On the Navier–Stokes equations

Daniel Thomas Hayes

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The problem on the existence and smoothness of the Navier–Stokes equations is considered.

1. Problem description

The Navier–Stokes equations are thought to govern the motion of a viscous incompressible fluid in $\mathbb{R}^3$, see Batchelor 1967. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^3$, $p = p(\mathbf{x}, t) \in \mathbb{R}$, and $\mathbf{f} = \mathbf{f}(\mathbf{x}, t) \in \mathbb{R}^3$ be the velocity, pressure, and given externally applied force respectively, each dependent on position $\mathbf{x} \in \mathbb{R}^3$ and time $t \geqslant 0$. The fluid is assumed to be incompressible with constant viscosity $\nu > 0$ and to fill all of $\mathbb{R}^3$. The Navier–Stokes equations can then be written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla p + \mathbf{f}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

with initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 \tag{3}$$

where $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x}) \in \mathbb{R}^3$. In these equations ∇ is the gradient operator and ∇^2 is the Laplacian operator. When $\nu = 0$, equations (1), (2), (3) are called the Euler equations. Solutions of (1), (2), (3) are to be found with

$$\mathbf{u}_0(\mathbf{x} + e_j) = \mathbf{u}_0(\mathbf{x}), \ \ \mathbf{f}(\mathbf{x} + e_j, t) = \mathbf{f}(\mathbf{x}, t) \ \text{ for } \ 1 \leqslant j \leqslant 3 \tag{4}$$

where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. The initial condition $\mathbf{u}_0$ is a given C^∞ divergence-free vector field on $\mathbb{R}^3$ and

$$|\partial_{\mathbf{x}}^{\alpha} \partial_t^{\beta} \mathbf{f}| \leq C_{\alpha\beta\gamma}(1 + |t|)^{-\gamma} \ \text{ on } \ \mathbb{R}^3 \times [0, \infty) \ \text{ for any } \ \alpha, \beta, \gamma. \tag{5}$$

A solution of (1), (2), (3) would then be accepted to be physically reasonable if

$$\mathbf{u}(\mathbf{x} + e_j, t) = \mathbf{u}(\mathbf{x}, t), \ \ p(\mathbf{x} + e_j, t) = p(\mathbf{x}, t) \ \text{ on } \ \mathbb{R}^3 \times [0, \infty) \ \text{ for } \ 1 \leqslant j \leqslant 3 \tag{6}$$

and

$$\mathbf{u}, p \in C^\infty(\mathbb{R}^3 \times [0, \infty)). \tag{7}$$

I consider a proof of the following statement (D), see Fefferman 2000.

(D) Breakdown of Navier–Stokes Solutions on $\mathbb{R}^3/\mathbb{Z}^3$.

Take $\nu > 0$. Then there exist a smooth, divergence-free vector field $\mathbf{u}_0$ on $\mathbb{R}^3$ and a smooth $\mathbf{f}$ on $\mathbb{R}^3 \times [0, \infty)$, satisfying (4), (5), for which there exist no solutions $(\mathbf{u}, p)$ of (1), (2), (3), (6), (7) on $\mathbb{R}^3 \times [0, \infty)$.

2. Proof of statement (D)

Herein I take $\mathbf{f} = \mathbf{0}$. I seek the approximation of the form

$$\mathbf{u} = \sum_{\mathbf{L}=-\mathbf{1}}^{\mathbf{1}} \sum_{l=0}^{n} \frac{\partial^l \mathbf{u}_{\mathbf{L}}}{\partial t^l}|_{t=0} \frac{t^l}{l!} e^{ik\mathbf{L}\cdot\mathbf{x}}, \tag{8}$$

$$p = \sum_{\mathbf{L}=-\mathbf{1}}^{\mathbf{1}} \sum_{l=0}^{n} \frac{\partial^l p_{\mathbf{L}}}{\partial t^l}|_{t=0} \frac{t^l}{l!} e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{9}$$

to the solution of (1), (2), (3), (4), (5), (6) in light of Theorem 1 and Theorem 2 in the Appendix. Here $\mathbf{u}_{\mathbf{L}} = \mathbf{u}_{\mathbf{L}}(t)$, $p_{\mathbf{L}} = p_{\mathbf{L}}(t)$, $k = 2\pi$, and $\sum_{\mathbf{L}=-\mathbf{H}}^{\mathbf{H}}$ denotes the sum over all $\mathbf{L} \in \mathbb{Z}^3$ with $-H \leqslant \mathbf{L}_j \leqslant H, 1 \leqslant j \leqslant 3$. Herein the smooth divergence-free initial condition $\mathbf{u}_0$ on $\mathbb{R}^3$ is chosen to be

$$\mathbf{u}_0 = \sum_{\mathbf{L}=-\mathbf{1}}^{\mathbf{1}} \mathbf{L} \times (\mathbf{L} \times \mathbf{1}) a_{\mathbf{L}} \delta_{|\mathbf{L}|, \sqrt{3}} e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{10}$$

where $\mathbf{1} = (1, 1, 1)$, $\delta_{i,j}$ is the Kronecker delta defined by

$$\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \tag{11}$$

and $a_{\mathbf{L}}$ are constants that are chosen such that $\mathbf{u}_0 \in \mathbb{R}^3$.

Method 1

Let

$$\mathbf{u} = \sum_{l=0}^{n} \frac{\partial^l \mathbf{u}}{\partial t^l}|_{t=0} \frac{t^l}{l!}, \tag{12}$$

$$p = \sum_{l=0}^{n} \frac{\partial^l p}{\partial t^l}|_{t=0} \frac{t^l}{l!}. \tag{13}$$

Substituting (12), (13) into (1) and equating like powers of t in accordance with Theorem 1 yields

$$\frac{\partial^{l+1} \mathbf{u}}{\partial t^{l+1}}|_{t=0} + \sum_{m=0}^{l} (\frac{\partial^{l-m} \mathbf{u}}{\partial t^{l-m}}|_{t=0} \cdot \nabla) \frac{\partial^m \mathbf{u}}{\partial t^m}|_{t=0} \binom{l}{m} = \nu \nabla^2 \frac{\partial^l \mathbf{u}}{\partial t^l}|_{t=0} - \nabla \frac{\partial^l p}{\partial t^l}|_{t=0}. \tag{14}$$

Substituting (12) into (2) and equating like powers of t in accordance with Theorem 1 yields

$$\nabla \cdot \frac{\partial^l \mathbf{u}}{\partial t^l}|_{t=0} = 0. \tag{15}$$

Applying $\nabla \times \nabla \times$ to (14) and using the identities

$$\nabla \times \nabla \times \mathbf{a} = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}, \tag{16}$$

$$\nabla \times \nabla a = \mathbf{0} \tag{17}$$

along with (15) gives

$$\nabla^2 \frac{\partial^{l+1}\mathbf{u}}{\partial t^{l+1}}|_{t=0} = \nabla \times \nabla \times \sum_{m=0}^{l} (\frac{\partial^{l-m}\mathbf{u}}{\partial t^{l-m}}|_{t=0} \cdot \nabla) \frac{\partial^m \mathbf{u}}{\partial t^m}|_{t=0} \binom{l}{m} + \nu \nabla^4 \frac{\partial^l \mathbf{u}}{\partial t^l}|_{t=0}. \tag{18}$$

Applying the inverse Laplacian ∇^{-2} to (18) gives

$$\frac{\partial^{l+1}\mathbf{u}}{\partial t^{l+1}}|_{t=0} = \nabla^{-2} \nabla \times \nabla \times \sum_{m=0}^{l} (\frac{\partial^{l-m}\mathbf{u}}{\partial t^{l-m}}|_{t=0} \cdot \nabla) \frac{\partial^m \mathbf{u}}{\partial t^m}|_{t=0} \binom{l}{m} + \nu \nabla^2 \frac{\partial^l \mathbf{u}}{\partial t^l}|_{t=0} + \mathbf{\Phi}_l \tag{19}$$

where $\mathbf{\Phi}_l$ must satisfy the Laplace equation

$$\nabla^2 \mathbf{\Phi}_l = \mathbf{0}. \tag{20}$$

The required solution to (20) is $\mathbf{\Phi}_l = \mathbf{0}$ in light of (4), (6). Equation (19) is then solved for $\frac{\partial^{l+1}\mathbf{u}}{\partial t^{l+1}}|_{t=0}$ where $l = 0, 1, \ldots, n-1$. Applying $\nabla \cdot$ to (14) and noting (15) yields

$$\nabla^2 \frac{\partial^l p}{\partial t^l}|_{t=0} = -\nabla \cdot \sum_{m=0}^{l} (\frac{\partial^{l-m}\mathbf{u}}{\partial t^{l-m}}|_{t=0} \cdot \nabla) \frac{\partial^m \mathbf{u}}{\partial t^m}|_{t=0} \binom{l}{m}. \tag{21}$$

Applying ∇^{-2} to (21) gives

$$\frac{\partial^l p}{\partial t^l}|_{t=0} = -\nabla^{-2} \nabla \cdot \sum_{m=0}^{l} (\frac{\partial^{l-m}\mathbf{u}}{\partial t^{l-m}}|_{t=0} \cdot \nabla) \frac{\partial^m \mathbf{u}}{\partial t^m}|_{t=0} \binom{l}{m} + \psi_l \tag{22}$$

where

$$\nabla^2 \psi_l = 0. \tag{23}$$

Arbitrary constant $\psi_l \in \mathbb{R}$ is the solution to (23) in light of (4), (6). Equation (22) is then solved for $\frac{\partial^l p}{\partial t^l}|_{t=0}$ where $l = 0, 1, \ldots, n$. After truncating (12), (13) in their modes, expressions for (8), (9) from Method 1 are then known in terms of given functions. Note that for the Fourier series

$$\mathbf{g} = \sum_{\mathbf{L} \neq \mathbf{0}} \mathbf{g}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{24}$$

where $\sum_{\mathbf{L} \neq \mathbf{0}}$ denotes the sum over all $\mathbf{L} \in \mathbb{Z}^3$ with $\mathbf{L} \neq \mathbf{0}$, the ∇^{-2} operator is defined herein as

$$\nabla^{-2} \sum_{\mathbf{L} \neq \mathbf{0}} \mathbf{g}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}} = \sum_{\mathbf{L} \neq \mathbf{0}} \frac{\mathbf{g}_{\mathbf{L}} e^{ik\mathbf{L}\cdot\mathbf{x}}}{-k^2 |\mathbf{L}|^2}. \tag{25}$$

In Method 1 the assumption of smoothness is only on $\mathbf{u}_0$.

Method 2

Let

$$\mathbf{u} = \sum_{\mathbf{L}=-\mathbf{1}}^{\mathbf{1}} \mathbf{u}_\mathbf{L} e^{ik\mathbf{L}\cdot\mathbf{x}}, \tag{26}$$

$$p = \sum_{\mathbf{L}=-\mathbf{1}}^{\mathbf{1}} p_\mathbf{L} e^{ik\mathbf{L}\cdot\mathbf{x}}. \tag{27}$$

Substituting (26), (27) into (1) and equating like powers of e in accordance with Theorem 2 yields

$$\frac{\partial \mathbf{u}_\mathbf{L}}{\partial t} + \sum_{\mathbf{M}} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M})\mathbf{u}_\mathbf{M} = -\nu k^2 |\mathbf{L}|^2 \mathbf{u}_\mathbf{L} - ik\mathbf{L} p_\mathbf{L}. \tag{28}$$

Substituting (26) into (2) and equating like powers of e in accordance with Theorem 2 yields

$$\mathbf{L} \cdot \mathbf{u}_\mathbf{L} = 0. \tag{29}$$

Applying $\mathbf{L} \times \mathbf{L} \times$ to (28) and noting the vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c} \tag{30}$$

along with (29) yields

$$|\mathbf{L}|^2 \frac{\partial \mathbf{u}_\mathbf{L}}{\partial t} = \sum_{\mathbf{M}} \mathbf{L} \times (\mathbf{L} \times (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M})\mathbf{u}_\mathbf{M}) - \nu k^2 |\mathbf{L}|^4 \mathbf{u}_\mathbf{L}. \tag{31}$$

Equation (31) implies

$$\frac{\partial \mathbf{u}_\mathbf{L}}{\partial t} = \sum_{\mathbf{M}} \hat{\mathbf{L}} \times (\hat{\mathbf{L}} \times (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M})\mathbf{u}_\mathbf{M}) - \nu k^2 |\mathbf{L}|^2 \mathbf{u}_\mathbf{L} \tag{32}$$

where the right hand side of (32) is $\mathbf{0}$ when $\mathbf{L} = \mathbf{0}$ and $\hat{\mathbf{L}} = \mathbf{L}/|\mathbf{L}|$ is the unit vector in the direction of $\mathbf{L}$.

Applying $\mathbf{L}\cdot$ to (28) and noting (29) gives

$$ik|\mathbf{L}|^2 p_\mathbf{L} = -\sum_{\mathbf{M}} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot ik\mathbf{M})(\mathbf{u}_\mathbf{M} \cdot \mathbf{L}) \tag{33}$$

implying that

$$p_\mathbf{L} = -\sum_{\mathbf{M}} (\mathbf{u}_{\mathbf{L}-\mathbf{M}} \cdot \hat{\mathbf{L}})(\mathbf{u}_\mathbf{M} \cdot \hat{\mathbf{L}}) \tag{34}$$

where $p_\mathbf{0} \in \mathbb{R}$ is an arbitrary function of t. Let

$$\mathbf{u}_\mathbf{L} = \sum_{l=0}^{n} \frac{\partial^l \mathbf{u}_\mathbf{L}}{\partial t^l}\Big|_{t=0} \frac{t^l}{l!}, \tag{35}$$

$$p_\mathbf{L} = \sum_{l=0}^{n} \frac{\partial^l p_\mathbf{L}}{\partial t^l}\Big|_{t=0} \frac{t^l}{l!}. \tag{36}$$

Substituting (35) into (32) and equating like powers of t in accordance with Theorem 1 yields

$$\frac{\partial^{l+1} \mathbf{u}_\mathbf{L}}{\partial t^{l+1}}\Big|_{t=0} = \sum_{m=0}^{l} \sum_{\mathbf{M}} \hat{\mathbf{L}} \times (\hat{\mathbf{L}} \times (\frac{\partial^{l-m} \mathbf{u}_{\mathbf{L}-\mathbf{M}}}{\partial t^{l-m}}\Big|_{t=0} \cdot ik\mathbf{M}) \frac{\partial^m \mathbf{u}_\mathbf{M}}{\partial t^m}\Big|_{t=0}) \binom{l}{m} - \nu k^2 |\mathbf{L}|^2 \frac{\partial^l \mathbf{u}_\mathbf{L}}{\partial t^l}\Big|_{t=0}. \tag{37}$$

Equation (37) is then solved for $\frac{\partial^{l+1}\mathbf{u}_{\mathbf{L}}}{\partial t^{l+1}}|_{t=0}$ where $l = 0, 1, \ldots, n-1$ and $-1 \leqslant \mathbf{L}_j \leqslant 1, 1 \leqslant j \leqslant 3$. Substituting (35), (36) into (34) and equating like powers of t in accordance with Theorem 1 yields

$$\frac{\partial^l p_{\mathbf{L}}}{\partial t^l}|_{t=0} = -\sum_{m=0}^{l}\sum_{\mathbf{M}}(\frac{\partial^{l-m}\mathbf{u}_{\mathbf{L-M}}}{\partial t^{l-m}}|_{t=0}\cdot\hat{\mathbf{L}})(\frac{\partial^m\mathbf{u}_{\mathbf{M}}}{\partial t^m}|_{t=0}\cdot\hat{\mathbf{L}})\binom{l}{m}. \tag{38}$$

Equation (38) is then solved for $\frac{\partial^l p_{\mathbf{L}}}{\partial t^l}|_{t=0}$ where $l = 0, 1, \ldots, n$ and $-1 \leqslant \mathbf{L}_j \leqslant 1, 1 \leqslant j \leqslant 3$. Expressions for (8), (9) from Method 2 are then known in terms of given functions.

With $n = 2$, I found that the approximation (8) found from Method 1 is different to the approximation (8) found from Method 2. The difference occurs at $O(t^2)$. Because of this nonuniqueness at least one of the assumptions used was invalid. The only assumptions used are those required for use of Theorem 1 and Theorem 2. Therefore the only way statement (D) could not be true is if the smoothness of $\mathbf{u}$ can break down at an $\mathbf{x} \in \mathbb{R}^3$ where $t \in \mathbb{C}\backslash\{0\}$ but with $t \not> 0$.

It is found that $(\mathbf{u}(\mathbf{x}-\mathbf{\Omega}t, t) + \mathbf{\Omega}, p(\mathbf{x}-\mathbf{\Omega}t, t))$ is a solution to (1), (2), (3), (4), (5), (6) if $(\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t))$ is a solution to (1), (2), (3), (4), (5), (6) where $\mathbf{\Omega} \in \mathbb{R}^3$ is a constant. If there exists an $\mathbf{x} = \mathbf{\Xi}(t) \in \mathbb{R}^3$ at which the smoothness of $\mathbf{u}(\mathbf{x}, t)$ breaks down where $t \in \mathbb{C}\backslash\{0\}$ then the smoothness of $\mathbf{u}(\mathbf{x}-\mathbf{\Omega}t, t)+\mathbf{\Omega}$ breaks down at an $\mathbf{x} = \mathbf{\Theta}(t) \in \mathbb{R}^3$ with $t \in \mathbb{C}\backslash\{0\}$. It is possible to write $\mathbf{\Theta}(t) - \mathbf{\Omega}t = \mathbf{\Xi}(t)$ and therefore the smoothness of $\mathbf{u}$ can then break down at an $\mathbf{x} \in \mathbb{R}^3$ where $t \in \mathbb{R}\backslash\{0\}$.

For $\nu = 0$, it is found that $(\zeta\mathbf{u}(\mathbf{x}, \zeta t), \zeta^2 p(\mathbf{x}, \zeta t))$ is a solution to (1), (2), (3), (4), (5), (6) if $(\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t))$ is a solution to (1), (2), (3), (4), (5), (6) where $\zeta \in \mathbb{R}$ is a constant, so if the smoothness of $\mathbf{u}$ breaks down at $t < 0$ where $\mathbf{u}_0 = \mathbf{U}_0 \in \mathbb{R}^3$ then the smoothness of $\mathbf{u}$ breaks down at $t > 0$ where $\mathbf{u}_0 = -\mathbf{U}_0 \in \mathbb{R}^3$. Therefore statement (D) is true when $\nu > 0$ is replaced with $\nu = 0$.

For $\nu > 0$, when applying Method 1 for $n = 2$ and Method 2 for all $n \in \mathbb{N}$, it is found that the governing equation for $\mathbf{u}$ is effectively

$$\frac{\partial\mathbf{u}}{\partial t} = \nabla^{-2}\nabla\times\nabla\times((\mathbf{u}\cdot\nabla)\mathbf{u}) + \nu\lambda\mathbf{u} \tag{39}$$

where $\lambda = -3k^2$. Equation (39) implies

$$\frac{\partial}{\partial t}(\mathbf{u}e^{-\nu\lambda t}) = \nabla^{-2}\nabla\times\nabla\times((\mathbf{u}\cdot\nabla)\mathbf{u})e^{-\nu\lambda t}. \tag{40}$$

Then a change of variables

$$\tau = e^{\nu\lambda t} - 1, \tag{41}$$

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, \tau)\frac{\partial\tau}{\partial t} \tag{42}$$

yields

$$\frac{\partial\mathbf{v}}{\partial\tau} = \nabla^{-2}\nabla\times\nabla\times((\mathbf{v}\cdot\nabla)\mathbf{v}). \tag{43}$$

Equation (2) becomes

$$\nabla \cdot \mathbf{v} = 0 \tag{44}$$

and the initial condition (3) becomes

$$\mathbf{v}(\mathbf{x}, 0) = \frac{\mathbf{u}_0}{\nu\lambda}. \tag{45}$$

Equations (43), (44), (45) define an Euler problem. If the smoothness of $\mathbf{v}$ breaks down at an $\mathbf{x} \in \mathbb{R}^3$ with $\tau \in \mathbb{R}\backslash\{0\}$, then the smoothness of $\mathbf{u}$ can break down at an $\mathbf{x} \in \mathbb{R}^3$ with $t > 0$. Therefore statement (D) is true. □

Appendix

Theorem 1

Providing that the Maclaurin series

$$\overline{\mathbf{A}} = \sum_{l=0}^{n} \frac{\partial^l \mathbf{A}}{\partial t^l}\Big|_{t=0} \frac{t^l}{l!} = \sum_{l=0}^{n} \frac{\partial^l \overline{\mathbf{A}}}{\partial t^l}\Big|_{t=0} \frac{t^l}{l!} \tag{46}$$

of the exact general solution to a Q^{th} order partial differential equation

$$\frac{\partial^Q \mathbf{A}}{\partial t^Q} = \mathbf{\Psi} \tag{47}$$

exists, it will solve the coefficients of t^l for all $l = 0, 1, \ldots, n - Q$ in (47) with $\mathbf{A} = \overline{\mathbf{A}}$ provided $\mathbf{\Psi}|_{\mathbf{A}=\overline{\mathbf{A}}}$ is expandable in Maclaurin series as

$$\mathbf{\Psi}|_{\mathbf{A}=\overline{\mathbf{A}}} = \sum_{l=0}^{m} \frac{\partial^l \mathbf{\Psi}|_{\mathbf{A}=\overline{\mathbf{A}}}}{\partial t^l}\Big|_{t=0} \frac{t^l}{l!} \tag{48}$$

where $m \geqslant n$. Here all of the partial derivatives of $\mathbf{A}$ with respect to t are defined at $t = 0$.

Proof of Theorem 1

Since the Maclaurin series of $\mathbf{A}$ exists and all of the partial derivatives of $\mathbf{A}$ with respect to t are defined at $t = 0$, one can integrate (47) Q times and then substitute the result into (46) to find

$$\overline{\mathbf{A}} = \sum_{l=0}^{n} \frac{\partial^{l-Q} \mathbf{\Psi}}{\partial t^{l-Q}}\Big|_{t=0} \frac{t^l}{l!} = \sum_{l=0}^{n} \frac{\partial^l \int_Q \mathbf{\Psi}\, dt|_{\mathbf{A}=\overline{\mathbf{A}}}}{\partial t^l}\Big|_{t=0} \frac{t^l}{l!} \tag{49}$$

where $\int_Q \mathbf{\Psi}\, dt$ denotes the Q^{th} integral of $\mathbf{\Psi}$ with respect to t. Substituting $\mathbf{A} = \overline{\mathbf{A}}$ into the residual $\mathbf{r}$ of (47) then gives

$$\mathbf{r} = \sum_{l=0}^{n} \frac{\partial^{l-Q} \mathbf{\Psi}|_{\mathbf{A}=\overline{\mathbf{A}}}}{\partial t^{l-Q}}\Big|_{t=0} \frac{t^{l-Q}}{(l-Q)!} - \sum_{l=0}^{m} \frac{\partial^l \mathbf{\Psi}|_{\mathbf{A}=\overline{\mathbf{A}}}}{\partial t^l}\Big|_{t=0} \frac{t^l}{l!} \tag{50}$$

providing $\mathbf{\Psi}|_{\mathbf{A}=\overline{\mathbf{A}}}$ is expanded in Maclaurin series as in (48). Collecting like powers of t in (50) yields

$$\mathbf{r} = \sum_{l=0}^{n-Q} \frac{\partial^l \mathbf{\Psi}|_{\mathbf{A}=\overline{\mathbf{A}}}}{\partial t^l}\Big|_{t=0} \frac{t^l}{l!} - \sum_{l=0}^{m} \frac{\partial^l \mathbf{\Psi}|_{\mathbf{A}=\overline{\mathbf{A}}}}{\partial t^l}\Big|_{t=0} \frac{t^l}{l!} \tag{51}$$

which shows that Theorem 1 is true. □

Theorem 2

Providing that the Fourier series

$$\widetilde{\mathbf{A}} = \sum_{\mathbf{L}=-\mathbf{N}}^{\mathbf{N}} P(\mathbf{A}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} = \sum_{\mathbf{L}=-\mathbf{N}}^{\mathbf{N}} P(\widetilde{\mathbf{A}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{52}$$

of the exact general solution to a Q^{th} order partial differential equation

$$\frac{\partial^Q \mathbf{A}}{\partial t^Q} = \mathbf{\Psi} \tag{53}$$

exists, it will solve the coefficients of $e^{ik\mathbf{L}\cdot\mathbf{x}}$ for all $-N \leqslant \mathbf{L}_j \leqslant N, 1 \leqslant j \leqslant 3$ in (53) with $\mathbf{A} = \widetilde{\mathbf{A}}$ provided $\mathbf{\Psi}|_{\mathbf{A}=\widetilde{\mathbf{A}}}$ is expandable in Fourier series as

$$\mathbf{\Psi}|_{\mathbf{A}=\widetilde{\mathbf{A}}} = \sum_{\mathbf{L}=-\mathbf{M}}^{\mathbf{M}} P(\mathbf{\Psi}|_{\mathbf{A}=\widetilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{54}$$

where $M \geqslant N$. Here $\mathbf{A}$ is spatially periodic and smooth for all $\mathbf{x} \in \mathbb{R}^3$, $k > 0$ is a constant, and $P(\mathbf{h}, e^{ik\mathbf{L}\cdot\mathbf{x}})$ denotes the projection of $\mathbf{h}$ onto $e^{ik\mathbf{L}\cdot\mathbf{x}}$.

Proof of Theorem 2

Since the Fourier series of $\mathbf{A}$ exists where $\mathbf{A}$ is spatially periodic and smooth for all $\mathbf{x} \in \mathbb{R}^3$, one can integrate (53) Q times and then substitute the result into (52) to find

$$\widetilde{\mathbf{A}} = \sum_{\mathbf{L}=-\mathbf{N}}^{\mathbf{N}} P(\int_Q \mathbf{\Psi}\, dt, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} = \sum_{\mathbf{L}=-\mathbf{N}}^{\mathbf{N}} P(\int_Q \mathbf{\Psi}\, dt|_{\mathbf{A}=\widetilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}}. \tag{55}$$

Substituting $\mathbf{A} = \widetilde{\mathbf{A}}$ into the residual $\mathbf{r}$ of (53) then gives

$$\mathbf{r} = \frac{\partial^Q}{\partial t^Q} \sum_{\mathbf{L}=-\mathbf{N}}^{\mathbf{N}} P(\int_Q \mathbf{\Psi}\, dt|_{\mathbf{A}=\widetilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} - \sum_{\mathbf{L}=-\mathbf{M}}^{\mathbf{M}} P(\mathbf{\Psi}|_{\mathbf{A}=\widetilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{56}$$

providing $\mathbf{\Psi}|_{\mathbf{A}=\widetilde{\mathbf{A}}}$ is expanded in Fourier series as in (54). Equation (56) can be written as

$$\mathbf{r} = \sum_{\mathbf{L}=-\mathbf{N}}^{\mathbf{N}} P(\mathbf{\Psi}|_{\mathbf{A}=\widetilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} - \sum_{\mathbf{L}=-\mathbf{M}}^{\mathbf{M}} P(\mathbf{\Psi}|_{\mathbf{A}=\widetilde{\mathbf{A}}}, e^{ik\mathbf{L}\cdot\mathbf{x}})e^{ik\mathbf{L}\cdot\mathbf{x}} \tag{57}$$

which shows that Theorem 2 is true. □

References

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