On the Navier–Stokes equations

Daniel Thomas Hayes

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The problem on the existence and smoothness of the Navier–Stokes equations is solved.

1. Introduction

The Navier–Stokes equations are thought to govern the motion of a fluid in $\mathbb{R}^d$ where $d \in \mathbb{N}$, see [1,3]. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^d$ be the velocity and let $p = p(\mathbf{x}, t) \in \mathbb{R}$ be the pressure, each dependent on position $\mathbf{x} \in \mathbb{R}^d$ and time $t \geqslant 0$. We take the externally applied force to be identically zero. The fluid is assumed to be incompressible with constant viscosity $\nu \geqslant 0$ and to fill all of $\mathbb{R}^d$. The Navier–Stokes equations can then be written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla p, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

with initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 \tag{3}$$

where $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x}) \in \mathbb{R}^d$. In these equations

$$\nabla = (\frac{\partial}{\partial \mathbf{x}_1}, \frac{\partial}{\partial \mathbf{x}_2}, \dots, \frac{\partial}{\partial \mathbf{x}_d}) \tag{4}$$

is the gradient operator and

$$\nabla^2 = \sum_{i=1}^{d} \frac{\partial^2}{\partial {\mathbf{x}_i}^2} \tag{5}$$

is the Laplacian operator. When $\nu = 0$, equations (1), (2), (3) are called the Euler equations. Solutions of (1), (2), (3) are to be found with

$$\mathbf{u}_0(\mathbf{x} + e_i) = \mathbf{u}_0(\mathbf{x}) \tag{6}$$

for $1 \leqslant i \leqslant d$ where e_i is the i^{th} unit vector in $\mathbb{R}^d$. The initial condition $\mathbf{u}_0$ is a given C^∞ divergence-free vector field on $\mathbb{R}^d$. A solution of (1), (2), (3) is then accepted to be physically reasonable [3] if

$$\mathbf{u}(\mathbf{x} + e_i, t) = \mathbf{u}(\mathbf{x}, t), \quad p(\mathbf{x} + e_i, t) = p(\mathbf{x}, t) \tag{7}$$

on $\mathbb{R}^d \times [0, \infty)$ for $1 \leqslant i \leqslant d$ and

$$\mathbf{u}, p \in C^\infty(\mathbb{R}^d \times [0, \infty)). \tag{8}$$

2. Solution to the Navier–Stokes problem

I provide a proof of the following theorem [2,3,6].
Theorem. Let $\mathbf{u}_0$ be any smooth, divergence-free vector field satisfying (6). Then there exist smooth functions $\mathbf{u}$, p on $\mathbb{R}^d \times [0, \infty)$ that satisfy (1), (2), (3), (7), (8).
Proof. It is sufficient to rule out the possibility that there is a smooth, divergence-free $\mathbf{u}_0$ for which (1), (2), (3) have a solution with a finite blowup time [3].
Let the *exponential series* of $\mathbf{u}$, p be

$$\tilde{\mathbf{u}} = \sum_{\mathbf{L}=\mathbf{0}}^{\infty} \mathbf{a}_{\mathbf{L}} \mathrm{e}^{-k\mathbf{L}\cdot\mathbf{x}}, \tag{9}$$

$$\tilde{p} = \sum_{\mathbf{L}=\mathbf{0}}^{\infty} b_{\mathbf{L}} \mathrm{e}^{-k\mathbf{L}\cdot\mathbf{x}} \tag{10}$$

respectively. Here $\mathbf{a}_{\mathbf{L}} = \mathbf{a}_{\mathbf{L}}(t) \in \mathbb{R}^d$, $b_{\mathbf{L}} = b_{\mathbf{L}}(t) \in \mathbb{R}$, $k > 0$ is a constant, and $\sum_{\mathbf{L}=\mathbf{0}}^{\infty}$ denotes the sum over all $\mathbf{L} \in (\mathbb{N} \cup \{0\})^d$. The *exponential series* is similar to a Taylor series. It is equivalent to a Laplace transform of a sampled signal with sampling period k. The initial condition $\mathbf{u}_0$ is a Fourier series [2] of which is convergent for all $\mathbf{x} \in \mathbb{R}^d$. Since $\mathbf{u}_0$ is a Fourier series this then implies that $\mathbf{u}_0$ at complex values of $\mathbf{x}$ is irrelevant and that $\mathbf{u}_0$ can be taken to be smooth for all $\mathbf{x}$. The *exponential series* $\tilde{\mathbf{u}}|_{t=0}$ certainly converges for all $\mathbf{x} \in \mathbb{R}^d$ when $0 < k \ll 1$ as it then becomes a Taylor series of which would converge for all $\mathbf{x} \in \mathbb{R}^d$. Substituting $\mathbf{u} = \tilde{\mathbf{u}}$, $p = \tilde{p}$ into (1) gives

$$\sum_{\mathbf{L}=\mathbf{0}}^{\infty} \frac{\partial \mathbf{a}_{\mathbf{L}}}{\partial t} \mathrm{e}^{-k\mathbf{L}\cdot\mathbf{x}} - \sum_{\mathbf{L}=\mathbf{0}}^{\infty} \sum_{\mathbf{M}=\mathbf{0}}^{\infty} (\mathbf{a}_{\mathbf{L}} \cdot k\mathbf{M}) \mathbf{a}_{\mathbf{M}} \mathrm{e}^{-k(\mathbf{L}+\mathbf{M})\cdot\mathbf{x}} = \sum_{\mathbf{L}=\mathbf{0}}^{\infty} \nu k^2 |\mathbf{L}|^2 \mathbf{a}_{\mathbf{L}} \mathrm{e}^{-k\mathbf{L}\cdot\mathbf{x}} + \sum_{\mathbf{L}=\mathbf{0}}^{\infty} k\mathbf{L} b_{\mathbf{L}} \mathrm{e}^{-k\mathbf{L}\cdot\mathbf{x}}. \tag{11}$$

Equating like powers of the exponentials in (11) yields

$$\frac{\partial \mathbf{a}_{\mathbf{L}}}{\partial t} - \sum_{\mathbf{M}=\mathbf{0}}^{\infty} (\mathbf{a}_{\mathbf{L}-\mathbf{M}} \cdot k\mathbf{M}) \mathbf{a}_{\mathbf{M}} = \nu k^2 |\mathbf{L}|^2 \mathbf{a}_{\mathbf{L}} + k\mathbf{L} b_{\mathbf{L}} \tag{12}$$

on using the Cauchy product formula [4]

$$\sum_{l=0}^{\infty} a_l x^l \sum_{m=0}^{\infty} b_m x^m = \sum_{l=0}^{\infty} \sum_{m=0}^{l} a_{l-m} b_m x^l. \tag{13}$$

Herein $\mathbf{a}_{\mathbf{L}} = \mathbf{0}$ if any component of $\mathbf{L}$ is negative. Substituting $\mathbf{u} = \tilde{\mathbf{u}}$ into (2) gives

$$-\sum_{\mathbf{L}=\mathbf{0}}^{\infty} k\mathbf{L} \cdot \mathbf{a}_{\mathbf{L}} \mathrm{e}^{-k\mathbf{L}\cdot\mathbf{x}} = 0. \tag{14}$$

Equating like powers of the exponentials in (14) yields

$$\mathbf{L} \cdot \mathbf{a_L} = 0. \tag{15}$$

Applying $\mathbf{L}\cdot$ to (12) and noting (15) leads to

$$b_\mathbf{L} = -\sum_{\mathbf{M}=\mathbf{0}}^{\infty} (\mathbf{a_{L-M}} \cdot \hat{\mathbf{L}})(\mathbf{a_M} \cdot \hat{\mathbf{L}}) \tag{16}$$

where $b_\mathbf{0}$ is arbitrary and $\hat{\mathbf{L}} = \mathbf{L}/|\mathbf{L}|$ is the unit vector in the direction of $\mathbf{L}$. Then substituting (16) into (12) gives

$$\frac{\partial \mathbf{a_L}}{\partial t} = \sum_{\mathbf{M}=\mathbf{0}}^{\infty} (\mathbf{a_{L-M}} \cdot k\mathbf{M})\mathbf{a_M} + \nu k^2 |\mathbf{L}|^2 \mathbf{a_L} - \sum_{\mathbf{M}=\mathbf{0}}^{\infty} k\mathbf{L}(\mathbf{a_{L-M}} \cdot \hat{\mathbf{L}})(\mathbf{a_M} \cdot \hat{\mathbf{L}}) \tag{17}$$

where $\mathbf{a_0} = \mathbf{a_0}(0)$. The equations for $\mathbf{a_L}$ can then be solved for all $\mathbf{L} \in (\mathbb{N} \cup \{0\})^d$. Note that there is an invariance that applies to $\tilde{\mathbf{u}}|_{t=0}$ due to (6). Also (16), (17) are invariant if $\mathbf{a_L} \to \mathrm{e}^{\mathbf{L}\cdot\mathbf{c}}\mathbf{a_L}$ and $b_\mathbf{L} \to \mathrm{e}^{\mathbf{L}\cdot\mathbf{c}} b_\mathbf{L}$ where $\mathbf{c} \in \mathbb{R}^d$ is a constant vector. The problem is invariant if $\mathbf{x} \to \mathbf{x} + \mathbf{c}$. From (12) and in light of (15) it is possible to write

$$\frac{\partial \mathbf{a_L}}{\partial t} \cdot \hat{\mathbf{a}}_\mathbf{L} = \sum_{\mathbf{M}=\mathbf{0}}^{\infty} (\mathbf{a_{L-M}} \cdot k\mathbf{M})\mathbf{a_M} \cdot \hat{\mathbf{a}}_\mathbf{L} + \nu k^2 |\mathbf{L}|^2 \mathbf{a_L} \cdot \hat{\mathbf{a}}_\mathbf{L} \tag{18}$$

where $\hat{\mathbf{a}}_\mathbf{L} = \mathbf{a_L}/|\mathbf{a_L}|$ is the unit vector in the direction of $\mathbf{a_L}$. Then (18) implies

$$\frac{\partial |\mathbf{a_L}|}{\partial t} = \sum_{\mathbf{M}=\mathbf{0}}^{\infty} (\mathbf{a_{L-M}} \cdot k\mathbf{M})\mathbf{a_M} \cdot \hat{\mathbf{a}}_\mathbf{L} + \nu k^2 |\mathbf{L}|^2 |\mathbf{a_L}|. \tag{19}$$

From (19) it is possible to write

$$\frac{\partial |\mathbf{a_L}|}{\partial t} \leqslant \sum_{\mathbf{M}=\mathbf{0}}^{\infty} |\mathbf{a_{L-M}}| k |\mathbf{M}| |\mathbf{a_M}| + \nu k^2 |\mathbf{L}|^2 |\mathbf{a_L}| \tag{20}$$

on using the Cauchy–Schwarz inequality [5]

$$|\mathbf{a} \cdot \mathbf{b}| \leqslant |\mathbf{a}||\mathbf{b}|. \tag{21}$$

It then follows from (20) that

$$\sum_{\mathbf{L}=\mathbf{0}}^{\infty} \frac{\partial |\mathbf{a_L}|}{\partial t} \mathrm{e}^{k|\mathbf{L}||\mathbf{x}|} \leqslant \sum_{\mathbf{L}=\mathbf{0}}^{\infty} \sum_{\mathbf{M}=\mathbf{0}}^{\infty} |\mathbf{a_{L-M}}| k |\mathbf{M}| |\mathbf{a_M}| \mathrm{e}^{k|\mathbf{L}||\mathbf{x}|} + \sum_{\mathbf{L}=\mathbf{0}}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{a_L}| \mathrm{e}^{k|\mathbf{L}||\mathbf{x}|} \tag{22}$$

implying that

$$\sum_{\mathbf{L}=\mathbf{0}}^{\infty} \frac{\partial|\mathbf{a_L}|}{\partial t} \mathrm{e}^{k|\mathbf{L}||\mathbf{x}|} \leqslant \sum_{\mathbf{L}=\mathbf{0}}^{\infty} \sum_{\mathbf{M}=\mathbf{0}}^{\infty} |\mathbf{a_L}| k |\mathbf{M}||\mathbf{a_M}| \mathrm{e}^{k|\mathbf{L}+\mathbf{M}||\mathbf{x}|} + \sum_{\mathbf{L}=\mathbf{0}}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{a_L}| \mathrm{e}^{k|\mathbf{L}||\mathbf{x}|} \tag{23}$$

in light of (13), which yields

$$\sum_{\mathbf{L}=\mathbf{0}}^{\infty} \frac{\partial|\mathbf{a_L}|}{\partial t} \mathrm{e}^{k|\mathbf{L}||\mathbf{x}|} \leqslant \sum_{\mathbf{L}=\mathbf{0}}^{\infty} \sum_{\mathbf{M}=\mathbf{0}}^{\infty} |\mathbf{a_L}| k |\mathbf{M}||\mathbf{a_M}| \mathrm{e}^{k(|\mathbf{L}|+|\mathbf{M}|)|\mathbf{x}|} + \sum_{\mathbf{L}=\mathbf{0}}^{\infty} \nu k^2 |\mathbf{L}|^2 |\mathbf{a_L}| \mathrm{e}^{k|\mathbf{L}||\mathbf{x}|} \tag{24}$$

on using the triangle inequality [5]

$$|\mathbf{a} + \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|. \tag{25}$$

Let

$$\psi = \sum_{\mathbf{L}=\mathbf{0}}^{\infty} |\mathbf{a_L}| \mathrm{e}^{k|\mathbf{L}|X} \tag{26}$$

where $X = |\mathbf{x}|$ and note that

$$|\tilde{\mathbf{u}}| \leqslant \psi. \tag{27}$$

Then (24) can be written as

$$\frac{\partial \psi}{\partial t} \leqslant \psi \frac{\partial \psi}{\partial X} + \nu \frac{\partial^2 \psi}{\partial X^2}. \tag{28}$$

Here $\psi|_{t=0}$ converges for all $X \in \mathbb{R}$ since $\tilde{\mathbf{u}}|_{t=0}$ converges for all $\mathbf{x} \in \mathbb{R}^d$. In light of [8] it is found that (28) is globally regular for $\nu \geqslant 0$. Therefore blowup is ruled out via Taylor's theorem [7] for functions of several variables. □

References

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