x}_1,x_2,\widetilde{x}_2;\delta,\varepsilon))_{\varepsilon} = \prod_{i=1}^{2}(\Re_i(x_i,\widetilde{x}_i;\delta,\varepsilon))_{\varepsilon}, \tag{4.3.8}$$

$$\Re_i(x,\widetilde{x}_i;\delta,\varepsilon) = \begin{cases} (\pi_{\delta}\delta^2)^{-1/4}\exp\left[-\frac{(x_i-\widetilde{x}_i)^2}{2\delta^2}\right] \text{ iff } \|x_i-\widetilde{x}_i\| \leq \varepsilon, \\ 0 \text{ iff } \|x_i-\widetilde{x}_i\| > \varepsilon. \end{cases}$$

in region $\Gamma \subseteq \mathbb{R}^2$ if there exist an solution $\Psi(x_1,x_2)$ of Schrödinger equation (4.3.7) such that the estimate

$$\left(\int_{\Gamma}\widehat{H}\Psi^{\#}_{\varepsilon}(x_1,x_2)dx_1dx_2\right)_{\varepsilon} = (O(\hbar^{\alpha}))_{\varepsilon}, \tag{4.3.9}$$

$$(x_1,x_2)\in\Gamma^2,$$

with $1/2 \leq \alpha$, is satisfied.

Definition 4.3.5. Equation (4.3.9) with a boundary conditions

$$\begin{aligned} \langle x_B \rangle &\simeq \langle x_A \rangle + x_0, \\ \langle x_A \rangle &= \left(\int x_A|\Psi^{\#}_{\varepsilon}(x_A,x_B)|^2 dx_A dx_B\right)_{\varepsilon}, \\ \langle x_B \rangle &= \left(\int x_B|\Psi^{\#}_{\varepsilon}(x_A,x_B)|^2 dx_A dx_B\right)_{\varepsilon}, \end{aligned} \tag{4.3.10}$$

that is time-independent nonlocal Schrödinger equation corresponding to EPR entangled state.

Definition 4.3.6.(i) The time-independent integral equation (4.3.9) with a boundary conditions (4.3.10) is colled the time-independent nonlocal Schrödinger equation of the order $\hbar^{\alpha}$ corresponding to EPR entangled state.

(ii) Such collapsed wave function $\Psi^{\#}(x_1,x_2)$ as mentioned in Definition 4.3.5 is colled the

$\hbar^{\alpha}$- solution of the nonlocal Schrödinger equation (4.3.9)-(4.3.10) of the order α.

Lemma 4.3.1.Let $\Phi(\lambda)$ be a function

$$\Phi(\lambda) = \int_0^a x^{\beta-1}\exp(-\lambda x^{\alpha})f(x)dx, \qquad (4.3.11)$$

where $\lambda \gg 1,\ 0 < a < \infty, 0 < \beta, 0 < \alpha$. Assume that $f(x)$ is continuous on $[0,a]$. Then

$$\Phi(\lambda) = \alpha^{-1}\Gamma\left(\frac{\beta}{\alpha}\right)[f(0) + o(1)]\lambda^{-\beta/\alpha} \qquad (4.3.12)$$

Lemma 4.3.2.Let $f(x)$ be a function such that $f \in C^2(x < x_0)$ and $f \in C^2(x > x_0)$. Then

$$\begin{gathered} f'(x) = \left\{f'(x)\right\}_{x \neq x_0} + [f]_{x_0}\delta(x - x_0), \\ f''(x) = \left\{f''(x)\right\}_{x \neq x_0} + \left[f'\right]_{x_0}\delta(x - x_0) + [f]_{x_0}\delta'(x - x_0), \\ [f]_{x_0} = f(x_0 + 0) - f(x_0 - 0), \\ \left[f'\right]_{x_0} = f'(x_0 + 0) - f'(x_0 - 0). \end{gathered} \qquad (4.3.13)$$

Theorem 4.3.1. Assume that there exist an classical solution $\Psi(x_1,x_2)$ of the Schrödinger equation (4.3.7) such that

$$\begin{gathered} \sup_{(x_1,x_2)\in\Gamma} |\Psi(x_1,x_2)| = O(\hbar^{-1/2}), \\ \sup_{(x_1,x_2)\in\Gamma} |\partial\Psi(x_1,x_2)/\partial x_1| = O(\hbar^{-3/2}), \quad \sup_{(x_1,x_2)\in\Gamma} |\partial\Psi(x_1,x_2)/\partial x_2| = O(\hbar^{-3/2}). \end{gathered} \qquad (4.3.14)$$

Then any collapsed wave function $\Psi^{\#}(x)$ given by Eq.(4.3.8) with $\sqrt{\hbar/\delta} = \hbar^{\alpha}, 1/4 < \alpha < 1/2$ that is $\hbar^{\alpha}$-solution of the time-independent nonlocal Schrödinger equation (4.3.9)-(4.3.10) of the order α.

Proof. The Schrödinger equation (4.3.7) has the following form

$$H\Psi(x_1,x_2) = \hbar^2\frac{\partial^2\Psi(x_1,x_2)}{\partial x_1^2} + \hbar^2\frac{\partial^2\Psi(x_1,x_2)}{\partial x_2^2} + V(x_1,x_2)\Psi(x_1,x_2) = 0. \qquad (4.3.15)$$

Let $\Psi_\delta^{\#}(x_1,x_2)$ be a function

$$\Psi_\delta^{\#}(x_1,x_2) = R_\delta(x_1,\widetilde{x}_1)R_\delta(x_2,\widetilde{x}_2)\Psi(x_1,x_2), \qquad (4.3.16)$$

where

$$R_\delta(x_i,\widetilde{x}_i) = \begin{cases} (\pi_\delta\delta^2)^{-1/4}\exp\left[-\dfrac{(x_i - \widetilde{x}_i)^2}{2\delta^2}\right] \text{ iff } \|x_i - \widetilde{x}_i\| \leq \varepsilon, \\ 0 \text{ iff } \|x_i - \widetilde{x}_i\| > \varepsilon. \end{cases} \qquad (4.3.17)$$

From Eq.(4.3.17) by using Eq.(4.3.13) we obtain

$$\frac{\partial R_\delta(x_1,\widetilde{x}_1)}{\partial x_1} = -(\pi_\delta\delta)^{-1/4}\delta^{-1}(x_1-\widetilde{x}_1)\exp\left[-\frac{(x_1-\widetilde{x}_1)^2}{2\delta}\right]+$$
$$+\left([R_\delta(x_1,\widetilde{x}_1)]_{\widetilde{x}_1-\varepsilon}\right)\delta(x_1-\widetilde{x}_1+\varepsilon)+([R_\delta(x_1,\widetilde{x}_1)]_{\widetilde{x}+\varepsilon})\delta(x_1-\widetilde{x}_1-\varepsilon),$$

$$\begin{aligned}
\frac{\partial^2 R_\delta(x_1,\widetilde{x}_1)}{\partial x_1^2} &= -(\pi_\delta\delta)^{-1/4}\delta^{-1}\exp\left[-\frac{(x_1-\widetilde{x}_1)^2}{2\delta}\right]+\\
&(\pi_\delta\delta)^{-1/4}\delta^{-2}(x_1-\widetilde{x}_1)^2\exp\left[-\frac{(x_1-\widetilde{x}_1)^2}{2\delta}\right]+\\
&\left(\left[\frac{\partial R_\delta(x_1,\widetilde{x}_1}{\partial x_1}\right]_{\widetilde{x}-\varepsilon}\right)\delta(x_1-\widetilde{x}_1+\varepsilon)+\left(\left[\frac{R_\delta(x_1,\widetilde{x}_1}{\partial x_1}\right]_{\widetilde{x}+\varepsilon}\right)\delta(x_1-\widetilde{x}_1-\varepsilon)+\\
&([R_\delta(x_1,\widetilde{x}_1]_{\widetilde{x}-\varepsilon})\delta'(x_1-\widetilde{x}_1+\varepsilon)+([R_\delta(x_1,\widetilde{x}_1]_{\widetilde{x}+\varepsilon})\delta'(x_1-\widetilde{x}_1-\varepsilon)
\end{aligned} \quad (4.3.18)$$

and

$$\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_2} = -(\pi_\delta\delta)^{-1/4}\delta^{-1}(x_2-\widetilde{x}_2)\exp\left[-\frac{(x_2-\widetilde{x}_2)^2}{2\delta}\right]+$$
$$+\left([R_\delta(x_2,\widetilde{x}_2)]_{\widetilde{x}_1-\varepsilon}\right)\delta(x_2-\widetilde{x}_2+\varepsilon)+([R_\delta(x_2,\widetilde{x}_2)]_{\widetilde{x}+\varepsilon})\delta(x_2-\widetilde{x}_2-\varepsilon),$$

$$\begin{aligned}
\frac{\partial^2 R_\delta(x_2,\widetilde{x}_2)}{\partial x_2^2} &= -(\pi_\delta\delta)^{-1/4}\delta^{-1}\exp\left[-\frac{(x_2-\widetilde{x}_2)^2}{2\delta}\right]+\\
&(\pi_\delta\delta)^{-1/4}\delta^{-2}(x_2-\widetilde{x}_2)^2\exp\left[-\frac{(x_2-\widetilde{x}_2)^2}{2\delta}\right]+\\
&\left(\left[\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_2}\right]_{\widetilde{x}-\varepsilon}\right)\delta(x_2-\widetilde{x}_2+\varepsilon)+\left(\left[\frac{R_\delta(x_2,\widetilde{x}_2)}{\partial x_2}\right]_{\widetilde{x}+\varepsilon}\right)\delta(x_2-\widetilde{x}_2-\varepsilon)+\\
&+([R_\delta(x_2,\widetilde{x}_2)]_{\widetilde{x}-\varepsilon})\delta'(x_2-\widetilde{x}_2+\varepsilon)+([R_\delta(x_2,\widetilde{x}_2)]_{\widetilde{x}+\varepsilon})\delta'(x_2-\widetilde{x}_2-\varepsilon).
\end{aligned} \quad (4.3.19)$$

From Eq.(4.3.16) by differentiation we obtain

$$\begin{aligned}
\frac{\partial^2\Psi^{\#}_\delta(x_1,x_2)}{\partial x_1^2} &= \frac{\partial^2[R_\delta(x_1,\widetilde{x}_1)R_\delta(x_2,\widetilde{x}_2)\Psi(x_1,x_2)]}{\partial x_1^2} = \\
&\frac{\partial}{\partial x_1}\left[\Psi(x_1,x_2)R_\delta(x_2,\widetilde{x}_2)\frac{\partial R_\delta(x_1,\widetilde{x}_1)}{\partial x_1}+R_\delta(x_1,\widetilde{x}_1)R_\delta(x_2,\widetilde{x}_2)\frac{\partial\Psi(x_1,x_2)}{\partial x_1}\right] = \\
&2\frac{\partial\Psi(x_1,x_2)}{\partial x_1}R_\delta(x_2,\widetilde{x}_2)\frac{\partial R_\delta(x_1,\widetilde{x}_1)}{\partial x_1}+\\
&+\Psi(x_1,x_2)R_\delta(x_2,\widetilde{x}_2)\frac{\partial^2 R_\delta(x_1,\widetilde{x}_1)}{\partial x_1^2}+R_\delta(x_1,\widetilde{x}_1)R_\delta(x_2,\widetilde{x}_2)\frac{\partial^2\Psi(x_1,x_2)}{\partial x_1^2}
\end{aligned} \quad (4.3.20)$$

and

$$\frac{\partial^2 \Psi_\delta^\#(x_1,x_2)}{\partial x_2^2} = \frac{\partial^2 [R_\delta(x_1) R_\delta(x_2,\widetilde{x}_2)\Psi(x_1,x_2)]}{\partial x_2^2} =$$

$$\frac{\partial}{\partial x_2}\left[\Psi(x_1,x_2) R_\delta(x_1,\widetilde{x}_1)\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_2} + R_\delta(x_1,\widetilde{x}_1) R_\delta(x_2,\widetilde{x}_2)\frac{\partial \Psi(x_1,x_2)}{\partial x_2}\right] = \tag{4.3.21}$$

$$2\frac{\partial \Psi(x_1,x_2)}{\partial x_2} R_\delta(x_1,\widetilde{x}_1)\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_2} +$$

$$+\Psi(x_1,x_2) R_\delta(x_1,\widetilde{x}_1)\frac{\partial^2 R_\delta(x_2,\widetilde{x}_2)}{\partial x_2^2} + R_\delta(x_1,\widetilde{x}_1) R_\delta(x_2,\widetilde{x}_2)\frac{\partial^2 \Psi(x_1,x_2)}{\partial x_2^2}.$$

By substitution Eq.(4.3.15) and Eq.(4.3.20)-Eq.(4.3.21) into LHS of the Eq.(4.3.9) we obtain

$$\int_\Gamma \widehat{H}\Psi_\delta^\#(x_1,x_2)dx_1dx_2 =$$

$$\int_\Gamma dx_1dx_2 R_\delta(x_1,\widetilde{x}_1) R_\delta(x_2,\widetilde{x}_2) \times$$

$$\left[\hbar^2\frac{\partial^2 \Psi(x_1,x_2)}{\partial x_1^2} + \hbar^2\frac{\partial^2 \Psi(x_1,x_2)}{\partial x_2^2} + V(x_1,x_2)\Psi(x_1,x_2)\right] +$$

$$+\hbar^2\int_\Gamma dx_1dx_2 \times \tag{4.3.22}$$

$$\left[2\frac{\partial \Psi(x_1,x_2)}{\partial x_1} R_\delta(x_2,\widetilde{x}_2)\frac{\partial R_\delta(x_1,\widetilde{x}_1)}{\partial x_1} + \Psi(x_1,x_2) R_\delta(x_2,\widetilde{x}_2)\frac{\partial^2 R_\delta(x_1,\widetilde{x}_1)}{\partial x_1^2}\right] +$$

$$+\hbar^2\int_\Gamma dx_1dx_2 \times$$

$$\left[2\frac{\partial \Psi(x_1,x_2)}{\partial x_2} R_\delta(x_1,\widetilde{x}_1)\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_2} + \Psi(x_1,x_2) R_\delta(x_1,\widetilde{x}_1)\frac{\partial^2 R_\delta(x_2,\widetilde{x}_2)}{\partial x_2^2}\right] =$$

$$= \Sigma_1(\hbar,\delta) + \Sigma_2(\hbar,\delta).$$

Now we go to estimate the quantities

$$\Sigma_1(\hbar,\delta) = \hbar^2\int_\Gamma dx_1dx_2 \times \tag{4.3.23}$$

$$\left[2\frac{\partial \Psi(x_1,x_2)}{\partial x_1} R_\delta(x_2,\widetilde{x}_2)\frac{\partial R_\delta(x_1,\widetilde{x}_1)}{\partial x_1} + \Psi(x_1,x_2) R_\delta(x_2,\widetilde{x}_2)\frac{\partial^2 R_\delta(x_1,\widetilde{x}_1)}{\partial x_1^2}\right]$$

and

$$\Sigma_2(\hbar,\delta) = \hbar^2 \int_\Gamma dx_1 dx_2 \times$$
$$\left[2\frac{\partial\Psi(x_1,x_2)}{\partial x_2} R_\delta(x_1,\widetilde{x}_1) \frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_2} + \Psi(x_1,x_2) R_\delta(x_1,\widetilde{x}_1) \frac{\partial^2 R_\delta(x_2,\widetilde{x}_2)}{\partial x_2^2} \right] \tag{4.3.24}$$

From Eq.(4.3.23) using Eq.(4.3.14) we obtain

$$|\Sigma_1(\hbar,\delta)| \leq \hbar^2 \int_\Gamma dx_1 dx_2 \times$$
$$\left[2\left|\frac{\partial\Psi(x_1,x_2)}{\partial x_1}\right| R_\delta(x_1,\widetilde{x}_1) \left|\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_1}\right| + |\Psi(x_1,x_2)| R_\delta(x_1,\widetilde{x}_1) \left|\frac{\partial^2 R_\delta(x_2,\widetilde{x}_2)}{\partial x_1^2}\right| \right]$$
$$\leq 2O(\hbar^{1/2}) \int_\Gamma R_\delta(x_2,\widetilde{x}_2) \left|\frac{\partial R_\delta(x_1,\widetilde{x}_1)}{\partial x_1}\right| dx_1 dx_2 +$$
$$+ O(\hbar^{3/2}) \int_\Gamma R_\delta(x_2,\widetilde{x}_2) \frac{\partial^2 R_\delta(x_1,\widetilde{x}_1)}{\partial x_1^2} dx_1 dx_2 = \tag{4.3.25}$$
$$= 2O(\hbar^{1/2}) \int_{\mathbb{R}} R_\delta(x_2,\widetilde{x}_2) dx_2 \int_{\mathbb{R}} \left|\frac{\partial R_\delta(x_1,\widetilde{x}_1)}{\partial x_1}\right| dx_1 +$$
$$O(\hbar^{3/2}) \int_{\mathbb{R}} R_\delta(x_2,\widetilde{x}_2) dx_2 \int_{\mathbb{R}} \left|\frac{\partial^2 R_\delta(x_1,\widetilde{x}_1)}{\partial x_1^2}\right| dx_1 =$$
$$= \int_{\mathbb{R}} R_\delta(x_2,\widetilde{x}_2) dx_2 \left[2O(\hbar^{1/2}) \int_{\mathbb{R}} \left|\frac{\partial R_\delta(x_1,\widetilde{x}_1)}{\partial x_1}\right| dx_1 + O(\hbar^{3/2}) \int_{\mathbb{R}} \left|\frac{\partial^2 R_\delta(x_1,\widetilde{x}_1)}{\partial x_1^2}\right| dx_1 \right].$$

From Eq.(4.3.24) using Eq.(4.3.14) we obtain

$$|\Sigma_2(\hbar,\delta)| \leq \hbar^2 \int_\Gamma dx_1 dx_2 \times$$
$$\left[2\left|\frac{\partial\Psi(x_1,x_2)}{\partial x_2}\right| R_\delta(x_1,\widetilde{x}_1) \left|\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_2}\right| + |\Psi(x_1,x_2)| R_\delta(x_1,\widetilde{x}_1) \left|\frac{\partial^2 R_\delta(x_2,\widetilde{x}_2)}{\partial x_2^2}\right| \right] \leq$$
$$2O(\hbar^{1/2}) \int_\Gamma R_\delta(x_1,\widetilde{x}_1) \left|\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_2}\right| dx_1 dx_2 +$$
$$O(\hbar^{3/2}) \int_\Gamma R_\delta(x_1,\widetilde{x}_1) \left|\frac{\partial^2 R_\delta(x_2,\widetilde{x}_2)}{\partial x_2^2}\right| dx_1 dx_2 \tag{4.3.26}$$
$$2O(\hbar^{1/2}) \int_{\mathbb{R}} R_\delta(x_1,\widetilde{x}_1) dx_1 \int_{\mathbb{R}} \left|\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_2}\right| dx_2 +$$
$$O(\hbar^{3/2}) \int_{\mathbb{R}} R_\delta(x_1,\widetilde{x}_1) dx_1 \int_{\mathbb{R}} \left|\frac{\partial^2 R_\delta(x_2,\widetilde{x}_2)}{\partial x_2^2}\right| dx_2 =$$
$$\int_{\mathbb{R}} R_\delta(x_1,\widetilde{x}_1) dx_1 \left[2O(\hbar^{1/2}) \int_{\mathbb{R}} \left|\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_2}\right| dx_2 + O(\hbar^{3/2}) \int_{\mathbb{R}} \left|\frac{\partial^2 R_\delta(x_2,\widetilde{x}_2)}{\partial x_2^2}\right| dx_2 \right].$$

Having substituted Eq.(4.3.18) into Eq.(4.3.25) and Eq.(4.3.19) into Eq.(4.3.26) and having applied Lemma 4.3.1 we have finalized the proof of the Eq.(4.3.9).

We assume now that

$$N\int_{\mathbb{R}}\int_{\mathbb{R}} R_{\delta}^2(x_1,\widetilde{x}_1)R_{\delta}^2(x_2,\widetilde{x}_2)|\Psi(x_1,x_2)|^2 dx_1 dx_2 = 1 \tag{4.3.27}$$

From Eq.(4.3.27) and Eq.(4.3.17) by Lemma 4.3.1 we obtain

$$N\int_{\mathbb{R}}\int_{\mathbb{R}} R_{\delta}^2(x_1,\widetilde{x}_1)R_{\delta}^2(x_2,\widetilde{x}_2)|\Psi(x_1,x_2)|^2 dx_1 dx_2 = \tag{4.3.28}$$

From Eq.(4.3.16)-Eq.(4.3.17) we obtain

$$\langle x_A\rangle = \int\int x_1 \Psi_{\delta}^{\#}(x_1,x_2)dx_1dx_2 = \int\int x_1 R_{\delta}^2(x_1,\widetilde{x}_1)R_{\delta}^2(x_2,\widetilde{x}_2)|\Psi(x_1,x_2)|^2 dx_1 dx_2, \quad (4.3.$$

IV.4.Position-momentum entangled photon pairs in non-linear wave-guide

The physical system where we expect the entangled photon states to appear include: (A) a Kerr-type nonlinear single-mode wave-guide characterized by strong photon-photon coupling [56], [57], or (B) a chain of coupled non-linear resonators [58–61]. For two photons with momenta $\hbar k_1 = \hbar(k_0 - \delta k) and \hbar k^2 = \hbar(k_0 + \delta_k)$ and dispersion

$$\omega k_0 + \delta k \approx \omega k_0 + v\delta k + \beta\delta k^2/2, \tag{4.4.1}$$

where v is the photon group velocity, the variation of the energy of a photon pair

$$\Delta^{(2)}\omega = \omega k_0 - \delta k + \omega k_0 + \delta k - 2\omega k_0 \approx \beta\delta k^2. \tag{4.4.2}$$

As the photon-photon interaction conserves both energy and longitudinal momentum, the two-photon states propagating along the non-linear transmission line can be described by the Fock function

$$|\psi\rangle_{2k_0} = \int dk_1 dk_2 \delta(k_1 + k_2 - 2k_0) f(k_1 - k_2)|k_1,k_2\rangle \tag{4.4.3}$$

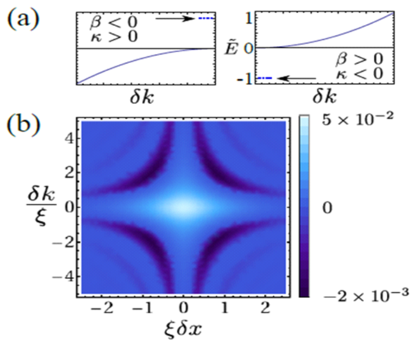


Fig. 4.4.1: Entangled two-photon states in non-linear wave guides.

(a)Spectrum of a two-photon state, $\tilde{E} = (E - 2\omega k_0)|\beta|/\kappa^2$, with total momentum $2k^0$ in awave-guide with quadratic dispersion (4.3.1) for $\beta < 0, \kappa > 0$ (left) and $\beta > 0, \kappa < 0$ (right).Solid line corresponds to the continuous spectrum,while the single eigenvalue corresponding to the entangled state is shown by dashed line.(b)Wigner function of the two-photon entangled state. It takes negative values,which is a hallmark of non-Gaussian entangled states.

(A) To demonstrate the principle of position-momentum entanglement of photons in Kerr-nonlinear systems, we, first, consider the entangled photon pairs in non-linear optical wave-guides. Classically, Kerr nonlinearity in an isotopic medium manifests itself in the third-order polarisation $\mathbf{P}^{(3)(+)} = \chi^{(3)}[(\mathbf{E}^{(-)}\cdot\mathbf{E}^{(+)})\mathbf{E}^{(+)} + \alpha(\mathbf{E}^{(+)}\cdot\mathbf{E}^{(+)})\mathbf{E}^{(-)}]$, where ”+” and ”-” correspond to positive and negative frequency parts, E is electric field, $\chi^{(3)}$ is the susceptibility of the medium $\chi^{(3)} = \chi^{(3)}_{xyxy}$, $\alpha = \chi^{(3)}_{xxyy}/(2\chi^{(3)})$. Quantizing electromagnetic field, integrating over transverse degrees of freedom, and neglecting magneto-optical effects ($\alpha = 0$) leading to entanglement over polarization degrees of freedom, one obtains the following Hamiltonian ($\hbar = c = 1$):

$$\begin{aligned} &H = H_0 + H_{int}, H_0 = \sum_k k\omega k a_k^\dagger a_k, \\ &H_{int} = \frac{k}{L}\sum_{k_1,k_2,k_3,k_4} \delta(k_1 + k_2, k_3 + k_4) a_{k_4}^\dagger a_{k_3}^\dagger a_{k_1} a_{k_2}, \end{aligned} \tag{4.4.4}$$

where $a_k(a_k^\dagger)$ is the annihilation (creation) operator of a photon with longitudinal momentum k and energy ω_k, L is the length of the system. The non-linear term H_{int} in Eq. (4.3.4) describes photon-photon interaction with coupling $\kappa = \pi\omega^2\chi^{(3)}/2n_r^4 A\epsilon_0$, where n_r is refraction index, A is the area occupied by the wave-guide mode and ϵ_0 is the vacuum permittivity. Hamiltonian (4.3.4) can be diagonalized exactly in the case of $\Delta^{(2)}\omega \propto \delta k^2$. We consider a sector of the Hilbert space,which consists of allt he two-photon states with the total pair momentum $2k_0$ and assume the effective mass approximation for th ewave-guide dispersion given by Eq. (4.3.1). In the coordinate domain, $a_x = 1/\sqrt{L}\sum_k a_k \exp[i(k - k_0)x]$,the Hamilton Eq. (4.3.4) takes the form

$$H = \int dx\Big(\omega_{k_0} a_x^\dagger a_x - iva_x^\dagger \partial_x a_x - \frac{1}{2}\beta a_x^\dagger \partial^2 a_x\Big) + \frac{1}{2}\int dx_1 dx_2 a_{x_1}^\dagger a_{x_2}^\dagger U(x_1 - x_2) a_{x_1} a_{x_2}, \quad (4.4.5)$$

where $U(x_1 - x_2) = 2\kappa\delta(x_1 - x_2)$. For a two-photon state, described by the wave-function

$$|\,\psi\rangle = \int dx_1 dx_2 f(x_1,x_2) a_{x_1}^\dagger a_{x_2}^\dagger |0\rangle,$$

one obtains the following Schrödinger equation:

$$[2\omega_{k_0} - iv(\partial_{x_1} + \partial_{x_2}) - \frac{1}{2}\beta(\partial_{x_1}^2 + \partial_{x_2}^2) + 2\kappa\delta(x_1 - x_2)]f(x_1,x_2) = Ef(x_1,x_2), \quad (4.4.6)$$

where E is the energy of a two-photon state. Equation (4.4.6) has scattering state solutions, which correspond to the continuous spectrum of non-interacting photons with energies given by Eq. (4.4.2) (See Fig.4.4.1(a)). When the curvature of the wave-guide dispersion β and the photonphoton coupling constant κ are of opposite signs, $\beta_\kappa < 0$, there exists a bound state solution with

$$f(x_1,x_2) = \sqrt{\frac{\xi}{2L}}\, exp[-|x_1 - x_2|\xi], \xi = |\kappa/\beta| \quad (4.4.7)$$

The energy of this state is split from the continuum of weakly correlated scattering states, as we show in Fig. 4.4.1(a), and it is given by

$$E_b = 2\omega_{k_0} - \kappa^2/\beta, \quad (4.4.8)$$

as expected from binding of a one-dimensional massive particle to an attractive δ-functional potential well [30]. In the momentum domain, the two-photon bound state wave-function is given by Eq. (4.4.3) with

$$f(k_1 - k_2) = \frac{8\xi^{3/2}}{\sqrt{2L}\,[(k_1 - k_2)^2 + 4\xi^2]} \quad (4.4.9)$$

The state (4.4.9) can be characterised by the Wigner function defined as the expectation value

$$\mathbf{W}(x_1,k_1;x_2,k_2) = \pi^{-2}\langle\psi|\Pi(x_1,k_1) \otimes \Pi(x_2,k_2)|\psi\rangle$$

of the parity operator

$$\Pi(x,k) = \int d\zeta e^{-2ix\zeta} a_{k+\zeta}^\dagger |0\rangle\langle 0| a_{k-\zeta}.$$

After straightforward calculations, one obtains

$$\mathbf{W}(x_1,k_1;x_2,k_2) = \frac{\xi^2 e^{-2\xi|\delta x|}}{2\pi^2(\delta k^2 + \xi^2)}\cos(2\delta k|\delta x|) + \frac{\xi}{\delta k}\sin(2\delta k|\delta x|)\delta(k_1 + k_2; 2k_0), \quad (4.4.10)$$

where $\delta x = x_1 - x_2$. This function is negative for $\cos(2\delta k|\delta x|) + (\xi/\delta k)\sin(2\delta k|\delta x|) < 0$, as shown in Fig. 4.4.1(b), which implies that the state (4.4.9) is entangled in position-momentum degrees of freedom. Moreover, for $\xi \to \infty$, the two-photon wave-function approaches the ideal Einstein-Podolsky-Rosen state in which position and momenta are perfectly (anti-) correlated:

$$|\,\psi\rangle = \int d(\delta k)|k_0 + \delta k, k_0 - \delta k\rangle = \int dx e^{2ik_0 x}|x,x\rangle.$$

Alternatively, to demonstrate that the state (4.4.9) is entangled in position-momentum degrees of freedom, one can find the uncertainties $\Delta(x_1 - x_2)$ and $\Delta(k_1 + k_2)$ calculated over the joint probability distributions $\mathbf{P}(x_1, x_2)$ and $\mathbf{P}(k_1, k_2)$ respectively, for which, the separability criterion:

$$[\Delta(x_2 - x_1)]^2[\Delta(k_2 + k)]^2 \geq 1, \tag{4.4.11}$$

can be applied. Although, the states for which the inequality (4.4.11) is violated are inseparable, they do not necessarily lead to EPR paradox. In order for an EPR "paradox" to arise, correlations must violate a more strict inequality []:

$$[\Delta(x_2 - x_1)]^2[\Delta(k_2 + k)]^2 \geq 1/4, \tag{4.4.12}$$

which can be accessible experimentally [].

IV.5.Position-momentum entangled photon pairs and the experimental verification of the postulate of nonlocality.

In paper [23] is reported on a demonstration of the EPR paradox using position- and momentum-entangled photon pairs produced by spontaneous parametric down conversion. Transverse correlations from parametric down conversion have been studied both theoretically and experimentally. It was find experimentally that the position and momentum correlations are strong enough to allow the position or momentum of a photon to be inferred from that of its partner with a product of variances $\leq 0.01\hbar^2$, which violates the separability bound by 2 orders of magnitude. In the idealized entangled state proposed by EPR, the positions and momenta of the two particles are perfectly correlated.However such idealized entangled state is non-normalizable and cannot be realized inthe laboratory. However,the state of the light produced in parametric down conversion can be made to approximate the EPR state under suitable conditions. In parametric down conversion, a pump photon is absorbed by a nonlinear medium and reemitted as two photons (conventionally called signal and idler photons), each with approximately half the energy of the pump photon. Considering only the transverse components, the momentum conservation of the down conversion process requires $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_p$, where $1, 2,$ and p refer to the signal, idler, and pump photons, respectively. Provided the uncertainty in the pump transverse momentum is small, the transverse momenta of the signal and idler photons are highly anticorrelated. The exact degree of correlation depends on the structure of the signal idler state. In the regime of weak generation, this state has the form

$$|\psi\rangle_{1,2} = |\text{vac}\rangle + \int d\mathbf{p}_1 d\mathbf{p}_2 A(\mathbf{p}_1, \mathbf{p}_2)|\mathbf{p}_1, \mathbf{p}_2\rangle, \tag{4.5.1}$$

where $|\text{vac}\rangle$ denotes the vacuum state and the two-photon amplitude $A(\mathbf{p}_1, \mathbf{p}_2)$ is

$$A(\mathbf{p}_1, \mathbf{p}_2) = \chi E_p(\mathbf{p}_1, \mathbf{p}_2)\frac{\exp(i\Delta\mathbf{k}_z L) - 1}{i\Delta\mathbf{k}_z}. \tag{4.5.2}$$

Here is the coefficient of the nonlinear interaction, E_p is the amplitude of the plane-wave component of the pump with transverse momentum $p_1 p_2, L$ is the length of the nonlinear medium, and $\Delta \mathbf{k}_z = \mathbf{k}_{p,z} - \mathbf{k}_{1,z} - \mathbf{k}_{2,z}$ (where $\mathbf{k} = \mathbf{p}/\hbar$) is the longitudinal wave vector mismatch,which generally increases with transverse momentum and limits the angular spread of signal and idler photons. The vacuum component of the state makes no contribution to photon counting measurements and may be ignored. Also, there is no inherent difference between different transverse components; so without loss of generality, we consider scalar position and momentum. The narrower the angular spectrum of the pump field and the wider the angular spectrum of the generated light, the more closely the integral (4.4.1) approximates $\int dp_1 dp_2 \delta(p_1 + p_2)|p_1, p_2\rangle = |EPR\rangle$ and the stronger the correlations in position and momentum become. The experimental setup used to determine position and momentum correlations is portrayed in Fig. 4.5.1(a)-(b).The idea is to measure the positions and momenta by measuring the down converted photons in the near and far fields, respectively [34]. The source of entangled photons is spontaneous parametric down conversion generated by pumping a 2 mm thick type-II -barium-borate (BBO) crystal with a 30 mW, cw, 390 nm laser beam. A prism separates the pump light from the down converted light. The signal and idler photons have orthogonal polarizations and are separated by a polarizing beam splitter. In each arm, the light passes through a narrow 40 m vertical slit, a 10 nm spectral filter, and a microscope objective.The objective focuses the transmitted light onto a multimode fiber which is coupled to an avalanche photodiode single-photon counting module. The spectral filter ensures that only

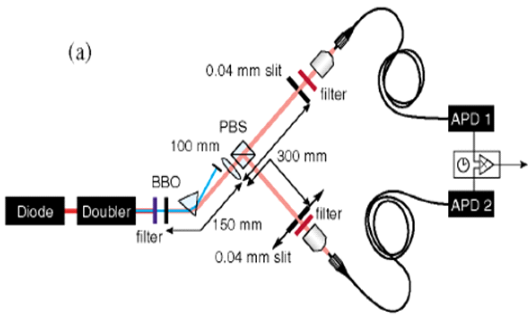


Fig.4.4.1(a) [23]. Experimental setup for measuring position photon correlations. Position correlations are obtained by imaging the birth place of each photon of a pair onto a separate detector.

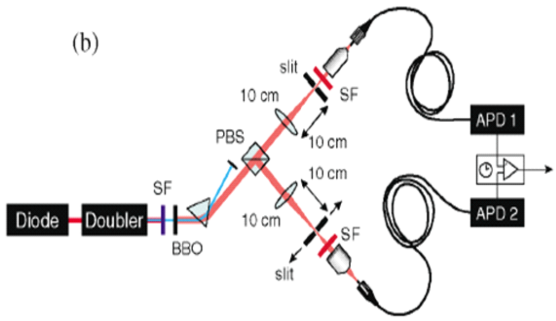


Fig.4.4.1(b) [23].Experimental setup for measuring correlations in transverse momentum.Correlations in transverse momentum are obtained by imaging the the propagation direction of each photon of a pair onto a separate detector.

photons with nearlyequalenergies are detected. To measure correlations in the positions of the photons, a lens of focal length 100 mm (placed prior to the beam splitter) is used to image the exit face of the crystal onto the planes of the two slits [Fig. 4.4.1(a)]. One slit is fixed at the location of peak signal intensity.The other slit is mounted on a translation stage.The photon coincidence rate is then recorded as a function of the displacement of the second slit. To measure correlations in the transverse momenta of the photons, the imaging lens is replaced by two lenses of focal length 100 mm, one in each arm, a distance f from the planes of the two slits [Fig.4.5.1(b)].These lenses map transverse momenta to transverse positions, such that a photon with transverse momentum $\hbar k_{\perp}$ comes to a focus at the point $x = fk_{\perp}/k$ in the plane of the slit. Again, one slit is fixed at the location of the peak count rate while the other is translated to obtain the coincidence distribution. By normalizing the coincidence distributions, the conditional probability density functions $p(x_2|x_1)$ and $p(p_2|p_1)$ was obtained (see Fig. 4.5.2-Fig. 4.5.3). These probability densities are then used to calculate the uncertainty in the inferred position or momentum of photon 2 given the position or momentum of photon 1:

$$\begin{aligned} \Delta x_2^2(x_1) &= \int x_2^2 p(x_2|x_1)dx_2 - \left(\int x_2 p(x_2|x_1)dx_2\right)^2, \\ \Delta p_2^2(p_1) &= \int p_2^2 p(p_2|p_1)dx_2 - \left(\int p_2 p(p_2|p_1)dx_2\right)^2. \end{aligned} \tag{4.5.3}$$

Because of the finite width of the slits, the raw data in Fig. 4.5.2-Fig.4.5.3 describe a slightly broader distribution than is associated with the down conversion process itself. By adjusting the computed values of Δx_2 and Δp_2 to account for this broadening (an adjustment smaller than 10%), we obtainthe correlation uncertainties $\Delta x_2 = 0.027$ mm and $\Delta p_2 == 3.7\hbar$mm^{-1}.

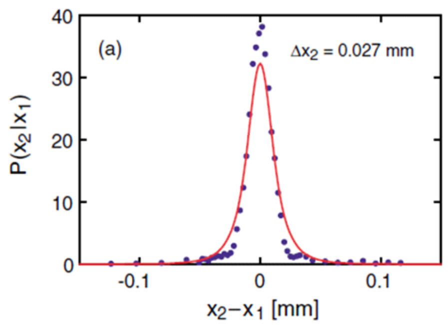


Fig.4.5.2.[23].The conditional probability distribution of the relative birthplace of the entangled photons.The solid line are the theoretical prediction and the dot are the experimental data.

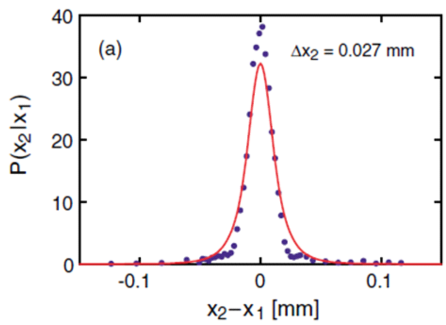


Fig.4.5.3.[23].The conditional probability distribution of the relative transverse momentum of the entangled photons. The solid line are the theoretical prediction and the dot are the experimental data.

The widths of the distributions determine the uncertainties in inferring the position or the momentum of one photon from that of the other. The experimentally measured variance product is then [23]

$$[\Delta^{\exp}x_2^2(x_1)][\Delta^{\exp}p_2^2(p_1)] = 0.01\hbar^2. \quad (4.5.4)$$

Also shown in Fig.4.5.2-Fig.4.5.3 are the predicted probability densities. These curves contain no free parameters and are obtained directly from the two-photon amplitude $A(p_1,p_2)$ [23], which is determined by the optical properties of BBO and the measured profile of the pump beam. Figure 4.5.2 indicates that the correlation widths was obtained are intrinsic to the down conversion process and are limited only by the degree to which it deviates from the idealized EPR state (4.5.1).The value of $\Delta(p_2+p_1)$ is limited by the finite width of the pump beam. The pump photons in a Gaussian beam of width w have an uncertainty $\hbar/2w$ in transverse momentum which, due to conservation of momentum, is imparted to the total momentum p_1+p_2 of the signal and idler photons. The value of $\Delta(x_2-x_1)$ is limited by the range of angles over which the crystal generates signal and idler photons. If the angular width of emission is $\Delta\phi$,then the principle of diffraction indicates that the photons cannot have a smaller transverse dimension than $\sim(k_{s,i}\Delta\phi)^{-1}$.Careful analysis based on the angular distribution of emission yields $\Delta(x_2-x_1) = 1.88(k_{s,i}\Delta\phi)^{-1}$ [23].With the measured beam width of $w = 0.17$mm and predicted angular width 0.012 rad, the theory predicts [23]:

$$[\Delta^{\mathbf{th}}x_2^2(x_1)][\Delta^{\mathbf{th}}p_2^2(p_1)] = 0.0036\hbar^2. \quad (4.5.5)$$

Remark 4.5.1.This is somewhat smaller than the experimentally calculated value of $0.01\hbar^2$,
even though the data appear to closely match the theoretical curves.

$$\Delta^{\exp}x_2^2(x_1)\Delta^{\exp}p_2^2(p_1) - \Delta^{\mathbf{th}}x_2^2(x_1)\Delta^{\mathbf{th}}p_2^2(p_1) = 0.01\hbar^2 - 0.0036\hbar^2 = 0.0064\hbar^2. \quad (4.5.6)$$

Remark 4.5.2.The reason for this discrepancy is that the experimental distributions have
small (1% of the peak) but very broad wings.

Remark 4.5.3.The origin of this uncoincidence counts is unknown [23].

Remark 4.5.4.In paper [23] it was assumed that this counts are perhaps due to scattering
from optical components. If these counts are treated as a noise background and
subtracted, the experimentally obtained uncertainties come into somewhat better
agreement with the theoretically predicted values, yielding an uncertainty product of
$0.004\hbar^2$:

$$\delta_{\mathbf{EPR}}^{\mathbf{nonloc.}}(x_2-x_1,p_2+p_1) = \Delta^{\exp}x_2^2(x_1)\Delta^{\exp}p_2^2(p_1) - \Delta^{\mathbf{th}}x_2^2(x_1)\Delta^{\mathbf{th}}p_2^2(p_1) = 0.006\hbar^2. \quad (4.5.7)$$

Thus final value of uncoincidence counts is

$$\delta_{\mathbf{EPR}}^{\mathbf{nonloc.}}(x_2-x_1,p_2+p_1) = 0.006\hbar^2. \quad (4.5.8)$$

Remark 4.5.5.Note that the separability criterion derived by Mancini et al. [35] is more
useful here. We remind that it states that separable systems satisfy the joint uncertainty product

$$\Delta(x_2 - x_1)\Delta(p_2 + p_1) \geq \hbar^2, \qquad (4.5.9)$$

where the uncertainties are calculated over the joint probability distributions $P(x_1,x_2)$
and $P(p_1,p_2)$, respectively.

In this experiments the widths of the conditional probability distributions P

Therefore our results constitute a 2-order-of-magnitude violation of Mancini's separability
criterion as well as a strong violation of EPR's criterion.

IV.6.EPR-B experiment

The EPR-B, the spin version of the Einstein-Podolsky-Rosen experiment proposed by Bohm, see [63],[64]

[64]:"We consider a molecule of total spin zero consisting of two atoms, each of spin one-half. The wave function of the system is therefore

$$\psi = \frac{1}{\sqrt{2}}[\psi_+(1)\psi_-(2) - \psi_-(1)\psi_+(2)] \qquad (\quad)$$

where $\psi_+(1)$ refers to the wave function of the atomic state in which one particle (A) has spin $+\hbar/2$, etc. The two atoms are then separated by a method that does not influence the total spin. After they have separated enough so that they cease to interact, any desired component of the spin of the first particle (A) is measured. Then, because the total spin is still zero, it can immediately be concluded that the same component of the spin of the other particle (B) is opposite to that of A.

If this were a classical system, there would be no difficulty in interpreting the above results, because all components of the spin of each particle are well defined at each instant of time. Thus, in the molecule, each component of the spin of particle A has, from the very beginning, a value opposite to that of the same component of B; and this relationship does not change when the atom disintegrates. In other words, the two spin vectors are correlated. Hence, the measurement of any component of the spin of A permits us to conclude also that the same component of B is opposite in value. **The possibility of obtaining knowledge of the spin of particle B in this way evidently does not imply any interaction of the apparatus with particle B or any interaction between A and B**.

In quantum theory, a difficulty arises, in the interpretation of the above experiment, because only one component of the spin of each particle can have a definite value at a given time. Thus, if the x component is definite, then the y and z components are indeterminate and we may regard them more or less as in a kind of random fluctuation.

In spite of the effective fluctuation described above, however, the quantum theory still implies that no matter which component of the spin of A may be measured the same component of the spin of B will have a definite and opposite value when the measurement is over. Of course, the wave function then reduces to $\psi_{+}(1)\psi_{-}(2)$ or $\psi_{-}(1)\psi_{+}(2)$, in accordance with the result of the measurement. Hence, there will then be no correlations between the remaining components of the spins of the two atoms. Nevertheless, before the measurement has taken place (even while the atoms are still in flight) we are free to choose any direction as the one in which the spin of particle A (and therefore of particle B) will become definite.

In order to bring out the difficulty of interpreting the result, let us recall that originally, the indeterminacy principle was regarded as representing the effects of the disturbance of the observed system by the indivisible quanta connecting it with the measuring apparatus. This interpretation leads to no serious difhculties for the case of a single particle. For example, we could say that on measuring the z component of the spin of particle A, we disturb the x and y components and make them fluctuate. This point of view more generally implies that the definiteness of any desired component of the spin is (along with the indefiniteness of the other two components) a potentiality' which can be realized with the aid of a suitably oriented spinmeasuring apparatus.

In the case of complementary pairs of continuous variables, such as position and momentum, one obtains from this point of view the well known wave-particle duality. In other words, the electron, for example, has potentialities for mutually incompatible wave-like and particle-like behavior, which are realized under suitable external conditions. In the laboratory those conditions are generally determined by the measuring apparatus although, more generally, they may be determined by any arrangement of matter with which the electron interacts. But in any case, it is essential that there must be an external interaction, which disturbs the observed system in such a way as to bring about the realization of one of its various mutually incompatible potentialities. As a result of this disturbance, when any one variable is made definite, other (noncommuting) variables must necessarily become indefinite and undergo fluctuation".

Evidently, the foregoing interpretation is not satisfactory when applied to the experiment of ERP. It is of course acceptable for particle A alone (the particle whose spin is measured directly). But it does not explain why particle B (which does not interact with A or with the measuring apparatus) realizes its potentiality for a definite spin in precisely the same direction as that of A. Moreover, it cannot explain the fluctuations of the other two components of the spin of particle B as the result of disturbances due to the measuring apparatus.

In this subsection we explain EPR-B experiment using reduction to an sort of generic EPR correlations for two particles A and B with maximally correlated position z_A and z_B. This explanation avoid the EPR-Bohm paradox.

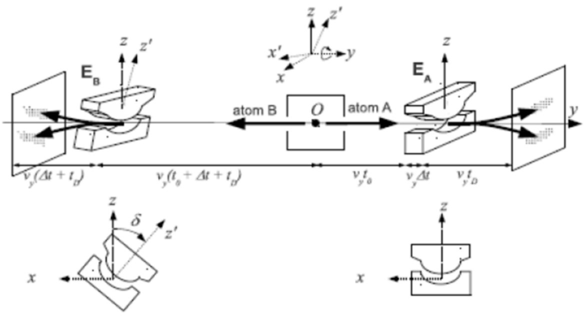


Fig.4.5.1.Einstein-Podolsky-Rosen-Bohm experiment.

Figure 4.5.1 presents the Einstein-Podolsky-Rosen-Bohm experiment. A source S created in O pairs of identical atoms A and B, but with opposite spins. The atoms A and B split following the y-axis in opposite directions, and head towards two identical Stern-Gerlach apparatus $\mathbf{E_A}$ and $\mathbf{E_B}$. The electromagnet $\mathbf{E_A}$ "measures" the spin of A along the z-axis and the electromagnet $\mathbf{E_B}$ "measures" the spin of B along the z'-axis, which is obtained after a rotation of an angle δ around the y-axis.

Remark 4.5.1. So far we have consistently made use of the idea that if we know something definite about the state of a such physical system, say that we know the z component of the spin of a spin half particle is $S_z = \pm\frac{1}{2}\hbar$, then we assign to the system the state $|S_z\rangle = |\pm\frac{1}{2}\hbar\rangle$, or, more simply, $|\pm\rangle$.

Remark 4.5.2. We can also note that these two states $|+\rangle$ and $|-\rangle$ are mutually exclusive, i.e. if atom in the state $|+\rangle$, then the result $S_z = -\frac{1}{2}\hbar$ is never observed, and furthermore, we note that the two states $|+\rangle$ and $|-\rangle$ cover all possible values for S_z.

Remark 4.5.3. When we say that we 'know' the value of some physical observable of a quantum system, we are presumably implying that some kind of measurement has been made that provided us with this knowledge. It is furthermore assumed that in the process of acquiring this knowledge, the system, after the measurement has been performed, survives the measurement, and moreover if we were to immediately remeasure the same quantity, we would get the same result. This is certainly the situation with the measurement of spin in a Stern-Gerlach experiment. If an atom emerges from one such set of apparatus in a beam that indicates that $S_z = \frac{1}{2}\hbar$ for that atom, and we were to pass the atom through a second apparatus, also with its magnetic field oriented in the z direction, we would find the atom emerging in the $S_z = \frac{1}{2}\hbar$ beam once again. Under such circumstances, we would be justified in saying that the *atom has been prepared* in the state $|S_z = \frac{1}{2}\hbar\rangle$,etc.

Definition 4.5.1. Assume that *atom* **A** *has been prepared in the state* $|S_z = \frac{1}{2}\hbar\rangle$,

$|S_z = -\frac{1}{2}\hbar\rangle$,etc. Then we will say that these events $|S_z = \frac{1}{2}\hbar\rangle, |S_z = -\frac{1}{2}\hbar\rangle$,etc. occurs. We will be denoted these events by symbols $|S_z = \frac{1}{2}\hbar\rangle^{\mathbf{A}}, |S_z = -\frac{1}{2}\hbar\rangle^{\mathbf{A}}$,etc., or $|\frac{1}{2}\hbar\rangle^{\mathbf{A}}, |-\frac{1}{2}\hbar\rangle^{\mathbf{A}}$, etc.

Definition 4.5.2. Assume that we know exactly that *atom* **A** *in the state* $|\frac{1}{2}\hbar\rangle, |-\frac{1}{2}\hbar\rangle$, etc.

Then we will say that these events $|\frac{1}{2}\hbar\rangle, |-\frac{1}{2}\hbar\rangle$, etc. occurs and we will be denoted these events again by symbols $|\frac{1}{2}\hbar\rangle^{\mathbf{A}}, |-\frac{1}{2}\hbar\rangle^{\mathbf{A}}$, etc.

Definition 4.5.3. Assume that these events $|\frac{1}{2}\hbar\rangle^{\mathbf{A}}, |-\frac{1}{2}\hbar\rangle^{\mathbf{A}}$, etc. occurs in point $\mathbf{x} = (t, x_1, x_2, x_3) = (t, \mathbf{r}) \in M_4$ of Minkowski spacetime M_4. Then we will be denoted these

events by symbols $|\frac{1}{2}\hbar\rangle_{\mathbf{x}}^{\mathbf{A}}, |-\frac{1}{2}\hbar\rangle_{\mathbf{x}}^{\mathbf{A}}$, etc. or $|\frac{1}{2}\hbar\rangle_{(t_A, z_A)}^{\mathbf{A}}, |-\frac{1}{2}\hbar\rangle_{(t_A, z_A)}^{\mathbf{A}}$, etc.

Assumption 4.5.1. We claim for any $\mathbf{x} \in M_4$ that:

$$|\tfrac{1}{2}\hbar\rangle_{\mathbf{x}}^{\mathbf{A}} \in \mathcal{F}_{M_4}, |-\tfrac{1}{2}\hbar\rangle_{\mathbf{x}}^{\mathbf{A}} \in \mathcal{F}_{M_4}, etc. \tag{4.5.1}$$

Here $\mathcal{F}_{M_4}$ is a measure algebra $\mathcal{F}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ with a probability measure $\mathbf{P}$, see

subsection **II**.2, Definition 2.2.2.

Remark 4.5.4. Note that for any $\mathbf{x} \in M_4$ and for any atom **A** these events $|\frac{1}{2}\hbar\rangle_{\mathbf{x}}^{\mathbf{A}}, |-\frac{1}{2}\hbar\rangle_{\mathbf{x}}^{\mathbf{A}}$

are mutually exclusive, see Remark 4.5.2, and therefore for any $\mathbf{x} \in M_4$

$$\mathbf{P}\left(|\tfrac{1}{2}\hbar\rangle_{\mathbf{x}}^{\mathbf{A}} \bigwedge |-\tfrac{1}{2}\hbar\rangle_{\mathbf{x}}^{\mathbf{A}}\right) = 0. \tag{4.5.2}$$

Remark 4.5.5. We remind that if an atom is prepared in an arbitrary initial state $|S\rangle$, then

the probability amplitude of finding it in some other state $|S'\rangle$ is given by

$$\langle S'|S\rangle = \langle S'|+\rangle\langle +|S\rangle + \langle S'|-\rangle\langle -|S\rangle \tag{4.5.3}$$

which leads, by the cancellation trick to

$$|S\rangle = |+\rangle\langle +|S\rangle + |-\rangle\langle -|S\rangle \tag{4.5.4}$$

and therefore the states $|\pm\rangle$ form a complete set of orthonormal basis states for the state space of the system.

Suppose we have an n-dimensional quantum system which contains only a quantum observable with discrete values such as S_z, etc.

II.Then we claim the following:

$\mathbf{Q_d}$.I.1. Any given n-dimensional quantum system which contains only a quantum observable with discrete values such that mentioned above is identified by a set $\mathbf{Q_d}$:

$$\mathbf{Q_d} \triangleq \langle \mathbf{H_d}, \mathfrak{I}_\mathbf{d}, \mathfrak{R}_\mathbf{d}, \mathcal{L}_{2,1}^{\mathbf{d}}, \mathbf{G_d}, |\psi_t\rangle\rangle \tag{4.5.5}$$

where:

(i) $\mathbf{H_d}$ that is some finite-dimensional complex Hilbert space,

(ii) $\mathfrak{I}_\mathbf{d} = (\Omega_\mathbf{d}, \mathcal{F}_\mathbf{d}, \mathbf{P_d})$ that is complete probability space,

(iii) $\mathfrak{R}_\mathbf{d} = (\mathbb{R}^n, \Sigma_\mathbf{d})$ that is measurable space ,

(iv) $\mathcal{L}_{2,1}^{\mathbf{d}}(\Omega_\mathbf{d})$ that is complete space of discrete random variables $X_\mathbf{d} : \Omega_\mathbf{d} \to \mathbb{R}^n$

such that

$$\int_{\Omega_{\mathbf{d}}} \|X_{\mathbf{d}}(\omega)\| d\mathbf{P}_{\mathbf{d}} < \infty, \int_{\Omega_{\mathbf{d}}} \|X_{\mathbf{d}}(\omega)\|^2 d\mathbf{P}_{\mathbf{d}} < \infty \tag{4.5.6}$$

(v) $\mathbf{G}_{\mathbf{d}} : \mathbf{H}_{\mathbf{d}} \to \mathcal{L}_{2,1}(\Omega_{\mathbf{d}})$ that is one to one correspondence such that

$$\langle\psi|\widehat{Q}_{\mathbf{d}}|\psi\rangle = \int_{\Omega_{\mathbf{d}}} \left(\mathbf{G}_{\mathbf{d}}\left[\widehat{Q}_{\mathbf{d}}|\psi\rangle\right](\omega)\right) d\mathbf{P}_{\mathbf{d}} = \mathbf{E}_{\Omega_{\mathbf{d}}}\left(\mathbf{G}_{\mathbf{d}}\left[\widehat{Q}_{\mathbf{d}}|\psi\rangle\right](\omega)\right) \tag{4.5.7}$$

for any $|\psi\rangle \in \mathbf{H}_{\mathbf{d}}$ and for any Hermitian operator with discrete spectrum $\widehat{Q}_{\mathbf{d}} : \mathbf{H}_{\mathbf{d}} \to \mathbf{H}_{\mathbf{d}}$,

(vi) $|\psi_t\rangle$ is an continuous vector function $|\psi_t\rangle : \mathbb{R}_+ \to \mathbf{H}_{\mathbf{d}}$ which represented the evolution of the quantum system $\mathbf{Q}_{\mathbf{d}}$.

$\mathbf{Q}_{\mathbf{d}}$.I.2. For any $|\psi_1\rangle, |\psi_2\rangle \in \mathbf{H}_{\mathbf{d}}$ and for any Hermitian operator $\widehat{Q}_{\mathbf{d}} : \mathbf{H}_{\mathbf{d}} \to \mathbf{H}_{\mathbf{d}}$

such that

$$\left\langle\psi_1\middle|\widehat{Q}_{\mathbf{d}}\middle|\psi_2\right\rangle = \left\langle\psi_2\middle|\widehat{Q}_{\mathbf{d}}\middle|\psi_1\right\rangle = 0 \tag{4.5.8}$$

valid the equality

$$\mathbf{G}_{\mathbf{d}}\left[\widehat{Q}_{\mathbf{d}}(|\psi_1\rangle + |\psi_2\rangle)\right](\omega) = \mathbf{G}_{\mathbf{d}}\left[\widehat{Q}_{\mathbf{d}}|\psi_1\rangle\right](\omega) + \mathbf{G}_{\mathbf{d}}\left[\widehat{Q}_{\mathbf{d}}|\psi_2\rangle\right](\omega). \tag{4.5.9}$$

Remark 4.5.6. Let $S_z^+(\omega)$ and $S_z^-(\omega)$ be discrete random variables $S_z^+ : \Omega_{\mathbf{d}} \to \{1,-1\}$,

$S_z^- : \Omega_{\mathbf{d}} \to \{-1,1\}$ correspondingly such that:

(i) $S_z^+(\omega) = \mathbf{G}[|+\rangle]$, (ii) $\mathbf{P}_{\mathbf{d}}(\Delta_+^{=+1}) = 1$, where $\Delta_+^{=+1} \triangleq \left\{\omega | S_z^+(\omega) = 1\right\}$,

(iii) $\mathbf{P}_{\mathbf{d}}(\Delta_+^{=-1}) = 0$, where $\Delta_+^{=-1} \triangleq \left\{\omega | S_z^+(\omega) = -1\right\}$

and (4.5.10)

(i) $S_z^-(\omega) = \mathbf{G}[|-\rangle]$, (ii) $\mathbf{P}_{\mathbf{d}}(\Delta_-^{=-1}) = 1$, where $\Delta_-^{=-1} \triangleq \left\{\omega | S_z^-(\omega) = -1\right\}$,

(iii) $\mathbf{P}_{\mathbf{d}}(\Delta_-^{=+1}) = 0$, where $\Delta_-^{=+1} \triangleq \left\{\omega | S_z^-(\omega) = 1\right\}$.

Let $\mathbf{Q}_{\mathbf{c}}$ be any n-dimensional quantum system which contains only a quantum observable with continuous values. We remind that such quantum system is identified by a set $\mathbf{Q}$

$$\mathbf{Q} \triangleq \langle \mathbf{H}, \mathfrak{I}, \mathfrak{R}, \mathcal{L}_{2,1}, \mathbf{G}, |\psi_t\rangle\rangle, \tag{4.5.11}$$

see subsection I.7.1.

Definition 4.5.4. We define now a composite quantum system $\mathbf{Q}_{\mathbf{c},\mathbf{d}}$ which contains both sort of quantum observables by a set $\mathbf{Q}_{\mathbf{c},\mathbf{d}}$

$$\mathbf{Q}_{\mathbf{c},\mathbf{d}} \triangleq \left\langle \mathbf{H}_{\mathbf{c},\mathbf{d}}, \mathfrak{I}_{\mathbf{c},\mathbf{d}}, \mathfrak{R}_{\mathbf{c},\mathbf{d}}, \mathcal{L}_{2,1}^{\mathbf{c},\mathbf{d}}, \mathbf{G}_{\mathbf{c},\mathbf{d}}, |\psi_t\rangle \right\rangle \tag{4.5.12}$$

where:

(i) $\mathbf{H}_{\mathbf{c},\mathbf{d}} = \mathbf{H}_{\mathbf{c}} \times \mathbf{H}_{\mathbf{d}}$ that is composite complex Hilbert space,

(ii) $\mathfrak{I}_{\mathbf{c},\mathbf{d}} = (\Omega_{\mathbf{c},\mathbf{d}}, \mathcal{F}_{\mathbf{c},\mathbf{d}}, \mathbf{P}_{\mathbf{d}})$ that is complete probability space,

with
$\Omega_{\mathbf{c,d}} = \Omega_{\mathbf{c}} \times \Omega_{\mathbf{d}}, \mathcal{F}_{\mathbf{c,d}} = \mathcal{F}_{\mathbf{c}} \times \mathcal{F}_{\mathbf{d}}, \Re_{\mathbf{c,d}} = \Re_{\mathbf{c}} \times \Re_{\mathbf{d}}, \mathcal{L}_{2,1}^{\mathbf{c,d}} = \mathcal{L}_{2,1}^{\mathbf{c}} \times \mathcal{L}_{2,1}^{\mathbf{d}}, \mathbf{G}_{\mathbf{c,d}} = \mathbf{G}_{\mathbf{c}} \times \mathbf{G}_{\mathbf{d}},$

(iii) $\Re_{\mathbf{c,d}} = (\mathbb{R}^n, \Sigma_{\mathbf{c,d}})$ that is measurable space with $\Sigma_{\mathbf{c,d}} = \Sigma_{\mathbf{c}} \times \Sigma_{\mathbf{d}}$,

(iv) $\mathcal{L}_{2,1}^{\mathbf{c,d}}(\Omega_{\mathbf{d}})$ that is complete space of random variables $X_{\mathbf{c,d}} : \Omega_{\mathbf{c,d}} \to \mathbb{R}^n$ such that

$$\int_{\Omega_{\mathbf{c,d}}} \|X_{\mathbf{c,d}}(\omega)\| d\mathbf{P}_{\mathbf{c}} \times d\mathbf{P}_{\mathbf{d}} < \infty, \int_{\Omega_{\mathbf{c,d}}} \|X_{\mathbf{c,d}}(\omega)\|^2 d\mathbf{P}_{\mathbf{c}} \times d\mathbf{P}_{\mathbf{d}} < \infty, \omega \in \Omega_{\mathbf{c,d}} \tag{4.5.13}$$

(v) $\mathbf{G}_{\mathbf{c,d}} : \mathbf{H}_{\mathbf{c,d}} \to \mathcal{L}_{2,1}^{c,\mathbf{d}}(\Omega_{\mathbf{d}})$ that is one to one correspondence such that

$$\langle\psi|\widehat{Q}_{\mathbf{c,d}}|\psi\rangle = \int_{\Omega_{\mathbf{c,d}}} \left(\mathbf{G}_{\mathbf{c,d}}\left[\widehat{Q}_{\mathbf{c,d}}|\psi\rangle\right](\omega)\right) d\mathbf{P}_{\mathbf{c}} \times d\mathbf{P}_{\mathbf{d}} = \mathbf{E}_{\Omega_{\mathbf{c,d}}}\left(\mathbf{G}_{\mathbf{c,d}}\left[\widehat{Q}_{\mathbf{c,d}}|\psi\rangle\right](\omega)\right) \tag{4.5.14}$$

IV.6.EPR-B paradox resolution

The usual conclusion of EPR-B experiment is to reject the non-local realism for two reasons: the impossibility of decomposing a pair of entangled atoms into two states, one for each atom, and the impossibility of interaction faster than the speed of light.

Remark.4.6.1. We find that the EPRB-paradox can be resolved by nonprincipal and convenient relaxing of the Einstein's locality principle. That is the "relaxed locality principle" introduced in subsection IV.1.

Remark 4.6.2. The solution to the entangled state is obtained by resolving the Pauli

equation from an initial singlet wave function with a spatial extension as [55]:

$$\Psi_0(\mathbf{r}_A, \mathbf{r}_B) = \frac{1}{\sqrt{2}} f(\mathbf{r}_A) f(\mathbf{r}_B)(|+_A\rangle \otimes |-_B\rangle - |-_A\rangle \otimes |+_B\rangle), \tag{4.6.1}$$

Remark 4.6.2. The initial wave function of the entangled state is the singlet state (4.6.1)

with

$$f(\mathbf{r}) \asymp \begin{cases} (2\pi\sigma_0^2)^{-\frac{1}{2}} e^{-\dfrac{x^2+y^2+z^2}{4\sigma_0^2}} \text{ iff } \|\mathbf{r}\| \leq \epsilon, \\ 0 \text{ iff } \|\mathbf{r}\| > \epsilon \end{cases} \tag{4.6.2}$$

$$\mathbf{r} = (x,y,z), \sigma_0 \ll 1, \epsilon \ll 1$$

and where $|\pm_A\rangle$ and $|\pm_B\rangle$ are the eigenvectors of the operators σ_{z_A} and σ_{z_B} :

$$\sigma_{z_A}|\pm_A\rangle = \pm|\pm_A\rangle, \sigma_{z_B}|\pm_B\rangle = \pm|\pm_B\rangle. \tag{4.6.3}$$

Remark 4.6.3. We treat the dependence with y strictly quasiclassically as in subsection **III.4**,i.e. with speed $-\mathbf{v}_{y\pm}^A(\mathbf{v}_0, \theta_0^A)$ for **A** and $\mathbf{v}_{y\pm}^B(\mathbf{v}_0, \theta_0^B)$ for **B** such that

$$
\begin{aligned}
&\mathbf{P}\left\{ \left| y + \mathbf{v}^{A}_{y\pm}(\mathbf{v}_0, \theta^{A}_0)t \right| \le \epsilon \right\} = 1, \\
&\mathbf{P}\left\{ \left| y + \mathbf{v}^{A}_{y\pm}(\mathbf{v}_0, \theta^{A}_0)t \right| > \epsilon \right\} = 0, \\
&\mathbf{P}\left\{ \left| y - \mathbf{v}^{B}_{y\pm}(\mathbf{v}_0, \theta^{B}_0)t \right| \le \epsilon \right\} = 1, \\
&\mathbf{P}\left\{ \left| y - \mathbf{v}^{B}_{y\pm}(\mathbf{v}_0, \theta^{B}_0)t \right| > \epsilon \right\} = 0, \\
&\epsilon \ll 1,
\end{aligned}
\tag{4.6.4}
$$

where

$$
\begin{aligned}
&\mathbf{v}^{A}_{y\pm}(\mathbf{v}_0, \theta^{A}_0) = \eta^{\pm}_{\theta^{A}_0}\mathbf{v}_0, \mathbf{v}^{B}_{y\pm}(\mathbf{v}_0, \theta^{B}_0) = \eta^{\pm}_{\theta^{B}_0}\mathbf{v}_0, \\
&\eta^{+}_{\theta^{A}_0} = \cos^2\frac{\theta^{A}_0}{2}, \eta^{-}_{\theta^{A}_0} = \sin^2\frac{\theta^{A}_0}{2}, \\
&\eta^{+}_{\theta^{B}_0} = \cos^2\frac{\theta^{B}_0}{2}, \eta^{-}_{\theta^{B}_0} = \sin^2\frac{\theta^{B}_0}{2}.
\end{aligned}
\tag{4.6.5}
$$

The wave function $\Psi(\mathbf{r}_A, \mathbf{r}_B, t)$ of the two identical particles **A** and **B**, electrically neutral and with magnetic moments μ_0, subject to magnetic fields $\mathbf{E_A}$ and $\mathbf{E_B}$, admits on the basis of $|\pm_A\rangle$ and $|\pm_B\rangle$ four components $\Psi^{a,b}(\mathbf{r}_A, \mathbf{r}_B, t)$ and satisfies the two-body Pauli equation

$$
i\hbar\frac{\partial\Psi^{a,b}(t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\Delta_A - \frac{\hbar^2}{2m}\Delta_B\right)\Psi^{a,b}(t) + \mu B^{\mathbf{E_A}}_j(\sigma_j)^a_c\Psi^{c,b}(t) + \mu B^{\mathbf{E_B}}_j(\sigma_j)^b_d\Psi^{a,d}(t) \tag{4.6.6}
$$

with the initial conditions:

$$
\Psi^{a,b}(0, \mathbf{r}_A, \mathbf{r}_B) = \Psi^{a,b}_0(\mathbf{r}_A, \mathbf{r}_B), \tag{4.6.7}
$$

where $\Psi^{a,b}_0(\mathbf{r}_A, \mathbf{r}_B)$ corresponds to the singlet state (4.6.1).To obtain an explicit solution of the EPR-B experiment, we take the numerical values of the Stern-Gerlach experiment, see subsection III.4

Below we explain the EPR-B experiment by using nonlocal two-body Pauli equation

$$
\begin{aligned}
&\int d\mathbf{r}_A d\mathbf{r}_B \int dt\Big[-i\hbar\frac{\partial\Psi^{\#a,b}(t, t', \mathbf{r}_A, \mathbf{r}_B)}{\partial t} + \left(-\frac{\hbar^2}{2m}\Delta_A - \frac{\hbar^2}{2m}\Delta_B\right)\Psi^{\#a,b}(t, t', \mathbf{r}_A, \mathbf{r}_B) \\
&+\mu B^{\mathbf{E_A}}_j(\sigma_j)^a_c\Psi^{\#c,b}(t, t', \mathbf{r}_A, \mathbf{r}_B) + \mu B^{\mathbf{E_B}}_j(\sigma_j)^b_d\Psi^{\#a,d}(t, t', \mathbf{r}_A, \mathbf{r}_B)\Big] = O(\hbar^\alpha), \\
&d\mathbf{r}_A = dx_A dy_A z_A, d\mathbf{r}_B = dx_B dy_B z_B
\end{aligned}
\tag{4.6.8}
$$

with a boundary condition

$$
\int d\mathbf{r}_A d\mathbf{r}_B z_A(t_1)|\Psi^{\#}(t_1, t', \mathbf{r}_A, \mathbf{r}_B)|^2 = -\int d\mathbf{r}_A d\mathbf{r}_B z_B(t_2)|\Psi^{\#}(t_2, t', \mathbf{r}_A, \mathbf{r}_B)|^2. \tag{4.6.9}
$$

One of the difficulties of the canonical interpretation of the EPR-B experiment is the existence of two simultaneous measurements.By doing these measurements one after the other, the interpretation of the experiment will be facilitated. That is the purpose of the two-step version of the experiment EPR-B studied below.

A. First step EPR-B: Spin measurement of A

Consider that at time t_0 the particle **A** arrives at the entrance of electromagnet $\mathbf{E_A}$.

Remark 4.6.5. We assume that a particle **A** collapses in a magnetic field $\mathbf{E_A}$ at some instant t' into two particles $\mathbf{A}_+$ and $\mathbf{A}_-$,i.e. the spinor $\Psi(z,y,t)$ collapses in a magnetic field $\mathbf{E_A}$ at some instant t' into two spinors $\Psi_+(z,y,t,t',\delta)$ and $\Psi_-(z,y,t,t',\delta)$ given by Eq.(3.4.9.a)-Eq.(3.4.9.b), see Assumption 3.4.1.

Remark 4.6.6. The particles $\mathbf{A}_+$ and $\mathbf{A}_-$ stays within the magnetic field for a time $\Delta t' \leq \Delta t = \frac{\Delta l}{\mathbf{v}_0}$.

Thus after exit of the magnetic field $\mathbf{E_A}$, at time $t_1 = t_0 + \triangle t + t$, the wave functions $\Psi_+(z,y,t_0 + \Delta t + t,\delta)$ and $\Psi_-(z,y,t_0 + \Delta t + t,\delta)$ becomes

$$\Psi_+(\mathbf{r}_{\mathbf{A}_+},\mathbf{r}_{\mathbf{B}_-},t_0 + \triangle t + t) = f(\mathbf{r}_{\mathbf{B}_-}) \times \left(f^+(\mathbf{r}_{\mathbf{A}_+},t)|+_{\mathbf{A}}\rangle \otimes |-_{\mathbf{B}}\rangle\right) \tag{4.6.10.a}$$

and

$$\Psi_-(\mathbf{r}_{\mathbf{A}_-},\mathbf{r}_{\mathbf{B}_+},t_0 + \triangle t + t) = f(\mathbf{r}_{\mathbf{B}_+}) \times \left(f^-(\mathbf{r}_{\mathbf{A}_-},t)|-_{\mathbf{A}}\rangle \otimes |+_{\mathbf{B}}\rangle\right) \tag{4.6.10.b}$$

respectively, with

$$\begin{aligned} f^+(\mathbf{r},t) &= \cos\frac{\theta_0}{2} f(x,z - z_\triangle - ut)\exp\left[i\left(\frac{muz}{\hbar} + \varphi^+(t)\right)\right] \\ f^-(\mathbf{r},t) &= \sin\frac{\theta_0}{2} f(x,z + z_\triangle + ut)\exp\left[i\left(-\frac{muz}{\hbar} + \varphi^-(t)\right)\right] \end{aligned} \tag{4.6.11}$$

where $z_\triangle$ and u are given by

$$z_\triangle = \frac{\mu_B B_0'(\Delta t)^2}{2m} = 10^{-5}m, \qquad u = \frac{\mu_B B_0'(\Delta t)}{m} = 1m/s. \tag{4.6.12}$$

Remark 4.6.7. We deduce that: the beam of particle **A** is divided into two $\mathbf{A}_+$ and $\mathbf{A}_-$, and the beam of particle **B** is divided into two $\mathbf{B}_+$ and $\mathbf{B}_-$.

Remark 4.6.8. Our first conclusion is: the position of $\mathbf{B}_+$ and $\mathbf{B}_-$ does not depend on the spin measurement of $\mathbf{A}_+$ and $\mathbf{A}_-$, only the spins are involved. We conclude from equation (4.6.10) that the spins of $\mathbf{A}_+$ and $\mathbf{B}_-$ ($\mathbf{A}_-$ and $\mathbf{B}_+$)remain opposite throughout the experiment. These are the two properties used in the relaxed causal interpretation.

Remark 4.6.9. By "relaxed locality principle" and decoherence it follows that the interection between $\mathbf{A}_+,\mathbf{A}_-$ $\mathbf{B}_+$,and $\mathbf{B}_-$ is absens, we assume the existance of wave functions

$$\Psi_0^{\mathbf{A}_+}(\mathbf{r}_{\mathbf{A}_+},\theta_0^{\mathbf{A}_+},\varphi_0^{\mathbf{A}_+}),\Psi_0^{\mathbf{A}_-}(\mathbf{r}_{\mathbf{A}_+},\theta_0^{\mathbf{A}_-},\varphi_0^{\mathbf{A}_-}),\Psi_0^{\mathbf{B}_+}(\mathbf{r}_{\mathbf{B}},\theta_0^{\mathbf{B}_+},\varphi_0^{\mathbf{B}_+}),\Psi_0^{\mathbf{B}_-}(\mathbf{r}_{\mathbf{B}},\theta_0^{\mathbf{B}_-},\varphi_0^{\mathbf{B}_-}). \tag{4.6.13}$$

B. Second step EPR-B: Spin measurement of B

The second step is a continuation of the first and corresponds to the EPR-B experiment broken down into two steps. On a pairs of particles $\mathbf{A}_+,\mathbf{B}_-$ and $\mathbf{A}_-$,$\mathbf{B}_+$ in a singlet state, first we made a Stern and Gerlach measurement on the $\mathbf{A}_+$and $\mathbf{A}_-$ atom at instant t_1 between t_0 and $t_0 + \triangle t + t_D$:

$$t_0 < t_1 < t_0 + \triangle t + t_D. \tag{4.6.14}$$

Secondly, we make a Stern and Gerlach measurement on the $\mathbf{B}_+$ and $\mathbf{B}_-$ atom with an electromagnet $\mathbf{E_B}$ forming an angle δ with $\mathbf{E_A}$ at instant t_2 between $t_0 + \triangle t + t_D$ and $t_0 + 2(\triangle t + t_D)$:

$$t_0 + \triangle t + t_D < t_2 \leq t_0 + 2(\triangle t + t_D) \tag{4.6.15}$$

At the exit of magnetic field $\mathbf{E_A}$, at time $t_0 + \triangle t + t_D$, the pair of particles wave functions is given by Eq.(4.6.10.a) and Eq.(4.6.10.b) respectively.Immediately after the measurements of **A**$_+$ and **A**$_-$, still at time $t_0 + \triangle t + t_D$, the wave functions of **B**$_+$ and **B**$_-$ depends on the measurements ± of **A** respectively such that:

$$\Psi_{\mathbf{B}_{-}/+\mathbf{A}}(\mathbf{r}_{\mathbf{B}_-}, t_0 + \triangle t + t_1) = f(\mathbf{r}_{\mathbf{B}_-})|-_{\mathbf{B}}\rangle, \tag{4.6.16.a}$$

and

$$\Psi_{\mathbf{B}_{+}/-\mathbf{A}}(\mathbf{r}_{\mathbf{B}_+}, t_0 + \triangle t + t_1) = f(\mathbf{r}_{\mathbf{B}_+})|+_{\mathbf{B}}\rangle. \tag{4.6.16.b}$$

Then, the measurement of **B**$_+$ and **B**$_-$ at time $t_2 > t_0 + 2(\triangle t + t_D)$ yields, in this two-step version of the EPR-B experiment, the same results for spatial quantization and correlations of spins as in the EPR-B experiment.

Resolution of the EPR-B experiment in de Broglie-Bohm interpretation by the "relaxed locality principle"

We assume, at the creation of the two entangled particles **A** and **B**, that each of the two particles **A** and **B** has an initial wave function with opposite spins:

$$\Psi_0^A(\mathbf{r}_A, \theta_0^A, \varphi_0^A) = f(\mathbf{r}_A)\left(\cos\frac{\theta_0^A}{2}|+_A\rangle + \sin\frac{\theta_0^A}{2}e^{i\varphi_0^A}|-_A\rangle\right) \tag{4.6.17}$$

and

$$\begin{aligned}\Psi_0^B(\mathbf{r}_B, \theta_0^B, \varphi_0^B) = f(\mathbf{r}_B)\left(\cos\frac{\theta_0^B}{2}|+_B\rangle + \sin\frac{\theta_0^B}{2}e^{i\varphi_0^B}|-_B\rangle\right) = \\ f(\mathbf{r}_B)\left[\cos\left(\frac{\pi}{2} - \frac{\theta_0^A}{2}\right)|+_B\rangle + \sin\left(\frac{\pi}{2} - \frac{\theta_0^A}{2}\right)e^{i(\varphi_0^A - \pi)}|-_B\rangle\right] = \\ \Psi_0^B(\mathbf{r}_B, \theta_0^B, \varphi_0^B) = f(\mathbf{r}_B)\left(\sin\frac{\theta_0^A}{2}|+_B\rangle + \cos\frac{\theta_0^A}{2}e^{i\varphi_0^A}|-_B\rangle\right)\end{aligned} \tag{4.6.18}$$

with $\theta_0^B = \pi - \theta_0^A$ and $\varphi_0^B = \varphi_0^A - \pi$, see Remark 4.6.4. The two particles **A** and **B** are statistically prepared as in the Stern and Gerlach experiment. Then the Pauli principle tells us that the two-body wave function must be antisymmetric; after calculation we find the same singlet state (4.6.1):

$$\Psi_0(\mathbf{r_A}, \theta^{\mathbf{A}}, \varphi^{\mathbf{A}}, \mathbf{r_B}, \theta^{\mathbf{B}}, \varphi^{\mathbf{B}}) = -e^{i\varphi^{\mathbf{A}}}f(\mathbf{r_A})f(\mathbf{r_B}) \times (|+_\mathbf{A}\rangle \otimes |-_\mathbf{B}\rangle - |-_\mathbf{A}\rangle \otimes |+_\mathbf{B}\rangle). \tag{4.6.19}$$

Thus, we can consider that the singlet wave function is the wave function of a family of two fermions **A** and **B** with opposite spins: the direction of initial spin **A** and **B** exists, but is not *known*. It is a local hidden variable which is therefore necessary to add in the initial conditions of the model.

Here, we assume that at the initial time we know the spin of each particle (given by each initial wave function) and the initial position of each particle.

Step 1: **spin measurement of A in de Broglie-Bohm interpretation**

In the equation (4.6.19) particle **A** can be considered independent of **B**. We can therefore give it the wave function

$$\Psi^{A}(\mathbf{r}_A, t_0 + \triangle t + t) = \cos\frac{\theta_0^A}{2} f^{+}(\mathbf{r}_A, t)|+_A\rangle + \sin\frac{\theta_0^A}{2} e^{i\varphi_0^A} f^{-}(\mathbf{r}_A, t)|-_A\rangle \tag{4.6.20}$$

which is the wave function of a free particle in a Stern Gerlach apparatus and whose initial spin is given by $(\theta_0^A, \varphi_0^A)$. For an initial polarization $(\theta_0^A, \varphi_0^A)$ and an initial position z_0^A, we obtain, in the de Broglie-Bohm interpretation [63] of the Stern and Gerlach experiment, an evolution of the position $z_{A_\pm}(t)$ and of the spin orientation of **A**$_\pm$,$\theta^{A_\pm}(z_{A_\pm}(t), t)$, see [65].

The case of particles **B**$_\pm$ is different. **B**$_\pm$ follows a rectilinear trajectories with $y_{B_\pm}(t) = \mathbf{v}_y^{\pm}(\mathbf{v}_0, \theta_0)t$, $z_{B_\pm}(t) = z_0^B$ and $x_{B_\pm}(t) = x_0^B$. By contrast, the orientation of its spin moves with the orientation of the spin of **A**$_\pm$:

$$\theta^{\mathbf{B}_\mp}(t) = \pi - \theta^{\mathbf{A}_\pm}(z_{\mathbf{A}_\pm}(t), t) \tag{4.6.21}$$

and

$$\varphi^{\mathbf{B}_\mp}(t) = \varphi^{\mathbf{A}_\pm}(z_{\mathbf{A}_\pm}(t), t) - \pi. \tag{4.6.22}$$

Remark 4.6.10. Let $\mathbf{A}_\pm(t, \mathbf{r}_{\mathbf{A}_\pm}(t), \theta^{\mathbf{A}_\pm}(z_{\mathbf{A}_\pm}(t), t), \varphi^{\mathbf{A}_\pm}(z_{\mathbf{A}_\pm}(t), t))$ denote events such that: "at instant t particle **A**$_\pm$ obtain the position coordinates $\mathbf{r}_{\mathbf{A}_\pm}(t) = \{x_{\mathbf{A}_\pm}(t), y_{\mathbf{A}_\pm}(t), z_{\mathbf{A}_\pm}(t)\}$ and spin orientation $\theta^{\mathbf{A}_\pm}(t) = \theta^{\mathbf{A}_\pm}(z_{\mathbf{A}_\pm}(t), t)$ and $\varphi^{\mathbf{A}_\pm}(t) = \varphi^{\mathbf{A}_\pm}(z_{\mathbf{A}_\pm}(t), t)$. Let $\mathbf{B}_\mp(t, \mathbf{r}_{\mathbf{B}_\mp}(t), \theta^{\mathbf{B}_\mp}(z_{\mathbf{B}_\mp}(t), t))$ denote events such that: "at instant t particle $\mathbf{B}_\mp$ obtain the position coordinates $\mathbf{r}_{\mathbf{B}_\mp}(t) = \{x_{\mathbf{B}_\mp}(t), y_{\mathbf{B}_\mp}(t), z_{\mathbf{B}_\mp}(t)\}$ and spin orientation $\theta^{\mathbf{B}_\mp} = \theta^{\mathbf{B}_\mp}(z_{\mathbf{B}_\mp}(t), t)$ and $\varphi^{\mathbf{B}_\mp}(z_{\mathbf{B}_\mp}(t), t)$. Then in accordance with the relaxed principle of locality (see subsection IV.1) we assume that

$$\begin{aligned}&\{\mathbf{A}_\pm(t_1, \mathbf{r}_{\mathbf{A}_\pm}(t), \theta^{\mathbf{A}_\pm}, \varphi^{\mathbf{A}_\pm}(t)), \mathbf{B}_\mp(t, \mathbf{r}_{\mathbf{B}_\mp}(t), \theta^{\mathbf{B}_\mp}(t)), \varphi^{\mathbf{B}_\mp}(t)\}_{\mathbf{s.l.s.}} \in \\ &\in [\mathcal{F}_{M_4}^{\#}, \{(t_1, \mathbf{r}_1), (t_2, \mathbf{r}_2)\}]_{\mathbf{s.l.s.}}^{\Leftrightarrow},\end{aligned} \tag{4.6.23}$$

see subsection IV.1, Definition 4.1.2. We can then associate the wave functions:

$$\Psi^{B_+}(\mathbf{r}_{B_+}, t_0 + \triangle t + t) = f(\mathbf{r}_{B_+}) \cos\frac{\theta^{B_+}(t)}{2}|+_B\rangle \tag{4.6.24}$$

and

$$\Psi^{B_-}(\mathbf{r}_{B_-}, t_0 + \triangle t + t) = f(\mathbf{r}_{B_-}) \sin\frac{\theta^{B_-}(t)}{2} e^{i\varphi^{B_-}(t)}|-_{B_-}\rangle \tag{4.6.25}$$

This wave functions is specific, because it depends upon initial conditions of **A** (position and spin). The orientation of spin of the particles **B**$_\pm$ is driven by the particles **A**$_\mp$ respectively through the singlet wave functions.

Step 2: Spin measurement of B$_\mp$ in de Broglie-Bohm interpretation

(I) The prediction of the result of the spin measurement of B$_\mp$ under assumption of canonical postulate of locality

At the time $t_0 + \Delta t + t_D$, immediately after the measurement of **A**,

$\theta^{B_{\mp}}(t_0 + \Delta t + t_D) = \pi$ or 0 in accordance with the value of $\theta^{A_{\pm}}(z_{A_{\pm}}(t), t)$ and the wave functions of **B**$_{\mp}$ is given by Eq.(4.6.16.a) and Eq.(4.6.16.b) respectively. The frame $(Ox'yz')$ corresponds to the frame $(Oxyz)$ after a rotation of an angle δ around the y-axis (see Fig.4.5.1). $\theta^{B_{\pm}}$ corresponds to the **B**$_{\mp}$-spin angle with the z-axis, and $\theta'^{B_{\pm}}$ to the **B**-spin angle with the z'-axis, then $\theta'^{B_{\pm}}(t_0 + \Delta t + t_D) = \pi + \delta$ or δ. In this second step, we are exactly in the case of a particle in a simple Stern and Gerlach experiment (with magnet $\mathbf{E_B}$) with a specific initial polarization equal to $\pi + \delta$ or δ and not random like in step 1. Then, the measurement of **B**, at time $t_0 + 2(\triangle t + t_D)$), gives, in this interpretation of the two-step version of the EPR-B experiment, the same results as in the EPR-B experiment above, III.4.1. Thus we obtain EPR-B paradox again in de Broglie-Bohm interpretation.

Remark 4.6.11.Note that derivation EPR-B paradox in de Broglie-Bohm interpretation completely based on canonical postulate of locality

Step 2: Spin measurement of B$_{\mp}$ in de Broglie-Bohm interpretation

(II) The prediction of the result of the spin measurement of B$_{\mp}$ under assumption of postulate of nonlocality

We assume now a weak or strong postulate of nonlocality,see subsection I.9. At the time $t_1 = t_0 + \Delta t + t_D$, immediately after the spin measurement of **A**$_{\pm}$, $\theta^{B_{\mp}}(t_0 + \Delta t + t_D) = \pi$ or 0 in accordance with the value of $\theta^{A_{\pm}}(z_{A_{\pm}}(t), t)$ and the wave functions of **B**$_{\mp}$ is given by Eq.(4.6.16.a) and Eq.(4.6.16.b) respectively.

Remark 4.6.12.In accordance with postulate of nonlocality it follows:

(**i**) Whenever a measurement of the spin of a particle **A**$_{+}$ is performed at instant t_1 and

particle **A**$_{+}$ is found in the state $|\uparrow\rangle_z$, i.e., a state $|\psi_{t_1}\rangle_{\mathbf{A}_+}$ collapses at instant t_1 to the state

$|\uparrow\rangle_{z,\mathbf{A}_+}$ with respect of the Heisenberg spin uncertainty relations (1.9.5), then a state $|\psi_{t_1}\rangle_{\mathbf{B}_-}$

immediately collapses at instant t_1 to the state $|\downarrow\rangle_{z,\mathbf{B}_-}$ with respect of the Heisenberg spin

uncertainty relations (1.9.5), and this is true independent of the distance in Minkovski

spacetime that separates the particles,e.g.,

$$|\psi_{t_1}\rangle_{\mathbf{A}_+} \overset{\text{col}\,\textbf{lapse}}{\rightarrow} |\uparrow\rangle_{z,\mathbf{A}_+} \Rightarrow |\psi_{t_1}\rangle_{\mathbf{B}_-} \overset{\text{col}\,\textbf{lapse}}{\rightarrow} |\downarrow\rangle_{z,\mathbf{B}_-} \tag{4.6.26}$$

In accordance with Heisenberg spin uncertainty relations (1.9.5) spin of a particle $\mathbf{B}_-$ obtain

an uncertainty along direction Oz' (see Fig.4.5.1) and therefore EPR-B paradox disappears.

(**ii**) Whenever a measurement of the spin of a particle **A** is performed at instant

t_1 and

particle $\mathbf{A}_-$ is found in the state $|\downarrow\rangle_z$, i.e., a state $|\psi_{t_1}\rangle_{\mathbf{A}_-}$ collapses at instant t_1 to the state

$|\downarrow\rangle_{z,\mathbf{A}_-}$ with respect of the Heisenberg spin uncertainty relations (1.9.5), then a state $|\psi_{t_1}\rangle_{\mathbf{B}_+}$

immediately collapses at instant t_1 to the state $|\uparrow\rangle_{z,\mathbf{B}_+}$ with respect of the Heisenberg

spin uncertainty relations (1.9.5), and this is true independent of the distance in Minkovski

spacetime that separates the particles,e.g.,

$$|\psi_{t_1}\rangle_{\mathbf{A}_-} \overset{\text{collapse}}{\rightarrow} |\downarrow\rangle_{z,\mathbf{A}_-} \Rightarrow |\psi_{t_1}\rangle_{\mathbf{B}_+} \overset{\text{collapse}}{\rightarrow} |\uparrow\rangle_{z,\mathbf{B}_+}. \tag{4.6.27}$$

In accordance with Heisenberg spin uncertainty relations (1.9.5) spin of a particle $\mathbf{B}_-$ obtain

an uncertainty along direction Oz' (see Fig.4.5.1) and therefore EPR-B paradox disappears.

Physical explanation of non-local influences using the relaxed principle of locality

From the wave function of two entangled particles, we find spins, trajectories and also a wave function for each of the two particles. In this interpretation, the quantum particle has a local position like a classical particle, but it has also a non-local behavior through the wave function. So, it is the wave function that creates the non classical properties. We can keep a view of a local realist world for the particle, but we should add a non-local vision through the wave function. As we saw in step 1, the non-local influences in the EPR-B experiment only concern the spin orientation, not the motion of the particles themselves. Indeed only spins are entangled in the wave function but not positions and motions like in the initial EPR experiment. This is a key point in the search for a physical explanation of non-local influences.

The simplest explanation of this non-local influence given above by using the relaxed principle of locality (see subsection IV.1)

Conclusion: A new quantum mechanical formalism based on the probability representation of quantum states is proposed. This paper in particular deals with the special case of the measurement problem, known as Schrödinger’s cat paradox. We pointed out that Schrödinger’s cat demands to reconcile Born’s rule. Using new quantum mechanical formalism we find the collapsed state of the Schrödinger’s cat always shows definite and predictable outcomes even if cat also consists of a superposition

$$|\text{cat}\rangle = c_1|\text{live cat}\rangle + c_2|\text{death cat}\rangle$$
$$|c_1|^2 + |c_2|^2 = 1.$$

Using new quantum mechanical formalism the EPRB-paradox is considered successfully. We find that the EPRB-paradox can be resolved by nonprincipal and convenient relaxing of the Einstein’s locality principle.

Appendix. **A**.

The time-dependent Schrödinger equation governs the time evolution of a quantum mechanical system is:

$$i\hbar\frac{\partial\Psi(\mathbf{x},t)}{\partial t} = \widehat{\mathbf{H}}\Psi(\mathbf{x},t). \qquad (A.1)$$

The average, or expectation, value $\langle x_i\rangle$ of an observable x_i corresponding to a quantum mechanical operator $\widehat{x}_i$ is given by:

$$\langle x_i\rangle(t,\mathbf{x}_0,t_0;\hbar) = \frac{\int\limits_{\mathbb{R}^d} x_i|\Psi(\mathbf{x},t,\mathbf{x}_0,t_0;\hbar)|^2 d^dx}{\int\limits_{\mathbb{R}^d}|\Psi(\mathbf{x},t,\mathbf{x}_0,t_0;\hbar)|^2 d^dx}. \qquad (A.2)$$

$$i = 1,\ldots,d.$$

Remark A.1. We assume now that: the solution $\Psi(\mathbf{x},t,\mathbf{x}_0,t_0;\hbar)$ of the time-dependent Schrödinger equation (A.1) has a good approximation by a delta function such that

$$|\Psi(\mathbf{x},t,\mathbf{x}_0,t_0;\hbar)|^2 \simeq \prod_{i=1}^{d}\delta(x_i - x_i(t,\mathbf{x}_0,t_0)),$$
$$x_i(t,\mathbf{x}_0,t_0) = x_{i,0}, \qquad (A.3)$$
$$i = 1,\ldots,d.$$

Remark A.2. Note that under conditions given by Eq.(A.3) QM-system which governed by Schrödinger equation Eq.(A.1) completely evolve quasiclassically i.e. estimating the position $\{x_i(t,\mathbf{x}_0,t_0;\hbar)\}_{i=1}^{d}$ at each instant t with final error δ gives $|\langle x_i\rangle(t,\mathbf{x}_0,t_0;\hbar) - x_i(t,\mathbf{x}_0,t_0)| \leq \delta, i = 1,\ldots,d$ with a probability

$$\mathbf{P}\{|\langle x_i\rangle(t,\mathbf{x}_0,t_0;\hbar) - x_i(t,\mathbf{x}_0,t_0)| \leq \delta\} \simeq 1.$$

Thus from Eq.(A.2) and Eq.(A.3) we obtain

$$\langle x_i\rangle(t,\mathbf{x}_0,t_0;\hbar) \simeq$$

$$\simeq \frac{\int\limits_{\mathbb{R}^d} x_i \prod\limits_{i=1}^{d=1} \delta(x_i - x_i(t,\mathbf{x}_0,t_0))d^d x}{\int\limits_{\mathbb{R}^d} \prod\limits_{i=1}^{d=1} \delta(x_i - x_i(t,\mathbf{x}_0,t_0))d^d x} = x_i(t,\mathbf{x}_0,t_0). \tag{A.4}$$

$$i = 1,\ldots,d.$$

Thus under condition given by Eq.(A.3) one obtain

$$\langle x_{i,t}\rangle(t,\mathbf{x}_0,t_0;\hbar) \simeq x_i(t,\mathbf{x}_0,t_0), \tag{A.5}$$

$$i = 1,\ldots,d.$$

Remark A.3.Let $\Psi_i(\mathbf{x},t,\mathbf{x}_0,t_0), i = 1,2$ be the solutions of the time-dependent Schrödinger equation (A.1). We assume now that $\Phi(\mathbf{x},t,\mathbf{x}_0,\mathbf{y}_0,t_0)$ is a linear superposition such that

$$\Phi(\mathbf{x},t,\mathbf{x}_0,\mathbf{y}_0,t_0) = c_1\Psi_1(\mathbf{x},t,\mathbf{x}_0,t_0) + c_2\Psi_2(\mathbf{x},t,\mathbf{y}_0,t_0). \tag{A.6}$$

$$|c_1|^2 + |c_2|^2 = 1.$$

Then we obtain

$$|\Phi(\mathbf{x},t,\mathbf{x}_0,\mathbf{y}_0,t_0)|^2 = (\Phi(\mathbf{x},t,\mathbf{x}_0,\mathbf{y}_0,t_0)\Phi^*(\mathbf{x},t,\mathbf{x}_0,\mathbf{y}_0,t_0)) =$$

$$= ([c_1\Psi_1(\mathbf{x},t,\mathbf{x}_0,t_0) + c_2\Psi_2(\mathbf{x},t,\mathbf{y}_0,t_0)]) \times$$

$$\times([c_1^*\Psi_1^*(\mathbf{x},t,\mathbf{x}_0,t_0) + c_2^*\Psi_2^*(\mathbf{x},t,\mathbf{x}_0,\mathbf{y}_0,t_0)]) = \tag{A.7}$$

$$= |c_1|^2\left(|\Psi_1(\mathbf{x},t,\mathbf{x}_0,t_0)|^2\right) + c_1^*c_2(\Psi_1^*(\mathbf{x},t,\mathbf{x}_0)\Psi_2(\mathbf{x},t,\mathbf{y}_0,t_0)) +$$

$$|c_2|^2\left(|\Psi_2(\mathbf{x},t,\mathbf{y}_0,t_0)|^2\right) + c_1c_2^*(\Psi_1(\mathbf{x},t,\mathbf{x}_0)\Psi_2^*(\mathbf{x},t,\mathbf{y}_0,t_0)).$$

Definition A.1. Let $\langle\mathbf{x}\rangle(t,\mathbf{x}_0,\mathbf{y}_0,t_0)$ be a vector-function

$$\langle\mathbf{x}\rangle(t,\mathbf{x}_0,\mathbf{y}_0,t_0) : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times[0,T]\to\mathbb{R}^d$$

$$\langle\mathbf{x}\rangle(t,\mathbf{x}_0,\mathbf{y}_0,t_0) = \{\langle x_1\rangle(t,\mathbf{x}_0,\mathbf{y}_0,t_0),\ldots,\langle x_d\rangle(t,\mathbf{x}_0,\mathbf{y}_0,t_0)\}, \qquad (A.8)$$

where

$$\langle x_i\rangle(t,\mathbf{x}_0,\mathbf{y}_0,t_0) = \int\limits_{\mathbb{R}^d} x_i|\Phi(\mathbf{x},t,\mathbf{x}_0,\mathbf{y}_0,t_0)|^2 d^dx =$$

$$= |c_1|^2\int\limits_{\mathbb{R}^d} x_i|\Psi_1(\mathbf{x},t,\mathbf{x}_0,t_0)|^2 d^dx +$$

$$+|c_2|^2\int\limits_{\mathbb{R}^d} x_i|\Psi_2(\mathbf{x},t,\mathbf{y}_0,t_0)|^2 d^dx + \qquad (A.9)$$

$$+c_1^*c_2\int\limits_{\mathbb{R}^d} x_i\Psi_1^*(\mathbf{x},t,\mathbf{x}_0,t_0)\Psi_2(\mathbf{x},t,\mathbf{y}_0,t_0)d^dx +$$

$$+c_1c_2^*\int\limits_{\mathbb{R}^d} x_i\Psi_1(\mathbf{x},t,\mathbf{x}_0,t_0)\Psi_2^*(\mathbf{x},t,\mathbf{y}_0,t_0)d^dx.$$

Definition A.2. Let $\Delta(t,\mathbf{x}_0,\mathbf{y}_0,t_0)$ be a vector-function

$$\Delta(t,\mathbf{x}_0,\mathbf{y}_0,t_0) : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$$

$$(\Delta(t,\mathbf{x}_0,\mathbf{y}_0,t_0)) = \{\delta_1(t,\mathbf{x}_0,\mathbf{y}_0,t_0),\ldots,\delta_d(t,\mathbf{x}_0,\mathbf{y}_0,t_0)\}, \qquad (A.10)$$

where

$$\delta_i(t,\mathbf{x}_0,\mathbf{y}_0,t_0) = \delta[x_i(t,\mathbf{x}_0,\mathbf{y}_0,t_0)] =$$

$$= c_1^* c_2 \int\limits_{\mathbb{R}^d} x_i \Psi_1^*(\mathbf{x},t,\mathbf{x}_0,t_0)\Psi_2(\mathbf{x},t,\mathbf{y}_0,t_0)d^dx + \qquad (A.11)$$

$$+ c_1 c_2^* \int\limits_{\mathbb{R}^d} x_i \Psi_1(\mathbf{x},t,\mathbf{x}_0,t_0)\Psi_2^*(\mathbf{x},t,\mathbf{y}_0,t_0)d^dx.$$

Substituting Eqs.(A.11) into Eqs.(A.9) gives

$$\langle x_i\rangle(t,\mathbf{x}_0,\mathbf{y}_0,t_0) = \int\limits_{\mathbb{R}^d} x_i|\Phi(\mathbf{x},t,\mathbf{x}_0,\mathbf{y}_0,t_0)|^2 d^dx =$$

$$= |c_1|^2 \int\limits_{\mathbb{R}^d} x_i|\Psi_1(\mathbf{x},t,\mathbf{x}_0,t_0)|^2 d^dx + \qquad (A.12)$$

$$+ |c_2|^2 \int\limits_{\mathbb{R}^d} x_i|\Psi_2(\mathbf{x},t,\mathbf{y}_0,t_0)|^2 d^dx + \delta_i(t,\mathbf{x}_0,\mathbf{y}_0,t_0) =$$

$$= |c_1|^2\langle x_i\rangle(t,\mathbf{x}_0,t_0) + |c_2|^2\langle x_i\rangle(t,\mathbf{y}_0,t_0) + \delta_i(t,\mathbf{x}_0,\mathbf{y}_0,t_0).$$

Substitution equations (A.5) into equations (A.12) gives

$$\langle x_i\rangle(t,\mathbf{x}_0,\mathbf{y}_0,t_0) = \int\limits_{\mathbb{R}^d} x_i|\Phi(\mathbf{x},t,\mathbf{x}_0,\mathbf{y}_0,t_0)|^2 d^dx =$$

$$= |c_1|^2\langle x_i\rangle(t,\mathbf{x}_0,t_0) + |c_2|^2\langle x_i\rangle(t,\mathbf{y}_0,t_0) + \delta_i(t,\mathbf{x}_0,\mathbf{y}_0,t_0) \qquad (A.13)$$

$$\simeq |c_1|^2 x_i(t,\mathbf{x}_0,t_0) + |c_2|^2 x_i(t,\mathbf{y}_0,t_0) + \delta_i(t,\mathbf{x}_0,\mathbf{y}_0,t_0).$$

Appendix. B.

The Schrödinger equation (2.1) in region $\mathbf{I} = \{x|x < 0\}$ has the folloving form

$$\hbar^2 \frac{\partial^2 \Psi_{\mathbf{I}}(x)}{\partial x^2} + 2mE\Psi_{\mathbf{I}}(x) = 0. \qquad (B.1)$$

From Schrödinger equation (B.1) follows

$$\hbar^2 \int_{-\infty}^{0} \frac{\partial^2 \Psi_{\mathbf{I}}(x)}{\partial x^2} dx + 2mE \int_{-\infty}^{0} \Psi_{\mathbf{I}}(x) dx = 0. \qquad (B.2)$$

Let $\Psi_{\mathbf{I}}^{\#}(x)$ be a function

$$\Psi_{\mathbf{I}}^{\#}(x) = \phi(x)\Psi_{\mathbf{I}}(x), \qquad (B.3)$$

where

$$\phi(x) = (\pi r_c^2)^{-1/4} \exp\left(-\frac{x^2}{2r_c^2}\right) \qquad (B.4)$$

see Eq.(2.9). Note that

$$\frac{\partial^2[\phi(x)\Psi_{\mathbf{I}}(x)]}{\partial x^2} = \frac{\partial}{\partial x}\left[\Psi_{\mathbf{I}}(x)\frac{\partial \phi(x)}{\partial x} + \phi(x)\frac{\partial \Psi_{\mathbf{I}}(x)}{\partial x}\right] =$$

$$2\frac{\partial \Psi_{\mathbf{I}}(x)}{\partial x}\frac{\partial \phi(x)}{\partial x} + \Psi_{\mathbf{I}}(x)\frac{\partial^2 \phi(x)}{\partial x^2} + \phi(x)\frac{\partial^2 \Psi_{\mathbf{I}}(x)}{\partial x^2}. \qquad (B.5)$$

Therefore substitution (B.2) into LHS of the Schrödinger equation (B.1) gives

$$\hbar^2 \int\limits_{-\infty}^{0} \frac{\partial^2 \Psi_{\mathbf{I}}^{\#}(x)}{\partial x^2} dx + 2mE \int\limits_{-\infty}^{0} \Psi_{\mathbf{I}}^{\#}(x) dx =$$

$$\hbar^2 \int\limits_{-\infty}^{0} \frac{\partial^2 \phi(x) \Psi_{\mathbf{I}}(x)}{\partial x^2} dx + 2Em \int\limits_{-\infty}^{0} \phi(x) \Psi_{\mathbf{I}}(x) dx =$$

$$(B.6)$$

$$2\hbar^2 \int\limits_{-\infty}^{0} \frac{\partial \Psi_{\mathbf{I}}(x)}{\partial x} \frac{\partial \phi(x)}{\partial x} dx + \hbar^2 \int\limits_{-\infty}^{0} \Psi_{\mathbf{I}}(x) \frac{\partial^2 \phi(x)}{\partial x^2} dx +$$

$$+ \int\limits_{-\infty}^{0} \phi(x) \left\{ \hbar^2 \frac{\partial^2 \Psi_{\mathbf{I}}(x)}{\partial x^2} + 2Em \int\limits_{-\infty}^{0} \Psi_{\mathbf{I}}(x) \right\} dx.$$

Note that

$$\int\limits_{-\infty}^{0} \phi(x) \left\{ \hbar^2 \frac{\partial^2 \Psi_{\mathbf{I}}(x)}{\partial x^2} + 2Em \int\limits_{-\infty}^{0} \Psi_{\mathbf{I}}(x) \right\} dx = 0. \qquad (B.7)$$

Therefore from Eq.(B.6) and Eq.(2.3)-Eq.(2.4) one obtains

$$\hbar^2 \int\limits_{-\infty}^{0} \frac{\partial^2 \Psi_{\mathbf{I}}^{\#}(x)}{\partial x^2} dx + 2mE \int\limits_{-\infty}^{0} \Psi_{\mathbf{I}}^{\#}(x) dx =$$

$$\hbar^2 \int\limits_{-\infty}^{0} \frac{\partial^2 \phi(x) \Psi_{\mathbf{I}}(x)}{\partial x^2} dx + 2Em \int\limits_{-\infty}^{0} \phi(x) \Psi_{\mathbf{I}}(x) dx = \qquad (B.8)$$

$$= 2\hbar^2 \int\limits_{l}^{\infty} \frac{\partial \Psi_{\mathbf{I}}(x)}{\partial x} \frac{\partial \phi(x)}{\partial x} dx + \hbar^2 \int\limits_{l}^{\infty} \Psi_{\mathbf{I}}(x) \frac{\partial^2 \phi(x)}{\partial x^2} dx.$$

From Eq.(B.6) one obtains

$$\frac{\partial\phi(x)}{\partial x} = (\pi r_c^2)^{-1/4}\frac{\partial}{\partial x}\exp\left[-\frac{x^2}{2r_c^2}\right] = -(\pi r_c^2)^{-1/4} r_c^{-2} x \exp\left[-\frac{x^2}{2r_c^2}\right],$$

$$\frac{\partial^2\phi(x)}{\partial x^2} = -(\pi r_c^2)^{-1/4} r_c^{-2}\exp\left[-\frac{x^2}{2r_c^2}\right] + \qquad (B.9)$$

$$+(\pi r_c^2)^{-1/4} r_c^{-4} x^2 \exp\left[-\frac{x^2}{2r_c^2}\right].$$

From Eq.(B.9) and Eq.(2.3)-Eq.(2.4) one obtains

$$\hbar^2\int_{-\infty}^{0}\frac{\partial\Psi_{\mathbf{I}}(x)}{\partial x}\frac{\partial\phi(x)}{\partial x}dx =$$

$$-\frac{\hbar^2}{(\pi r_c^2)^{1/4} r_c^2}\int_{-\infty}^{0}\frac{\partial\exp(ikx)}{\partial x}x\exp\left[-\frac{x^2}{2r_c^2}\right]dx = \qquad (B.10)$$

$$-\frac{2\pi\sqrt{2mE}\,\hbar}{(\pi r_c^2)^{1/4} r_c^2}\int_{-\infty}^{0}x\exp\left(i\frac{2\pi\sqrt{2mE}}{\hbar}x\right)\exp\left[-\frac{x^2}{2r_c^2}\right]dx,$$

$$k = \frac{2\pi}{\hbar}\sqrt{2mE}.$$

and

$$\hbar^2\int_{-\infty}^{0}\Psi_{\mathbf{I}}(x)\frac{\partial^2\phi(x)}{\partial x^2}dx = -\frac{\hbar^2}{(\pi r_c^2)^{3/4} r_c^2}\int_{-\infty}^{0}\exp(ikx)\exp\left[-\frac{x^2}{2r_c^2}\right]dx + \qquad (B.11)$$

$$+\frac{\hbar^2}{(\pi r_c^2)^{1/4} r_c^2}\int_{-\infty}^{0}x^2\exp(ikx)\exp\left[-\frac{x^2}{2r_c^2}\right]dx.$$

Appendix E. Calculating the spinor evolution in the Stern-Gerlach experiment

In the magnetic field $B = (B_x, 0, B_z)$, the Pauli equation gives coupled Schrödinger equations for each spinor component

$$i\hbar\frac{\partial\psi_\pm}{\partial t}(x,y,z,t) = -\frac{\hbar^2}{2m}\nabla^2\psi_\pm(x,y,z,t) \pm \mu_B(B_0 - B_0'z)\psi_\pm(x,y,z,t) \mp i\mu_B B_0' x\psi_\mp(x,y,z,t). \qquad (E.1)$$

Under transformation

$$\psi_\pm(x,z,t) = \exp\left(\pm\frac{i\mu_B B_0 t}{\hbar}\right)\overline{\psi}_\pm(x,y,z,t) \qquad (E.2)$$

equation (E.1) becomes

$$\begin{aligned} i\hbar\frac{\partial\overline{\psi}_\pm}{\partial t}(x,y,z,t) &= -\frac{\hbar^2}{2m}\nabla^2\overline{\psi}_\pm(x,y,z,t) \mp \mu_B B_0' z\overline{\psi}_\pm(x,y,z,t) \\ &\mp i\mu_B B_0' x\overline{\psi}_\mp(x,y,z,t)\exp\left(\pm i\frac{2\mu_B B_0 t}{\hbar}\right) \end{aligned} \qquad (E.3)$$

The coupling term oscillates rapidly with the Larmor frequency $\omega_L = \frac{2\mu_B B_0}{\hbar} = 1,4\times 10^{11}s^{-1}$. Since $|B_0| \gg |B_0'z|$ and $|B_0| \gg |B_0'x|$, the period of oscillation is short compared to the motion of the wave function. Averaging over a period that is long compared to the oscillation period, the coupling term vanishes, which entails

$$i\hbar\frac{\partial\overline{\psi}_\pm}{\partial t}(x,y,z,t) = -\frac{\hbar^2}{2m}\nabla^2\overline{\psi}_\pm(x,y,z,t) \mp \mu_B B_0' z\overline{\psi}_\pm(x,y,z,t). \qquad (E.4)$$

The initial wave function $\overline{\psi}^0_\pm(x,y,z) = \psi^0_\pm(x,y,z) = \psi^0_x(x)\psi^0_y(y)\psi^0_\pm(z)$ with

$$\begin{aligned}
\psi^0_x(x) &= \begin{cases} (2\pi\sigma_0^2)^{-\frac{1}{4}}e^{-\frac{x^2}{4\sigma_0^2}} \text{ if } |x| \le \delta \\ 0 \text{ if } |x| > \delta \end{cases} \\
\psi^0_y(y) &= \begin{cases} (2\pi\sigma_0^2)^{-\frac{1}{4}}e^{-\frac{y^2}{4\sigma_0^2}} \text{ if } |x| \le \delta \\ 0 \text{ if } |x| > \delta \end{cases} \\
\psi^0_+(z) &= \begin{cases} (2\pi\sigma_0^2)^{-\frac{1}{4}}e^{-\frac{z^2}{4\sigma_0^2}}\cos\frac{\theta_0}{2}e^{i\frac{\varphi_0}{2}} \text{ if } |x| \le \delta \\ 0 \text{ if } |x| > \delta \end{cases} \\
\psi^0_-(z) &= \begin{cases} (2\pi\sigma_0^2)^{-\frac{1}{2}}e^{-\frac{z^2}{4\sigma_0^2}}i\sin\frac{\theta_0}{2}e^{-i\frac{\varphi_0}{2}} \text{ if } |x| \le \delta \\ 0 \text{ if } |x| > \delta \end{cases}
\end{aligned} \qquad (E.5)$$

allows a separation of variables x,y and z. Then we can compute explicitly the preceding equations for $t \in [0,\Delta t]$. We obtain: $\overline{\psi}_\pm(x,y,z,t) = \overline{\psi}_x(x,t)\overline{\psi}_y(y,t)\overline{\psi}_\pm(z,t)$ with

$$
\begin{aligned}
\overline{\psi}_x(x,t) &= (2\pi\sigma_t^2)^{-\frac{1}{4}} e^{-\frac{x^2}{4\sigma_t^2}} \exp\left\{ \frac{i}{\hbar} \left[-\frac{\hbar}{2} \tan^{-1}\left(\frac{\hbar t}{2m\sigma_0^2} \right) + \frac{x^2\hbar^2 t^2}{8m\sigma_0^2\sigma_t^2} \right] \right\}, \\
\overline{\psi}_+(z,t) &= \psi_K(z,t) \cos\frac{\theta_0}{2} e^{i\frac{\varphi_0}{2}} \quad \text{and } K = -\mu_B B_0', \\
\overline{\psi}_-(z,t) &= \psi_K(z,t) i \sin\frac{\theta_0}{2} e^{-i\frac{\varphi_0}{2}} \quad \text{and } K = +\mu_B B_0', \\
\sigma_t^2 &= \sigma_0^2 + \left(\frac{\hbar t}{2m\sigma_0} \right)^2
\end{aligned} \qquad (E.6)
$$

and

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