Quaternionic formulation in symmetry breaking mechanism

Gaurav Karnatak*

May 1, 2016

*Department of Physics, Kumaun University, S. S. J. Campus, Almora-263601 (Uttarakhand) India
Email - gauravkarnatak2009@yahoo.in

Abstract

In this formalism the covariant derivative contains the four potentials associated with four charges and thus leads the different gauge strength for the particles containing electric, magnetic, gravitational and Heavisidian charges. Quaternions representation in spontaneously symmetry of breaking and Higg's mechanics and the equation of motion are derived for free particles (i.e. electric, magnetic, gravitational and Heavisidian charges). The local gauge invariance in order to explain the Yang-Mill's field equation and spontaneous symmetry breaking mechanism. The quaternionic gauge theory of quantum electrodynamics has also been developed in presence of electric, magnetic, gravitational and Heavisidian charges.

Keywords: dyons, quaternions, symmetry breaking, gauge fields

1 Introduction

The asymmetry between electricity and magnetism became very clear at the end of 19^{th} century with the formulation of Maxwell's equations. Magnetic monopoles were advocated [1, 2] to symmetrize these equations in a manifest way that the mere existence of an isolated magnetic charge implies the quantization of electric charge and accordingly the considerable literature [3]-[14] has come in force. The fresh interests in this subject have been enhanced by 't Hooft [15] and Polyakov [16] with the idea that the classical solutions having the properties of magnetic monopoles may be found in Yang-Mill's gauge theories. Now it has become clear that monopoles are better understood in grand unified theories and supersymmetric gauge theories. Julia and Zee [17] extended the ' t Hooft - Polyakov theory [15, 16] of monopoles and constructed the theory of non -Abelian dyons (particles carrying simultaneously electric and magnetic charges). The quantum mechanical excitation of fundamental monopoles include dyons which are automatically arisen [18] from the semi-classical quantization of global charge rotation degree of freedom of monopoles. In view of the explanation of CP-violation in terms of non-zero vacuum angle of world [19], the monopoles are necessary dyons and Dirac quantization condition permits dyons to have analogous electric charge. Renewed interests in the subject of monopole has gathered enormous potential importance in connection of quark confinement problem [20] in quantum chromo dynamics, possible magnetic condensation [21, 22] of vacuum leading to absolute color confinement in QCD, its role as catalyst in proton decay [23, 24], CP-violation [19], current grand unified theories [25] and supersymmetric gauge theories [26, 27, 28, 29]. There has been a revival in the formulation of natural laws within the frame work of general quaternion algebra and basic physical equations. Quaternions [30] were very first example of hyper complex numbers having the significant impacts on Mathematics and Physics. Moreover, quaternions are already used in the context of special relativity [31], electrodynamics [32, 33], Maxwell's equation [34], quantum mechanics [35, 36], gauge theories [37, 38], supersymmetry [39, 40] and other branches of Physics [41] and Mathematics [42]. Symmetry plays the central role in determining its dynamical structure. The Lagrangian exhibits invariance under $SU(2)\times U(1)$ gauge transformations for the electroweak interactions. Since the imposition of local symmetry implies the existence of massless vector particles [43], Higg's mechanism is used for the spontaneous breaking of gauge symmetry to generate masses for the weak gauge bosons charged as well as neutral particle [44]. If these features of the gauge theory are avoided, we obtain massive vector bosons and hence the gauge symmetry must be broken. In the Higg's mechanism a larger symmetry is spontaneously broken into a smaller symmetry through the vacuum expectation value of the Higg's field and accordingly gauge bosons become massive. The simplest way of introducing spontaneous symmetry breakdown is to include scalar Higg's fields by hand into the Lagrangian [45]. Recently, we [46] have made an attempt to develop the quaternionic formulation of Yang-Mill's field equations and octonion reformulation of quantum chromo dynamics (QCD) by taking magnetic monopole into account [47]. The quaternion gauge theory of spontaneously symmetry breaking mechanism already developed by others [48, 49, 50, 51, 52] in terms of gauge groups and methodology adopted by them in different manners.In the previous paper we have already develop [53] a meaningful gauge theory which may purport a model for massive gauge particles. Starting with the definition of quaternion gauge theory, we have undertaken the study of $SU(2)_e\times SU(2)_m\times U(1)_e\times U(1)_m$ in terms of the simultaneous existence of electric and magnetic charges along with their Yang-Mill's counterparts[54, 55, 56, 57]. As such, we have developed the gauge theory in terms of four coupling constants associated with four-gauge symmetry $SU(2)_e\times SU(2)_m\times U(1)_e\times U(1)_m$. Accordingly, we have made an attempt to obtain the Abelian and non-Abelian gauge structures for the particles carrying simultaneously the electric and magnetic charges (namely dyons). In this paper the covariant derivative contains the four-potentials associated with these four charges and thus leads the different gauge strength for the particles containing electric, magnetic, gravitational and Heavisidian charges. Quaternions representation in spontaneously symmetry of breaking and Higg's mechanics and the equation of motion are derived for free particles (i.e. electric, magnetic, gravitational and Heavisidian charges). We have extended the local gauge invariance in order to explain spontaneous symmetry breaking mechanism. The quaternionic gauge theory of quantum electrodynamics has also been developed in presence of electric, magnetic, gravitational and Heavisidian charges.

2 Quaternion Preliminaries

The algebra $\mathbb{H}$ of quaternion is a four - dimensional algebra over the field of real numbers

$\mathbb{R}$ and a quaternion ϕ is expressed in terms of its four base elements as [55]

$$\phi=\phi_{\mu}e_{\mu}=\phi_{0}+e_{1}\phi_{1}+e_{2}\phi_{2}+e_{3}\phi_{3}\ \ (\mu=0,1,2,3), \tag{1}$$

where ϕ_0,ϕ_1,ϕ_2,ϕ_3 are the real quartets of a quaternion and e_0, e_1, e_2, e_3 are called quaternion units and satisfies the following relations,

$$\begin{aligned} e_{0}^{2} &= e_{0} = 1\ , e_{j}^{2}=-e_{0}, \\ e_{0}\,e_{i}=e_{i}\,e_{0} &= e_{i}(i=1,2,3)\ , \\ e_{i}\,e_{j} &= -\delta_{ij}+\varepsilon_{ijk}\,e_{k}(\forall\,i,j,k=1,2,3)\ , \end{aligned} \tag{2}$$

where δ_{ij} is the Kronecker delta symbol and ε_{ijk} is the Levi Civita three index symbol having value $(\varepsilon_{ijk}=+1)$ for cyclic permutation , $(\varepsilon_{ijk}=-1)$ for anti cyclic permutation and $(\varepsilon_{ijk}=0)$ for any two repeated indices. Addition and multiplication are defined by the usual distribution law $(e_j\,e_k)\,e_l=e_j(e_k e_l)$ along with the multiplication rules given by equation (2). $\mathbb{H}$ is an associative but non commutative algebra. If ϕ_0, ϕ_1, ϕ_2, ϕ_3 are taken as complex quantities, the quaternion is said to be a bi-quaternion. Alternatively, a quaternion is defined as a two dimensional algebra over the field of complex numbers $\mathbb{C}$. We thus have $\phi=\upsilon+e_2\omega(\upsilon,\omega\in\mathbb{C})$ and $\upsilon=\phi_0+e_1\phi_1$, $\omega=\phi_2-e_1\phi_3$ with the basic multiplication law changes to $\upsilon e_2=-e_2\bar{\upsilon}$. The quaternion conjugate $\overline{\phi}$ is defined as

$$\overline{\phi}=\phi_{\mu}\bar{e_{\mu}}=\phi_{0}-e_{1}\phi_{1}-e_{2}\phi_{2}-e_{3}\phi_{3}\ . \tag{3}$$

In practice ϕ is often represented as a 2×2 matrix $\phi=\phi_0-i\,\vec{\sigma}\cdot\vec{\phi}$ where $e_0=I, e_j=-i\,\sigma_j(j=1,2,3)$ and σ_j are the usual Pauli spin matrices. Then $\overline{\phi}=\sigma_2\phi^T\sigma_2$ with ϕ^T the trans pose of ϕ. The real part of the quaternion ϕ_0 is also defined as

$$Re\,\phi = \frac{1}{2}(\overline{\phi}+\phi)\ , \tag{4}$$

where Re denotes the real part and if $Re\,\phi=0$ then we have $\phi=-\overline{\phi}$ and imaginary ϕ is known as pure quaternion written as

$$\phi=e_{1}\phi_{1}+e_{2}\phi_{2}+e_{3}\phi_{3}\ . \tag{5}$$

The norm of a quaternion is expressed as $N(\phi)=\phi\overline{\phi}=\overline{\phi}\phi=\sum_{j=0}^{3}\phi_j^2$ which is non negative i.e.

$$N(\phi)=|\phi|=\phi_{0}^{2}+\phi_{1}^{2}+\phi_{2}^{2}+\phi_{3}^{2}=Det.(\phi)\geq0. \tag{6}$$

Since there exists the norm of a quaternion, we have a division i.e. every ϕ has an inverse of a quaternion and is described as

$$\phi^{-1}=\frac{\overline{\phi}}{|\phi|}\ . \tag{7}$$

While the quaternion conjugation satisfies the following property

$$\overline{\phi_1\phi_2} = \overline{\phi_2}\,\overline{\phi_1} \ . \tag{8}$$

The norm of the quaternion (6) is positive definite and enjoys the composition law

$$N(\phi_1\phi_2) = N(\phi_1)N(\phi_2) \ . \tag{9}$$

Quaternion (1) is also written as $\phi = (\phi_0, \vec{\phi})$ where $\vec{\phi} = e_1\phi_1 + e_2\phi_2 + e_3\phi_3$ is its vector part and ϕ_0 is its scalar part. So, the sum and product of two quaternions are described as

$$\begin{aligned} (\alpha_0, \vec{\alpha}) + (\beta_0, \vec{\beta}) &= (\alpha_0 + \beta_0, \, \vec{\alpha} + \vec{\beta}) \ , \\ (\alpha_0, \vec{\alpha}) \cdot (\beta_0, \vec{\beta}) &= (\alpha_0\beta_0 - \overrightarrow{\alpha} \cdot \overrightarrow{\beta} \, , \alpha_0 \overrightarrow{\beta} + \beta_0 \overrightarrow{\alpha} + \overrightarrow{\alpha} \times \overrightarrow{\beta}) \ . \end{aligned} \tag{10}$$

Quaternion elements are non -Abelian in nature and thus represent a non commutative division ring.

3 Spontaneous symmetry breaking in the form of Quaternions

The Lagrangian of a complex scalar field, ϕ which carries a scalar electric charge (e), and magnetic charge(g), gravitational(m) and Heavisidian (h) must be gauged with respect to both the vector and pseudovector potentials $(A_\mu, B_\mu, C_\mu, D_\mu)$ is [47],

$$L = \left(\overline{D_\mu\phi}\right)(D_\mu\phi) - V(\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}M_{\mu\nu}M^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} - \frac{1}{4}N_{\mu\nu}N^{\mu\nu}; \tag{11}$$

then the Lagrangian for unified charges (electric,magnetic,gravitational, Heavisidian) of the scalar field is [47],

$$\begin{aligned} L_s = & \left(\partial_\mu + ieA_\mu + igB_\mu + imC_\mu + ih\mathcal{D}_\mu\right)\bar{\phi}\left(\partial^\mu - ieA^\mu - igB^\mu - imC^\mu - ih\mathcal{D}^\mu\right)\phi \\ & -V(\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}M_{\mu\nu}M^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} - \frac{1}{4}N_{\mu\nu}N^{\mu\nu}; \end{aligned} \tag{12}$$

where $\bar{\phi}$ is the complex conjugate of ϕ. The electric, magnetic, gravitational and Heavisidian four currents $J_\mu^{(e)}$, $J_\mu^{(m)}$,$J_\mu^{(G)}$, $J_\mu^{(H)}$ of scalar field can be written in terms as,

$$\begin{aligned} J_\mu^{(e)} &= ie\left[\bar{\phi}\left(D^\mu\phi\right) - \phi\left(\overline{D_\mu\phi}\right)\right]; \\ J_\mu^{(m)} &= ig\left[\bar{\phi}\left(D^\mu\phi\right) - \phi\left(\overline{D_\mu\phi}\right)\right]; \\ J_\mu^{(G)} &= im\left[\bar{\phi}\left(D^\mu\phi\right) - \phi\left(\overline{D_\mu\phi}\right)\right]; \\ J_\mu^{(H)} &= ih\left[\bar{\phi}\left(D^\mu\phi\right) - \phi\left(\overline{D_\mu\phi}\right)\right]. \end{aligned} \tag{13}$$

Since e, g, m, h are scalar quantities, then the potential terms described as [47],

$$V(\phi)^2 = m^2\left(\bar{\phi}\phi\right) + \lambda\left(\bar{\phi}\phi\right)^2 . \tag{14}$$

where m^2 and λ are real constant parameters and λ should be positive to ensure the stable vacuum. If the potential energy in the vacuum state of minimum energy can be found by minimizing potential $V(\phi)$. Then for the vacuum state,

$$\begin{aligned} \frac{dV}{d\phi} = 0 \qquad \Longrightarrow \qquad & \frac{dV}{d\bar{\phi}} = 0 \\ \Longrightarrow \quad m^2\bar{\phi} + 2\lambda\left(\bar{\phi}\phi\right)\bar{\phi} \quad & = 0; \end{aligned} \tag{15}$$

where

$$\phi = \pm\frac{\upsilon}{\sqrt{2}} = \sqrt{\frac{-m^2}{2\lambda}}; \tag{16}$$

substituting the value of $V(\phi)^2$ in the equation (12), then equation (12) can be written as,

$$\begin{aligned} L_s = & \left(\partial_\mu + ieA_\mu + igB_\mu + imC_\mu + ih\mathcal{D}_\mu\right)\bar{\phi}\left(\partial^\mu - ieA^\mu - igB^\mu - imC^\mu - ih\mathcal{D}^\mu\right)\phi \\ & - m^2\left(\bar{\phi}\phi\right) - \lambda\left(\bar{\phi}\phi\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}M_{\mu\nu}M^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} - \frac{1}{4}N_{\mu\nu}N^{\mu\nu}; \end{aligned} \tag{17}$$

The self interaction coupling constant (λ) is taken to be positive definite and for $m^2 \geq 0$ the potential acquires a vacuum expectation value of

$$\begin{aligned} \phi(x) &= \frac{1}{\sqrt{2}}\left(\upsilon + \eta(x) + i\xi(x)\right) \\ &= \frac{1}{\sqrt{2}}(\upsilon + \eta(x))e^{\frac{i\xi(x)}{\nu}}; \end{aligned} \tag{18}$$

one can transform the $\xi(x)$ field away by making the gauge transformation under the following conditions,

$$\phi^{'}(x) = \phi(x) = \frac{1}{\sqrt{2}}(\upsilon + \eta(x)); \tag{19}$$

and the potentials has the following form

$$\begin{aligned} A_\mu^{'}(x) = & A_\mu(x) - \frac{1}{2e\upsilon}\partial_\mu\kappa(x); \\ B_\mu^{'}(x) = & B_\mu(x) - \frac{1}{2g\upsilon}\partial_\mu\kappa(x); \\ C_\mu^{'}(x) = & C_\mu(x) - \frac{1}{2m\upsilon}\partial_\mu\kappa(x); \\ D_\mu^{'}(x) = & D_\mu(x) - \frac{1}{2h\upsilon}\partial_\mu\kappa(x). \end{aligned} \tag{20}$$

The Lagrangian L_s is invariant under the above transformations, substituting these unitary gauge transformations in the equation (17), the Lagrangian becomes [47]

$$\begin{aligned}
L_s =& \frac{1}{2}\left(\partial_\mu \eta\right)\left(\partial^\mu \eta\right)+\frac{1}{2}\left(2m^2\right)\eta^2-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{4}M_{\mu\nu}M^{\mu\nu}-\frac{1}{4}f_{\mu\nu}f^{\mu\nu}-\frac{1}{4}N_{\mu\nu}N^{\mu\nu}\\
&+\frac{1}{2}\nu^2(e^2A_\mu A^\mu+g^2B_\mu B^\mu+m^2C_\mu C^\mu+h^2\mathcal{D}_\mu\mathcal{D}^\mu+2egA_\mu B^\mu+2emA_\mu C^\mu+2ehA_\mu\,\mathcal{D}^\mu\\
&+2gmB_\mu C^\mu+2ghB_\mu\mathcal{D}_\mu+2mhC_\mu\mathcal{D}_\mu)-\lambda\nu\eta^3-\frac{\lambda}{4}\eta^4\\
&+\nu\eta(e^2A_\mu A^\mu+g^2B_\mu B^\mu+m^2C_\mu C^\mu+h^2\mathcal{D}_\mu\mathcal{D}^\mu\\
&+2emA_\mu C^\mu+2ehA_\mu\mathcal{D}^\mu+2gmB_\mu C^\mu+2ghB_\mu\mathcal{D}^\mu+2mhC_\mu\mathcal{D}^\mu);
\end{aligned} \tag{21}$$

where

$$L_s = L_0 + L_I. \tag{22}$$

If the Lagrangian is free from kinetic and mass terms, then [47]

$$\begin{aligned}
L_0 =& \frac{1}{2}\left(\partial_\mu \eta\right)\left(\partial^\mu \eta\right)+\frac{1}{2}\left(2m^2\right)\eta^2-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{4}M_{\mu\nu}M^{\mu\nu}-\frac{1}{4}f_{\mu\nu}f^{\mu\nu}-\frac{1}{4}N_{\mu\nu}N^{\mu\nu}\\
&+\frac{1}{2}\nu^2(e^2A_\mu A^\mu+g^2B_\mu B^\mu+m^2C_\mu C^\mu+h^2\mathcal{D}_\mu\mathcal{D}^\mu+2egA_\mu B^\mu+2emA_\mu C^\mu\\
&+2ehA_\mu\mathcal{D}^\mu+2gmB_\mu C^\mu+2ghB_\mu\mathcal{D}^\mu+2mhC_\mu\mathcal{D}^\mu);
\end{aligned} \tag{23}$$

and the interaction terms of Lagrangian L_I becomes [47]

$$\begin{aligned}
L_I =& -\lambda\nu\eta^3-\frac{\lambda}{4}\eta^4+\nu\eta(e^2A_\mu A^\mu+g^2B_\mu B^\mu+m^2C_\mu C^\mu+h^2\mathcal{D}_\mu\mathcal{D}^\mu\\
&+2egA_\mu B^\mu+2emA_\mu C^\mu+2ehA_\mu\mathcal{D}^\mu+2gmB_\mu C^\mu+2ghB_\mu\mathcal{D}^\mu+2mhC_\mu\mathcal{D}^\mu)\\
&+\frac{\eta^2}{2}(e^2A_\mu A^\mu+g^2B_\mu B^\mu+m^2C_\mu C^\mu+h^2\mathcal{D}_\mu\mathcal{D}^\mu+2egA_\mu B^\mu+2emA_\mu C^\mu\\
&+2ehA_\mu\mathcal{D}^\mu+2gmB_\mu C^\mu+2ghB_\mu\mathcal{D}^\mu+2mhC_\mu\mathcal{D}^\mu).
\end{aligned} \tag{24}$$

One can write the gauge boson mass and their scalar interaction terms in the form of of 4×4 matrices as,

$$\beta=\mathcal{K}\left(A_\mu B_\mu C_\mu D_\mu\right)\begin{pmatrix} e^2 & eg & em & eh\\ ge & g^2 & gm & gh\\ me & mg & m^2 & mh\\ he & hg & hm & h^2\end{pmatrix}\begin{pmatrix}A^\mu\\ B^\mu\\ C^\mu\\ \mathcal{D}^\mu\end{pmatrix}; \tag{25}$$

$\mathcal{K}=\frac{\upsilon^2}{2},\upsilon,\frac{1}{2}$ for the gauge boson mass, tri-linear interaction and quaternionic action terms respectively. Now applying the duality transformation $\mathcal{E}\Rightarrow\mathcal{E}\cos\eta+\mathcal{H}\sin\eta$, $\mathcal{H}\Rightarrow\mathcal{E}\cos\eta-\mathcal{H}\sin\eta$ and $\mathcal{G}\Rightarrow\mathcal{G}\cos\eta+\mathcal{M}\sin\eta$, $\mathcal{G}\Rightarrow\mathcal{G}\cos\eta-\mathcal{M}\sin\eta$. if the mass matrix and integration matrices are diagonalized, then

$$\beta=\mathcal{K}\left(A_{\mu}B_{\mu}C_{\mu}\mathcal{D}_{\mu}\right)\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \left(e^{2}+g^{2}+m^{2}+h^{2}\right)\end{pmatrix}\begin{pmatrix}A^{\mu}\\ B^{\mu}\\ C^{\mu}\\ \mathcal{D}^{\mu}\end{pmatrix}. \tag{26}$$

The equation (17) can also be written as,

$$\begin{aligned}L_{s}= & \overline{D_{\mu}\phi}D^{\mu}\phi-m^{2}\overline{\phi}\phi-\lambda\left(\overline{\phi}\phi\right)^{2}\\ & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{4}M_{\mu\nu}M^{\mu\nu}-\frac{1}{4}f_{\mu\nu}f^{\mu\nu}-\frac{1}{4}N_{\mu\nu}N^{\mu\nu};\end{aligned} \tag{27}$$

applying the following gauge transformations ,

$$\phi\rightarrow\omega\phi\overline{\omega}; \tag{28}$$

the unified gauge fields for electric, magnetic,gravitational,Heavisidian A_{μ}, B_{μ}, C_{μ} and D_{μ} are transformed [53] as

$$\begin{aligned}A_{\mu}= & \omega A_{\mu}\overline{\omega}+\omega\partial_{\mu}\overline{\omega};\\ B_{\mu}= & \omega B_{\mu}\overline{\omega}+\omega\partial_{\mu}\overline{\omega};\\ C_{\mu}= & \omega C_{\mu}\overline{\omega}+\omega\partial_{\mu}\overline{\omega};\\ \mathcal{D}_{\mu}= & \omega\mathcal{D}_{\mu}\overline{\omega}+\omega\partial_{\mu}\overline{\omega}.\end{aligned} \tag{29}$$

applying the condition (28), the covariant derivatives for unified charges reduces to

$$D_{\mu}\phi=\partial_{\mu}\phi-ieA_{\mu}\phi-igB_{\mu}\phi-imC_{\mu}\phi-ih\mathcal{D}_{\mu}\phi; \tag{30}$$

taking the variation in the covariant derivative, the equation (30) becomes,

$$\begin{aligned}\delta\left(D_{\mu}\phi\right)= & \partial_{\mu}\delta\phi-ieA_{\mu}\delta\phi-igB_{\mu}\delta\phi-imC\delta\phi-ih\mathcal{D}\delta\phi\\ & -ie\delta A_{\mu}\phi-ig\delta B_{\mu}\phi-im\delta C_{\mu}\phi-ih\delta\mathcal{D}_{\mu}\phi\\ = & \left(\partial_{\mu}-ieA_{\mu}-igB_{\mu}-imC_{\mu}-ih\mathcal{D}_{\mu}\right)\delta\phi\\ & -ie\delta A_{\mu}\phi-ig\delta B_{\mu}\phi-im\delta C_{\mu}\phi-ih\delta\mathcal{D}_{\mu}\phi\\ = & D_{\mu}\delta\phi--ie\delta A_{\mu}\phi-ig\delta B_{\mu}\phi-im\delta C_{\mu}\phi-ih\delta\mathcal{D}_{\mu}\phi.\end{aligned} \tag{31}$$

Similarly,

$$\delta\left(D_{\nu}\phi\right)=D_{\nu}\delta\phi--ie\delta A_{\nu}\phi-ig\delta B_{\nu}\phi-im\delta C_{\nu}\phi-ih\delta\mathcal{D}_{\nu}\phi; \tag{32}$$

where the energy momentum field strength for different charges such as electric, magnetic, gravitational and Heavisidian [53]

$$\begin{aligned}
F_{\mu\nu} =&\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}-ie\left[A_{\mu},A_{\nu}\right];\\
M_{\mu\nu} =&\partial_{\mu}B_{\nu}-\partial_{\nu}B_{\mu}-ig\left[B_{\mu},B_{\nu}\right];\\
f_{\mu\nu} =&\partial_{\mu}C_{\nu}-\partial_{\nu}C_{\mu}-im\left[C_{\mu},C_{\nu}\right];\\
N_{\mu\nu} =&\partial_{\mu}D_{\nu}-\partial_{\nu}D_{\mu}-ih\left[\mathcal{D}_{\mu},\mathcal{D}_{\nu}\right].
\end{aligned} \tag{33}$$

and correspondingly the variation in equation (33)

$$\begin{aligned}
\delta F_{\mu\nu} =&D_{\mu}\delta A_{\nu}-D_{\nu}\delta A_{\mu};\\
\delta M_{\mu\nu} =&D_{\mu}\delta B_{\nu}-D_{\nu}\delta B_{\mu};\\
\delta f_{\mu\nu} =&D_{\mu}\delta C_{\nu}-D_{\nu}\delta C_{\mu};\\
\delta N_{\mu\nu} =&D_{\mu}\delta\mathcal{D}_{\nu}-\Delta_{\nu}\delta\mathcal{D}_{\mu}.
\end{aligned} \tag{34}$$

Applying the variational principle, then the Lagrangian L in equation (27) becomes [53],

$$\begin{aligned}
\delta L =& \left(\overline{D_{\mu}\delta\phi--ie\delta A_{\mu}\phi-ig\delta B_{\mu}\phi-im\delta C_{\mu}\phi-ih\delta\mathcal{D}_{\mu}\phi}\right)D^{\mu}\phi\\
&+\overline{D_{\mu}\phi}(D^{\mu}\delta\phi-ie\delta A^{\mu}\phi-ig\delta B^{\mu}\phi-im\delta C^{\mu}\phi-ih\delta\mathcal{D}^{\mu}\phi)\\
&-\left(m^{2}+2\lambda\bar{\phi}\phi\right)\left[\phi\delta\bar{\phi}+\bar{\phi}\delta\phi\right]\\
&+\frac{1}{4}F_{\nu\mu}\left(D^{\mu}\delta A^{\nu}-D^{\nu}\delta A^{\mu}\right)+\frac{1}{4}M_{\nu\mu}\left(D^{\mu}\delta B^{\nu}-D^{\nu}\delta B^{\mu}\right)\\
&+\frac{1}{4}f_{\nu\mu}\left(D^{\mu}\delta C^{\nu}-D^{\nu}\delta C^{\mu}\right)+\frac{1}{4}N_{\nu\mu}\left(D^{\mu}\delta\mathcal{D}^{\nu}-D^{\nu}\delta\mathcal{D}^{\mu}\right);
\end{aligned} \tag{35}$$

where the transformation are,

$$\begin{aligned}
F_{\mu\nu} =&-F_{\nu\mu};\\
M_{\mu\nu} =&-M_{\nu\mu};\\
f_{\mu\nu} =&-f_{\nu\mu};\\
N_{\mu\nu} =&-N_{\nu\mu}.
\end{aligned} \tag{36}$$

rearranging the terms of equation (35), we get

$$
\begin{aligned}
\delta L = & \left[\left\{\delta\bar{\phi}\left(D_{\mu}D^{\mu}\phi-(m^{2}+2\lambda\bar{\phi}\phi)\phi\right)\right\}\right]+\left[\left\{\delta\phi\left(D_{\mu}D^{\mu}\bar{\phi}-(m^{2}+2\lambda\bar{\phi}\phi)\bar{\phi}\right)\right\}\right] \\
& +\left[\delta A^{\mu}(D_{\mu}\phi)\bar{\phi}-\phi(\overline{D_{\mu}\phi})+\frac{1}{ie}D^{\mu}F_{\nu\mu}\right] \\
& +\left[\delta B^{\mu}(D_{\mu}\phi)\bar{\phi}-\phi(\overline{D_{\mu}\phi})+\frac{1}{ig}D^{\mu}M_{\nu\mu}\right]+\left[\delta C^{\mu}(D_{\mu}\phi)\bar{\phi}-\phi(\overline{D_{\mu}\phi})+\frac{1}{im}D^{\mu}f_{\nu\mu}\right] \\
& +\left[\delta\mathcal{D}^{\mu}(D_{\mu}\phi)\bar{\phi}-\phi(\overline{D_{\mu}\phi})+\frac{1}{ih}D^{\mu}N_{\nu\mu}\right]; \qquad (37)
\end{aligned}
$$

we get the equation of motion for unified charge as,

$$
\begin{aligned}
& D_{\mu}D^{\mu}\phi-\left(m^{2}+2\lambda\bar{\phi}\phi\right)\phi=0 \\
& \qquad \Rightarrow D_{\mu}D^{\mu}\bar{\phi}-\left(m^{2}+2\lambda\bar{\phi}\phi\right)\bar{\phi}. \qquad (38)
\end{aligned}
$$

Then the current equation for the unified charges are described as,

$$
\begin{aligned}
D^{\nu}F_{\nu\mu}=J_{\mu}^{(e)} &= ie\left[\phi(\overline{D_{\mu}\phi})-(D_{\mu}\phi)\bar{\phi}\right]; \\
D^{\nu}M_{\nu\mu}=J_{\mu}^{(m)} &= ig\left[\phi(\overline{D_{\mu}\phi})-(D_{\mu}\phi)\bar{\phi}\right]; \\
D^{\nu}f_{\nu\mu}=J_{\mu}^{(G)} &= im\left[\phi(\overline{D_{\mu}\phi})-(D_{\mu}\phi)\bar{\phi}\right]; \\
D^{\nu}N_{\nu\mu}=J_{\mu}^{(H)} &= ih\left[\phi(\overline{D_{\mu}\phi})-(D_{\mu}\phi)\bar{\phi}\right]. \qquad (39)
\end{aligned}
$$

4 The Electroweak interactions of Quaternions

The Lagrangian for a free Dirac fermion [58]

$$
L_{0}=i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi-m\bar{\psi}\psi; \qquad (40)
$$

L_0 is invariant under global $U(1)$ transformations,

$$
\psi \quad\rightarrow\quad \psi^{'}=exp\left\{iQ\theta\right\}\psi; \qquad (41)
$$

where $Q\theta$ is arbitrary real constant. Then

$$
\partial_{\mu}\psi(x) \quad\longrightarrow\quad exp\left\{iQ\theta\right\}\left(\partial_{\mu}+iQ\partial_{\mu}\theta\right); \qquad (42)
$$

If one introduces a new spin-1, since $\partial_{\mu}\theta$ has a Lorentz index field A_{μ} ,B_{μ}, C_{μ} and $\mathcal{D}_{\mu}$ transforming as ,

$$\begin{aligned}
A_\mu &\to A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\theta; \\
B_\mu &\to B'_\mu = B_\mu - \frac{1}{g}\partial_\mu\theta; \\
C_\mu &\to C'_\mu = C_\mu - \frac{1}{m}\partial_\mu\theta; \\
\mathcal{D}_\mu &\to \mathcal{D}'_\mu = \mathcal{D}_\mu - \frac{1}{h}\partial_\mu\theta.
\end{aligned} \tag{43}$$

and defines the covariant derivative of unified charges (electric,magnetic,gravitational and Heavisidian),

$$D_\mu\psi = (\partial_\mu + ieA_\mu + igB_\mu + imC_\mu + ih\mathcal{D}_\mu)\,\psi; \tag{44}$$

which is required property of the transforming ,

$$D_\mu\psi \to D_\mu\psi' = exp\,\{iQ\theta\}\,D_\mu\psi; \tag{45}$$

Then the Lagrangian of unified charges (electric,magnetic,gravitational and Heavisidian) becomes [58],

$$L = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi; \tag{46}$$

substitute the value of D_μ in the equation (46),

$$\begin{aligned}
L =& i\bar{\psi}\gamma^\mu(\partial_\mu + ieA_\mu + igB_\mu + imC_\mu + ih\mathcal{D}_\mu)\psi - m\bar{\psi}\psi \\
=& (i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi) - eQA_\mu\bar{\psi}\gamma^\mu\psi - gQB_\mu\bar{\psi}\gamma^\mu\psi \\
&- mQC_\mu\bar{\psi}\gamma^\mu\psi - hQ\mathcal{D}_\mu\bar{\psi}\gamma^\mu - m\bar{\psi}\psi \\
=& L_0 - eQA_\mu\bar{\psi}\gamma^\mu\psi - gQB_\mu\bar{\psi}\gamma^\mu\psi - mQC_\mu\bar{\psi}\gamma^\mu\psi - hQD_\mu\bar{\psi}\gamma^\mu\psi;
\end{aligned} \tag{47}$$

where L_0 is the free Lagrangian and it is invariant under local $U(1)$transformations, described as

$$L_0 = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi. \tag{48}$$

The electromagnetic charge Q is completely arbitrary. If $A_\mu, B_\mu, C_\mu, \mathcal{D}_\mu$ to be true propagating field, then the gauge invariant kinetic term of unified charges i.e.

$$L_{kin} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}M_{\mu\nu}M^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} - \frac{1}{4}N_{\mu\nu}N^{\mu\nu}; \tag{49}$$

where the electromagnetic field strength of unified charges i.e. electric, magnetic, gravitational and Heavisidian is given by

$$\begin{aligned} F_{\mu\nu} =& \partial_\mu A_\nu - \partial_\nu A_\mu; \\ M_{\mu\nu} =& \partial_\mu B_\nu - \partial_\nu B_\mu; \\ f_{\mu\nu} =& \partial_\mu C_\nu - \partial_\nu C_\mu; \\ N_{\mu\nu} =& \partial_\mu D_\nu - \partial_\nu D_\mu. \end{aligned} \tag{50}$$

is usual electromagnetic field strength in the form of quaternion. A possible mass term for the unified gauge field [58]

$$L_m = \frac{1}{2}m^2 A_\mu A^\mu + \frac{1}{2}m^2 B_\mu B^\mu + \frac{1}{2}m^2 C_\mu C^\mu + \frac{1}{2}m^2 \mathcal{D}_\mu \mathcal{D}^\mu; \tag{51}$$

Then the total Lagrangian for unified charges (electric,magnetic,gravitational,Heavisidian) becomes,

$$\begin{aligned} L_T =& L + L_{kin} \\ =& L_0 - eQA_\mu \bar{\psi}\gamma^\mu\psi - gQB_\mu\bar{\psi}\gamma^\mu\psi \\ & - mQC_\mu\bar{\psi}\gamma^\mu\psi - hQD_\mu\bar{\psi}\gamma^\mu\psi \\ & - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}M_{\mu\nu}M^{\mu\nu} - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} - \frac{1}{4}N_{\mu\nu}N^{\mu\nu}; \end{aligned} \tag{52}$$

then the equation of motion for unified charges (electric, magnetic, gravitational, Heavisidian) [56],

$$\begin{aligned} \frac{\partial L_T}{\partial A_\mu} - \partial_\nu \frac{\partial L_T}{\partial(\partial_\nu A_\mu)} =& 0; \\ \frac{\partial L_T}{\partial B_\mu} - \partial_\nu \frac{\partial L_T}{\partial(\partial_\nu B_\mu)} =& 0; \\ \frac{\partial L_T}{\partial C_\mu} - \partial_\nu \frac{\partial L_T}{\partial(\partial_\nu C_\mu)} =& 0; \\ \frac{\partial L_T}{\partial D_\mu} - \partial_\nu \frac{\partial L_T}{\partial(\partial_\nu D_\mu)} =& 0. \end{aligned} \tag{53}$$

Solving the equation52, then the equation of motion for unified charges (electric, magnetic, gravitational, Heavisidian) [58],

$$\begin{aligned} eQ\bar{\psi}\gamma^{\mu}\psi =&\partial_{\nu}F^{\mu\nu} = J^{\mu(e)};\\ gQ\bar{\psi}\gamma^{\mu}\psi =&\partial_{\nu}M^{\mu\nu} = J^{\mu(g)};\\ mQ\bar{\psi}\gamma^{\mu}\psi =&\partial_{\nu}M^{\mu\nu} = J^{\mu(G)};\\ hQ\bar{\psi}\gamma^{\mu}\psi =&\partial_{\nu}N^{\mu\nu} = J^{\mu(H)}; \end{aligned} \tag{54}$$

References

[1] P. A. M. Dirac, Proc. Roy. Soc. London, **A133**, 60(1931).

[2] P. A. M. Dirac, Phys. Rev., **74**, 817(1948).

[3] J. Schwinger, Science, **151**, 1048(1966).

[4] J. Schwinger, Science, **151**, 1055(1966).

[5] J. Schwinger, Phys. Rev., **144**, 1087(1966).

[6] J. Schwinger, Science, **173**, 1536(1968).

[7] J. Schwinger, Science, **165**, 757(1969).

[8] D. Zwanzinger, Phys. Rev., **176** , 1489(1968).

[9] D. Zwanzinger, Phys. Rev., **D3**, 880(1971).

[10] C. N. Yang and T. T. Wu, Nucl. Phys., **B107**, 365(1976).

[11] C. N. Yang and T. T. Wu, Phys. Rev., **D14**, 437(1976).

[12] F. Röhrlich, Phys. Rev., **150**, 1104(1966).

[13] P. Goddard and D. Olive, Rep. on Progress in Phys., **41**, 1357(1978).

[14] N. Craigie, World Scientific, Singapur, (1986).

[15] G.' t Hooft, Nucl. Phys., **B79**, 276(1974).

[16] A. M. Polyakov, JETP lett., **20**, 194(1974).

[17] B. Julia and A. Zee, Phys. Rev., **D11**, 2227(1975).

[18] J. Preskill, Ann. Rev. Nucl. Sci., **34**, 461(1984).

[19] E. Witten, Phys. Lett., **B86**, 283(1979).

[20] Y. M. Cho, Phys. Rev., **D21,** 1080(1980).

[21] G. ' t Hooft, Nucl. Phys., **B153** (1979), 141(1980).

[22] G. ' t Hooft, Nucl. Phys., **B190**, 455(1981).

[23] V. A. Rubakov, JETP Lett., **33**, 644(1981).

[24] V. A. Rubakov, Nucl. Phys., **B203**, 311(1982).

[25] C. Dokos and T. Tomaras, Phys. Rev., **D21**, 2940(1980).

[26] N. Seiberg and E. Witten, Nucl. Phys., **B426**, 19(1994).

[27] K. Intriligator and N. Seiberg, Nucl. Phys.(Proc. Suppl.), **BC 45**, 1(1996).

[28] M. E. Peskin, *"Dualities In Supersymmetric Yang-Mills Theories*, hep-th/9703136.

[29] A. Kapustin and E. Witten, *"Electric-Magnetic Duality And The Geometric Lang lands Program"*, hep-th/0604151.

[30] W. R. Hamilton, *"Elements of quaternions"*, Chelsea Publications Co., New York, (1969).

[31] L. Silberstein, Phil. Mag., **23**, 790(1912).

[32] P. G. Tait, *"An elementary Treatise on Quaternions",* Oxford Univ. Press (1875).

[33] B. S. Rajput, S. R. Kumar and O. P. S. Negi, Lett. Nuovo Cimento, **34,** 180(1982).

[34] V. Majernik and M. Nagy, Nuovo Comento Lett., **16**, 265(1976).

[35] S. L. Adler, *"Quaternion Quantum Mechanics and Quantum Fields",* Oxford Univ. Press, NY, (1995).

[36] D. Finklestein, J. M. Jauch, S. Schiminovich and D. Speiser, J. Math. Phys., **4** , 788(1963).

[37] K. Morita, Prog. Theor. Phys., **68**, 2159(1982).

[38] P. S. Bisht, O. P. S. Negi and B. S. Rajput, Int. J. Theor. Phys., **32**, 2099(1993).

[39] J. Lukiersky and A. Nowicki, Fortsch. Physik, **30,** 75(1982).

[40] A. J. Davies, Phys. Rev.,_**A49**, 714(1994).

[41] V. V. Kravchenkov, *"Applied Quaternion Analysis"*, Helderman Verlag, Germany (2003).

[42] A. Waser, *"Quaternions in Electrodynamics",* AW-Verlag; www.aw-verlag.ch (2000).

[43] T. P. Cheng and L. F. Li, *"Gauge Theory of Elementary Particle Physics",* Clarendon, Oxford (1984).

[44] P. H. Frampton, *"Gauge Field Theories",* Wiley - VCH Verlag, GmbH & Co. KGaA, Weinheim (2008).

[45] C. Quigg, *"Gauge Theories of the Strong, Weak and Electromagnetic Interaction",* Westview Press, Boulder (1997).

[46] P. S. Bisht, Pushpa, T. Li and O. P. S. Negi, Int. J. Theor. Phys., **50**, 594(2011) .

[47] D. Singleton, *"Magnetic charge as a "hidden" gauge symmetry"*, arXiv:hep-th/9701044v1.

[48] C. Nash and G. C. Joshi, , J. Math. Phys., **28** , 463(1987).

[49] K. Morita, Prog. Theor. Phys,. **65**, 2071(1981).

[50] F. N. Ndili, Int. J. Theor. Phys., **15**, 265(1976).

[51] S. De Leo, Int. J. Theor. Phys. **35**, 1821(1996).

[52] A. S. Rawat, O. P. S. Negi, P. S. Bisht and B. S. Rajput, Indian J. Pure Appl. Phys., **38**, 390(2000).

[53] Pushpa, P. S. Bisht and O. P. S. Negi, Int. J. Theor. Phys., **50**, 1927(2011).

[54] Shalini Dangwal, P. S. Bisht and O. P. S. Negi, Russian Physics Journal, **49**, 1274(2006).

[55] P. S. Bisht, O. P. S. Negi and B. S. Rajput, Ind. J. Pure and Applied Math., **31**, 543(1993).

[56] Gaurav Karnatak, P. S. Bisht and O. P. S. Negi, Inter. J. of Theor. Phys., **49**, 1344(2010).

[57] O. P. S. Negi and H. Dehnen, Intern. Journ. Theor. Phys., **50**, 2446(2011).

[58] A. Pich, *"The standard model of electroweak interactions"*, arXiv hep-ph/0502010v1.