An Algebraic Representation for Action-at-a-Distance
I

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ABSTRACT

A new algebra is analyzed that is found to be suitable for representing **asynchronous** action-at-a-distance. This "***Natural Vector***" representation is based on Hamilton's quaternions with the four bases constructed from **real** (4x4) matrices with the scalar component explicitly including the standard square root of minus one, so that natural vectors operate like *a simple non-commuting, complex algebra*. This algebra directly relates the *Wave* equation, the *Continuity* equation and the *Flow* equation together, which are at the center of mathematical physics. This new representation forms the mathematical foundation for a new research programme that challenges several of the basic assumptions of modern physics that are directly related to the metaphysical "*Continuum Hypothesis*". This new form of quaternion algebra described here is found to be mathematically simpler than the similar bi-quaternions and appears to be the natural algebraic form for describing 'relativistic' interactions that implicitly incorporate the asynchronous delays occurring in the *electromagnetic interaction*. These complex four-component vectors are inherently simpler than other alternatives, such as Clifford algebra or Minkowski 4-vectors – they are more physically transparent than tensor calculus. Natural vectors are always anti-symmetric for two point-particles, so they form a natural **fermionic** representation even for 'classical' electrons; their anti-commutative properties lead naturally into a suitable representation for quantum mechanics. The new focus is on the **interaction** between two electrons at **two** different times, rather than other standard theories centered on a single particle or field-point in empty space, at *one single time*. Asynchronous inter-particle interactions, like those found in electromagnetism, are here represented by natural vectors that are both separable and temporally invariant. This is the first paper in a new research programme investigating the foundations of physics.

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1. INTRODUCTION & OVERVIEW

1.1 INTRODUCTION

This paper is the first in an extended series that will report on a new independent research programme into the foundations of physics. The focus of this investigation has been the basic nature of the electromagnetic interaction.

1.1.1 RESEARCH PROGRAMME: OVERVIEW

This research programme has several **objectives**, that can be grouped into three areas: a) methodological b) mathematical c) physical. Methodologically, this programme wishes to re-establish an approach to conducting theoretical physics that formed the successful foundation for physics since Newton but has fallen out of fashion in the 20th century. This older approach emphasized the philosophical investigation of nature and, like all scholarly pursuits during this period, was grounded in a thorough knowledge of the historical developments of the subject, not just the most recent, "modern" viewpoint. As Confucius said, "*study the past if you want to define the future*". The benefits of this historical perspective were that the many contributions of earlier researchers were reviewed, re-analyzed in terms of their assumptions and the results given their due significance in the overall development of the science. Challenging the **assumptions** of modern physics has been a critical part of the present programme. Another reason for this approach is that the history of physics demonstrates that whenever philosophical concepts have preceded mathematics then stronger theories have resulted. Modern physics shows the profound problems that arise when mathematical innovation precedes conceptual innovation – scientists are left with equations that cannot be interpreted coherently and theoretical progress stalls for lack of intuitive and especially visualizable inspiration.

Mathematically, but again related to the historical perspective, this programme will focus on the almost forgotten area of discrete mathematics, as this is believed to better reflect the real nature of the world. In contrast, for over 300 years physics has thoroughly explored the consequences of the "***Continuum Hypothesis***" – the metaphysical position that reality is best described in terms of the "plenum" (the continuous inter-connectedness of the world). This has been reflected in both the extensive use of the infinitesimal calculus and continuous group theory. The present work extends the almost forgotten work of Sir William Rowan Hamilton, namely, real **quaternions** into a specialized form that is referred to here as "**Natural Vectors**". So, one of the principal objectives is to demonstrate that this new form of discrete mathematics is ideally suited to the study of the fundamental interactions between the basic entities that together constitute the nature of the world. Since this representation is a linear algebra it should prove easier to apply than the problematic areas of mathematics grounded in continuous change that can readily throw up unwanted infinities. As will be shown later, Hamilton's original form of quaternions, involving only real parameters, although suited for geometry, does not reflect nature, whereas Natural Vectors (which are the imaginary scalar version of Hamilton's quaternions) do correspond closely to reality; this is why **Maxwell** could not make Hamilton's real quaternions fit his theory of electromagnetism. Natural Vectors apply to all systems where the interaction occurs **asynchronously** between pairs of point-like objects – a revised Newtonian particle-based model.
It might prove helpful to clarify the thrust of this research programme by revising an explanation used by Einstein to explain the origins of the theory of special relativity that has been quoted by Holton [1]: "*the investigation into the phenomenon of electromagnetic induction has forced the abandonment of relativity*." Indeed, this new programme reinterprets Minkowski's famous quotation [2]: "*Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality*." In contrast, following Newton, reality is here viewed as always consisting separately of both space and time, with time itself inexorably evolving at a constant rate everywhere while spatial separations also remain invariant to all frames of reference, no matter what type of motion they undergo relative to one another. Since these two concepts are here taken to be distinct, invariant and passive, they are viewed, like **Kant**, as the foundational framework of reality. This invariance and distinctiveness of the spatial location at any time, for every electron, is mapped here by the fundamental location natural vector, $\mathbf{X}$; and especially, the two-point separation: $\mathbf{S} \equiv \mathbf{X}_1 - \mathbf{X}_2$.

In this programme, there is no fundamental difference assumed between the nature of the macro world of human experience and the micro world of atoms and electrons, except for differences in the numbers of objects involved and the time-scales of the relative phenomena – the macro world just aggregates and time-averages the details.

So, for example, human beings consist of the same material "*stuff*" as the micro world, namely **electrons**, just a lot more of them – there is no defined 'boundary' where the real material of the macro world transforms into the mysterious 'wavicles' of the quantum world. Ultimately, this programme will demonstrate that matter is synonymous with electrons – no more, no less. However, in contrast to Newton's circular **definition** of mass as the quantity of matter, this programme views mass as Newton **used** the concept, namely as a dynamical measure of a body's motion: the ratio of its long-term average change in momentum relative to the change in the standard body's momentum in response to a unit of interaction; in this case, a single electron's response to one single interaction with a second electron. Finally, this programme does not place **energy** on a pedestal, as it does not view this key concept as reflecting any fundamental entity in reality. As the great Oliver Heaviside wrote [3] in 1881 (sharply contradicting his overly enthusiastic fellow-Maxwellian, Oliver Lodge): "*We need not go so far as to assume the objectivity of energy, which seems to be inadmissible by the mere fact of the relativity of motion, on which the kinetic energy idea depends. We cannot, therefore, definitely individualize energy in the same way as is done with matter*." Only an identifiable 'parcel' of energy could be said to move through space but light (or a **photon**) has no identity. Recently [4], Marc Lange, a philosopher of physics, has done an excellent job of destroying the simplistic interpretations of the famous equation $E = mc^2$ where mass and energy are viewed as the same "thing" (like materially, steam and water are the same thing) by focusing on the Lorentz invariance of 'rest' mass and noting that energy is only one of the components of the energy-momentum 4-vector, which by itself is not invariant. Lange does not even allow Feynman (who disliked philosophy) to get away with phrases such as "associated with" in explanations like: "*the energy associated with the existence of the rest mass of a particle*" (p. 229), describing such terms as "*weasel words*" that are too vague to be worthy of philosophical, especially ontological, consideration. The precedence here for philosophy is based on the observation that natural languages are the richest, most extensive and powerful medium of communication between humans while all mathematics is deductive and narrowly focused; its symbols and significance must always be discussed in a natural language. The analysis of meaning and concepts are at the heart of **philosophy** and have been central to the thinking of Western Civilization since the very earliest attempts at producing a natural understanding of the world. Moreover, the philosophical subject of **ontology** (or the study of existents) has always been at the very foundation of physics. Metaphysical propositions of what are the fundamental objects that constitute the world and how they might behave must be made explicit, in the style of Newton in his *Principia*, rather than the implicit approach that is hidden when "positivistic" physics is formally pursued or when the 'Pythagorean' programme is silently being promoted by academics of a mathematical orientation.

The following table contrasts the view of key concepts in this research programme with those found today in the mainstream of physics ("Standard Physics"); the critical 'continuum versus discrete' is marked by an asterisk (*).

CONCEPT	**STANDARD PHYSICS**	*	**RESEARCH PROGRAMME**
World View	Positivistic		Realist
Space	Active (plenum)		Inert
Time	Continuous	*	Discrete
Temporal Focus	Single Time	*	Two-Time
Interaction Range	Spatio-temporal Locality	*	Action-at-a-Distance
Metaphysics	Pythagorean (math)	*	Newtonian
Core Entity	Field	*	Particle
Cause	Force	*	Impulse
Focus	Space-Time Point	*	Pair-wise Interaction
Energy Model	Photon (entity)	*	Kinetic (quality)
Mass	Variable (Planck)	*	Fixed (Newton)
Dynamics	Hamiltonian	*	Newton's Impulse Law (II)
Mathematics of Change	Differential Equations	*	Difference Equations
Algebra	Vectors		Natural Vectors (quaternions)
N-Body Interactions	Vector Summation		Pair-wise (saturated)
Velocity Invariance	Lorentz (relativity)		Galilean (Newtonian)

In summary, where physics has obsessed on continuous "**forces**" on a single particle, this programme focuses on the **discrete interaction** between **two** electrons. The research programme's objectives can therefore be summarized as:

1. Restore the roles of metaphysics and visualization to the evolution of the foundations of physics.
2. Emphasize the value of the history of physics, as a source of understanding and new ideas.
3. Revive the reputation of many physicists, whose historical contributions have been overlooked.
4. Demonstrate the value of returning to Newton's metaphysical and mathematical approach to theoretical physics: emphasizing the **discrete** nature of the world.
5. Demonstrate that Natural Vectors should be one of the most powerful mathematical techniques in the 'tool-bag' of physicists.
6. Remove the mysteries and paradoxes of 20^{th} century physics, especially those arising directly from the present theories of relativity and quantum mechanics.
7. Solve the dynamics of systems involving a small number (less than 10) of electrons, across a whole range of relative velocities.

1.1.2 MOTIVATION

Dirac has stated in lectures he gave in Australia in 1975, describing the origins of the modern quantum theory [5]: "*In any physical theory one usually knows just what one's equations mean before one sets them up. But (in 1926) we had the equations before we knew how to apply them*." He ended his lecture with his often-repeated complaint about the current state of quantum mechanics and especially quantum electrodynamics (which he invented): "*I just cannot accept that the present foundations are correct. People are, I believe, too complacent in accepting a theory, which contains basic imperfections and a true advance will be made only when some fundamental alteration is made. ... I must say I am very dissatisfied with this situation because this theory involves neglecting infinities which appear in its equations – this is just not sensible mathematics*."

Classical electromagnetism is still being used to justify all subsequent field theories although it fails to reflect the discrete nature of electrical charge and **cannot** even be used to solve the exact dynamics of just two charges. It was also the justification for Einstein's special theory of relativity, which has imposed constraints such as Lorentz invariance on all areas of theoretical physics. Subsequent papers will show that Einstein's theory is a misreading of the fundamental electron interaction. Additionally, it will be shown that the Coulomb force is not a fundamental view of nature but is itself a macro, time-averaged approximation that should not be extrapolated to the microcosm.

A physical theory always reflects some world-view that almost always has a basis in **metaphysics**. When a theory makes its metaphysical assumptions clear, as Newton did, it allows others to judge whether these assumptions are reasonable or not. This explicit, but out-of-fashion, approach will be followed throughout this research programme.

1.1.3 OBJECTIVES OF THIS PAPER

The present objectives of this first paper can be summarized as:

A Re-awaken interest in Hamilton's greatest mathematical innovation - quaternions.
B Introduce the concept of natural vectors and emphasize the significance of Voigt vectors.
C Demonstrate the role and value of natural vectors in theoretical physics.
D Re-direct attention from the single object focus in physics to the two-particle interaction.
E Demonstrate that natural vectors are an excellent representation of two-particle interactions.
F Announce a new research programme using natural vectors to investigate the foundations of physics.

1.2 OVERVIEW

In this overview, the contents of the paper will be summarized to include a brief description of each section and the major reasons the material has been included.

1.2.1 QUATERNIONS

The next section begins with a brief review of Sir William Hamilton, the creator of quaternions that form the central focus of the mathematics described herein; it is included to give the reader a sense of one of the intellectual giants of the 19th Century. This is followed by a quick summary of Hamilton's mathematical contributions, particularly as they have impacted theoretical physics and especially quantum mechanics. Hamilton's 1843 breakthrough in the evolution of **algebra** is described next, emphasizing the metaphysical viewpoint that enabled him to dispense with the commutative law of multiplication that moved algebra beyond the rules of arithmetic. The main mathematical features of quaternions are revealed through a brief review of Hamilton's quaternion publications. These papers illustrate his highly original and creative imagination, as well as his immense mathematical productivity. The papers selected describe how Hamilton introduced several major foundations of modern theoretical physics such as the concepts of scalars and vectors and, especially, the vector partial derivative, ∇. Quaternions are then situated from the perspective of modern algebra with an emphasis on the (2x2) and (4x4) matrix representations that are the main foundation for the remainder of the paper. Such matrices were used by Pauli (spin) and Dirac (relativistic electrons).

Section three describes the use of quaternions in physics over the last 150 years, beginning with the contributions of two of Hamilton's greatest admirers, P. G. Tait and J. C. Maxwell, both leading natural philosophers of the 19th Century; they were each convinced that this new algebraic breakthrough would clarify both mechanics and the study of electricity and magnetism. The remainder of this section discusses the fact that it was the 'complexified' form of Hamilton's quaternions that have proven most useful in electromagnetism and in the areas of modern physics such as relativity, quantum mechanics and Dirac's theory of the electron; which is surprising, as all of these areas were completely unknown in Hamilton's time. Although this section reveals that quaternions have been mainly ignored in the development of physics, their occasional use (in the form of bi-quaternions) illustrates the astonishing power of this representation in the area of physics associated with the modern view of the electron – a foreshadowing of the results, which will be presented later in this series, in terms of their logical successor: 'Natural Vectors' (**NV**s).

1.2.2 NATURAL VECTORS

Section four begins with a brief discussion of how Maxwell might have failed to summarize all his electromagnetic equations when he tried to use Hamilton's quaternions. This leads into the discussion of how Hamilton's intuition about space and time applied to the four generalized quaternion bases $\mathbf{I}_\mu$ suggests the idea of a new group, referred to here as the '**Interaction Group**'. A formal sub-set of Hamilton's bi-quaternions is selected to form mathematical objects called 'Natural Vectors' – these may be viewed simply as **imaginary, scalar quaternions**. By requiring that the 3D part of natural vectors remain real under time reversal, represented by standard complex conjugation, then only a real (4x4) matrix representation can be used for the four bases. Therefore only complex conjugation is needed here, as with complex numbers – a much greater simplification than is usually used with bi-quaternions. A set of "mapping-rules" is introduced to set up a one-to-one correspondence between a Newtonian-like 'reality' and natural vectors; these include the explicit introduction of a universal space-time constant ratio (*c*) for homogenizing all natural vectors. This will be shown to be the constant 'speed' of **interactions** between particles.

This paper focuses on the "Continuum Hypothesis" so that in section five the sub-set referred to as Continuous Natural Vectors (**CNV**s) is developed; a later paper will introduce **discrete** versions that will subsequently play a central role in this programme. In order to investigate Classical Electro-Magnetism it is necessary to emphasize total-time differentials – these are used extensively in the next paper investigating a Natural Vector Model of EM. Here we first define the concept of 'Flow Vectors' in terms of a set of 'Zero Conditions' that must apply to CNVs.

1.2.3 VOIGT VECTORS

In section six, a specific functional form of CNVs is defined, called Voigt Vectors, in honor of one of the pioneers of wave equations and their invariant transforms. These mathematical objects will later figure prominently in *The CNV Theory of Electromagnetism* [6]; their components always satisfy a velocity condition, called here the Lorenz Equation, as it re-appears in classical electromagnetism as the Lorenz gauge condition for the electromagnetic potentials. Harmonic Voigt vectors exhibit interesting characteristics, including ray-like, retarded and advanced, propagating solutions. The CNV gradients of Voigt vectors and scalars also exhibit features that will prove directly relevant to classical electromagnetism, including force 'field' intensities and gauge transforms. It is also proved here that there are no Voigt vectors that are functions only of a single object's local velocity.

1.2.4 NATURAL VECTORS IN PHYSICS

Section seven applies some of these general formulations of natural vectors to several well-known situations in physics, beginning with a kinematic analysis of the motion of a particle represented by a *Spatial Displacement* CNV. The famous *Wave Equation*, that has been at the heart of mathematical physics for over 150 years now, is an immediate consequence of proposing that the (CNV) Gradient of any other CNV is zero. The *Continuity Equation*, arising in the study of moving continuous media, is shown to be a direct consequence of the Voigt *Lorenz Equation*; these three equations are shown to be intimately inter-related. A major sub-section (7.4) is devoted to natural vector **invariants** distinguishing scalar and vector forms as these play a central role in modern physics. Here the *Separation* CNV is shown to be the central concept in the physics of two-particle interactions, leading to the invariant space-time interval of **special relativity**. Natural vectors that are both separable and form temporal invariants lead to the idea of conserved physical quantities or the interaction between two remote particles across space and time – in other words, **asynchronous action-at-a-distance**. Natural vectors immediately generate several invariants in a system of two interacting particles. A comparison is made between the fully functional CNVs and Minkowski 4-vectors, since it might be readily misunderstood that these two concepts are synonymous; in fact, one might think of natural vectors as "covariant Minkowski quaternions".

Several examples of fundamental physical quantities in classical physics are identified with Voigt vectors by choosing simple forms for the Voigt functional parameter; these are further classified into three sub-groups, whose properties are briefly examined. One class of these Voigt-type vectors maps directly to Planck's 1907 proposed form for a particle's **relativistic momentum**; this immediately results in all the well-known formulae of relativistic mechanics.

1.2.5 ASYNCHRONOUS INTERACTIONS

Section 8 discusses the concept of asynchronous interactions between two particles at two different times, as this idea is at the very heart of this research programme. Momentum exchange between the particles is shown to be central to this analysis, suggesting that the two-particle Natural Vector (NV) representation can be viewed as the 'momentum exchange' group. The concept of asynchronous conservation is introduced and this challenges the nearly universal assumption of continuous conservation at all times that is one of the major justifications for the use of field theories in modern physics. The traditional Euler-Lagrangian (continuum) re-formulation of Newtonian mechanics centered on the concept of **time-independent** spatially sensitive potentials has also been by-passed in this programme. Such potentials are a direct consequence of assuming instantaneous transfer of finite momentum from one particle to another remote particle: the foundation of the Galileo/Newton dynamical theory of physics. In fact, the present representation may be viewed as the generalization of the instantaneous Galilean group to one more closely representing the reality of asynchronous interactions. Schematic diagrams are also presented here to assist with a visual understanding the 4D nature of NVs. This section concludes with a discussion of various 're-labeling' operations on two-particle NVs – this will be used later to establish important symmetry constraints on the resulting particle dynamics.

2. HAMILTON'S QUATERNIONS

2.1 W.R. HAMILTON

William Rowan Hamilton was born in Dublin, Ireland in 1805. He was a child prodigy who could read English by three, Hebrew, Latin & Greek by the age of five and most of the major European languages plus Arabic, Persian, Sanskrit, Malay & Bengali by ten. His arithmetic skills did not begin until he was six but he had mastered algebra by thirteen. He graduated from Ireland's leading university (Trinity College, Dublin) with the highest honors (two gold medals) in science and classics – an achievement previously unheard of. At age **22** before he had even finished his degree, he was appointed Professor of Astronomy at Trinity with the title of Astronomer Royal of Ireland, a position he held for the rest of his life. He was knighted in 1835 for his scientific achievements and remained active intellectually until the very end of his life in 1865. Hamilton's life and scientific work have been well described in the comprehensive biography [7] by Thomas Hankins, one of the leading, modern historians of science.

Hamilton submitted his first paper ("On Caustics") when he was **19** based on an idea he developed two years earlier. This paper was so advanced that it confounded even his referees; he revised it and resubmitted it to the Irish Royal Academy two years later as "Theory of Rays": a work, which made his reputation and set his research course for the next eight years. His "**characteristic**" function summarized the paths of light through any optical system based on Fermat's Principle of **Least Time**, whether light itself was viewed as waves or particles. In 1834, he announced the extension of his characteristic function from geometric optics to particle mechanics in his two most famous papers entitled "General Method in Dynamics" [8] invoking Maupertuis's Principle of **Least Action**. Although Hertz later remarked this was still only a mathematical theory, soon after, in 1916 Sommerfeld clearly saw its value in the development of the "old" quantum mechanics. Indeed, Schrödinger later wrote that in 1926 he was directly inspired by this technique when he took it over directly into "wave" mechanics.

Hamilton, like Lagrange, had indeed found another way to write down Newton's Equations of motion for a system involving **instantaneous** forces or for conservative systems using spatially sensitive potentials. In particular, for any dynamical variable 'q' in a system characterized by a Hamiltonian function, 'H' the central dynamical equation now became: $dq/dt = \{q, H\}$ where $\{a, b\}$ is the classical Poisson bracket. This was a technique re-introduced by **Dirac** in 1926 and led directly to the modern formulation of quantum mechanics; the only one based on classical mechanics.

In 1832, Hamilton used his theory to predict conical refraction in biaxial crystals and this was soon confirmed experimentally – this confirmation "*electrified the scientific community*" [7, p.95] and won him the Royal Society's Royal Medal in 1855 (Faraday was the other winner that year). This achievement was the justification for his early knighthood by the Lord Lieutenant of Ireland, as interest in optics was at the centre of British physics at this time. Like most 19^{th} Century supporters of the wave theory of light, Hamilton believed in the reality of the **aether**, but for metaphysical reasons he believed it to consist of myriads of microscopic attracting and repelling points. He failed to develop an aetherial model of light but in 1839 he was the first to distinguish between the phase and group velocities of traveling waves – a discovery that history later, and erroneously, credited to Lord Rayleigh in 1877.

2.2 QUATERNIONS

Although negative and imaginary numbers had been admitted into algebra since the 17^{th} Century (so that every polynomial equation could be proven to have at least one root), it was still a challenge to mathematicians in the early 1800s to understand what these "*impossible*" quantities represented – this led into an extensive investigation into the foundations of algebra. Hamilton was very active in this area and was inspired by studying Kant to view algebra as "*the science of time*", complementing geometry as the science of space. It is usually a surprise to modern readers to realize that Hamilton's metaphysics was more productive for his mathematical research than for his researches in physics.

Accordingly, Hamilton invented a non-arithmetic form of complex numbers that he called "number couples", effectively: $z = x + i\,y \equiv (x;y)$. This resulted in Hamilton's first major work on algebra ("*Theory of Conjugate Functions or Algebraic Couples, with an essay on Algebra as the science of Pure Time*") that was published in 1835. This was a major step away from the universal idea (at that time) that algebra is always related to the operations of arithmetic. A theory of "triplets" was the obvious extension of the theory of "couples", partly because three was the next number and partly because complex numbers had recently been mapped into 2D space (Argand diagrams).

For over 13 years, Hamilton tried to generalize his ordered couples to similar triplets until finally on October 16, 1843 while walking by the canal in Dublin near the Brougham Bridge did he suddenly realize that only an ordered quartet of real numbers could form a suitable algebra. This led to his major mathematical innovation: **quaternions**. It was then that he realized that needed two extra imaginaries to form his new triplet of unit imaginaries $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. The biggest surprise was to discover that these imaginaries did not obey the commutative law of multiplication, so that, unlike the normal algebra of real numbers where $x\,y = y\,x$, for quaternions it was found that $\mathbf{i}\,\mathbf{j} = -\,\mathbf{j}\,\mathbf{i}$.

Hankins describes at the end of his chapter on *The Creation of Quaternions* how on the same day that Hamilton discovered quaternions he tried to find a geometrical interpretation for them. He decided that the quaternion's triplet set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ represented the three dimensions of space while the real term represented time. He found immediately what we now call the scalar and vector products of two directed lines in space. He was immediately convinced that these insights had great significance for the future of mathematics and physics. As Hankins writes [7]: "*The actual path that he followed was algebraic, yet his metaphysical speculations had given him insights that were not so obvious to his rivals. It was algebra as 'the science of pure time' that allowed him to dispense with the commutative law*." His rivals in the search to extend complex numbers (Augustus De Morgan and John Graves) were shocked to see how Hamilton had simply "imagined new imaginaries" – the road was blasted open to a wide variety of algebras that did **not** follow the rules of ordinary arithmetic. As author John Darbyshire quotes [9] in his recent best selling history of algebra – this is now viewed as "*one of the most important revelations in the history of mathematics*".

2.2.1 HAMILTON'S QUATERNION PUBLICATIONS

All references here refer to actual publication dates, although Hamilton usually had presented his papers to The Irish Royal Academy several months, if not years earlier. All of the following papers can be obtained through the website established by Professor D. R. Wilkins, Trinity College, Dublin (*Hamilton's Research on Quaternions*) [10].

2.2.1.1 On a New Species of Imaginary Quantities

Hamilton first announced his invention of quaternions to the Irish Academy on November 13, 1843, this speech was subsequently printed as a short paper in the following year. In this first quaternion paper, Hamilton introduces his new extensions of the single square root of negative one as three linearly independent imaginary quantities $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ [11] by his famous definitions:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\,\mathbf{j}\,\mathbf{k} = -1$$

He uses these to define a real quaternion, $\mathbf{Q}$ with real coefficients (w, x, y, z) that satisfies the rules of arithmetic (add, subtract, multiply & divide):

$$\mathbf{Q} \equiv w + \mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z$$

He defines a positive, real number M, the modulus of $\mathbf{Q}$ as : $M^2 \equiv w^2 + x^2 + y^2 + z^2 \geq 0$

So every quaternion, $\mathbf{Q}$ has an inverse $\mathbf{Q}^{-1}$, thus supporting division, defined as: $\mathbf{Q}^{-1} \equiv (w - \mathbf{i}\,x - \mathbf{j}\,y - \mathbf{k}\,z) / M^2$

Hamilton also notes that $\mathbf{Q}$ is a solution of the real quadratic equation: $\mathbf{Q}^2 - 2\,w\,\mathbf{Q} + M^2 = 0$

He defines an angle θ as the amplitude of any quaternion, $\mathbf{Q}$ by:

$$\mathbf{Q} \equiv M (\cos\theta + \mathbf{I}_n \sin\theta) \quad \text{where } \mathbf{I}_n \text{ is any } \textbf{one} \text{ of unit square roots of minus one } \{ \mathbf{i}, \mathbf{j}, \mathbf{k} \}.$$

Hamilton explores the relationship between rotations in spherical trigonometry by defining points (x, y, z) on the unit sphere centered on the origin with polar co-ordinates ϕ , ψ :

$$w = M \cos\theta \ , \ x = M \sin\theta \cos\phi \ , \ y = M \sin\theta \sin\phi \cos\psi \ , \ z = M \sin\theta \sin\phi \sin\psi$$

The imaginary units, **I** form an infinite family of points on this unit sphere when M = 1 and w = 0. Quaternions can be used to define polynomials, exponentials and logarithmic functions based on the identity:

$$\mathbf{Q}^K = M^K \{ \cos K(\theta + 2\pi m) + \mathbf{I}_n \sin K(\theta + 2\pi m) \} \text{ where K is real and m is an integer.}$$

Most importantly for the subsequent developments of modern algebra, Hamilton now explicitly points out for this new triple of imaginary quantities (**i, j, k**) that "*the commutative character is lost*", e.g. $\mathbf{i}\,\mathbf{j} \neq \mathbf{j}\,\mathbf{i}$.

2.2.1.2 On Quaternions

A year later, Hamilton presented his second paper [12] to the Irish Academy where he introduced the concepts of **scalars** and **vectors** as the separable parts of every real quaternion, **Q** :

$$\mathbf{Q} = a + \alpha \quad \text{where} \quad \alpha = \mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z \ \text{ and a, x, y, z are all real.}$$

After identifying the imaginary part ('α') with a straight line in space (the '*vector*') he rhetorically asks about the real part 'a' (the '*scalar*'): "*in this case, what does the scalar correspond to in geometry?*" This question will be answered later in this paper when the concept of a 'Natural Vector' is introduced. These ideas lead directly to the important mathematical concepts of scalar and vector products arising from multiplying two vectors together in forming his '**Quaternion Calculus**' in terms of a vector β_1 parallel to any vector α and β_2 normal to the vector α:

$$\alpha\,\beta = \alpha\,(\beta_1 + \beta_2) \cong \alpha \bullet \beta_1 + \alpha \wedge \beta_2 \quad \text{(modern notation)}$$

Letting A and B define the lengths of the lines α and β and (A, B) representing the angle between these lines with the symbol γ representing a unit vector perpendicular to both:

$$\alpha\,\beta + \alpha\,\beta = 2\ \alpha\,\beta_1 = 2\,A\,B\cos(A, B) \qquad \alpha\,\beta - \alpha\,\beta = 2\ \alpha\,\beta_2 = 2\,A\,B\sin(A, B)\,\gamma$$

This new approach allows Hamilton to easily derive several 3D geometrical relationships that are now trivial with vector algebra but are quite cumbersome with traditional trigonometry, such as calculating the resultant of several rectilinear displacements or summing several static forces acting at a single point. He also shows the power of his new 'calculus' by analyzing several successive rotations to a rigid body in 3D space, deriving the surprising 'half-angles' that always result [13] from any analysis of finite 3D rotations in classical mechanics.

2.2.1.3 Researches respecting Quaternions

Hamilton's first full account (requiring almost 100 pages) of the results of his first few years of research [14] on his quaternions is found in this paper. He now discusses the algebraic foundations of quaternions as a particular instance of ordered n-tuples (n = 4) as an extension of regular imaginary numbers as ordered couples (n = 2). Since Hamilton viewed algebra as "*The Science of Time*", he viewed sets of ordered numbers as examples of sequences of "moments of time" which were to be treated as a single unitary object, subject to four "Moment operators", $\boldsymbol{M}_j$:

$$\mathbf{Q} \equiv (A_0, A_1, A_2, A_3) \ \& \ \boldsymbol{M}_j\,\mathbf{Q} \equiv A_j \ \text{ so } \ \mathbf{Q} = (\boldsymbol{M}_0\mathbf{Q}, \boldsymbol{M}_1\mathbf{Q}, \boldsymbol{M}_2\mathbf{Q}, \boldsymbol{M}_3\mathbf{Q})$$

Hamilton examines permutations of these four basic components. He anticipates the extension to eight components 'biquaternions', when each one of the factors is a complex number (not to be confused with Graves's octonians).

2.2.1.4 On a New System of Imaginaries in Algebra

Hamilton subsequently published many papers developing the theory of quaternions including a 90 page paper [15] published in 18 installments in what was later known as *The Philosophical Magazine*. He repeats his discussion on spherical trigonometry and his decomposition of quaternions into scalar and vector parts to prove several theorems in 3D geometry, especially involving ellipsoids. Most importantly, he introduces his 3D '***nabla***' operator (the vector quaternion version of the 3D gradient operator, ∇) and identifies the square of this with the **Laplacian** operator, ∇^2.

2.2.1.5 On Symbolical Geometry

Hamilton also published another large (72 page) paper in 10 installments [16]. In this series, he develops the concept of vectors in 3D space as an algebra of directed lines, avoiding completely the use of Cartesian coordinates or any trigonometry. He maps points in 3D space to single letters (e.g. A), while directed lines are represented by ordered pairs of letters, like AB to represent the line from point A to point B. Hamilton is interested in directed lengths, like Grassmann, rather than point-to-point specific lines (i.e. independent of any coordinate origin); we can now recognize that here, he was introducing the modern concept of 3D **vectors**.

2.2.1.6 Books

Hamilton published one major book *Lectures on Quaternions* consisting of over 700 pages of detailed mathematics describing over ten years [17] of research on quaternions in 1853. Only a few copies were sold. Hamilton's second book *Elements of Quaternions* at over 800 pages was almost complete at the time of his death and was published posthumously [18] in 1866 – this too, was **not** a bestseller.

2.2.2 MODERN VIEWPOINT

2.2.2.1 Quaternions

Today, quaternions are viewed mathematically [19] from several, complementary viewpoints. Following Hamilton, who treated a complex number as an ordered pair of real numbers, quaternions are now seen as a non-commutative extension of complex numbers. The Cayley-Dickson construction generates a real quaternion from an ordered pair of complex numbers, each using a different root (i & j) of minus one, with the non-commutative rule that $i\,j = -\,ji$. Algebraically, quaternions are viewed as forming a 4-dimensional normed-division algebra over the real numbers. According to the Frobenius theorem, quaternions were proven to be one of the only three finite-dimensional division rings involving the real numbers as a sub-ring, where a ring is an algebraic structure, like a field, with operations of addition and multiplication, except for the commutativity of multiplication. The quaternions, along with real and complex numbers, are the **only** associative division algebra over the field of real numbers. John Graves had found that octonions were not associative under multiplication. The set of all quaternions can also be viewed as forming a 4-dimensional vector space over the real numbers. Under multiplication the four unit quaternion bases {1, **i**, **j**, **k**} along with their negatives, form the quaternion group of order 8, usually denoted $\mathcal{Q}_8$.

Quaternions are subject to addition and negation (and thus subtraction) with a **spatial** inverse (or 'conjugate'), which is defined as negation (and indicated by an asterisk) only on the imaginary or vector components:

$$\mathbf{Q} = q_0 + \mathbf{i}\,q_1 + \mathbf{j}\,q_2 + \mathbf{k}\,q_3 = q_0 + \underline{q} \quad \text{so that} \quad \mathbf{Q}^* \equiv q_0 - \underline{q}$$

There are several possible binary products that can be defined between two quaternions, **A** and **B** :

1. Grassmann Product: $\mathbf{A\,B} \equiv (A_0\,B_0 - \underline{A}\bullet\underline{B}) + (A_0\mathbf{B} + B_0\mathbf{A}) + (\mathbf{A} \wedge \mathbf{B})$
2. Inner Product: $\mathbf{A}\bullet\mathbf{B} \equiv (A_0\,B_0 + \underline{A}\bullet\underline{B}) = \tfrac{1}{2}(\mathbf{A}^*\mathbf{B} + \mathbf{B}^*\mathbf{A})$
3. Outer Product: $\mathbf{A}\otimes\mathbf{B} \equiv (A_0\,\mathbf{B} - B_0\,\mathbf{A}) - (\mathbf{A} \wedge \mathbf{B}) = \tfrac{1}{2}(\mathbf{A}^*\mathbf{B} - \mathbf{B}^*\mathbf{A})$
4. Grassmann Outer Product: $\mathbf{A}\wedge\mathbf{B} \equiv (\mathbf{A} \wedge \mathbf{B})$

There are at least two ways of representing quaternions as **matrices**, in such a manner that quaternion addition and multiplication correspond with matrix addition and multiplication. The first uses (**2x2**) complex matrices:

$$\mathbf{Q} \cong \begin{vmatrix} q_0 + iq_1 & q_2 + iq_3 \\ -q_2 + iq_3 & q_0 - iq_1 \end{vmatrix}$$

When q_2 and q_3 are zero the quaternion is reduced to a complex number, which corresponds to a **diagonal** matrix.

The second matrix representation uses (**4x4**) real matrices:

$$\mathbf{Q} \cong \begin{vmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{vmatrix}$$

In this representation, the conjugate of a quaternion corresponds to the transpose of the matrix.

2.2.2.2 Related Approaches

In 1878, William Kingdon Clifford used Hermann Grassmann's ideas to generalize Hamilton's quaternions into a whole family of n-dimensional algebras; these became the direct ancestors of modern **spinors**, which are ordered pairs of complex numbers. In the early 1880s, both Josiah Willard Gibbs and Oliver Heaviside independently separated Hamilton's vectors from his quaternions and treated these as independent mathematical objects, founding **modern vector analysis**, which has completely over-shadowed the quaternions from which they were derived.

The Clifford / Geometric Algebra of Hestenes [20] augments the familiar vector space with a structure that permits processing geometric objects that are more complex than points or line-segments. Hestenes' space-time algebra [21] is defined in terms [22] of the four Clifford algebra $\mathcal{C}(1,3)$ generated by the four vectors γ_μ which are isomorphic to the Dirac algebra or the Clifford algebra $\mathcal{C}(4,0)$ generated by the four basis vectors $\mathbf{e}_\mu$ satisfying: $\mathbf{e}_i\,\mathbf{e}_j + \mathbf{e}_j\,\mathbf{e}_i = \delta_{ij}$. These are equivalent to the mappings: $\gamma_0 = \mathbf{e}_0$, $\gamma_j = \mathbf{e}_j\,\mathbf{e}_0$ so $\underline{\mathbf{e}} \cong \underline{\sigma}$.

The fundamental objects of Clifford algebra are vectors, corresponding to physical 3D displacements. This research programme believes that this older approach is more suitable to the time-less, scale-less subject of geometry, whereas the world studied by physics is dynamical (**time-based**), where special points in space (i.e. point particles) moving through time form a simpler conceptual basis. These special points are not the universal points of space itself (i.e. a 3D manifold) but the finite number of points that correspond to the actual location of real particles at any time.

Although Hamilton dedicated the last 22 years of his life researching quaternions, they have never been viewed by modern mathematicians as more than a backwater, taught to math undergraduates only as brief diversion to a course on group theory or linear algebra. Hamilton soon generalized his real quaternions in 1844 to **bi-quaternions**, where each co-efficient was a complex number: these are sometimes referred to as 'complexified quaternions' needing eight real numbers. We shall see in the next section, how this 'after-thought' has proved useful in mathematical physics and leads directly to **Natural Vectors**, the subject of this paper.

3. QUATERNIONS IN PHYSICS

The impact, that Hamilton's quaternions have had in physics, has been very well described in the invitation paper *The Physical Heritage of Sir W. R. Hamilton* [23] contributed by Gsponer and Hurni to mark the 150th anniversary celebrations of the invention of quaternions held in 1993 at Hamilton's alma mater, Trinity College, Dublin. As the authors write in their introduction "*contrary to what is often said, Hamilton was right to have spent the last 22 years of his life studying all possible aspects of quaternions*" – an opinion shared by this research programme. As these two authors correctly point out, it has not been the real quaternions but those explicitly involving complex numbers (bi-quaternions) that have made significant contributions to classical electromagnetism, quantum mechanics and "*spin*". They also mention that quaternion algebra yields more efficient processing algorithms than matrix algebra for applications in 3 and 4 dimensions; the use of quaternions in computer simulations & graphics, numerical and symbolic calculations, robotics, navigation etc has become much more frequent in the last few decades. These two authors have also published an enormous online bibliography of over 1200 publications using (bi-)quaternions in mathematical physics, including a further 50 references to octonions [24] in physics.

3.1 HAMILTON'S FOLLOWERS

3.1.1 P.G. TAIT

Peter Guthrie Tait (1831-1901) was a rival and life-long friend of Maxwell, both attending the same high-school (Edinburgh Academy) and universities (Edinburgh and Cambridge): both were brilliant students. At 20, Tait achieved the top results in the annual undergraduate mathematical examinations in Cambridge ("*Senior Wrangler*"). In competition with Maxwell, Tait was awarded the professorship in Natural Philosophy at Edinburgh in 1858, a year after Tait started corresponding with Hamilton on quaternions. Tait published *Quaternion Investigations connected with Electrodynamics* in 1860 when he used quaternions to simplify Helmholtz's fluid analogy with electrodynamics. Tait continued the "***quaternion-crusade***" after Hamilton's death in 1865, including writing two important mathematical texts *Elementary Treatise on Quaternions* (1867) and *Introduction to Quaternions* (1873). The failure of Tait to convince William Thomson (later Lord Kelvin), his co-author of the book *Treatise on Natural Philosophy* (1867) meant that quaternions failed to get the wide attention that this very influential text generated. Tait continued to strongly support the superiority of quaternions for the rest of his life but lost out in the court of intellectual fashion to the simplified, vector ideas of Heaviside and Gibbs, introduced in 1881 and 1882.

3.1.2 MAXWELL

Maxwell was also very impressed with the power of this new algebra and calculus that were being promoted by both Hamilton and Tait. This is illustrated in a letter to Thomson in 1871 "*The volume-, surface- and line-integrals of vectors and quaternions, as being worked out by Tait, is worth all that is going on in other seats of learning*." Maxwell, who never communicated directly with Hamilton (but encouraged strongly by Tait) presented the final version of his electromagnetic equations in his *Treatise* (§618) in the form of partial quaternion equations by using a single letter prefix (S or V) to indicate whether he was referring to the scalar or vector part of the quaternion; so for example, two of his equations were rewritten as: $\mathbf{B} = \mathrm{V}.\nabla\mathbf{A}$ and $\rho = \mathrm{S}.\nabla\mathbf{D}$. It is not surprising that **Heaviside** would eventually rewrite these equations as simple vector equations, into the form they are known today.

3.2 ELECTROMAGNETISM & RELATIVITY

It was Hamilton's insight that the scalar part of the basic point-location quaternion corresponded to the **time** dimension that inspired post-1905 authors to use quaternions for re-writing the results of special relativity. Conway in 1911 [25] and (independently) Silberstein in 1912 [26] rewrote Maxwell's Equations using bi-quaternions. They deliberately introduced 4-gradient forms that acted on the 4-potential (both with a judicious use of the imaginary factor (i) to produce the electric and magnetic vector fields, whose 4-gradient then related them to the 4-current. Rastall has written a comprehensive review article [27] describing the history of using quaternions in relativity.

3.3 DIRAC'S RELATIVISTIC EQUATION

In 1929, just one year after Dirac established his relativistic equation for the electron, Lanczos published three articles [28] where he generated this famous equation from a pair of coupled bi-quaternion fields, an approach that made the spin explicit. He then continued this innovative approach for many years thereafter using bi-quaternions. The 1937 paper by Conway [29] proposed a completely quaternion form of this equation. This same approach has been recently [30] been revisited by Davies.

3.4 QUANTUM MECHANICS

In 1936 Birkhoff and von Neumann [31] presented a propositional basis for quantum mechanics, which indicated that a QM system may be represented as a vector space over the real, complex and quaternionic fields: the quantum superposition principle for probability amplitudes (wave functions) need only obey a division algebra. The use of a quaternion wave function $\mathbf{\Psi} = \psi_0 + \underline{\psi}$, results in a doubling of the components from four (Dirac) to eight. Adler has used this surprising feature to develop [32] a new form of quantum electrodynamics that eliminates Dirac's 'sea' of negative energy electrons by combining both particle and anti-particle states into a combined, single fermionic state. In the 1960s, Finkelstein et al wrote a series of papers [33] exploring the foundations and implications of quaternion quantum mechanics.

This brief summary reveals that quaternions have been mainly ignored in the development of physics. However, although modern physics has made a major commitment to Hamilton's Principal Function (Hamiltonian Mechanics) while almost completely ignoring his quaternions, this researcher believes that a scientist of Hamilton's rare genius would not have invested so much of his life on such a mistaken endeavor. Accordingly, this programme will ignore Hamiltonian mechanics as unsuitable for **asynchronous** interactions since this was designed for <u>single-time</u> physics, but we will show in this paper that a simple, evolutionary development, now referred to here as '**Natural Vectors**', promises to fulfill the great expectations in physics that Hamilton had hoped for his quaternions.

4. NATURAL VECTORS

4.1 MAXWELL'S FAILURE

Although Maxwell was impressed with this new algebra, his final attempt to use Hamilton's quaternions simply used the vector-style representation to condense three similar equations expressed in Cartesian coordinates into a single 3D vector-like equation. As a result, Maxwell was able to reduce his 20 equations summarizing his theory of electromagnetism down to eight: six in vector form plus two in scalar form. (Although, universally known today as Maxwell's Equations; their **vector** form is due to Heaviside.) It is quite likely that Maxwell knew that Hamilton's quaternions in their original form could not be used to generate these equations, as will now be demonstrated.

The following vector-quaternion notation will be used here based on Hamilton's basic quaternions (see 2.2.2).

$$\text{Vector: } \underline{A} \equiv \mathbf{i}\,A_1 + \mathbf{j}\,A_2 + \mathbf{k}\,A_3 \qquad \text{Quaternion: } \mathbf{A} \equiv A_0 + \underline{A}$$

$$\underline{\nabla} \equiv \mathbf{i}\,\partial_1 + \mathbf{j}\,\partial_2 + \mathbf{k}\,\partial_3 \qquad \mathbf{\nabla} \equiv \partial_0 + \underline{\nabla} \qquad \partial_0 \equiv \partial / c\,\partial t \qquad \partial_1 \equiv \partial / \partial x_1 \text{ etc}$$

Standard multiplication of two quaternions gives: $\mathbf{A\,B} = (A_0\,B_0 - \underline{A} \bullet \underline{B}) + (\underline{A}\,B_0 - A_0\,\underline{B}) + \underline{A} \wedge \underline{B}$

Defining two **electromagnetic field** quaternions: $\mathbf{E} \equiv 4\,\pi\,\rho + \underline{E}$ and $\mathbf{B} \equiv \underline{B}$ then:

$$\mathbf{\nabla}\,(\mathbf{E} + i\,\mathbf{B}) = (4\,\pi\,\partial_0\,\rho - \underline{\nabla} \bullet \underline{E} - i\,\underline{\nabla} \bullet \underline{B}) + \partial_0\,(\underline{E} + i\,\underline{B}) + 4\,\pi\,\underline{\nabla}\rho + \underline{\nabla} \wedge (\underline{E} + i\,\underline{B})$$

The Maxwell field **hypothesis** would then be equivalent to setting this equation always to **zero**. So equating the real and imaginary parts of the scalar and vector components generates four separate equations:

1) $\underline{\nabla} \bullet \underline{E} = 4\,\pi\,\partial_0\,\rho$ 2) $\underline{\nabla} \bullet \underline{B} = 0$ 3) $\underline{\nabla} \wedge \underline{E} = \partial_0\,\underline{E} - 4\,\pi\,\underline{\nabla}\rho$ 4) $\underline{\nabla} \wedge \underline{B} = \partial_0\,\underline{B}$

If this is compared with the Maxwell-Heaviside field equations:

a) $\underline{\nabla} \bullet \underline{E} = 4\,\pi\,\rho$ b) $\underline{\nabla} \bullet \underline{B} = 0$ c) $\underline{\nabla} \wedge \underline{E} = -\,\partial_0\,\underline{B}$ d) $\underline{\nabla} \wedge \underline{B} = \partial_0\,\underline{E} + 4\,\pi\,\underline{J} / c$

It can be seen that **only** the magnetic divergence equation is recovered, **not** a very promising match.

4.2 THE INTERACTION GROUP

The **motion** of two particles is determined by linear combinations of the **8** basic physical operators:

$$\{\, t,\ x,\ y,\ z,\ \partial/\partial t,\ \partial/\partial x,\ \partial/\partial y,\ \partial/\partial z \,\}$$

The generalized quaternion bases $\{\mathbf{I_0}, \mathbf{I_1}, \mathbf{I_2}, \mathbf{I_3}\}$ form the generators of the quaternion group under multiplication, based on the identity element $\mathbf{I_0}$; in other words, they satisfy the quaternion group defining multiplications:

$$\mathbf{I_0\,I_0} = +\,\mathbf{I_0} \qquad \mathbf{I_0\,I_j} = \mathbf{I_j} \qquad \mathbf{I_j\,I_j} = -\,\mathbf{I_0} \qquad \mathbf{I_1\,I_2} = \mathbf{I_3} \quad (\text{cyclic; } j = 1, 2, 3)$$

Previously, most other attempts to use quaternions in physics (see 3.3 & 3.4) have used "complexified" quaternions. These authors used the four "complex bases" $\{\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$ where $\mathcal{I}_0 = \mathbf{I_0}$ & $\mathcal{I}_j = i\,\mathbf{I_j}$ explicitly introducing the standard complex number (*i*) into the bases themselves. These four bases map to the **double** complex unitary group:

$$(\mathcal{I}_\mu) \cong (\mathbf{SU2}) \otimes (\mathbf{SU2})$$

As a result, these bases were defined in terms of Pauli (2x2) "spin" matrices, σ_j. The complexified quaternion group needs 16 elements to complete the group: $\{\mathbf{I_0}, \{\underline{\mathbf{I}}\}, i\,\mathbf{I_0}, i\{\underline{\mathbf{I}}\}, -\,\mathbf{I_0}, -\{\underline{\mathbf{I}}\}, -\,i\,\mathbf{I_0}, -\,i\{\underline{\mathbf{I}}\}\}$, where $\{\underline{\mathbf{I}}\}$ represents the sub-group $\{\mathbf{I_1}, \mathbf{I_2}, \mathbf{I_3}\}$. This results in a double mapping between the eight physical operators and this complex quaternion group.

In attempting to construct a representation of a two-particle interaction, it has proved useful to adopt a more physical view than the purely algebraic one adopted by Hamilton and his more mathematical followers. It is obvious that the characteristics of time itself, (namely linearity, sequencing and universality) are very different from the intuitive characteristics of space (3 independent directions, rotations, distinctions established by the location of particles, etc). Indeed, Hamilton was acutely aware of these differences, as demonstrated by the following quotation [34]:

"*The quaternion may be said to be 'time plus space' and in this sense involves a reference to **four** dimensions*."

Mathematically, 3D vectors can represent real displacements of any magnitude, so this suggests mapping the spatial locations of particles to the real, vector part of a quaternion. In order to emphasize the very different nature of time the purely imaginary number (the square root of minus one, symbolically represented in physics by 'i') will be used in the scalar part of quaternions to represent physical quantities that can exist in time. Its only properties that will be used here are its definition and that of complex conjugation, denoted here by an **asterisk** superscript:

$$i = \sqrt{(-1)} \quad \text{or} \quad i^2 = -1 \quad \text{and} \quad i^* = -i$$

The new representation developed here, which will be referred to as the *Natural Vector* representation, is a blend of these two well-known representations; the set of four bases are chosen to be: $\{i\ \mathbf{I_0},\ \mathbf{I_1},\ \mathbf{I_2},\ \mathbf{I_3}\}$. In this new group it takes either 4 or 6 operations to generate a return to any initial state, say |S> itself represented by a (4x1) column matrix; in other words:

$$((i\,\mathbf{I_0})(\mathbf{I_3})(i\,\mathbf{I_0})(\mathbf{I_3}))\,|S\rangle = (\mathbf{I_0})\,|S\rangle = |S\rangle \quad \text{and}$$

$$((i\,\mathbf{I_0})(\mathbf{I_1})(\mathbf{I_2})(i\,\mathbf{I_0})(\mathbf{I_1})(\mathbf{I_2}))\,|S\rangle = (\mathbf{I_0})\,|S\rangle = |S\rangle$$

In other words there exists a 12 element subgroup (**W**) within the Natural Vector group (the choice $\mathbf{I_3}$ is arbitrary), involving the parity group (**C**2) and the cyclic group (**C**4); this is the Weyl group: $(\mathbf{W}) \cong (\mathbf{C}2) \otimes (\mathbf{C}4)$. The (**C**4) group corresponds to the 4 roots of +1 based on the generator, i ; in other words: $\{1, i, i^2 = -1, i^3, = -\mathrm{i}\}$.

4.3 EXPONENTIALS

Quaternions can be used to define exponential functions based on Euler's (infinite) series definition of the basic exponential function:

$$\exp(x) = \sum x^n / n! \quad \text{with} \quad 0! = 1$$

Substituting a pure imaginary number for the real parameter ($x = i\,\theta$) generates Euler's famous formula:

$$\exp(i\,\theta) = \cos\theta + i\sin\theta$$

Similarly, using any one of the vector (non-zero) quaternion bases ($\mathbf{I_j}$) generates a comparable result:

$$\exp(\mathbf{I_j}\,\theta) = \mathbf{I_0}\cos\theta + \mathbf{I_j}\sin\theta$$

If we introduce the 'positional convention' (since these bases are non-commuting variables):

$$\exp(\mathbf{I_1}\,\theta)\exp(\mathbf{I_2}\,\theta)\exp(\mathbf{I_3}\,\theta) = \exp((\mathbf{I_1} + \mathbf{I_2} + \mathbf{I_3})\,\theta)$$

Then: $\exp((\mathbf{I_1} + \mathbf{I_2} + \mathbf{I_3})\,\theta) = \mathbf{I_0}(\cos^3\theta - \sin^3\theta) + ((\mathbf{I_1} + \mathbf{I_3})(\cos\theta + \sin\theta) + \mathbf{I_2}(\cos\theta - \sin\theta))\cos\theta\sin\theta$

So, 1) $\exp(-(\mathbf{I_1} + \mathbf{I_2} + \mathbf{I_3})\,\pi/2) = \mathbf{I_0}$ 2) $\exp((\mathbf{I_1} + \mathbf{I_2} + \mathbf{I_3})\,\pi/4) = (\mathbf{I_1} + \mathbf{I_3})/\sqrt{2}$

Since, for any quaternion, $\mathbf{Q} = \mathbf{I_0}Q_0 + \mathbf{I_1}Q_1 + \mathbf{I_2}Q_2 + \mathbf{I_3}Q_3 = R\exp((\mathbf{I_1} + \mathbf{I_2} + \mathbf{I_3})\,\theta)$

So, $Q_0 = R(\cos^3\theta - \sin^3\theta)$ $Q_2 = R(\cos\theta - \sin\theta)\cos\theta\sin\theta$ $Q_1 = Q_3 = R(\cos\theta + \sin\theta)\cos\theta\sin\theta$

Thus, $Q_1/Q_2 = (\cos\theta + \sin\theta)/(\cos\theta - \sin\theta)$ or $\tan\theta = (Q_1 - Q_2)/(Q_1 + Q_2)$

4.4 MATRIX REPRESENTATION

The new representation developed here, henceforth referred to as the *Natural Vector* representation, uses the set of four bases (see section 4.2): $\{i\, \mathbf{I_0}, \mathbf{I_1}, \mathbf{I_2}, \mathbf{I_3}\}$, where $\mathbf{I_0}$ is isomorphic with the unit number while $\{\mathbf{I_1}, \mathbf{I_2}, \mathbf{I_3}\}$ are isomorphic with Hamilton's three linearly independent imaginary quantities $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. So these bases satisfy the group multiplication rules, where j, k, l = 1, 2, 3 and μ = 0, j :

$$\mathbf{I_0\, I_\mu} = \mathbf{I_\mu\, I_0} = \mathbf{I_\mu} \qquad \mathbf{I_j\, I_k} = -\,\delta_{\mathbf{ij}}\, \mathbf{I_0} + \varepsilon_{jkl}\, \mathbf{I_l}$$

Here, $\delta_{\mathbf{ij}}$ is the Kronecker delta symbol with value +1 when both indices are equal or zero otherwise and ε_{jkl} is the cyclic permutation tensor whose value is zero unless all three indices are different when its value is +1 if the indices are cyclic (even permutation of 1,2,3) or –1 if anti-cyclic (odd permutation). This views the definition of a natural vector as an *imaginary, scalar quaternion*, symbolized by $\mathcal{Q}$ so as to distinguish it from a Hamilton quaternion **Q**.

$$\text{Definitions:} \qquad \mathcal{Q} \equiv i\, \mathbf{I_0}\, q_0 + \mathbf{I_1}\, q_1 + \mathbf{I_2}\, q_2 + \mathbf{I_3} q_3 \equiv \mathcal{Q}_0 + \textstyle\sum_{\mathbf{j}} \mathcal{Q}_{\mathbf{j}} \equiv \{i\, q_0 \,;\, \underline{q}\} = i\, q_0\, \mathbf{I_0} + \sum_{\mathbf{j}} q_{\mathbf{j}}\, \mathbf{I_j}$$

In this definition, the imaginary scalar component of $\mathcal{Q}$ is ($\mathcal{Q}_0 = i\, q_0$) while the vector component is ($\underline{\mathbf{Q}}$).
In terms of Grassmann's unit vectors $\mathbf{e_\mu}$,

$$\mathcal{Q} = \textstyle\sum_{\mu} q_{\mu}\, \mathbf{e_\mu} \quad \text{thus} \quad \mathbf{e}_0 \cong i\, \mathbf{I_0} \quad \mathbf{e}_j \cong \mathbf{I_j}$$

We require that ALL of the vector representation of a Natural Vector remains real AND unchanged under time reversal. This means that we cannot use the Pauli (2x2) or bi-quaternion representations of the three vector bases, $\mathbf{I_j}$ as we wish to represent time reversal (later) by complex conjugation, where this is satisfied by the scalar part of a natural vector. So we need a (4x4) matrix representation of each of these bases based on the 4 real one dimensional unit column matrices, $|\,\mu\rangle$, defined (as row matrices $\langle\,\mu\,|$):

$$\langle\,\mu\,| = \{\langle 0 |, \langle 1 |, \langle 2 |, \langle 3 |\} = \{[1\ 0\ 0\ 0],\ [0\ 1\ 0\ 0],\ [0\ 0\ 1\ 0],\ [0\ 0\ 0\ 1]\}$$

The representation chosen here, where the z-axis will be the 'spin-axis', is:

$$\mathbf{I_0} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \quad \mathbf{I_1} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix} \quad \mathbf{I_2} = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \quad \mathbf{I_3} = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}$$

So, any natural vector has a (4x4) representation:

$$\mathcal{Q} = \begin{vmatrix} iq_0 & q_1 & -q_3 & -q_2 \\ -q_1 & iq_0 & -q_2 & q_3 \\ q_3 & q_2 & iq_0 & q_1 \\ q_2 & -q_3 & -q_1 & iq_0 \end{vmatrix}$$

We can see that each quaternion basis $\mathbf{I_\mu}$ is represented by a real (4x4) matrix involving only the three foundational integers {+1, 0, –1}. In this new representation, any ordered set of 4 **real** parameters $\{q_0, q_1, q_2, q_3\}$ which are associated with **two** interacting particles can be mapped into a unique, imaginary-scalar quaternion, denoted by $\mathcal{Q}$. We can relate these (4x4) bases to the (2x2) Pauli spin matrices σ_j by introducing the anti-symmetric: $\mathbf{I_j = I^+_j - I^-_j}$. The (2x2) representation uses the unit matrix ***1***, the zero matrix ***0*** and the 'number' matrices η_+ and η_- defined as:

$$\boldsymbol{1} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \boldsymbol{1}^{\dagger} \qquad \eta_+ = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = (\eta_+)^{\dagger} \quad \eta_- = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = (\eta_-)^{\dagger} \quad \sigma_3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = \sigma_3^{\dagger} = \eta_+ - \eta_-$$

$$\boldsymbol{0} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \quad \sigma_+ = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = \sigma_-^{\dagger} \quad (\sigma_+ + \sigma_-) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = (\sigma_+ + \sigma_-)^{\dagger} \quad (\sigma_+ - \sigma_-) = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = -(\sigma_+ - \sigma_-)^{\dagger}$$

$$\text{Using the full representation:} \quad \mathcal{Q} = iq_0 \begin{vmatrix} \boldsymbol{1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{1} \end{vmatrix} + q_1 \begin{vmatrix} (\sigma_+ - \sigma_-) & \boldsymbol{0} \\ \boldsymbol{0} & (\sigma_+ - \sigma_-) \end{vmatrix} + q_2 \begin{vmatrix} (\sigma_+ + \sigma_-) & \boldsymbol{0} \\ \boldsymbol{0} & (\sigma_+ + \sigma_-) \end{vmatrix} + q_3 \begin{vmatrix} \boldsymbol{0} & \sigma_3 \\ \sigma_3 & \boldsymbol{0} \end{vmatrix}$$

So, $\mathbf{I_0} = \begin{vmatrix} \mathit{1} & \mathit{0} \\ \mathit{0} & \mathit{1} \end{vmatrix}$, $\mathbf{I_1}^{\lambda} = \begin{vmatrix} \sigma_\lambda & \mathit{0} \\ \mathit{0} & \sigma_\lambda \end{vmatrix}$, $\mathbf{I_2}^{\lambda} = \begin{vmatrix} \mathit{0} & -\lambda\,\sigma_\lambda \\ \lambda\sigma_\lambda & \mathit{0} \end{vmatrix}$, $\mathbf{I_3}^{\lambda} = \begin{vmatrix} \mathit{0} & -\eta_\lambda \\ \eta_\lambda & \mathit{0} \end{vmatrix}$ with $\lambda = +$ or $-$

The isomorphisms are: $\mathbf{I_0} \cong \mathit{1}$, $\mathbf{I_j} \cong -i\,\sigma_j$ giving : $\mathbf{Q} \cong i\,\mathit{1}\,q_0 - \underline{q} \cdot \underline{\boldsymbol{\sigma}}$ (called the spinor isomorphism).

The double-spinor representation is: $\mathbf{Q} \cong \begin{vmatrix} \psi & -\varphi^* \\ \varphi & \psi^* \end{vmatrix}$ where $\psi = q_0 - i\,q_3$ and $\varphi = q_2 - i\,q_1$

4.5 NATURAL VECTOR ALGEBRA

It is very important to note that in this representation of a Natural Vector **only** the explicit '*i*' is imaginary, so in terms of complex conjugation (denoted by a *):

$\mathbf{Q}_0^* = -\mathbf{Q}_0$ while $\mathbf{Q_j}^* = \mathbf{Q_j}$ but for (matrix) transposition: $\mathbf{Q}_0^{T} = \mathbf{Q}_0$ and $\mathbf{Q_j}^{T} = -\mathbf{Q_j}$

So, for the Hermitian conjugate (transposed conjugation) of a real Natural Vector: $\mathbf{Q}^{\dagger} = -\mathbf{Q}$

Thus, the Hermitian conjugate of a real Natural Vector $\mathbf{Q}$ is <u>not</u> independent of $\mathbf{Q}$ but its conjugate is; so, unlike quantum mechanics, the calculus of Natural Vectors will use just standard complex conjugation, not Hermitian conjugation.

In all quaternion calculus, addition is standard <u>and</u> associative: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

But multiplication is non-commutative, in particular: $(\underline{a} \bullet \underline{\mathbf{I}})\ (\underline{b} \bullet \underline{\mathbf{I}}) = -(\underline{a} \bullet \underline{b})\,\mathbf{I_0} + (\underline{a} \wedge \underline{b}) \bullet \underline{\mathbf{I}}$

So, the square of a Natural Vector is: $\mathbf{A}^2 \equiv \mathbf{AA} = -(a_0^2 + a_1^2 + a_2^2 + a_3^2)\,\mathbf{I_0} + 2\,i\,a_0\,(\underline{a} \bullet \underline{\mathbf{I}})$

This shows that simple (direct) multiplication of Natural Vectors mixes the imaginary scalar factors into the vector components but we require the imaginary scalar components to remain <u>distinct</u> when it comes to mapping physical quantities. Furthermore, we also require a positive, real "norm" for all Natural Vectors so we will <u>restrict</u> pair-wise multiplication to <u>conjugate multiplication</u> as in simple complex numbers ($z = x + i\,y$; $z^*\,z = x^2 + y^2$). These restrictions allow an inverse (and division) to be defined uniquely making Natural Vectors a mathematical group.

So the rules for addition and multiplication of two Natural Vectors $\mathbf{A}^*$ and $\mathbf{B}$ become:

Addition: $\mathbf{A}^* + \mathbf{B} = -i\,\mathbf{I_0}\,(a_0 - b_0) + (\underline{a} + \underline{b}) \bullet \underline{\mathbf{I}}$

Multiplication: $\mathbf{A}^*\mathbf{B} = \mathbf{I_0}\,(a_0\,b_0 - \underline{a} \bullet \underline{b}) + i\,\underline{\mathbf{I}} \bullet (b_0\,\underline{a} - a_0\,\underline{b}) + (\underline{a} \wedge \underline{b}) \bullet \underline{\mathbf{I}}$

For single vectors (**B** = **A**): $\mathbf{A}^* + \mathbf{A} = 2\,\underline{a} \bullet \underline{\mathbf{I}}$ $\quad \mathbf{A}^* - \mathbf{A} = -i\,2\,a_0\,\mathbf{I_0}$ $\quad \mathbf{A}^*\mathbf{A} = \mathbf{I_0}\,(a_0\,a_0 - \underline{a} \bullet \underline{a})$

This gives the "Grassmann" form of Natural Vectors: $S.\mathbf{A} = 1/2\,(\mathbf{A} - \mathbf{A}^*)$ $\quad V.\mathbf{A} = 1/2\,(\mathbf{A} + \mathbf{A}^*)$

This illustrates the feature that the vector part of a NV is symmetric, while the scalar part is anti-symmetric. It also should be pointed out that $\mathbf{X}^*$ corresponds to the standard "right-hand" reference-frame, while $\mathbf{X}$ corresponds to a "left-hand" reference-frame. Note, also, that the standard rule for operators applies to NVs, operands <u>always</u> stand to the **right** of operators, unlike matrices or bi-quaternions, so there is **no** ambiguity.

It is strongly recommended that the explicit form of NVs be used (e.g. $\mathbf{Q} = i q_0 \mathbf{I_0} + \underline{\mathbf{I}} \bullet \underline{q}$), not any of the implicit forms, like $\{i\ q_0\ ;\ \underline{q}\}$; these implicit forms (or worse, the conjugate product form) have been used too extensively in the history of quaternions and have hidden the algebra that is as <u>simple</u> as the algebra of complex numbers.

4.6 MAPPING REALITY

The form of the "norm" (that is to say the "square",) of a Natural Vector, **A*A** resembles the form of the space-time interval in Maxwell's electromagnetic theory, when $a_0 = c\,t$ and $\underline{a} = \underline{x}$. This suggests the following "mapping" rules between representations of reality and Natural Vectors.

1. Time will be mapped to a one dimensional real variable 't' with an arbitrary origin, $t_0 = 0$, allowing both positive and negative times relative to t_0.
2. Space will be mapped to three independent, one dimensional variables 'x_j' (j = 1, 2, 3) with a common origin $x_{j0} = 0$. In order to preserve their independence these real variables will be assigned to the three orthogonal (right-handed) unit vectors in space, $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$.
3. In order to work with Natural Vectors that are always physically homogenous (i.e. each component has the same physical dimensions) we will introduce the universal *space-to-time constant ratio* 'c', that has the dimensions of **speed**; this will later be identified with the "speed of light" in vacuo. Thus the product 'c t' has the same physical dimensions as each of the spatial variables 'x_j', namely length [L].
4. Since time is a scalar and ontologically separate and different from space then time will be mapped to the imaginary scalar variable with the spatial variables mapped to the vector components.
5. The **metaphysics** of all theories based on Natural Vectors is **Newtonian**. Here, the world is considered to consist of a very large but **finite** number of **point-particles** that are identified with those elementary particles known as "electrons". Since they are point-like they are each considered to have zero extent in space throughout all time. They are each considered both unique and eternal, so they can be 'labelled' by a unique integer identifier 'α'. Each electron will have a unique position in space ($\underline{x}$) at time, t. It is the fundamental hypothesis of this research programme that these two parameters of every electron can be mapped into their own Natural Vector, $\mathbf{X}(\alpha)$.

$$\text{Hypothesis:}\quad \{\, x_j(\alpha, t_\alpha)\,\} \;\cong\; \mathbf{X}(\alpha) \;\equiv\; i\,c\,t_\alpha\,\mathbf{I_0} + \underline{x}(\alpha)\bullet\underline{\mathbf{I}}$$

The square (or 'norm') of this 'positional' NV is: $\mathbf{X}(t)^*\,\mathbf{X}(t) = (c^2 t^2 - x^2)\,\mathbf{I_0}$

4.7 VELOCITY & GRADIENT

Since the electron is considered in this programme as eternal then when it is moving it always has a nearby location in space no matter how small (but finite) the difference in time (δt) between the two locations. This requirement defines the instantaneous velocity of the electron, $\underline{v}$ at all times:

$$\text{Definition: Velocity,}\ \underline{v}(t) \equiv \operatorname*{Limit}_{\delta t \to 0} \{\, (\,\underline{x}(t+\delta t) - \underline{x}(t)\,) / \delta t\,\}$$

In all physics since Newton onwards, the assumption has been made that velocity is a continuous variable; this "**Continuity Hypothesis**" will be adopted here until explicitly challenged, **later**. This is usually written in terms of Newton's total time derivative in the calculus he invented for analyzing motion (but using Leibniz's notation):

$$\underline{v}(t) = d\underline{x}(t)/dt = d/dt\,\{\,\underline{x}(t)\,\}$$

The same limiting definition may be applied to an electron's own Natural Vector to define its NV velocity (see 5.3):

$$\text{Definition:}\ \mathbf{V}(t) \equiv \operatorname*{Limit}_{\delta t \to 0} \{\, (\mathbf{X}(t+\delta t) - \mathbf{X}(t)\,) / \delta t\,\} = d\mathbf{X}(t)/dt = i\,c\,\mathbf{I_0} + \underline{v}\bullet\underline{\mathbf{I}}$$

The "square" of this velocity NV is: $\mathbf{V}(t)^*\,\mathbf{V}(t) = (c^2 - v^2)\,\mathbf{I_0}$

We can follow Hamilton and extend his 'nabla' (or 'gradient') 3D space operator ($\underline{\nabla}$ section 4.1) to the Natural Vector ***Gradient*** applied to any scalar function ψ that is continuous in the four space-time variables $\{t\,;\,\underline{x}\}$:

$$\text{Defn:}\ \nabla\psi(t\,;\underline{x}) \equiv i\,\mathbf{I_0}\,\partial_0\psi(t\,;\underline{x}) + \underline{\mathbf{I}}\bullet\underline{\nabla}\psi(t\,;\underline{x}) \;\;\&\;\; \underline{\nabla} \equiv \hat{\mathbf{e}}_1\partial_1 + \hat{\mathbf{e}}_2\partial_2 + \hat{\mathbf{e}}_3\partial_3 \quad \partial_0 \equiv \partial/c\,\partial t \quad \partial_1 \equiv \partial/\partial x_1\ \text{etc}$$

The conjugate of the NV gradient operator can be applied to any NV continuous function, $\mathbf{Q}(t\,;\,\underline{x})$:

$$\nabla^*\mathbf{Q}(t\,;\,\underline{x}) \;=\; \mathbf{I_0}\,(\partial_0 q_0 \;-\underline{\nabla}\bullet\underline{q}) \;+\; i\,\underline{\mathbf{I}}\bullet(\underline{\nabla}q_0 -\; \partial_0\underline{q}) \;+\; \underline{\mathbf{I}}\bullet(\underline{\nabla}\wedge\underline{q})$$

When the gradient of a CNV is zero, this is equivalent to three separate equations (real & imaginary parts of $\mathbf{I_\mu}$):

If $\nabla^*\mathbf{Q} \;=\; 0$ then: 1) $\underline{\nabla}\bullet\underline{q} \;=\; \partial_0 q_0$ 2) $\underline{\nabla}q_0 = \;\partial_0\underline{q}$ 3) $\underline{\nabla}\wedge\underline{q} = 0$

4.8 NATURAL VECTOR PRODUCTS

The Natural Vector gradient operator (∇) is itself a linear operator as it is constructed from the basic linear operators ∂_0 and ∂_k. We can use all the results of differential calculus and vector algebra to develop a set of NV identities that will be used throughout the remainder of this research programme. In particular, the derivatives of $\underline{x}$ and $\underline{v}$ have certain characteristics (see below) that make their use in their NV versions ($\mathbf{X}$,$\mathbf{V}$) very powerful.

Most importantly: a) $\partial x_j / \partial x_k \;=\; \delta_{jk}$ b) $\partial_0 x_j \;=\; 0$ c) $\partial f(t) / \partial x_k = \;0$ d) $\partial_0 f(\underline{x}) \;=\; 0$

So, e) $\underline{\nabla}\bullet\underline{x} \;=\; 3$ f) $\underline{\nabla}\wedge\underline{x} = 0$ g) $(\underline{v}\bullet\underline{\nabla})\,\underline{x} \;=\; \underline{v}$ h) $\underline{\nabla}\,v_j \;=\; 0$ i) $\underline{\nabla}\bullet\underline{v} \;=\; 0$ j) $\underline{\nabla}\wedge\underline{v} = 0$

These basic identities are used to generate more-complicated identities with **scalar** functions like $\alpha(\underline{x})$, thus:

1. $\underline{\nabla}\,(\alpha\, v_j\,) \;=\; v_j\,\underline{\nabla}\alpha$ 2. $\underline{\nabla}\bullet(\alpha\,\underline{v}\,) = \;\underline{v}\bullet\underline{\nabla}\alpha$ 3. $\underline{\nabla}\wedge(\alpha\,\underline{v}\,) = \;-\,\underline{v}\;\wedge\underline{\nabla}\alpha$

A more comprehensive set of useful 3D vector identities is provided in **Appendix 10.1**. A similar set of identities can be readily developed for the corresponding Natural Vectors. These all use the basic conjugate multiplication form introduced in section 4.4 and 4.6; again, when α is any scalar function of $\{t\ \&\ \underline{x}\}$.

1. $\nabla^*\mathbf{X} \;=\; -\,2\,\mathbf{I_0}$ 2. $\nabla^*\mathbf{V} \;=\; -\,i\,\underline{\mathbf{I}}\bullet\;\partial_0\underline{v}$ 3. $\nabla^*\alpha \;=\; -\,i\,\mathbf{I_0}\,\partial_0\alpha \;+\; \underline{\mathbf{I}}\bullet\underline{\nabla}\alpha$

4. $\nabla^*(\alpha\,\mathbf{Q}) \;=\; \alpha\,(\nabla^*\mathbf{Q}) + (\nabla^*\alpha)\mathbf{Q}$ 5. $\nabla^*\,\nabla\alpha \;=\; \nabla\,\nabla^*\,\alpha \;=\; \mathbf{I_0}\,(\partial_0^{\,2} -\; \underline{\nabla}^2\,)\;\alpha \;\equiv\; -\,\mathbf{I_0}\,\Box\alpha$

6. $\mathbf{V}^*\,\nabla^*(\alpha) \;=\; -\,\mathbf{I_0}\,(c\;\partial_0 + \underline{v}\bullet\underline{\nabla})\alpha - i\,\underline{\mathbf{I}}\bullet\;(c\;\underline{\nabla} + \underline{v}\;\partial_0\,)\;\alpha \;+\; \underline{\mathbf{I}}\bullet(\underline{v}\wedge\underline{\nabla}\alpha)$

A more comprehensive set of useful Natural Vector identities is also provided in **Appendix 10.2**.

5. CONTINUOUS NATURAL VECTORS

5.1 DEFINITION

When any of the parameters (q_μ) of a Natural Vector $\mathbf{Q}$ is a function of a continuous variable ξ, such that for any small change $\delta\xi$ in this variable, the following two conditions hold, then $\mathbf{Q}(\xi)$ will be referred to as a Continuous Natural Vector (CNV).

$$q_\mu(\xi + \delta\xi) = q_\mu(\xi) + \delta q_\mu(\xi) \text{ when } \delta q_\mu(\xi) \neq 0 \;\&\; \delta\xi \neq 0 \quad \text{and} \quad q_\mu(\xi) = \underset{\delta\xi\to 0}{\text{Limit}} \{ q_\mu(\xi + \delta\xi) \}$$

5.2 TOTAL DIFFERENTIALS

Any particle at **any** time (t) is defined to **occupy** only <u>one position in 3D space</u>, denoted by the ordered triple of real numbers $\{x_1, x_2, x_3\}$ relative to a given reference frame. This is equivalent to the definition of a location vector $\underline{q}$ defined relative to the origin of the reference frame. Note that although the value of $\underline{q}$ varies over time, the symbol 't' is a monotonic scalar that acts as an identity tag for the value at any particular time. The notation $\underline{q}(t)$ does not mean that the particle's spatial location is an explicit function of time, rather it is reflects the implicit relationship:

$$\underline{q}(t) = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3 \equiv \underline{x}(t)$$

For infinitesimal changes in time (δt) this particle is assumed to move smoothly to a nearby point $\underline{x}(t + \delta t)$; in other words, the particle is assumed to have a <u>well-defined</u>, instantaneous velocity $\underline{v}(t)$ at all times, t and a continuous change in velocity or acceleration $\underline{a}(t)$. For each spatial direction ($\hat{\mathbf{e}}_j$) these are defined by the limit conditions:

$$\text{Defs:} \quad v_j(t) \equiv \underset{\delta t \to 0}{\text{Limit}} \{(x_j(t + \delta t) - x_j(t))/ \delta t\} \equiv dx_j(t) / dt \;;\quad a_j(t) \equiv \underset{\delta t \to 0}{\text{Limit}} \{(v_j(t + \delta t) - v_j(t))/ \delta t \} \equiv dv_j(t) / dt$$

When a function ψ is <u>always</u> associated with every example of **one** type of particle then it is considered a universal property of this type of particle. Indeed, the totality of all of these types of properties (ψ_k) can be considered as the definition of this class of point objects. In other words, every member of a class of particles will have a set of these similar "quality" functions; these properties are intrinsic to their "owning" objects and have no separate ontological significance. The identity of all such objects in a given class is not significant with respect to the values of these functions otherwise "significant" particles would be distinguishable and, ipso facto, they would <u>not</u> be members of this class. Thus, the value of any such intrinsic "property" function must depend only on the location of its "parent" particle in space ($\underline{x}$) at any given time t, i.e. $\psi(t\,;\,\underline{x})$ or $\psi(x_\mu)$. There may well be characteristics that only manifest themselves when objects are interacting between themselves: these would be pair-wise properties of the objects.

Any physical theory that assumes that these property functions (ψ) are <u>continuous</u> over all space and time is an example of a "continuum" theory – an assumption made for almost all mathematical theories of physics since Newton invented the differential calculus. When these types of theories separate a property from any particle and simply assign continuous functions to "space" itself, then such theories are building on the mathematics of "**fields**", manifolds of real numbers ($\Re$), used to map the co-ordinates of time and space to arbitrary scales. But when the physical objects of interest are assumed to be **localized** at ONE point in space for each and <u>every</u> point in time, then the theory is classified as a **particle** theory: for example, Newton's dynamics with instantaneous forces. When a theory proposes a physical medium covering a region of space continuously then the focus of interest becomes <u>any</u> infinitesimally small spatial volume of this medium centered on <u>every</u> spatial point; such theories are known as classical "<u>medium</u>" theories, as exemplified by Maxwell's theory of the electromagnetic **aether** or Helmholtz's <u>hydrodynamic</u> model of electrodynamics. It is logically consistent to define a velocity property for either a particle or medium theory as one can readily imagine the focus of interest moving relative to the background spatial frame; it is **not** obvious that such concepts are legitimate for pure "field" theories, where space itself is the focus of interest. Mixed theories occur whenever a medium is proposed that interacts with distinct particles (this is equivalent to interactions occurring between particles that are "**carried**" by the intervening medium). Such mixed, particle-medium theories are best exemplified by H. A. Lorentz's *Theory of the Electron*.

Contemporary theories in physics use the mathematics and vocabulary of particles and medium theories but propose no physical medium, so that the key field functions become properties of space itself – the '**plenum**' view championed by Newton's nemesis, René DesCartes. When interactions are limited to the products of different fields acting at the same point in space and time then we have an example of a "**local field**" theory – this has been the centre of research in theoretical physics now for over 75 years. This is not the area of interest of the present research programme. There are thousands of physics papers already devoted to local field theories; it is time for a change.

All continuum theories assume that every property (ψ) has non-zero limits of the first (and usually second) partial derivatives with respect to all four space-time variables, x_μ.

$$\text{Definition:}\quad \partial\psi(t\,;\,\underline{x}) / \partial x_\mu \equiv \lim_{\delta x_\mu \to \pm 0} \{ (\psi(\ldots, x_\mu + \delta x_\mu, \ldots) - \psi(\ldots, x_\mu, \ldots)) / \delta x_\mu \}$$

The ***total-differential*** change in value of a point-object's property ψ is defined as the sum of all its possible differential variations in the 3 spatial directions ($\mathbf{e_j}$) plus the differential change due to its explicit change in time.

$$\text{Definition:}\quad d\psi \equiv \sum_\mu dx_\mu\, \partial\psi/\partial x_\mu \qquad \text{where } x_0 = t \text{ so } dx_0 / dt = 1 \;\&\; dx_j / dt = v_j$$

Thus, for the total-time derivative (also see Goldstein [35]): $d\psi/dt = \partial\psi/\partial t + \underline{v} \bullet \underline{\nabla}\psi$

This illustrates that the total-change in a point-object's property ψ is due to an explicit variation, at the same location, due to changes induced by time itself **plus** the sum of the variations due to the object moving through space (at velocity, $\underline{v}$) to an infinitesimally nearby location in space (at the same time). This confirms the previous claim that there is no explicit time dependence of the object's location by choosing the object's location $\underline{x}$ as ψ itself:

$$v_j = dx_j / dt = \partial x_j /\partial t + \underline{v} \bullet \underline{\nabla} x_j = \partial x_j /\partial t + \underline{v} \bullet \mathbf{e_j} = \partial x_j /\partial t + v_j \quad \therefore\ \partial x_j /\partial t = 0$$

If we substitute the particle's velocity $\underline{v}$ for ψ we see that its acceleration, $\underline{a}(t)$ has an interesting feature:

$$a_j = dv_j / dt = \partial v_j /\partial t + \underline{v} \bullet \underline{\nabla} v_j = \partial v_j /\partial t \qquad \therefore\ \underline{a}(t) = d\underline{v} / dt = \partial\underline{v} /\partial t = d^2\underline{x} / dt^2$$

5.3 FLOW VECTORS

The 3D position in space of a point-object at time t is denoted by $\underline{x}(t)$; it is the fundamental hypothesis of this research programme (section 4.5) that this is isomorphic with the vector part of its corresponding Natural Vector $\mathbf{X}$.

$$\mathbf{X}(t) = i\, c\, t\, \mathbf{I_0} + \underline{x}(t) \bullet \underline{\mathbf{I}} \quad \& \quad \mathbf{X}(t + \delta t) = i\, c\, (t + \delta t)\, \mathbf{I_0} + \underline{x}(t + \delta t) \bullet \underline{\mathbf{I}} = i\, c\, (t + \delta t)\, \mathbf{I_0} + (\underline{x}(t) + \delta x(t)) \bullet \underline{\mathbf{I}}$$

$$\therefore\ \delta\mathbf{X}(t) \equiv \mathbf{X}(t + \delta t) - \mathbf{X}(t) = i\, c\, \delta t\, \mathbf{I_0} + \delta\underline{x}(t) \bullet \underline{\mathbf{I}} \quad \therefore\ \delta\mathbf{X}(t) / \delta t = i\, c\, \mathbf{I_0} + \underline{\mathbf{I}} \bullet \delta\underline{x}(t) / \delta t$$

So, in the infinitesimal limit, the CNV Velocity is defined as: $\mathbf{V}(t) \equiv \lim_{\delta t \to 0} \{ \delta\mathbf{X}(t) / \delta t \} = i\, c\, \mathbf{I_0} + \underline{v}(t) \bullet \underline{\mathbf{I}}$

This is the result that we used earlier (section 4.6) to develop a set of identities involving the CNV Velocity. If we now use equation 6 of section 4.7 while substituting for the total-time differential, then:

$$\mathbf{V}^* \nabla^* \psi = - \mathbf{I_0}\, d\psi/dt - i\, \underline{\mathbf{I}} \bullet (c\, \underline{\nabla} + \underline{v}\, \partial_0)\psi + \underline{\mathbf{I}} \bullet (\underline{v} \wedge \underline{\nabla}\psi)$$

We shall define the "***Zero Condition***" CNV, $\mathbf{Z}$ as: $\mathbf{Z}\psi \equiv i\, \underline{\mathbf{I}} \bullet (c\, \underline{\nabla} + \underline{v}\, \partial_0)\psi - \underline{\mathbf{I}} \bullet (\underline{v} \wedge \underline{\nabla}\psi)$

So, we have the CNV equivalent of the total-time differential: $\mathbf{I_0}\, d\psi/dt = -\mathbf{V}^* \nabla^* \psi$ if $\mathbf{Z}\psi = 0$

If this result is applied to each continuous component of a CNV $\mathbf{Q}$ then we derive the conditional "***Flow***" equation:

$$d\mathbf{Q}/dt + \mathbf{V}^* \nabla^* \mathbf{Q} = 0 \qquad \text{if } \mathbf{Z}\mathbf{Q} = 0$$

5.4 ZERO CONDITIONS

The "Zero" condition for any CNV **Q** that satisfies the "flow" equation (section 5.3) is equivalent to two scalar and two vector (i.e. 8 Cartesian coordinate) equations by setting the real and imaginary parts of the scalar and vector parts of this CNV equation individually to zero, while using the vector identity: $\underline{A} \wedge \underline{B} \bullet \underline{C} = \underline{A} \bullet \underline{B} \wedge \underline{C}$

1) $\underline{v} \wedge \underline{\nabla} \bullet \underline{q} = 0$ or $\underline{v} \bullet (\underline{\nabla} \wedge \underline{q}) = 0$ 2) $c\, \underline{\nabla} \bullet \underline{q} + \underline{v} \bullet \partial_0 \underline{q} = 0$

3) $\underline{\mathbf{I}} \bullet (c\, \underline{\nabla} + \underline{v}\, \partial_0)q_0 - (\underline{v} \wedge \underline{\nabla}) \bullet (\underline{q} \wedge \underline{\mathbf{I}}) = 0$ 4) $(c\, \underline{\nabla} + \underline{v}\, \partial_0) \bullet (\underline{q} \wedge \underline{\mathbf{I}}) - \underline{\mathbf{I}} \bullet \underline{v} \wedge \underline{\nabla} q_0 = 0$

Since condition (4) can be re-organized into a form of $(\underline{\nabla} \wedge \underline{q})$ in terms of $\underline{v}$ then condition (1) is always satisfied; conditions (3) and (4) can be rewritten in the following forms:

3) $v^2 (\partial_0 q_0 + \underline{\nabla} \bullet \underline{q}) + \underline{v} \bullet \underline{\nabla}(c\, q_0 - \underline{v} \bullet \underline{q}) = 0$ 4) $c\, \underline{\nabla} \wedge \underline{q} = \underline{v} \wedge (\underline{\nabla} q_0 - \partial_0\, \underline{q})$

We can left-multiply the Flow Equation (section 5.3) by $\mathbf{V}$ to derive an alternative form of this important equation:

$$\mathbf{V}\, d\mathbf{Q}/dt + (c^2 - v^2)\, \nabla^* \mathbf{Q} = 0 \quad \text{if} \quad \mathbf{VZQ} = 0$$

This has two solutions; either: A) $v \neq c$ which has interesting properties and B) $d\mathbf{Q}/dt = 0$ when $v = c$.

$$(c^2 - v^2)\,(\mathbf{I_0}\,(\partial_0 q_0 - \underline{\nabla} \bullet \underline{q}) + i\, \underline{\mathbf{I}} \bullet (\underline{\nabla} q_0 - \partial_0 \underline{q}) + \underline{\mathbf{I}} \bullet (\underline{\nabla} \wedge \underline{q})) = (\mathbf{I_0}\,(c\, dq_0/dt + \underline{v} \bullet d\underline{q}/dt) - i\, \underline{\mathbf{I}} \bullet (c\, d\underline{q}/dt + \underline{v}\, dq_0/dt) - \underline{\mathbf{I}} \bullet \underline{v} \wedge d\underline{q}/dt)$$

A) 1) $c\, dq_0/dt + \underline{v} \bullet d\underline{q}/dt = (c^2 - v^2)\,(\partial_0 q_0 - \underline{\nabla} \bullet\bullet \underline{q})$ 2) $c\, d\underline{q}/dt + \underline{v}\, dq_0/dt = (c^2 - v^2)\,(\partial_0 \underline{q} - \underline{\nabla} q_0)$

3) $\underline{v} \wedge d\underline{q}/dt = -\,(c^2 - v^2)\,(\underline{\nabla} \wedge \underline{q})$

or

B) 1) $c\, dq_0/dt + \underline{v} \bullet d\underline{q}/dt = 0$ 2) $c\, d\underline{q}/dt + \underline{v}\, dq_0/dt = 0$ 3) $\underline{v} \wedge d\underline{q}/dt = 0$

Alternatively, if $v \neq c$ while $\mathbf{VZQ} = 0$ and $d\mathbf{Q}/dt = 0$ then: $\nabla^*\mathbf{Q} = 0$ giving the ***Zero-Gradient Conditions***:

ZG: 1) $\partial_0 q_0 - \underline{\nabla} \bullet \underline{q} = 0$ 2) $\partial_0 \underline{q} - \underline{\nabla} q_0 = 0$ 3) $\underline{\nabla} \wedge \underline{q} = 0$

If: $\mathbf{VZ}\,\psi = 0$ then $(c\, \underline{\nabla} + \underline{v}\, \partial_0)\psi = 0$ choosing $\psi = \alpha v_j$ we get the two ***Zero-Velocity Conditions***, $\mathbf{VZQ} = 0$

ZV: 1) $(c\, \underline{\nabla} + \underline{v}\, \partial_0) \wedge \underline{q} = 0$ 2) $\underline{v} \bullet (c\underline{\nabla} + \underline{v}\partial_0)q_0 + c\,(c\underline{\nabla} + \underline{v}\partial_0) \bullet \underline{q} = 0$

5.5 VELOCITY VECTORS

We can define a major sub-class of CNVs as ***Velocity-Class*** Natural Vectors if they have the general structure:

$$\text{Definition:} \quad \mathbf{Q}_v \equiv i\, q_0\, \mathbf{I_0} + \alpha(t;\underline{x})\, \underline{v}(t) \bullet \underline{\mathbf{I}}$$

This produces a general "***Equation of Motion***" for such CNVs: $\alpha\, \underline{v} \bullet \underline{a} = -(c^2\, \underline{v} \bullet \underline{\nabla}\alpha + v^2\, c\, \partial_0 \alpha)$

So, the acceleration ($\underline{a}$) is orthogonal to the velocity (v) for a ***Velocity Vector*** when: $c^2\, \underline{v} \bullet \underline{\nabla}\alpha = -v^2\, c\, \partial_0 \alpha$

These have the harmonic solutions: $\alpha(t;\underline{x}) = \alpha_0 \exp(\, i\,(\underline{K} \bullet \underline{x} - \Omega\, t)\,)$ where: $\underline{K} \bullet \underline{v} = \Omega\, v^2 / c^2$

If it is assumed that: $\underline{K} = \beta\, \underline{v}$ then: $\beta = \Omega / c^2$ or $v = c$ resulting in: $c^2\, \underline{v} \bullet \underline{\nabla} q_0 = -v^2\, c\, \partial_0\, q_0$

This has its own similar harmonic solution, so that a ***Velocity Vector*** has the general solution:

$$\mathbf{Q}_v(t; \underline{x}) = (i\,\beta_0\,\mathbf{I_0} + \alpha_0\,\underline{v}(t) \bullet \underline{\mathbf{I}}) \exp(i\,(\underline{K} \bullet \underline{x} - \Omega\, t))$$

Thus a CNV ***Velocity Vector*** with the following space-time periodic structure satisfies all 4 "Zero" conditions in section (5.4):

$$\mathbf{Q}_v(t; \underline{x}) = \alpha_0 \exp(i\,(\underline{K} \bullet \underline{x} - \Omega\, t)) (- i\, c\, \mathbf{I_0} + \underline{v}(t) \bullet \underline{\mathbf{I}}) \equiv \alpha_H(t; \underline{x})\, \mathbf{V}^*$$

So, the "harmonic" functional parameter, $\alpha_H(t; \underline{x})$ satisfies the ***Wave Equation***: $\Box\alpha_H \equiv (\underline{\nabla}^2 - \partial_0^{\,2})\alpha_H = 0$

The "wave" parameters must then satisfy: $\Omega = \underline{v} \bullet \underline{K} = +c\,K$ or $-c\,K$ so $\underline{v} = +\underline{c}$ or $-\underline{c}$

6. VOIGT VECTORS

6.1 DEFINITION

The discussion of the CNV *Velocity Vector* in section 5.5 above suggests the generalization of the functional parameter $\alpha_H(t; \underline{x})$ to any function of the particle's space and time co-ordinates as a scalar functional multiplier of the ***Velocity*** CNV . In honor of the 19th Century German scientist that first studied the invariance of the wave equation, namely Woldemar **Voigt** (1850-1919), who pioneered the study of coordinate transforms [36], we have named such CNVs, *Voigt Vectors*; they will prove central to much that follows in this programme.

$$\text{Definition:}\quad \mathfrak{V} \equiv -i\, c\, \alpha(t; \underline{x})\, \mathbf{I_0} + \alpha(t; \underline{x})\, \underline{v}(t) \bullet \underline{\mathbf{I}} = \alpha(t; \underline{x})\, \mathbf{V^*} = \{ i\, \mathfrak{V}_0; \underline{\mathfrak{V}} \}$$

Obviously, the scalar and vector components of a *Voigt Vector* satisfy the following equation, named in honor [37] of Ludvig V. **Lorenz** (1829-1891); this equation will also re-appear in many forms throughout this programme.

$$\text{The \textbf{\textit{Lorenz Equation}}:}\quad c\, \underline{\mathfrak{V}} + \underline{v}\, \mathfrak{V}_0 = 0$$

6.2 VOIGT ZERO CONDITIONS

Since every *Voigt Vector* is a type of *Velocity Vector* then it will satisfy the *Flow Equation* if it satisfies the *Zero Conditions* (see section 5.4), so substituting $\alpha\, \underline{v}$ for $\underline{\mathfrak{V}}$ and $-c\alpha$ for $\mathfrak{V}_0$ into conditions 2) or 3).

Then, $\mathfrak{V} = \alpha(t; \underline{x})\, \mathbf{V^*}$ is a ***Flow Vector*** if: 2) $\underline{v} \bullet (c\, \underline{\nabla}\alpha + \underline{v}\, \partial\alpha/\partial t) = 0$ or 3) $d\alpha/dt = (1 - v^2/c^2)\, \partial\alpha/\partial t$

6.3 HARMONIC VOIGT VECTORS

We will define a Harmonic Voigt Vector when the Voigt parameter $\alpha(t; \underline{x})$ has a separable time-dependence that is sinusoidal.

$$\text{Definition:}\quad \text{\textbf{\textit{Harmonic Voigt Vector}}}\quad \mathfrak{V}_h \equiv \exp(i\omega t)\, \alpha(\underline{x})\, \mathbf{V^*}$$

When the Harmonic Voigt parameter $\alpha(t; \underline{x})$ satisfies the Wave Equation, then the first and second partial time derivatives of the particle's velocity must be zero:

$$\text{So, if:}\quad \Box\alpha(t; \underline{x}) = 0 \quad\text{then}\quad \partial\underline{v}/\partial t = 0 \;\text{ and }\; \partial^2\underline{v}/\partial t^2 = 0 \quad\text{therefore}\quad \underline{v}(t; \underline{x}) = v_0$$

Furthermore, since: $\Box\underline{v} = 0$ then if $\Box\alpha = 0$ then $\Box\underline{\mathfrak{V}} = 0$ therefore $\Box\mathfrak{V} = 0$

These results can be generalized beyond the "harmonic" solutions to any function $\varphi(t; \underline{x})$ that satisfies the homogenous ***Wave Equation***:

$$\Box\varphi(t; \underline{x}) = 0 \quad\therefore\quad \underline{\nabla}^2\varphi = \partial_0{}^2\varphi \quad\therefore\quad \underline{\nabla} \bullet (\underline{\nabla}\varphi) = \partial_0(\partial_0\varphi)$$

Since $\partial_0\varphi$ is a scalar and $\underline{\nabla}\varphi$ is a vector we can identify these as the two components of a Voigt vector.

$$\mathfrak{V} = i\, \mathbf{I_0}\, \partial_0\varphi + \underline{\mathbf{I}} \bullet \underline{\nabla}\varphi = \nabla\varphi = i\, \mathfrak{V}_0\, \mathbf{I_0} + \underline{\mathbf{I}} \bullet \underline{\mathfrak{V}} = -i\, c\, \alpha\, \mathbf{I_0} + \alpha\, \underline{v}(t) \bullet \underline{\mathbf{I}}$$

$$\therefore \mathfrak{V}_0 = \partial_0\varphi = -c\,\alpha \quad ; \quad \underline{\mathfrak{V}} = \underline{\nabla}\varphi = \alpha\, \underline{v}$$

This implies: $d\alpha / dt = 0$ or $\partial_0 (d\varphi / dt) = 0 \;\therefore\; \partial_0\varphi = -c\, \alpha_0$ This suggests a generic solution: $\varphi(t; \underline{x}) = \varphi(\xi)$ where: $\xi = z - z_0 \pm c\, t$. This need not be a sine wave but any function, including a **pulse**, that propagates along the z axis, in either z-direction at speed c from the location z_0 at time zero.

6.5 VOIGT VECTOR DERIVATIVES

The most important CNV associated with a continuous Voigt Vector ($\mathfrak{V}$) is its local gradient, $\nabla^*\mathfrak{V}$; the gradients play the role of forces in the corresponding electromagnetic theory (see Conclusion). This has the explicit form:

$$\nabla^*\mathfrak{V} = -\mathbf{I_0}\ d\alpha/dt\ - i\ \underline{\mathbf{I}} \bullet\ (c\ \underline{\nabla}\alpha + \partial_0\underline{\mathfrak{V}})\ +\ \underline{\mathbf{I}} \bullet \underline{\nabla} \wedge \underline{\mathfrak{V}}$$

This has three elements, so the vector component of the CNV ***Gradient*** $\mathfrak{G}$ will have both real and imaginary parts.

Definition: $\nabla^*\mathfrak{V} \equiv i\,\mathfrak{G} = i\,(i\ G_0\ \mathbf{I_0} + \underline{\mathbf{I}} \bullet\ (\underline{G}_R - i\ \underline{G}_I))\ =\ -G_0\ \mathbf{I_0} + i\ \underline{\mathbf{I}} \bullet G_R + \underline{\mathbf{I}} \bullet G_I$

Comparing coefficients gives: $G_0 = d\alpha/dt$; $G_R = -(c\ \underline{\nabla}\alpha + \partial_0\underline{\mathfrak{V}})$; $G_I = \underline{\nabla} \wedge \underline{\mathfrak{V}}$

Now $\mathbf{V}^*\nabla^*\mathfrak{V} = \mathbf{I_0}\,(i\ c\ G_0 - i\ \underline{v} \bullet \underline{G}_R - \underline{v} \bullet \underline{G}_I) + \underline{\mathbf{I}} \bullet\ (c\ \underline{G}_R + \underline{v} \wedge \underline{G}_I) - i\ \underline{\mathbf{I}} \bullet (c\ G_I - \underline{v} \wedge G_R)$

But if $\mathfrak{V}$ is a ***Flow Vector*** then: $d/dt\ (\mathfrak{V}) = -\mathbf{V}^*\nabla^*\mathfrak{V} = d/dt\ [-i\ c\ \alpha\ \mathbf{I_0} + \underline{\mathbf{I}} \bullet\ \underline{\mathfrak{V}}]\ =\ -\mathbf{I_0}\ i\ c\ d\alpha/dt\ + \underline{\mathbf{I}} \bullet\ d/dt(\underline{\mathfrak{V}})$

This gives the ***Gradient Equations***: 1) $\underline{v} \bullet \underline{G}_R = 0$ 2) $\underline{v} \bullet \underline{G}_I = 0$
3) $c\ \underline{G}_R + \underline{v} \wedge \underline{G}_I) = -d/dt(\underline{\mathfrak{V}})$ 4) $c\ G_I - \underline{v} \wedge G_R = 0$

Finally, in differential terms: $-i\ \mathbf{V}^*\ \mathfrak{G} = -\mathbf{V}^*\nabla^*\mathfrak{V} = d/dt\ (\mathfrak{V}) = -i\ (d/dt\ \mathbf{X}^*)\ \mathfrak{G}\ \therefore\ i\ d\mathfrak{V} = d\mathbf{X}^*\ \mathfrak{G}$

6.6 GAUGE VECTORS

There are a class of "harmonic" functions $\psi(t; \underline{x})$ that always satisfy the *Wave Equation* : $\Box\psi(t; \underline{x}) = 0$

Each such function has its own corresponding ***Associate*** CNV, defined as (note, not conjugate): $\mathfrak{A} \equiv \nabla\psi(t; \underline{x})$

The addition of any Associate Vector to a Voigt Vector, $\mathfrak{V}$ defines its corresponding ***Gauge Vector***, $\mathfrak{V}'$.

Definition: ***Gauge Transform*** $\mathfrak{V}' \equiv \mathfrak{V} + \nabla\psi$ where $\Box\psi = 0$

Since $\nabla\psi = i\ \mathbf{I_0}\ \partial_0\psi + \underline{\mathbf{I}} \bullet \underline{\nabla}\psi$ then $\underline{\mathfrak{V}}' = \underline{\mathfrak{V}} + \underline{\nabla}\psi$; $\mathfrak{V}_0' = \mathfrak{V}_0 + \partial_0\psi$ $\therefore\ c\ \alpha' = c\ \alpha - \partial_0\psi$

The conjugate gradient of this Associate CNV is zero, since: $\nabla^*\nabla\psi = -\mathbf{I_0}\Box\psi = 0\ \therefore\ \nabla^*\mathfrak{V}' = \nabla^*\mathfrak{V}$ or $\mathfrak{G}' = \mathfrak{G}$

This harmonic property of the Associate CNV ensures that the Gradient of any Voigt Vector ($\mathfrak{G}$) remains invariant under a Gauge Transform. We will also investigate the hypothesis that the Gauge transform preserves the object's local velocity ($\underline{v}$), then:

$$\underline{\mathfrak{V}}' = \alpha\ \underline{v}' = \underline{\mathfrak{V}} + \underline{\nabla}\psi = \alpha\ \underline{v} + \underline{\nabla}\psi\ \therefore\ c\ \underline{\nabla}\psi = -\underline{v}\ \partial_0\psi\ \therefore\ \nabla\psi = -\partial_0\psi\ /c\ (-i\ c\ \mathbf{I_0} + \underline{\mathbf{I}} \bullet \underline{v})$$

So, an Associate CNV is itself a Voigt Vector with the form: $\mathfrak{A} = \nabla\psi = \beta\ \mathbf{V}^*$ where $\beta = -\partial_0\psi\ /c$. Thus, the addition of a Gauge Vector is just adding the function $\beta\ (t; \underline{x})$ to the original Voigt parameter $\alpha(t; \underline{x})$, thus preserving the Gauge transformed Lorenz Equation that characterizes all Voigt Vectors. Furthermore, under these assumptions, it can be shown that the Gauge transform is only consistent if, and only if, the particle's acceleration is zero. Additionally, if the time-dependence of the Associate CNV's functional parameter $\beta\ (t; \underline{x})$ is sinusoidal then the interaction's solution only propagates at "light-speed". In the long wavelength limit or static limit, this functional parameter reduces to just a constant, β_0. Similarly, it can be shown that rather than assuming the Gauge transform preserves the object's velocity, we assume that it has its own arbitrary velocity, '$\underline{w}$' then it can also be demonstrated that this velocity is also a positive or negative "ray" characterized also by its "light-speed", in other words, $w = \pm c$.

6.7 EULERIAN VECTORS

In this final section examining the general characteristics of Voigt Vectors, we will investigate the possibility that the Voigt parameter $\alpha(t; \underline{x})$ can be a function only the object's local velocity, in other words $\alpha_E(v)$. We shall define such a Voigt Vector as an **Eulerian** Vector if it is also incrementally transformable by a Gauge Vector.

Definition: ***Eulerian Vector*** $\mathfrak{E} \equiv \alpha_E(v)\, \mathbf{V}^*$ and $\Delta\mathfrak{E} = \Delta\mathfrak{A}$ where $\mathfrak{A} = \nabla\psi(t; \underline{x})$

By design, $\Delta\mathfrak{A}$ is a 'small' Gauge Vector in comparison with $\mathfrak{E}$, such that: $\Delta\mathfrak{A} << \mathfrak{E}$ and $\underline{v}' = \underline{v} + \Delta\underline{v}$

$$\Delta\mathfrak{A} = \Delta(\, \beta\, (-i\, c\, \mathbf{I_0} + \underline{\mathbf{I}} \bullet \underline{v})\,) = -\, i\, c\, \mathbf{I_0}\, \Delta\beta + \underline{\mathbf{I}} \bullet (\beta\, \Delta\underline{v} + \underline{v}\, \Delta\beta) = \Delta\beta\, (-i\, c\, \mathbf{I_0} + \underline{\mathbf{I}} \bullet \underline{v}') = \Delta\beta\, (-i\, c\, \mathbf{I_0} + \underline{\mathbf{I}} \bullet (\underline{v} + \Delta\underline{v}))$$

This demonstrates that: $\Delta\beta = \beta$ so that: $\Delta\mathfrak{A} = \Delta\beta\, (\mathbf{V}')^*$ leading to the conclusion that: $\alpha_E >> \Delta\beta$.

Writing: $\alpha_E(v') = \alpha_E' \equiv \alpha_E(v) + \Delta\alpha_E$ while $\mathfrak{E}' = \mathfrak{E} + \Delta\mathfrak{E} = \mathfrak{E} + \Delta\mathfrak{A}$ $\therefore$ $\alpha_E'(\mathbf{V}')^* = \alpha_E \mathbf{V}^* + \Delta\beta\, (\mathbf{V}')^*$

Expanding: $\alpha_E' = \alpha_E + \Delta\beta$ $\therefore$ $\Delta\alpha_E = \alpha_E' - \alpha_E = \Delta\beta$ (constant) $\therefore$ $\Delta\mathfrak{A} = \Delta\alpha\, (\mathbf{V}')^*$ $\therefore$ $\alpha_E\, \underline{v} = \alpha_E\, \underline{v}'$ $\therefore$ $\Delta\underline{v} = 0$

Therefore, there are NO Eulerian Vectors.

7. NATURAL VECTORS IN PHYSICS

7.1 KINEMATICS

A CNV $\mathbf{Q}$ has a "***Flow***" differential equivalent form of its total-time derivative $d\mathbf{Q}/dt$ if it satisfies the ***Zero Conditions*** (see sections 5.3 & 5.4). If we use the ***Spatial Displacement*** Vector, $\mathbf{X}$ for $\mathbf{Q}$ then we find:

$$\mathbf{Q} = \mathbf{X} = \{i\,c\,t\,;\ \underline{x}\} \quad \therefore\ q_0 = ct \ \ \&\ \ \underline{q} = \underline{x}$$

1) $\underline{v} \bullet (\underline{\nabla} \wedge \underline{x}) = 0$ as $\underline{\nabla} \wedge \underline{x} = 0$
2) $c\,\underline{\nabla} \bullet \underline{x} + \underline{v} \bullet \partial_0\underline{x} = 3 \neq 0$ as $\underline{\nabla} \bullet \underline{x} = 3$ & $\partial_0\underline{x} = 0$
3) $v^2(\partial_0 ct + \underline{\nabla} \bullet \underline{x}) + \underline{v} \bullet \underline{\nabla}(c^2 t - \underline{v} \bullet \underline{x}) = 3\,v^2 \neq 0$ as $\underline{\nabla} \bullet \underline{x} = 3$ & $\underline{\nabla}(\underline{v} \bullet \underline{x}) = \underline{v}$
4) $\underline{v} \wedge (\underline{\nabla}ct - \partial_0\,\underline{x}) - c\,\underline{\nabla} \wedge \underline{x} = 0$ as $\underline{\nabla} \wedge \underline{x} = 0$

So, the Spatial Displacement Vector, $\mathbf{X}$ is <u>not</u> a ***Flow Vector*** as it fails to meet conditions (2) and (3); this result is not surprising as: $\nabla^*\mathbf{X} = -2\,\mathbf{I_0} \quad \therefore\ -\mathbf{V}^*\nabla^*\mathbf{X} = 2\mathbf{V}^* \neq \mathbf{V} = d\mathbf{X}/dt$

However, if we use the ***Velocity*** Vector, $\mathbf{V}$ for $\mathbf{Q}$ then we find:

$$\mathbf{Q} = \mathbf{V} = \{i\,c\ ;\ \underline{v}\} \quad \therefore\ q_0 = c \ \ \&\ \ \underline{q} = \underline{v} \quad \text{with} \quad \underline{a} = d\underline{v}/dt = \partial\underline{v}/\partial t = c\,\partial_0\underline{v}$$

1) $\underline{v} \bullet (\underline{\nabla} \wedge \underline{v}) = 0$ as $\underline{\nabla} \wedge \underline{v} = 0$
2) $c\,\underline{\nabla} \bullet \underline{v} + \underline{v} \bullet \partial_0\underline{v} = \underline{v} \bullet \underline{a}$ as $\underline{\nabla} \bullet \underline{v} = 0$
3) $v^2(\partial_0 c + \underline{\nabla} \bullet \underline{v}) + \underline{v} \bullet \underline{\nabla}(c^2 - \underline{v} \bullet \underline{v}) = 0$ as $\underline{\nabla} \bullet \underline{v} = 0$ & $\underline{\nabla}(v \bullet \underline{v}) = \underline{0}$
4) $\underline{v} \wedge (\underline{\nabla}c - \partial_0\,\underline{v}) - c\,\underline{\nabla} \wedge \underline{v} = \underline{v} \wedge \underline{a}$ as $\underline{\nabla} \wedge \underline{v} = 0$

We can see that the Velocity Vector, $\mathbf{V}$ is a ***Flow Vector*** only if the **orthogonal** component of the <u>acceleration</u> relative to the velocity <u>and</u> the parallel component of the acceleration relative to the velocity are <u>both</u> zero. This usually means that the acceleration is always zero or, at least, statistically zero. Therefore the ***Acceleration*** CNV, $\mathbf{A}$ is only consistently defined under these two "zero" conditions:

Definition: $\mathbf{A} \equiv d\mathbf{V}/dt = \underline{\mathbf{I}} \bullet \underline{a} = \{\,0\,;\underline{a}\,\} = -\mathbf{V}^*\nabla^*\mathbf{V}$ IFF: $\underline{v} \bullet \underline{a} = 0$ & $\underline{v} \wedge \underline{a} = 0$

We can re-derive this surprising result directly by noting that $\underline{\nabla}v_j = 0$ and $c\,\nabla^*\mathbf{V} = -i\,c\,\underline{\mathbf{I}} \bullet \partial_0\underline{v} = -i\,\underline{\mathbf{I}} \bullet \underline{a}$:

$$\mathbf{A} \equiv d\mathbf{V}/dt = (\partial/\partial t + \underline{v} \bullet \underline{\nabla})\,[i\,c\,\mathbf{I_0} + \underline{\mathbf{I}} \bullet \underline{v}] = (\partial/\partial t + \underline{v} \bullet \underline{\nabla})\,(\underline{\mathbf{I}} \bullet \underline{v}) = \underline{\mathbf{I}} \bullet \underline{a} = \{\,0\,;\underline{a}\,\}$$

$$-c\,\mathbf{V}^*\nabla^*\mathbf{V} = \mathbf{V}^*(\,i\,\underline{\mathbf{I}} \bullet \underline{a}\,) = c\,\underline{\mathbf{I}} \bullet \underline{a} - i\,(\underline{v} \bullet \underline{a})\,\mathbf{I_0} + \underline{\mathbf{I}} \bullet (\underline{v} \wedge \underline{a})$$

7.2 THE WAVE EQUATION

In section 4.6 it was shown that if $\nabla^*\mathbf{Q} = 0$ then: 1) $\underline{\nabla} \bullet \underline{q} = \partial_0 q_0$ 2) $\underline{\nabla}q_0 = \partial_0\underline{q}$ 3) $\underline{\nabla} \wedge \underline{q} = 0$

If the divergence is taken of the second equation, then:

$$\underline{\nabla} \bullet (\underline{\nabla}q_0 - \partial_0\underline{q}) = \underline{\nabla} \bullet \underline{\nabla}q_0 - \partial_0\,\underline{\nabla} \bullet \underline{q} = \nabla^2 q_0 - \partial_0\partial_0 q_0 = (\,\nabla^2 - 1/c^2\,\partial^2/\partial t^2\,)q_0 = 0 \quad \therefore\ \Box q_0 = 0$$

If the gradient is taken of the first equation, then:

$$\underline{\nabla}(\underline{\nabla} \bullet \underline{q} - \partial_0 q_0) = \underline{\nabla}(\,\underline{\nabla} \bullet \underline{q}) - \partial_0(\underline{\nabla}q_0) = (\nabla^2\underline{q} - \underline{\nabla} \wedge (\underline{\nabla} \wedge \underline{q})) - \partial_0\partial_0\underline{q} = (\nabla^2 - 1/c^2\,\partial^2/\partial t^2\,)\underline{q} = 0 \quad \therefore\ \Box\underline{q} = 0$$

So, if $\nabla^*\mathbf{Q} = 0$ then $\Box\mathbf{Q} = 0$. In other words, a *Zero Gradient* CNV always satisfies the *Wave Equation*.

Further, if $\mathbf{Q}$ is a *Flow Vector* then $d/dt\,\mathbf{Q} = -\mathbf{V}^*\nabla^*\mathbf{Q}$; so if $\Box\mathbf{Q} = 0$ then $d/dt\,\mathbf{Q} = 0$

Finally, if the *Voi**gt*** CNV $\mathfrak{V}$ is a *Flow Vector* and if $\Box\mathfrak{V} = 0$ then $d\alpha/dt = 0$ so α is constant over time.

7.3 THE CONTINUITY EQUATION

The two components of all *Voigt* CNVs $\mathfrak{V}$ satisfy their own Lorenz Equation: $c\,\underline{\mathfrak{V}} + \underline{v}\,\mathfrak{V}_0 = 0$
If the divergence is taken of this equation, then:

$$\underline{\nabla} \bullet (c\,\underline{\mathfrak{V}} + \underline{v}\,\mathfrak{V}_0) = c\,\underline{\nabla} \bullet \underline{\mathfrak{V}} + \underline{v} \bullet \underline{\nabla}\,\mathfrak{V}_0 = c\,\underline{\nabla} \bullet \underline{\mathfrak{V}} + (d/dt - \partial/\partial t)\,\mathfrak{V}_0 = 0 \quad \therefore \quad \underline{\nabla} \bullet \underline{\mathfrak{V}} + \partial\alpha/\partial t = d\alpha/dt$$

Any *Voigt* CNV $\mathfrak{V}$ where $d\alpha/dt = 0$ or $\Box\,\mathfrak{V} = 0$ satisfies its own ***Continuity Equation***: $\underline{\nabla} \bullet \underline{\mathfrak{V}} + \partial\alpha/\partial t = 0$

In a particle theory, we might assume that each particle has a characteristic property α_0, if this scalar multiplies the particle's velocity $\underline{v}$ then we can investigate the hypothesis that this "scaled" vector $\underline{J} = \alpha_0\,\underline{v}$ is the vector component of a Voigt Vector, $\mathfrak{J}$. In which case, its Lorenz Equation would define its scalar component, $J_0 = -\,c\,\alpha_0$. If this key parameter is not an explicit function of time, then this characteristic value becomes a conserved property, such as the particle's inertial mass, *m* or electric charge *e*; where the corresponding 3D vectors would be particle momentum or electron current. In a classical "medium" theory the characteristic property α_0, would have to be associated with an infinitesimal volume element of the medium that has a local velocity, such a characteristic property would then be an invariant density, ρ while the corresponding vector would be a density-current, $\rho\,\underline{v}$. We will further develop these ideas in section 7.6 and in subsequent papers in this series.

7.4 MINKOWSKI 4-VECTORS

Lorenz covariant 4-vectors ("Minkowski Vectors") have some superficial resemblances to Natural Vectors. Usually written as an ordered, four component list, A_μ they may be viewed as a 4D extension to standard 3D vectors A_j by the addition of an extra component based on a new unit vector, $\hat{\mathbf{e}}_0$; namely:

$$\mathbb{A} \equiv A_0\,\hat{\mathbf{e}}_0 + \underline{A} = \textstyle\sum_\mu A_\mu$$

The set of four unit vectors $\{\hat{\mathbf{e}}_\mu\}$ have an ortho-normal scalar product across all 4 indices, μ and ν :

$$\hat{\mathbf{e}}_\mu \bullet \hat{\mathbf{e}}_\nu = \delta_{\mu\nu}$$

This is the **only** form of multiplication used with Minkowski vectors, so that:

$$\mathbb{A} \bullet \mathbb{B} \equiv \textstyle\sum_\mu A_\mu B_\mu$$

The basic Minkowski vector $\mathbb{X}$ represents any point in space $\underline{x}$ at an instant of time t:

$$\mathbb{X} \equiv i\,c\,t\,\hat{\mathbf{e}}_0 + x_1\,\hat{\mathbf{e}}_1 + x_2\,\hat{\mathbf{e}}_2 + x_3\,\hat{\mathbf{e}}_3 \qquad \text{i.e. } \underline{x} = \textstyle\sum_j x_j\,\hat{\mathbf{e}}_j \ \text{(space)} \ \& \ x_0 = i\,c\,t \ \text{(imaginary time)}$$

Since this Minkowski vector-space is defined as a continuous, Euclidean manifold across each component x_μ, then each dimension has a separate differential dx_μ ; where dx_j (space) & $dx_0 = i\,c\,dt$ (imaginary time).

$$d\mathbb{X} = \textstyle\sum_\mu dx_\mu\,\hat{\mathbf{e}}_\mu \qquad \therefore \quad d\mathbb{X} \bullet d\mathbb{X} = d\underline{x} \bullet d\underline{x} - c^2\,dt^2 = (v^2 - c^2)\,dt^2$$

A vector multiplication product <u>could</u> be defined for Minkowski vectors, by analogy with 3D vectors:

$$\hat{\mathbf{e}}_\mu \wedge \hat{\mathbf{e}}_\nu \equiv \textstyle\sum_\lambda \varepsilon_{\mu\nu\lambda}\,\hat{\mathbf{e}}_\lambda \qquad \text{where } \varepsilon_{\mu\nu\lambda} \text{ is the cyclic 4D permutation tensor.}$$

In contrast, the algebra of Natural Vectors has **only** <u>ONE type of universal multiplication</u>, combining both scalar and vector multiplication together into **one single operation**, based on Hamilton's original quaternion rules:

$$\mathbf{I_0}\,\mathbf{I}_\mu = \mathbf{I}_\mu\,\mathbf{I_0} = \mathbf{I}_\mu \ ; \ \mathbf{I_j}\,\mathbf{I_k} = -\,\delta_{ij}\,I_0 + \textstyle\sum_l \varepsilon_{jkl}\,\mathbf{I_l} \ ; \ \mathbf{I}_\mu{}^* = \mathbf{I}_\mu \ \text{ with } \ \mathbf{X} = i\,x_0\mathbf{I_0} + \textstyle\sum_j x_j\,\mathbf{I}_j \qquad \{j = 1,2,3 \ ; \ \mu = 0, j\}$$

So, Natural Vectors may be viewed as "**covariant Minkowski quaternions**"; as such, they eliminate the need for abstract tensor analysis and form a more natural extension from timeless (or instantaneous) 3D physics to the real world of space and time with interactions propagating at constant "light" speed, c.

7.5 PHYSICAL VOIGT VECTORS

As the first paper in this series, a great deal of time has been expended in establishing the new mathematics. This is viewed as important as this will prove to be the only representational theory used in what follows: so its foundations must be sound. Now is the time to show that this new mathematics is a suitable replacement for the simpler algebra that has served physics well for over 300 years. The non-commutative nature of Natural Vectors will prove central to the resolution of problems in 20th century physics that were introduced by the discovery of the universal electron.

Selecting different functional forms for the Voigt parameter (α) generates three distinct sub-classes of physical theory (each with its own unique equation of motion): 1) Newtonian 2) Conservative 3) Maxwellian. This illustrates the **centrality** of the canonical *Velocity Vector*, $\mathbf{V}^*$.

7.5.1 NEWTONIAN VECTORS

Newtonian Vectors ($\mathfrak{v}_N$) are Voigt vectors where the Voigt parameter is simply a scalar constant: $\alpha = \alpha_0$; these are not only the simplest Voigt vectors but play the central roles in the subsequent theory.

Definition: ***Newtonian Vector*** $\mathfrak{v}_N \equiv \alpha_0 \mathbf{V}^* = \alpha_0\{-i\,c\,;\,\underline{v}\}$

The point-particle model of the electron (with inertial mass, *m* and electric charge, *e*) is reflected in the electron ***Momentum*** CNV $\mathbf{P}_e$ and the electron ***Current*** CNV $\mathbf{J}_e$ where:

$$\mathbf{P}_e \equiv m\,\mathbf{V}^* \quad \text{and} \quad \mathbf{J}_e \equiv e\,\mathbf{V}^*$$

Obviously, in this model: $\mathbf{J}_e = (e/m)\,\mathbf{P}_e$ so that (e/m) becomes a universal **constant** characterizing the electron-electron **interaction**. This universal ratio will become central to the new theory of electromagnetic interactions.

Newtonian physics was centered on the concept of a solid object that could be represented by its center-of-mass that moved continuously under the instantaneous influence of other remote distinct objects, whose influence was always aggregated into a continuous force. The present theory can recover this older kinematics by introducing a single CNV, called here the *Activity* Natural Vector (that is **not** actually a Voigt Vector, as its multiplier is not a scalar).

Definition: ***Activity Natural Vector*** $\mathbf{R} \equiv i\,\mathbf{X}\,\mathbf{P} = i\,m\mathbf{X}\mathbf{V}^* = i\,R_0\,\mathbf{I_0} + \underline{\mathbf{I}} \bullet \underline{R}$

Where: $R_0 = m\,(c^2 t - \underline{x} \bullet \underline{v})$ & $\underline{R} = \underline{R}_R + i\,\underline{R}_I$ with $\underline{R}_R = m\,c\,(\underline{x} - \underline{v}\,t)$ & $\underline{R}_I = m\,(\underline{x} \wedge \underline{v})$

The condition: $d\,\mathbf{R}(t)/dt = 0$ implies that: i) $c^2 - v^2 = \underline{x} \bullet \underline{a}$ ii) $c\,t\,\underline{a} = 0$ iii) $m\,(\underline{x} \wedge \underline{a}) = 0$

These equations have the solutions: $\underline{a}(t) = 0$ so that $\underline{v}(t) = \underline{v}_0$ and $\underline{v}_0 = \underline{c}$ (i.e. freely moving at 'light-speed').

Here $\underline{a}(t)$ is the acceleration of the body induced by the external force $\underline{F}$ according to Newton's Second Law $\underline{F} = m\,\underline{a}$, so that the temporal invariance of the Activity NV is equivalent to the body being 'force-free' or a 'free-particle'. The 3D spatial location solution is: $\underline{x}(t) = \underline{x}_0 + \underline{v}_0\,t$ where $\underline{x}_0$ is the location of the particle at t = 0 and c is its speed.

The other (force-free) 3D invariants are $\underline{x} \bullet \underline{R}$ and $\underline{v} \bullet \underline{R}$, since :

$$d\,(\underline{x} \bullet \underline{R})/dt = m\,c\,\underline{x}_0 \bullet \underline{v}_0 = \underline{v} \bullet \underline{R} = \text{constant (zero, if } \underline{x}_0 = 0).$$

The explicit NV form of the Activity vector is: $\mathbf{R}(t) = i\,\mathbf{I}_0\,(mc^2 t - \underline{x} \bullet \underline{p})\,\mathbf{I_0} + \underline{\mathbf{I}} \bullet c(m\,\underline{x} - \underline{p}\,t) + i\,\underline{\mathbf{I}} \bullet (\underline{x} \wedge \underline{p})$

In the present electron theory there are **no** free-floating '**forces**' – all changes in motion are due to the interactions with other electrons, so that the one-electron situation is always 'force-free', when R is invariant, i.e. R(t) = R(0). Later papers will return to this concept where it will re-appear as one of the key ideas in this new theory.

7.5.2 CONSERVATIVE VECTORS

Conservative Vectors ($\mathfrak{V}_C$) are Voigt Vectors where the Voigt parameter is a function only of spatial location and independent of time, that is: $\alpha = \alpha(\underline{x})$.

Definition: ***Conservative Vector*** $\mathfrak{V}_C \equiv \alpha(\underline{x})\,\mathbf{V^*} = \alpha(\underline{x})\{-i\,c\,;\ \underline{v}\}$

Conservative Vectors are useful for analyzing situations that only vary slowly over time (**quasi-static**), where the approximation of introducing (pseudo) static potentials is "good enough", such as electrostatics $\phi(\underline{x})$ and magneto-statics or constant electric currents: $\underline{A}(\underline{x})$. Surprisingly, this was the mistaken choice made by Maxwell when he incorrectly imposed the so-called "Coulomb gauge" $\underline{\nabla} \bullet \underline{A} = 0$ which is equivalent to infinitely fast propagation of electromagnetic effects instead of with the finite "light-speed", that results from the correct "Lorenz gauge".

7.5.3 MAXWELLIAN VECTORS

Maxwellian Vectors ($\mathfrak{V}_M$) are Voigt Vectors where the Voigt parameter is a function only of the particle's local speed (v) or more often the square of its velocity, that is: $\alpha = \alpha(v^2)$.

Definition: ***Maxwellian Vector*** $\mathfrak{V}_M \equiv \alpha(v^2)\,\mathbf{V^*} = \alpha(v^2)\{-i\,c\,;\ \underline{v}\}$

It is useful to define two "scaling" factors that can "normalize" Maxwellian Vectors:

Definition: ***Voigt Factor*** $\mathcal{V} \equiv \sqrt{(1 - v^2/c^2)}$ and ***Lorentz Factor*** $\mathcal{L} \equiv 1/\mathcal{V}$

The two sub-classes of Maxwellian Vectors are Heaviside Vectors and Lorenz Vectors, defined simply by:

Definition: ***Heaviside Vector*** $\mathfrak{V}_H \equiv \alpha_0\,\mathcal{V}\,\mathbf{V^*}$ and ***Lorenz Vector*** $\mathfrak{V}_L \equiv \alpha_0\,\mathcal{L}\,\mathbf{V^*}$

7.6 RELATIVISTIC MECHANICS

Since, for all Voigt Vectors their "norm" is: $\mathfrak{V}^*\,\mathfrak{V} = c^2\,\mathcal{V}^2\,\alpha(t;\underline{x})^2\,\mathbf{I_0}$ then $\mathfrak{V}_L{}^*\,\mathfrak{V}_L = c^2\,\alpha_0{}^2\,\mathbf{I_0}$
So, all Lorenz Vectors have an invariant (constant) "norm". The most famous Lorenz Vector corresponds to Planck's relativistic momentum for point-particles, where $\alpha_0 = m$:

$$\mathfrak{P}_L \equiv M(v)\,\mathbf{V^*} = m\,\mathcal{L}\,\mathbf{V^*} = (-i\,c\,\mathbf{I_0} + \underline{v}(t) \bullet \underline{\mathbf{I}})\,M = -i\,P_0\,\mathbf{I_0} + \underline{P} \bullet \underline{\mathbf{I}} \quad \therefore\ P_0 = M\,c\,;\ \underline{P} = M\,\underline{v}$$

In his 1907 paper, Planck introduced [39] a **fictional**, mechanical force $\underline{F}_0$, which was <u>always parallel</u> to the particle's velocity and constant across space and time, that transferred an energy E from the source of this force to accelerate the particle continuously from rest to velocity $\underline{v}$ that was constant in direction, i.e. words, $\underline{v}(t) = v(t)\,\hat{\mathbf{e}}_1$. Planck identified this energy with the kinetic energy of his relativistic particle through the classical kinetic energy equation:

$dE = (\underline{v} \bullet \underline{F}_0)\,dt$. In terms of Voigt Vectors this gives:

$$d/dt\,(\mathfrak{P}_L{}^*\,\mathfrak{P}_L) = d/dt((P_0\,P_0 - \underline{P} \bullet \underline{P})\,\mathbf{I_0}) = 2(P_0\,dP_0/dt - \underline{P} \bullet d\underline{P}/dt)\,\mathbf{I_0} = 0 \quad \therefore\ c\,P_0\,dP_0 = c\,\underline{P} \bullet d\underline{P} = P_0\,\underline{v} \bullet d\underline{P}$$

$$\therefore\ c\,dP_0 = \underline{v} \bullet d\underline{P} = \underline{v} \bullet (\underline{F}_0\,dt) = (\underline{v} \bullet \underline{F}_0)\,dt = dE \quad \therefore\ c\,P_0 = E = M\,c^2 \text{ (the \textbf{Einstein} Equation).}$$

Where we used the Lorenz Equation: $c\,\underline{P} = \underline{v}\,P_0$ since the Planck relativistic momentum is also a Voigt Vector.
This gives the covariant form of the particle's Relativistic Momentum CNV: $\mathfrak{P}_L = -i\,\mathbf{I_0}\,E/c + \underline{P} \bullet \underline{\mathbf{I}}$

$$\therefore\ \mathfrak{P}_L{}^*\,\mathfrak{P}_L = (P_0\,P_0 - \underline{P} \bullet \underline{P})\,\mathbf{I_0} = (M^2\,c^2 - M^2\,v^2)\,\mathbf{I_0} = m^2\,c^2\,\mathbf{I_0} \quad \therefore\ M^2\,c^4 = m^2\,c^4 + P^2\,c^2 = E^2$$

All spatial distances are measured relative to the source of this fictional force (and action & reaction are ignored) and this "force" <u>propagates instantaneously</u> (as Maxwell **erroneously** assumed when he used the Coulomb gauge). It is obvious that this situation implicitly contradicts major assumptions of Einstein's special relativity (no acceleration in inertial reference frames and no transfers of energy at speeds exceeding "light-speed").

7.7 NATURAL VECTOR INVARIANTS

7.7.1 SCALAR INVARIANTS

It is sometimes possible to define two kinds of Natural Vectors that have 'constant' properties (i.e. invariants), which reflect basic physical properties of the system and preservation of the "form" of the associated equations (covariance). The simplest forms have no spatial component – these are referred to as "scalar invariants" NVs.

Definition: $\mathbf{A}_0$ is a ***Scalar Invariant*** if: $\mathbf{A}_0 \equiv A_0\,\mathbf{I_0}$

The mathematical significance of scalar invariants is that they are equation-position insensitive (commutative) when contributing to products of NVs. In other words, if the NVs do not involve operators such as ∂x_μ then scalar invariant NVs commute with the other NVs; thus: $\mathbf{A}_0\,\mathbf{B} = \mathbf{B}\,\mathbf{A}_0 = A_0\,\mathbf{B}$

Obviously, the "norm" (or self-product) of every NV is a scalar invariant: $\mathbf{Q}^*\mathbf{Q} = \mathbf{Q}\mathbf{Q}^* = (q_0^2 - \underline{q} \bullet \underline{q})\,\mathbf{I_0}$
This is universally recognized for the separation vector: $\mathbf{S}(t) = \mathbf{X}^*\mathbf{X} = (c^2\,t^2 - \underline{x} \bullet \underline{x})\,\mathbf{I_0} = S_0\,\mathbf{I_0}$

7.7.2 TEMPORAL INVARIANTS

A Natural Vector is a "temporal invariant" if it remains **unchanged** throughout time.

Definition: $\mathbf{A}$ is a ***Temporal Invariant*** if: $d/dt\,\mathbf{A}(t) \equiv 0$

The *Velocity* Natural Vector $\mathbf{V}$ is a *Flow Vector* when it satisfies the *Zero Conditions* (see section 5.3 & 5.4); these results imply that the acceleration $\underline{a}$ is zero or equivalently that the ***Acceleration*** Natural Vector $\mathbf{A}$ is zero (see 7.1). In this case, the velocity is a constant $\underline{V}_0$ so that the Velocity Natural Vector for a single ('free') particle is invariant.

$$\therefore\quad d/dt\,\mathbf{V}(t) = 0 \qquad \text{or} \qquad \mathbf{V}(t) = i\,c\,\mathbf{I_0} + \underline{\mathbf{I}} \bullet \underline{V}_0$$

The products $\mathbf{X}^*\mathbf{V}$ and $\mathbf{V}^*\mathbf{X}$ evaluate to:

$$\mathbf{X}^*\mathbf{V} = (c^2\,t - \underline{v} \bullet \underline{x})\,\mathbf{I_0} + i\,c\,\underline{\mathbf{I}} \bullet (\underline{x} - \underline{v}\,t) + \underline{\mathbf{I}} \bullet (\underline{x} \wedge \underline{v}) \qquad \mathbf{V}^*\mathbf{X} = (c^2\,t - \underline{v} \bullet \underline{x})\,\mathbf{I_0} - i\,c\,\underline{\mathbf{I}} \bullet (\underline{x} - \underline{v}\,t) - \underline{\mathbf{I}} \bullet (\underline{x} \wedge \underline{v})$$

So, $d/dt\,(\mathbf{X}^*\mathbf{V} + \mathbf{V}^*\mathbf{X}) = 2\,d/dt\,(c^2\,t - \underline{v} \bullet \underline{x})\,\mathbf{I_0} = 2(c^2 - \underline{v} \bullet \underline{v})\,\mathbf{I_0} = 2(c^2 - V_0^2)\,\mathbf{I_0} \quad \therefore\ \underline{v} \bullet \underline{x} = \underline{V}_0 \bullet (\underline{X}_0 + \underline{V}_0\,t)$

For a 'free' particle, mass: m: $\underline{x} = \underline{X}_0 + \underline{v}\,t$ then $(\underline{x} - \underline{v}\,t) = \underline{X}_0$ (constant) $\therefore\ (\underline{x} \wedge \underline{v}) = \underline{X}_0 \wedge \underline{V}_0$ (constant)

$$\therefore\ m(\mathbf{X}^*\mathbf{V} - \mathbf{V}^*\mathbf{X}) = 2\,\underline{\mathbf{I}} \bullet \underline{L}_0 \text{ where } \underline{L}_0 = i\,mc\,\underline{X}_0 + m\underline{X}_0 \wedge \underline{V}_0 \text{ (another constant) } \&\ \mathbf{V} - \mathbf{V}^* = 2\,i\,c\,\mathbf{I_0}$$

If the particle has an intrinsic, invariant **inertial** mass m then the following are dynamical constants of the 'free' particle, especially when choosing the origin so that: $\underline{X}_0 = 0 \quad \therefore\ \underline{L}_0 = 0$

1. NV Momentum $\quad \mathrm{P} \equiv m\,\mathbf{V}^* \quad = -i\,m\,c\,\mathbf{I_0} + \underline{\mathbf{I}} \bullet m\,\underline{V}_0$
2. Linear Momentum $\quad \mathcal{P} \equiv \tfrac{1}{2}(\mathrm{P}^* + \mathrm{P}) \quad = m\,\underline{V}_0 \bullet \underline{\mathbf{I}}$
3. Particle Energy $\quad \mathcal{E} \equiv \tfrac{1}{4}(\mathbf{V}^* - \mathbf{V})(\mathrm{P}^* - \mathrm{P}) \quad = m\,c^2\,\mathbf{I_0}$
4. Kinetic Energy $\quad \mathcal{K} \equiv \tfrac{1}{4}(\mathrm{P}\,\mathbf{V} + \mathbf{V}\,\mathrm{P}) \quad = \tfrac{1}{2}\,m\,(c^2 - V_0^2)\,\mathbf{I_0}$
5. Particle Action $\quad \mathcal{A} \equiv \tfrac{1}{2}(\mathbf{X}^*\mathrm{P}^* + \mathrm{P}\,\mathbf{X}) \quad = 2\,t\,\mathcal{K}\,\mathbf{I_0}$
6. Angular Momentum $\quad \mathcal{M} \equiv \tfrac{1}{2}(\mathbf{X}^*\mathrm{P}^* - \mathrm{P}\,\mathbf{X}) \quad = \underline{L}_0 \bullet \underline{\mathbf{I}}$
7. Galilean Momentum $\quad \mathcal{G} \equiv m\,\mathbf{X} - \mathrm{P}^*\,t = m\,(\underline{x} - \underline{v}\,t) \bullet \underline{\mathbf{I}} \quad = m\,\underline{X}_0 \bullet \underline{\mathbf{I}}$

7.8 TWO PARTICLE NATURAL VECTORS

The analysis so far has focused on representing a single point-particle by a single-time Natural Vector. These can be combined together to represent the joint (or **relative**) positions of two such particles. If the locations of the two point particles (which are labeled here as #1 and #2) at times t_1 and t_2) are $\underline{x}_1$ and $\underline{x}_2$ then they may each be represented by their own Natural Vector $\mathbf{X}_\alpha$, where $\alpha = 1$ or 2; each particle has its own velocity NV $\mathbf{V}_\alpha$.

Definition: ***Location Vector*** $\mathbf{X}_\alpha \equiv i\, c\, t_\alpha \mathbf{I_0} + \underline{\mathbf{I}} \bullet \underline{x}_\alpha = \{ i\, c\, t_\alpha ; \underline{x}_\alpha \}$ and $\mathbf{V}_\alpha \equiv d/dt_\alpha \mathbf{X}_\alpha = i\, c\, \mathbf{I_0} + \underline{\mathbf{I}} \bullet \underline{v}_\alpha$

7.8.1 SEPARABLE VECTORS

Since Natural Vectors are used to represent interactions between pairs of point particles across space and time, the most useful NVs are anti-symmetric and separable across the co-ordinates and properties of the two particles.

Definition: ***Separable Vector*** $\mathbf{S}(t) \equiv \mathbf{S}(t_1 - t_2) \equiv \mathbf{S}_1(t_1) - \mathbf{S}_2(t_2)$

The canonical example of a separable CNV is the ***Separation*** Vector $\mathbf{X}$ between two particles that interact when they are located in space at $\underline{x}_1$ and $\underline{x}_2$ at the respective interaction times t_1 and t_2: $\mathbf{X}_{12}(t_1 - t_2) \equiv \mathbf{X}_1(t_1) - \mathbf{X}_2(t_2)$

7.8.2 CONSERVED QUANTITIES

If a Natural Vector $\mathbf{S}$ is both separable and a "temporal invariant" then $\mathbf{S}$ represents a conserved physical quantity that remains unchanged throughout time.

$$d/dt\, \mathbf{S}(t) = d/dt\, \mathbf{S}(t_1 - t_2) = d/dt_1 \mathbf{S}_1(t_1) + d/dt_2 \mathbf{S}_2(t_2) = 0 \quad (\text{as } t = t_1 - t_2) \quad \therefore\ d/dt_1 \mathbf{S}_1(t_1) = - d/dt_2\, \mathbf{S}_2(t_2) = d\mathbf{S}_0$$

This represents the "**transfer**" across space of a property 'S' from particle #2 that at time t_2 "loses" an amount dS_0, such that particle #1 at time t_1 "increases" its amount of 'S' by the same amount per unit of time. This transfer between the two particles is just the asynchronous action-at-a-distance (AAAD) generated by their mutual interaction and does NOT imply the physical existence of any intermediate carrier of the particle property 'S'. These co-ordinates of the two particles are defined in some arbitrary 3D Euclidean frame-of-reference with the same common origin of universal relative time. Each particle is located at position $\underline{X}_\alpha$ when the 'master-clock' is zeroed and, at this time (from t = 0 until $t = \tau - \delta t$), each particle has a velocity of $\underline{V}_\alpha$. If neither particle has interacted with any other until their latest time t_1 (assuming $t_1 > t_2$) they both move like 'free' particles and retain their initial velocities, so that: $\underline{v}_\alpha(t_\alpha) = \underline{V}_\alpha$; this implies that their locations at t are given by: $\underline{x}_\alpha(t_\alpha) = \underline{X}_\alpha + \underline{V}_\alpha\, t_\alpha$.
Thus, for two 'free' particles: $\underline{x}_\alpha - \underline{v}_\alpha\, t_\alpha = \underline{X}_\alpha$ (constant) and $\underline{x}_\alpha \wedge \underline{v}_\alpha = \underline{X}_\alpha \wedge \underline{V}_\alpha$ (constant) .

Now $\quad \mathbf{X}_\alpha{*}\mathbf{V}_\alpha = (c^2\, t_\alpha - \underline{v}_\alpha \bullet \underline{x}_\alpha)\, \mathbf{I_0} + i\, c\, \underline{\mathbf{I}} \bullet (\underline{x}_\alpha - \underline{v}_\alpha\, t_\alpha) + \underline{\mathbf{I}} \bullet (\underline{x}_\alpha \wedge \underline{v}_\alpha) = (c^2\, t_\alpha - \underline{v}_\alpha \bullet \underline{x}_\alpha)\, \mathbf{I_0} + \underline{\mathbf{I}} \bullet \underline{L}_\alpha$

And $\quad \mathbf{V}_\alpha{*}\mathbf{X}_\alpha = (c^2\, t_\alpha - \underline{v}_\alpha \bullet \underline{x}_\alpha)\, \mathbf{I_0} - \underline{\mathbf{I}} \bullet \underline{L}_\alpha$ where $\underline{L}_\alpha = i\, c\, \underline{X}_\alpha + \underline{X}_\alpha \wedge \underline{V}_\alpha$ (complex vector constants).

$$\therefore\ \mathbf{X}_\alpha{*}\mathbf{V}_\alpha - \mathbf{V}_\alpha{*}\mathbf{X}_\alpha = 2\, \underline{\mathbf{I}} \bullet \underline{L}_\alpha \quad \therefore\ d/dt_\alpha\, (\mathbf{X}_\alpha{*}\mathbf{V}_\alpha - \mathbf{V}_\alpha{*}\mathbf{X}_\alpha) = 0 \quad \text{and} \quad \mathbf{V}_\alpha - \mathbf{V}_\alpha{*} = 2\, i\, c\, \mathbf{I_0}$$

$$\therefore\ d/dt_\alpha(\mathbf{X}_\alpha{*}\mathbf{V}_\alpha + \mathbf{V}_\alpha{*}\mathbf{X}_\alpha) = 2\, d/dt_\alpha\, (c^2\, t_\alpha - \underline{v}_\alpha \bullet \underline{x}_\alpha)\, \mathbf{I_0} = 2\, (c^2 - \underline{v}_\alpha \bullet \underline{v}_\alpha)\, \mathbf{I_0} = 2\, (c^2 - V_\alpha^{\,2})\, \mathbf{I_0}$$

The **average** velocity of the two particles is defined as $\underline{V} \equiv ½\, (\underline{v}_1 + \underline{v}_2)$ and $\mathbf{V} \equiv ½\, (\mathbf{V}_1 + \mathbf{V}_2) = i\, c\, \mathbf{I_0} + \underline{\mathbf{I}} \bullet \underline{V}$

Once again, it is possible to define two global constants: $\underline{X}_0 = \underline{X}_1 - \underline{X}_2$ and $\underline{L}_0 = i\, c\, \underline{X}_0 + \underline{X}_0 \wedge \underline{V}_0$

Writing: $\quad T = t_1 - t_2$ and $\underline{X} = \underline{x}_1 - \underline{x}_2$ then: $\mathbf{X} = \mathbf{X}_{12}(T) = \mathbf{X}_1(t_1) - \mathbf{X}_2(t_2) = i\, c\, T\, \mathbf{I_0} + \underline{X} \bullet \underline{\mathbf{I}}$

If each of the pair of particles has the same intrinsic, invariant inertial mass *m* then the **total** momentum at the two times t_1 and t_2 is proportional to their average velocity at these two times; so that: $\underline{V} = \underline{V}_{12}(t_1,t_2) \equiv ½\, \{\underline{v}_1(t_1) + \underline{v}_2(t_2)\}$.

Definition: ***Total Momentum*** $\mathbf{P}(t_1,t_2) \equiv \mathbf{P}_1(t_1) + \mathbf{P}_2(t_2) = m\,(\mathbf{V}_1+\mathbf{V}_2)^* = 2\,m\mathbf{V}^*(t_1,t_2) = -i\,2\,m\,c\,\mathbf{I_0} + 2\,m\,\underline{V}\bullet\underline{\mathbf{I}}$

Now $\quad \mathbf{P}\ \mathbf{X} = 2m(c^2\,T - \underline{V}\bullet\underline{X})\,\mathbf{I_0} - i\,2mc\,\underline{\mathbf{I}}\bullet(\underline{X} - \underline{V}\,T) - 2m(\underline{X}\wedge\underline{V})\bullet\underline{\mathbf{I}} = 2m(c^2\,T - \underline{V}\bullet\underline{X})\,\mathbf{I_0} - 2m\,\underline{L}\bullet\underline{\mathbf{I}}$

and $\quad \mathbf{X}^*\mathbf{P}^* = 2m(c^2\,T - \underline{V}\bullet\underline{X})\,\mathbf{I_0} + 2\,\underline{L}\bullet\underline{\mathbf{I}}$ where $\underline{L} = i\,mc\,(\underline{X} - \underline{V}\,T) + m\,\underline{X}\wedge\underline{V}$

It will be shown in a subsequent paper that when the two particles interact only "on their mutual light-cone" that their combined velocity: $\underline{v}_1(t_1) + \underline{v}_2(t_2) = \underline{c}$ (the 'speed of light' defined between their interacting spatial positions). Thus, when the two similar particles are interacting "on their (mutual) light-cones" $\underline{X} = \underline{c}\,T$ and $\underline{V} = ½\,\underline{c}$

$$\therefore\ \underline{V}\bullet\underline{X} = ½\,c^2\,T \quad \therefore\ (c^2\,T - \underline{V}\bullet\underline{X}) = ½\,c^2\,T \ \ \&\ \ (\underline{X} - \underline{V}\,T) = ½\,\underline{c}\,T \ \ \&\ \ \underline{X}\wedge\underline{V} = 0 \quad \therefore\ \underline{L} = ½\,i\,c^2\,T\,\underline{\hat{e}}_{12}$$

$$\therefore\ \mathbf{PX} = m\,c\,T\,(c\,\mathbf{I_0} - i\,\underline{\mathbf{I}}\bullet\underline{c}) = -\,i\,m\,c\,T\,\mathbf{C} \quad \& \quad \mathbf{X}^*\mathbf{P}^* = m\,c\,T\,(c\,\mathbf{I_0} + i\,\underline{\mathbf{I}}\bullet\underline{c}) = i\,m\,c\,T\,\mathbf{C}^*$$

where: $\mathbf{C} \equiv i\,c\,\mathbf{I}_0 + \underline{\mathbf{I}}\bullet\underline{c} = \{i\,c\,;\underline{c}\}$ is the 'Natural **Light** Vector'- the ultimate Voigt vector.

In this situation, both $\underline{V}$ and $\mathbf{V}$ are constants of the motion (dynamical constants); the other two-particle constants:

1. Joint NV Momentum $\quad \mathbf{P} \equiv 2\,m\,\mathbf{V}^* \quad = i\,\mathbf{P}_0\,\mathbf{I}_0 + \underline{\mathbf{P}}\bullet\underline{\mathbf{I}} \quad = -\,i\,2\,m\,c\,\mathbf{I_0} + m\,\underline{c}\bullet\underline{\mathbf{I}}$
2. Linear Momentum $\quad \mathcal{P}_T \equiv ½\,(\mathbf{P}^* + \mathbf{P}) \quad = \underline{\mathbf{P}}\bullet\underline{\mathbf{I}} \quad = m\,\underline{c}\bullet\underline{\mathbf{I}}$
3. Total Energy $\quad \mathcal{E}_T \equiv ¼\,(\mathbf{V}^* - \mathbf{V})\,(\mathbf{P}^* - \mathbf{P}) = -c\,\mathbf{P}_0\,\mathbf{I}_0 \quad = 2m\,c^2\,\mathbf{I}_0$
4. Kinetic Energy $\quad \mathcal{K}_T \equiv ¼\,(\mathbf{PV} + \mathbf{VP}) \quad = m\,(c^2 - V^2)\,\mathbf{I}_0$
5. Joint Action $\quad \mathcal{A}_T \equiv ½\,(\mathbf{X}^*\mathbf{P}^* + \mathbf{PX}) \quad = 2\,m(c^2T - \underline{V}\bullet\underline{X})\,\mathbf{I}_0 \quad = m\,X\,c\,\mathbf{I}_0$
6. Angular Momentum $\quad \mathcal{M}_T \equiv ½\,(\mathbf{X}^*\mathbf{P}^* - \mathbf{PX}) \quad = 2\,\underline{L}\bullet\underline{\mathbf{I}} \quad = i\,m\,X\,\underline{c}\bullet\underline{\mathbf{I}}$
7. Galilean Momentum $\quad \mathcal{G}_T \equiv m\,\mathbf{X} - \mathbf{P}^*T \quad = m(\mathbf{X} - \mathbf{V}\,T) \quad = m\,\underline{X}\bullet\underline{\mathbf{I}}$

This suggests a two-particle Voigt vector, the *InterActivity* Natural Vector $\mathbf{R}$ **combining** both action and angular momentum, that remains unchanged as the interaction continues between the two particles, as $t_\alpha' = t_\alpha+\Delta$ and $T' = T$.

Def'n: ***InterActivity*** $\mathbf{R}_{12}^*(T) \equiv -i\,(\mathcal{A}_T(T) + \mathcal{M}_T(T)) = -i\,\mathbf{X}^*\mathbf{P}^* = -i\,m\,\mathbf{X}^*\mathbf{V} = m\,c\,T\,\mathbf{C}^* = mc^2T(-i\,\mathbf{I_0} + \underline{\mathbf{I}}\bullet\underline{\hat{e}}_{12})$

7.8.3 NATURAL LIMITS

The conjugate self-product of the ***Separation*** Vector $\mathbf{X}$ (defined in section 7.4.1) is a scalar invariant $\mathbf{S}(t)$ that characterizes the interaction between two particles that interact when they are located in space at $\underline{x}_1$ and $\underline{x}_2$ at the respective interaction times t_1 and t_2, with $T = (t_1 - t_2)$:

$$\mathbf{S}(T) \equiv \mathbf{X}^*(T)\,\mathbf{X}(T) = (\,\mathbf{X}_1(t_1) - \mathbf{X}_2(t_2)\,)^*\,(\,\mathbf{X}_1(t_1) - \mathbf{X}_2(t_2)\,) = S_0\,\mathbf{I_0} \quad \therefore\ S_0(T) = c^2\,(t_1 - t_2)^2 - (\underline{x}_1 - \underline{x}_2)^2 = c^4$$

The requirement that interactions only occur when $S_0(T) = c^4$ is referred to as the ***Light-Cone*** condition.

We will assume that the "final" interaction between the two particles occurs when ever $t_1 = t_1'$ and $t_2 = t_2'$ and $\underline{x}_1 = \underline{x}_1'$ and $\underline{x}_2 = \underline{x}_2'$; we can define these maximal separations to be: $T = t_1' - t_2'$ and $D = |\underline{x}_1' - \underline{x}_2'|$ so, $D = \pm\,c\,T$.

$$d/dt\ S_0(t) = d/dt\,(c^2\,t^2 - \underline{x}^2) = 2\,(c^2\,t - \underline{x}\bullet\underline{v}) \ \therefore\ d/dt\ S_0(T) = 2\,(c^2\,T - \underline{D}\bullet\underline{v}') = 0 \ \text{ when } v' = c.$$

So, at $(T;\,\underline{D})$ $d/dt\,\mathbf{S}(t) = 0$ from this point onwards there will be no further interaction, so c is the **maximum** speed.

$$d^2/dt^2\ S_0(t) = d/dt\,(d/dt\ S_0(t)) = 2\ d/dt\,(c^2\,t - \underline{x}\bullet\underline{v}) = 2\,(c^2 - \underline{v}^2 - \underline{x}\bullet\underline{a}) \ \therefore\ d^2/dt^2\mathbf{S}(t) = 2(c^2 - V_0^2)\,\mathbf{I_0} => 0$$

8. ASYNCHRONOUS INTERACTIONS

8.1 TWO-PARTICLE INTERACTIONS

It is central to this research programme to show that the mathematics of natural vectors (NVs) is an appropriate representation of the interaction between two real particles, such as **electrons**. In particular, this representation exposes the subtle kinematics needed to represent the asynchronous exchange of momentum. Historically, almost all earlier studies of mechanics have focused on the action of a single particle at a single point in space at only one instant of time – represented by the normal algebraic real variables x and t. Newton first represented the particle's momentum by DesCartes' new symbolic algebra using the product of an invariant property of the particle (Newton's greatest conceptual innovation) inertial mass, *m* and its (directional or vector, instantaneous) velocity, v which he defined as the total, rate of change of its absolute location, x; in other words, Newton invented the extremely mathematically powerful and conceptually rich, definition of particle momentum: $\underline{p}(t; \underline{x}) \equiv m\, d/dt\, \underline{x}(t)$.

As a direct consequence of Einstein's analysis of Lorentz's model of the electron [38] interacting with Maxwell's electromagnetic fields, Planck proposed [39] that the response of a particle to a fixed, **collinear**, mechanical local force could be described by altering Newton's definition of momentum to include a variable scalar mass, M whose value varied with speed; i.e. $M(v) = \mathcal{L}(v)\ m$, where $\mathcal{L}(v)$ is the "Lorentz" factor defined in section 7.6.3. This change and the substitution of "local-time" for speed-invariant universal time, assumed by Newton, was forced on physics to accommodate the invariance of the electromagnetic equations of motion involving electric and magnetic fields using the Lorentz force law on charged particles with respect to observers moving in different inertial reference frames. In contrast, the NV representation is intrinsically **relational** – it only depends on the differences of the space and time parameters characterizing the two interacting particles: it is independent of all third-party "observers" who are not participating in the interaction. We shall show later that there is an alternative viewpoint to relativity theory when the electron/electron interaction is analyzed asynchronously.

8.2 ASYNCHRONOUS CONSERVATION

The NV representation maps the interaction of two electrons as the asynchronous exchange of linear and angular momentum plus the resulting energy exchange (temporal "momentum") between the two particles by assigning an invariant mass, *m* to every electron. The Galilean (or kinetic) momentum, G or $m(\underline{x} - \underline{v}\ t)$ may be defined, as this is also globally conserved across the complete interaction. It is this mapping which suggests that that another name for the two-particle NV representation might well be the "***momentum exchange***" (or **MX**) group. This approach does not view the traditional location-dependent potential as a useful concept, where the potential energy of a single particle varies only with local changes in spatial location – a technique introduced by Lagrange. This new, dynamic approach is also not one that can be redefined as an equivalent single particle conservative potential theory, where no net work is done in moving the particle around a closed path in 3D space. Since each electron is not infinitely massive, all interactions change the spatial location of each particle, so that even if one particle could return to its initial spatial location the other one would not, as reaction effects have moved the other electron. The NV interaction viewpoint is adopted where the potential energy is a system property (i.e. involving both of the two particles) that is a function only of the time-difference between the two interacting electrons, i.e. it is represented by another Natural Vector, the ***Potential*** NV, **U**(t). Physics has almost always investigated **conservative** systems. This has resulted in major simplifications in the mathematics by preserving the total energy $\mathcal{E}$ at all times, i.e. $\mathcal{E}(t) = \mathcal{E}_0$ or $d/dt\, \mathcal{E}(t) = 0$; it was this prejudice that prevented Maxwell supporting Lorenz's view of retarded interactions. Equally, modern quantum electrodynamics introduces virtual particles at each interaction 'node' to maintain total momentum at all times. An **asynchronous** conservative system only preserves the total energy and momentum across the duration of each interaction. Thus, if particle #2 interacts at time t_2 with particle #1 at a later time t_1 then an asynchronous interaction implies that $\mathcal{E}(t < t_2) = \mathcal{E}(t > t_1) = \mathcal{E}_0$ but this is not preserved during the finite time of the total interaction so that $\mathcal{E}(t_2 < t < t_1) \neq \mathcal{E}_0$. In this interaction view, no intermediate carriers (i.e. 'virtual particles') of energy or momentum are required. Third-party observers (i.e. measurements) cannot directly distinguish these two alternative views of the conservation of energy; only differential predictions based on these two sets of assumptions can determine empirically which hypothesis best describes the interaction of electrons: a goal of this research.

8.3 INTERACTION SPACE

Mathematically, it is very useful to define the "Interaction Space", $\mathscr{I}$ for describing the interaction between two electrons. This is a mapping from a sub-set of the linear representation of electron motion through physical space over time (both viewed, like Newton, as inert) and referred to here as "Reality", $\mathscr{R}$. Although $\mathscr{R}$ is used to represent the trajectories of each electron through 3D space, the mathematical 'space' $\mathscr{I}$ only consists of those mathematical points (in a 3D Euclidean 'space') when the two electrons actually interact with each other, without the intervention of any other electron; these points may not (and we will subsequently show, do not) form a dense set or manifold. The mapping is defined in terms of the two relative variables:

1) $T \equiv t_a - t_b$ 2) $\underline{X} \equiv \underline{x}_1(t_a) - \underline{x}_2(t_b)$ so: $\mathscr{I}(T; X) \cong \mathscr{R}(t_a, t_b; \underline{x}_1, \underline{x}_2)$

Here, we assume that **all** interactions are like the electromagnetic interaction, so that they only occur between two electrons when they are on each other's "light-cone" (although this is a necessary condition it is not a sufficient condition, additional constraints will be added later); mathematically this corresponds to the following condition for the n^{th} consecutive interaction between the two electrons:

Definition: ***Light-Cone*** Condition, $\underline{X}(T_n) \bullet \underline{X}(T_n) = c^2 T_n^2$

The dynamics of the uninterrupted set of interactions (i.e. no third-party interventions) between two electrons is to determine the extent of the interactions (from an initial time difference, $T = t_0$ until a final time difference, $T = T_0$), with corresponding spatial separations of d_0 and D_0. Traditional physics has assumed that $\underline{X}(T)$ is continuous across this extent; this research will show that there are only a finite number of consecutive interactions possible between any two electrons, i.e. the interaction index 'n' is finite.

These two 'spaces' may be visualized diagrammatically.

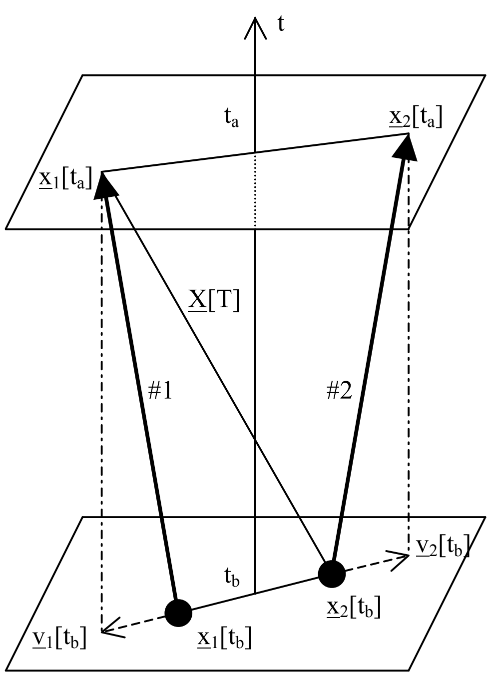

Fig. 1 Reality $\mathscr{R}$

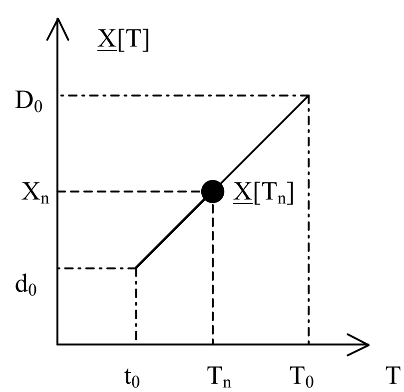

Fig. 2 Interaction Space $\mathscr{I}$

8.4 SPACE-TIME DIAGRAMS

Natural Vectors (NVs) may be visualized in a 4D "space-time" diagram viewed isometrically by suppressing one of the three space dimensions (say, z) or as planar diagram by suppressing two of the space dimensions (say, x and y).

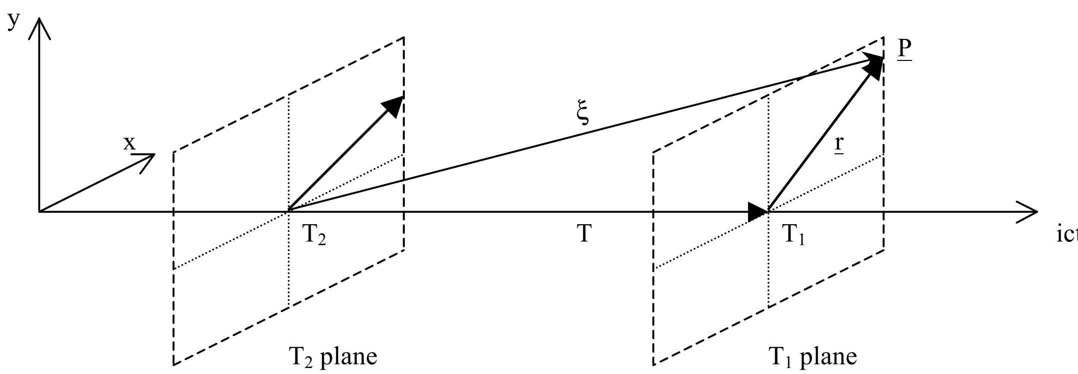

Fig.3 Space(2)-Time Diagram

In these diagrams, the temporal axis is always imaginary (and scaled by *c*) to indicate the ontological difference between one-dimensional time and the three orthogonal dimensions of space. The 4D point, P at time T_1 is located at the spatial point $\underline{r}$ relative to the origin at O (or location $\underline{x}_2$ and time t_2). Relative to O the point P is located via a (Minkowski) 4-vector ξ. Since ξ is time-like, it is characterized by a real parameter, ζ where $\xi = i\zeta$. Thus ξ satisfies:

$$\xi^2 = r^2 + T^2 \quad \text{where } T = i\,c\,(T_1 - T_2) = i\,c\,t \quad \therefore\ -\zeta^2 = r^2 - c^2 t^2 \ \text{ or } \ \zeta^2 = c^2 t^2 - \underline{x} \bullet \underline{x} \ \cong\ \mathbf{X}^* \mathbf{X}$$

In terms of 4D vectors: $\quad \underline{\xi} = \underline{T} + \underline{r} \quad$ or $\quad i\,\underline{\zeta} = i\,c\,t\,\underline{\hat{\mathbf{e}}}_0 + x\,\underline{\hat{\mathbf{e}}} \cong \mathbf{X}$

$$\therefore\ \mathbf{X} = i\,c\,t\,\mathbf{I_0} + \underline{x} \bullet \underline{\mathbf{I}} \cong i\,c\,t\,\hat{\mathbf{e}}_0 + x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3 \quad \therefore\ \mathbf{I_0} \cong \hat{\mathbf{e}}_0\,,\ \mathbf{I_j} \cong \hat{\mathbf{e}}_j \quad \therefore\ \text{Bases} \cong \text{Space \& Time axes.}$$

If the continuous interaction of two electrons is mapped across one spatial dimension (say, z) then:

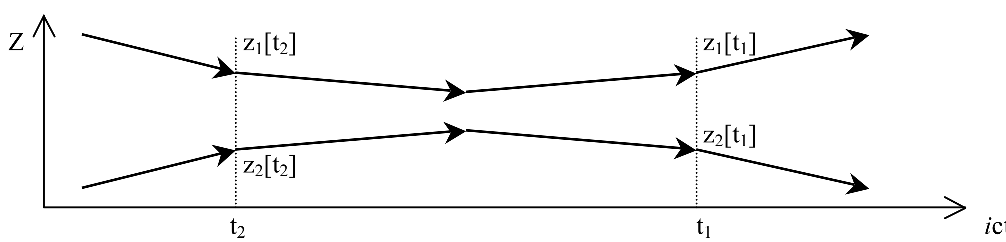

Fig. 4 Space (1) – Time Diagram

Following Newton, the electrons move inertially, outside the interaction time zone, i.e. ($t < t_2$ & $t > t_1$), in constant (straight-line) motion but piecewise linearly within ($t_2 > t > t_1$). Since electrons are here viewed as real particles, they always exist, so they have continuous trajectories across space over time. Defining $t = t_1 - t_2$ and the spatial separation between them as z(t), then (by inspection): $z(t) \equiv z_1(t_1) - z_2(t_2) = c\,t$ & $z(-t) = z_1(t_2) - z_2(t_1) = c\,t$

Defining the individual & joint NVs: $\quad \mathbf{X}(t\,;\,\underline{x}) \equiv i\,c\,t\,\mathbf{I_0} + \underline{\mathbf{I}} \bullet \underline{x} \quad \& \quad \mathbf{X}_n(t_n\,;\,\underline{x}_n) \equiv i\,c\,t_n\,\mathbf{I_0} + \underline{\mathbf{I}} \bullet \underline{x}_n(t_n)$

$$\therefore\ \mathbf{X}(t\,;\,\underline{x}) = \mathbf{X}(t_1 - t_2\,;\,\underline{x}_1 - \underline{x}_2) = \mathbf{X}_1(t_1) - \mathbf{X}_2(t_2) = -(-\mathbf{X}_1(t_1) + \mathbf{X}_2(t_2)) \quad \therefore\ \mathbf{X}(-t\,;\,-\underline{x}) = -\mathbf{X}(t\,;\,\underline{x})$$

Thus, two-electron ***Relative-Location*** Natural Vectors are anti-symmetric in their space and time arguments.

8.5 NATURAL VECTOR RELABELLING

Since natural vectors (NVs) are used to describe the interaction of pairs of electrons (say, #1 & #2) at two different locations ($\underline{x}_1$ & $\underline{x}_2$) at two different times (t_1 & t_2) subject to the *Light-Cone* condition (section 8.3), it is always possible to re-label these components. Unfortunately, simply exchanging the "gross" labels "1 & 2" disguises several important sub-exchanges of the labels. This can be made much clearer by labeling the times (t_a & t_b) since these are just any two times over which the interaction occurs and this labeling is independent of the particle labeling (i.e. "1 & 2"). Similarly, the points in space where the interaction takes place can be relabeled ($\underline{x}_A$ & $\underline{x}_B$); finally, the maximum clarity can be achieved by labeling even the electrons themselves (α & β). So, for the Relative-Location CNV, using these labeling conventions:

$$\mathbf{X}_{\alpha\beta}(t\,;\underline{x}) \;=\; \mathbf{X}_{\alpha\beta}(t_a - t_b\,;\underline{x}_A - \underline{x}_B) \;=\; i\,c\,(t_a - t_b)\,\mathbf{I_0} \;+\; \underline{\mathbf{I}} \bullet (\underline{x}_A - x_B)$$

$$\therefore\quad \mathbf{X}_{\alpha\beta}(t_a - t_b\,;\underline{x}_A - \underline{x}_B) \;=\; \mathbf{X}_{\alpha}(t_a;\underline{x}_A) - \mathbf{X}_{\beta}(t_b;\underline{x}_B)$$

Also, for the conjugates: $\mathbf{X}^*_{\alpha\beta}(t_a - t_b\,;\underline{x}_A - \underline{x}_B) \;=\; \mathbf{X}^*_{\alpha}(t_a;\underline{x}_A) - \mathbf{X}^*_{\beta}(t_b;\underline{x}_B)$

8.5.1 TEMPORAL EXCHANGE

The Temporal Exchange operator, $\mathscr{T}_x$ only re-labels the times at which the interaction occurs. So, if the earlier time was originally labeled 't_b' and the later time 't_a' then the $\mathscr{T}_x$ operator (sometimes referred to as "time-reversal") will re-label the earlier time 't_a' and the later one as 't_b' for any function ψ of these two time arguments:

Definition ***Temporal Exchange***: $\mathscr{T}_x\psi(t_a\,, t_b) \;\equiv\; \psi(t_b\,, t_a)$

Applying this temporal exchange operator to the Relative-Location NV:

$$\mathscr{T}_x\;\mathbf{X}_{\alpha\beta}(t_a - t_b\,;\underline{x}_A(t_a) - \underline{x}_B(t_b)) \;=\; \mathbf{X}_{\alpha\beta}(t_b - t_a\,;\underline{x}_A(t_b) - \underline{x}_B(t_a)) \quad \text{and} \quad \mathscr{T}_x\,\mathscr{T}_x = \mathbf{I_0}\ \text{(i.e. unitary.)}$$

8.5.2 PARITY EXCHANGE

A coordinate system can be converted from a right-handed reference frame ($\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 \wedge \hat{\mathbf{e}}_2$) to a left-handed reference frame ($\hat{\mathbf{e}}'_3 = -\,\hat{\mathbf{e}}'_1 \wedge \hat{\mathbf{e}}'_2$) by reflecting each spatial vector ($\hat{\mathbf{e}}_j$) in the origin; this operation defines the Parity Exchange operator, $\mathscr{P}_x$ for any electron, labeled 'α' at any time 't' , operating on its j^{th} spatial coordinate ($j = 1,2,3$):

Definition ***Parity Exchange***: $\mathscr{P}_x\,x_j(\alpha : t) \;\equiv -\,x_j(\alpha : t)$

So,

$$\mathscr{P}_x\;\mathbf{X}_{\alpha\beta}(t_a - t_b\,;\underline{x}_A - \underline{x}_B) \;=\; \mathbf{X}_{\alpha\beta}(t_a - t_b\,;-(\underline{x}_A - \underline{x}_B)) \;=-\,\mathbf{X}^*_{\alpha\beta}(t_a - t_b\,;\underline{x}_A - \underline{x}_B)$$

Sometimes, parity exchange is referred to as "space-reversal"; this is also a unitary operator: $\mathscr{P}_x\;\mathscr{P}_x = \mathbf{I_0}$

8.5.3 IDENTITY EXCHANGE

The interaction between two electrons must appear unchanged if the electrons are simply relabeled (β,α) instead of (α,β); the Identity Exchange operator, $\mathscr{I}_x$ is introduced for this purpose:

Definition ***Identity Exchange***: $\mathscr{I}_x\,\psi(\alpha,\beta) \;\equiv\; \psi(\beta,\alpha)$

$$\therefore\quad \mathscr{I}_x\;\mathbf{X}_{\alpha\beta}(t\,;\underline{x}) \;=\; \mathbf{X}_{\beta\alpha}(t\,;\underline{x}) \qquad \text{and} \qquad \mathscr{I}_x\;\mathscr{I}_x = \mathbf{I_0}$$

9. SUMMARY & CONCLUSIONS

In this final section, the results and conclusions of the main body of the paper will be briefly summarized so that the appropriate implications may be drawn. The paper concludes with summaries of future papers in this programme.

9.1 OBJECTIVES

It is hoped that the objectives of this paper introduced in section 1.1.3 have been met. The deliberate inclusion of the brief biography of **Hamilton** was designed to remind the modern physicist, who has rarely been exposed to the history of his subject, that physics was created by a handful of intellectual giants, many of them practicing in the 19^{th} Century and William Rowan Hamilton was in the first rank of these titans. A man of his obvious genius, who had devoted the major part of his life to the investigation of one of the major innovations in mathematics, has earned the respect of history. When such an individual believes so strongly that one innovative area of algebra – quaternions – is destined to play a major role in the future development of physics then others need pay attention; many of those who have later applied quaternions to physics have come to share this opinion. In this paper, the concept of 'Natural Vectors' (**NV**s) has been introduced by imposing a set of logical conditions on Hamilton's bi-quaternions, combined with the need to map 'reality' to a simple, algebraic form. The subsequent research has resulted in the realization that certain generic, algebraic structures, referred to here as '**Voigt Vectors**', have direct relevance to several major foundational areas of theoretical physics.

A major objective of this paper was to introduce the research programme that is the framework for all of the papers in this series. The most radical parts of this approach are the rejection both of the '**continuum** view' and the historical focus on the dynamics of single objects (particles or fields) moving under the influence of potentials evaluated always at a single point in time. This programme focuses on the interaction between the fundamental particles of physics at two different times; namely, asynchronous interactions. The establishment of natural vectors for representing these types of interactions is the mathematical foundation for this research. A complementary philosophical position will be developed that provides the **ontological** basis for the ideas needed for this programme.

9.2 HAMILTON & QUATERNIONS

Hamilton's introduction of quaternions, as reviewed in section two, was one of the major innovations in the history of mathematics. It directly led to the scalar / vector distinction and the vector gradient operator (∇) while being the inspiration for all of vector analysis. It was also the direct progenitor of hyper-complex numbers and the explosion of **modern algebras**; it can also be viewed as a critical step in the development of the idea of linear vector spaces.

Hamilton was a firm believer in the power of the **imagination**, which when united with the intellect, generated the creative results of poetry, mathematics, physics, etc. Hamilton's mathematical style was always to abstracting and generalizing. Until 1834, inspired by loyalty and patriotism, most of Hamilton's publications appeared only in the *Transactions of the Royal Irish Academy* that, unfortunately, were almost unread outside of Ireland; this resulted in an unwarranted obscurity that lasted until about 1900. The brief summary of Hamilton's quaternion papers is a small attempt [10] to redress this huge oversight; the Internet now provides universal access to all of his work.

9.3 QUATERNIONS IN MODERN PHYSICS

Section III described the use of quaternions in physics over the last 150 years, especially in the 20^{th} Century when relativity and quantum mechanics 'overthrew' so-called classical physics. One of the objectives of this research programme is to demonstrate that the evolution of Newtonian ideas provides a firmer foundation for the future of physics than the conceptual developments that have been grounded in the application of fields to physics. The fruitful uses of bi-quaternions (the 'complexified' form of Hamilton's quaternions) in these 'field' theories demonstrated that this powerful representation was not excluded from describing these phenomena appropriately.

9.4 NATURAL VECTORS

In contrast to all but the bi-quaternion representation, *Natural Vectors* (**NV**s), introduced in section IV, explicitly introduced the **time** dimension as a full and equal partner to the spatial dimensions. However, in order to distinguish the metaphysical difference between space and time, NVs emphasize this distinction by explicitly introducing the standard symbol for the square root of minus one ('*i*'). The use of (4x4) real representations for the four quaternion bases $\mathbf{I}_\mu$ means that the single NV symbol $\mathbf{X}_\alpha$ can represent the location of any one electron (labeled α) throughout any spatial location $\underline{x}$ at time t (in a given inertial frame), while a symbol $\mathbf{X}_{\alpha\beta}$ can similarly represent the difference between any pair of interacting electrons at two different times.

$$\mathbf{X}_\alpha(t\,;\underline{x}) = i\,c\,t\,\mathbf{I_0} + \underline{\mathbf{I}} \bullet \underline{x} \qquad \mathbf{X}_{\alpha\beta}(t\,;\underline{x}) = \mathbf{X}_{\alpha\beta}(t_a - t_b\,;\underline{x}_A - \underline{x}_B) = i\,c\,(t_a - t_b)\mathbf{I_0} + \underline{\mathbf{I}} \bullet (\underline{x}_A - \underline{x}_B)$$

The present interpretation of the invariance of the speed of light in all reference frames (or the universal appearance of the same constant, *c* in all **Voigt** vectors) is that the time difference between two electromagnetic events is only a function of the spatial separation between the events and both of these are invariant with respect to all third-party motion. For example, it is predicted that Einstein's "double-lightning" strikes will, in reality, be seen equally when the passenger on the '*Einstein Express*' passes the stationary observer in between the 'strikes'. In the present model, as in all action-at-a-distance theories, like the **Lorenz** theory of electric induction, there is no 'invisible' medium and no movement of an invisible entity (like a photon) "carrying the light". The classical electromagnetism theory of special relativity is a consequence of the metaphysical assumption that light is an entity that moves mysteriously through space at a constant speed in all reference frames. However, in this programme all Galilean foundational concepts, such as space and time differences, are unchanged with respect to any frame motion, including any form of accelerations. These ideas will be elaborated in a later paper.

Standard 3D vectors are timeless, scaleless representations of unchanging spatial relationships; when they are used as a representation of the interactions of particles it is necessary to imply time when they are interpreted as movement directed from one end of the vector to the other. Thus, when the mathematical vector equation $\underline{v}_1 = \underline{v}_2$ is interpreted physically for the velocities of two particles (1 & 2) then we must explicitly add the times to this equation, so that it reads $\underline{v}_1(t_1) = \underline{v}_2(t_2)$. Even here, all anyone can say is that the particles are moving parallel at the same speed, but one **cannot** infer that they are covering the same trajectory across space (A = B), which they may or may not have done. Indeed, physics has usually taken a single time ($t_1 = t_2$) approach, where these velocities are instantaneous and in fact the time parameter shrinks to insignificance, as it can be any time: the symbolic and **disposable** 't'. In other words, contra Descartes: geometry is not identical with physics. The explicit introduction of the standard imaginary (*i*) in natural vectors results in their norms always being positive or zero for real physical phenomena, overcoming an objection that goes all the way back to Heaviside in the criticism of quaternions with respect to vectors.

In contrast to bi-quaternions (**BQ**s), where **each** of the four components is a complex number, natural vectors (**NV**s), where only the time component is a complex number, offer several benefits. Firstly, and most importantly, there is only one product rule and one conjugation rule with NVs compared with the four conjugation types used with BQs, which further confound their usage by reversing the order of the products. Since NVs always carry their real bases explicitly with them there is never any need to separate out the scalar or vector parts as is done with BQs, which continues the approach and notation originally introduced by Hamilton; namely $\mathbb{S}[\mathbf{A}\ \mathbf{B}] = \mathbf{A} \bullet \mathbf{B}$ & $\mathbb{V}[\mathbf{A}\ \mathbf{B}] = \mathbf{A} \wedge \mathbf{B}$. Further, BQs allow left and right multiplication whereas NVs always follow the standard convention of implying that the operand remains only to the right of the operator. This ambiguous usage for BQ operators results in quite unattractive symbology so, for example, the Conway-Silberstein BQ expressions for Maxwell's Equations appear:

$$\nabla^+ \wedge\ \mathbf{A} = \underline{F}\ ; \qquad \tfrac{1}{2}\,(\,\nabla\ \underline{F} + \underline{F}^{\sim}\,\nabla\,) = -4\,\pi\,J$$

This can be contrasted with the simpler algebraic appearance of the NV equivalent formulation [6]:

$$\nabla^*\mathbf{A} = i\,\mathbf{F}\ ; \qquad i\,\nabla\ \mathbf{F} = 4\,\pi\,\mathbf{J}$$

Finally, the explicit appearance of the standard algebraic symbol '*i*' for the square root of minus one always appears in the same (first) position in NVs in contrast to BQs where its appearance reflects "informed" judgment, as in the respective formulations of classical electromagnetism, for the location **X** and potential **A** quaternion forms:

$$\text{NV:}\;\; \mathbf{X} = \{i\,c\,t\,;\underline{x}\}\;\; \mathbf{A} = \{-i\,\phi\,;\underline{A}\} \qquad \text{BQ:}\;\; \mathbf{X} = (i\,c\,t\,;\underline{x})\;\; \mathbf{A} = (\phi\,;-i\,\underline{A})$$

Most authors that have used quaternions successfully in physics, especially in relativity and classical EM, have used a more limited form of bi-quaternions by choosing the vector part to be imaginary so that the complex conjugate equals Hamilton's original "reverse vector" form of conjugation. Natural vectors have not adopted this convention as its resulting Voigt vectors (section 6) form a more natural representation of the interaction between two particles.

In summary, the use of bi-quaternions emphasizes the need for the use of explicit complex numbers in quaternions when representing physical quantities but this representation goes too far in assuming they are needed for all four components; they thus destroy the major benefit of simplicity of form found when using quaternions in physics.

The benefits of using Natural Vectors (NVs) in mathematical physics can be summarized under four categories.

1) NVs are <u>single</u> symbols consisting of one to <u>eight</u> real numbers representing a <u>single</u> physical concept that represents and *unifies the spatial and temporal aspects of the natural world*.
2) NV equations are very <u>compact</u> and can correspond to up to <u>eight</u> separate but related Cartesian equations; they are *simpler and more transparent* than similar formulas written in standard matrix, vector, tensor, bi-quaternion or higher rank Clifford number notation.
3) NVs provide a <u>unifying formalism</u> for the mathematics of classical, relativistic and quantum physics; they are ideal for describing the *asynchronous interactions between two particles*.
4) NVs form a *simple algebra* that is almost as familiar as that used for complex numbers; they can be manipulated using standard, complex number algebra (except for commutivity).

9.5 NATURAL VECTORS IN PHYSICS

Section 7 applied some of these general formulations of natural vectors to several well-known situations in physics, beginning with a kinematic analysis of the motion of a particle represented by a *Spatial Displacement* CNV. The important result was found that the **total** time derivative of this Spatial Displacement CNV does not comply with the simple 'Flow' form that applies to all Voigt vectors indicating that it does not play a fundamental role in the physics of interactions. However, the *Velocity* CNV (as the prototypical Voigt vector) does satisfy this condition whenever its <u>acceleration is zero</u>. Obviously, this is incorrect for a single particle but introduces intriguing possibilities for the 'joint' velocity of a pair of interacting, remote objects. This is explored further in the next paper in this series that uses this NV algebra in the description of classical electromagnetism [6] generating a fully relational mechanics.

In section 7.2, it was directly proved that any continuous natural vector (CNV) $\mathbf{Q}$ with a zero NV *Gradient* ($\nabla * \mathbf{Q}$) always satisfies the homogenous *Wave Equation*. Conversely, when $\mathbf{Q}$ is a *Flow Vector* then it not only satisfies the Wave Equation but its total time derivative is always zero, so that its scalar and vector components are always *conserved over time* – a key feature in asynchronous interactions. This intriguing result is enhanced when it is realized that when a ***Voigt*** CNV is a Flow Vector <u>and</u> it satisfies the Wave Equation then its Voigt parameter must be conserved. In this situation, the defining *Lorenz Equation* for this Voigt vector then implies that its scalar and vector components satisfy their own *Continuity Equation*. These considerations demonstrate that scalar properties of the individual electron, such as its inertial mass, *m* and its unit electric charge, *e* are <u>conserved universally</u> over time and become the defining characteristics of this fundamental particle. In a continuous "medium" theory, like classical electromagnetism, the need to preserve such a characteristic property (associated with an infinitesimal volume element of the medium that has a local velocity) and assign it an invariant value leads directly to the Lorentz transformations and the special theory of relativity. These ideas will be developed [6] in a subsequent paper.

Section 7.4 deliberately focused on **Minkowski** 4-vectors in order to emphasize that Natural Vectors are inherently more powerful than these older but simpler non-algebraic forms; however, it did show that NVs could be viewed as "covariant Minkowski quaternions".

Various special forms of the generic Voigt vectors were defined in section 7.5 to relate to earlier, mathematical objects that have been introduced to represent important physical quantities. A sub-class was defined that closely resembled Planck's suggested form for the 3D relativistic momentum (section 7.6); this immediately resulted in all of the usual equations for **relativistic mechanics**, including $E = Mc^2$. An analysis of the special conditions that Planck needed to assign to the 'mechanical' forces that would accelerate these particles to very high speeds showed that there are no such forces in Maxwell's theory of electromagnetism, so Planck's particles cannot be electrons or any other type of electrically charged particle. However, it was Kaufmann's experiments in 1901 and 1902 [40] on high-speed electrons moving under the influence of crossed electric and magnetic fields that first indicated that there was an increase in inertial resistance (which has always been interpreted ever since as an increase in mass). This was the first empirical evidence for the *validity* of relativistic mechanics and therefore for the theory of special relativity.

Section 7.7 introduced several concepts associated with the invariance properties of natural vectors. The critical idea of separability across space and time, especially involving the *Separation* NV between two particles, was defined to simplify the study of asynchronous interactions. Invariants were investigated that preserved the 'form' of NV equations (covariance) or were unchanged across time (temporal invariance). Here it was demonstrated that asynchronous action-at-a-distance (**AAAD**) could be represented by NVs that are both separable and temporally invariant. It was also shown that when both particles have the same inertial mass then seven mechanical NVs become dynamical constants of any asynchronous interaction that is separable. The well-known "*Light-Cone*" condition arising in classical electromagnetism corresponds here to the requirement that interactions only occur when the invariant 'norm' of the two-particle *Separation* CNV is constant. This directly leads to the idea that there are "natural limits" to any interaction corresponding to the minimum and maximum separations of two interacting particles associated with relative velocities of zero and "light-speed": these extrema neither occur at zero or infinite spatial or temporal separation, as is usually assumed.

9.6 ASYNCHRONOUS INTERACTIONS

Section 8 was devoted to a discussion of the concept of asynchronous interactions between two particles at two **different** times - the central idea at the very heart of this research programme. All earlier studies of mechanics have focused on the action of a single particle at a single point in space at only one instant of time: represented by the normal algebraic real variables $\underline{x}$ and t. This programme recognizes that these two aspects of reality are both equally important and must, in the case of particle mechanics, always be treated together equally – justifying the appellation "natural vector" for the representation of a particle's unique **existence** in space and time. A later paper on discrete electron dynamics will demonstrate that the peculiar "twisting" feature of quaternions maps the actual trajectories of electrons; this intrinsic motion of electrons has only been hinted at previously, when the motion of charged particles was investigated as they 'reacted to electromagnetic fields'. In this model it will be seen as the direct consequence of the asynchronous exchange of both linear and angular momentum, indicating why the two-particle natural vector (NV) representation should be viewed as nature's fundamental '**momentum exchange**' group.

Lagrange diverted attention away from Newton's original dynamical concept of discrete **impulse** as the cause of the change in any particle's straight-line inertial motion. A careful reading of Newton's *Principia* [41] indicates that the concept of '**force**' was introduced by Newton only as a calculational technique to sum many small, successive impulses together. Mathematicians seized on this powerful technique and **reified** it into an independent existent: Newton recognized that this was not the case by explicitly adding his Third Law of action equaling reaction. This 'Law' is further elaborated upon here as a primary characteristic of asynchronous interactions between electrons. Therefore, the traditional Lagrangian (continuum) reformulation of Newtonian mechanics has been by-passed in this programme by concentrating on changes (later extended to discrete) in the two-particle relative variables. Further, the Lagrangian concept of a temporally invariant, potential energy function that depends only on position is rejected here; instead, this older concept is now replaced by the system-wide concept of "interaction energy", represented by the *Potential* NV, **U**(t) which will be introduced later. This will lead to the concept of asynchronous conservation of energy and not the universal conservation at all times that has characterized physical models to date.

The idea of "interaction space" was introduced in section 8.3; this is an abstract mapping used for describing the interaction between the two particles that is represented algebraically by natural vectors. In contrast to continuum theories, with real manifolds, this viewpoint emphasizes **only** those discrete points where interactions actually occur. A later paper [6] will demonstrate that the *Light-Cone* condition **excludes** the possibility of continuous interactions between point particles with inertial mass. The evolution of experimental physics has repeatedly shown that the micro-world is intrinsically discontinuous (to the chagrin of the Platonic mathematicians): therefore, the symbolic representations used to map this discrete nature should be reflected in the associated discrete mathematics. Later papers will extend continuous natural vectors to **discrete** natural vectors, with differential operators being replaced by a new set of **difference** operators. A similar light-cone model will replicate the Wave-Mechanical results for the hydrogen atom: the best analytical result yet achieved with calculus-based quantum mechanics.

Various diagrammatic techniques for understanding the 4D nature of asynchronous interactions and natural vectors were described in section 8.4. This capability is considered very important as the power of our visual imagination vastly exceeds the linear thinking characterized by mathematical, deductive thought. Innovation is encouraged when suitable imagery can be invoked to assist our thinking about the world. These types of 'mental models' were used extensively in physics prior to the 20th Century and also account for the usefulness of geometrical thinking; the dramatic impact of this style of thinking on both innovation in art and science has been very well documented by various authors, such as Arthur I. Miller [42], in his *Insights of Genius*. The final sub-section (8.5) briefly covered the subject of 'labeling' the variables used to map the interaction between two particles. This is an important area that will allow new constraints to be introduced as the research programme progresses.

9.7 CONCLUDING REMARKS

Just as quaternions have better algebraic properties than vectors, so do natural vectors. Therefore, natural vectors are proposed as a very powerful representation for **physics** while vectors and geometric algebra are more suitable for **geometry**. Pythagoreans (like Galileo) will always prefer the timeless vision of geometry and its representations but Newtonians should favor the **dynamical** use of natural vectors. Maxwell himself was convinced that quaternions express the physics of electricity and magnetism much more directly than is possible with coordinates (i.e. vectors), revealing more clearly the nature of the phenomena. For over 100 years now, physics has been using vectors, which as Heaviside wrote: "*are the quaternions of the practical man*"; perhaps the time has come to return 'upstream' to their source, in the form of natural vectors, to investigate the heart of physics – **asynchronous interactions**. This new research programme hopes to show that Sir William Hamilton's strong intuition for the significance of quaternions in physics in 1843 and the efforts to refocus the attention of physicists on quaternions by Sir Edward Whittaker in 1940 will now be met [43] with natural vectors as the "*most natural expression of the new physics*".

The next three papers in this series will use natural vectors to investigate classical electromagnetism (CEM): the prototypical model for all subsequent 'field' theories. The first paper will rapidly recover all of Maxwell's famous results based on the popular 'fluid' model of continuous electricity in motion but from a charge / potential rather than a continuous field perspective. The second paper will focus on just the continuous interaction between only two point charges going beyond the simplistic Coulomb approximation. This will also show how certain implicit assumptions in earlier theories led to the dynamical results found in the special theory of relativity. The third paper in this CEM series will combine Gauss's original vision of asynchronous electromagnetic interaction with that of Newton's particle mechanics to produce a unified classical theory of matter. A later series will explore the dramatic consequences of **quantizing** the interaction between the electrons at three different scales of separation: nuclear, atomic and cosmic – **unifying the four fundamental 'forces' of nature**.

It is now a popular opinion among theoretical physicists that their role is to create mathematical models of reality. This programme views that goal as **too limiting** – the objective should be to create broader symbolic models. We must recognize that mathematics is just one of our intellectual tools, all of which are embedded within the most powerful tool of all human beings – our **natural languages**: hence this programme's emphasis on **philosophy**, so that the results of the progress in physics can be communicated to everyone. This programme also returns to the creation of models of nature that can be **visualized** – both explicitly and in our imaginations. The acceptance of this radical, new programme will not be easy, as new directions are not simple or obvious when a mind must make a radical break with accepted modes of explanation, in which it has heavily invested.

10. APPENDICES

10.1 3D VECTOR IDENTITIES

Let $\underline{x}$ be a location in 3D space (relative to a fixed origin) with components x_j (with $j = 1, 2, 3$) so that its complete time derivative is its 3D velocity $\underline{v}$ defined as: $v_j = dx_j / dt$ while $\underline{\nabla}$ is the vector gradient operator.

The basic partial derivative results are: a) $\partial x_j / \partial x_k = \delta_{jk}$ b) $\partial_0 x_j = 0$ c) $\partial f(t) / \partial x_k = 0$ d) $\partial_0 f(\underline{x}) = 0$

So, e) $\underline{\nabla}\, x = \underline{x} / x$ f) $\underline{\nabla} \bullet \underline{x} = 3$ g) $\underline{\nabla} \wedge \underline{x} = 0$ h) $(\underline{v} \bullet \underline{\nabla})\, \underline{x} = \underline{v}$

i) $\underline{\nabla}\, v_j = 0$ j) $\underline{\nabla} \bullet \underline{v} = 0$ k) $\underline{\nabla} \wedge \underline{v} = 0$ l) $\underline{\nabla}\, (v^2) = 0$

m) $\underline{\nabla}\, (1/x) = (-1/ x^3)\, \underline{x}$ n) $\underline{\nabla}\, (x^2) = 2\, x$ o) $\underline{\nabla} \bullet (\underline{x}/ x^3) = -\nabla^2(1/x) = 0$ {if $x \neq 0$}

These identities are used for more complicated identities with scalar functions like $\alpha(\underline{x})$ and vector functions $\underline{Q}(\underline{x})$:

1. $\underline{\nabla}\, (\alpha\, v_j) = v_j\, \underline{\nabla}\alpha$
2. $\underline{\nabla} \bullet (\alpha\, \underline{v}) = \underline{v} \bullet \underline{\nabla}\alpha$
3. $\underline{\nabla} \wedge (\alpha\, \underline{v}) = -\underline{v} \wedge \underline{\nabla}\alpha$
4. $\underline{\nabla}\, (\alpha / x) = (1/ x)\, \underline{\nabla}\alpha - (\alpha/ x^3)\, \underline{x}$
5. $\underline{\nabla}(v_j / x) = (-v_j / x^3)\, \underline{x}$
6. $\underline{\nabla} \bullet (\alpha\, \underline{Q}) = \underline{Q} \bullet \underline{\nabla}\alpha + \alpha\, \underline{\nabla} \bullet \underline{Q}$
7. $\underline{\nabla} \bullet (\alpha\, \underline{x}) = 3\alpha + \underline{x} \bullet \underline{\nabla}\alpha$
8. $\underline{\nabla}\, (v \bullet \underline{Q}) = \underline{v} \bullet \underline{\nabla}\underline{Q}$
9. $\underline{\nabla}(\underline{v} \bullet \alpha\, \underline{Q}) = \underline{Q}\, (\underline{v} \bullet \underline{\nabla}\alpha) + \alpha(\underline{v} \bullet \underline{\nabla})\underline{Q}$
10. $\underline{\nabla} \wedge (\alpha\underline{x}) = -\underline{x} \wedge \underline{\nabla}\alpha$
11. $\underline{\nabla} \wedge (f(v^2)\, \underline{v}) = 0$
12. $\underline{\nabla} \wedge (\alpha\, \underline{Q}) = -\underline{Q} \wedge \underline{\nabla}\alpha + \alpha\, \underline{\nabla} \wedge \underline{Q}$
13. $\underline{\nabla} \bullet (\underline{v} \wedge \underline{Q}) = -\underline{v} \bullet (\underline{\nabla} \wedge \underline{Q})$
14. $\underline{\nabla} \bullet \underline{\nabla}\alpha = \nabla^2\alpha$
15. $\underline{\nabla} \wedge (\underline{v} / x) = (1/ x^3)\, \underline{v} \wedge \underline{x}$
16. $\underline{\nabla} \wedge (\underline{Q} / x) = (1/ x^3)\, \underline{Q} \wedge \underline{x} + (1/x)\underline{\nabla} \wedge \underline{Q}$
17. $\underline{\nabla} \wedge (\underline{\nabla} \wedge \underline{Q}) = -\nabla^2\underline{Q} + \underline{\nabla}(\underline{\nabla} \bullet \underline{Q})$
18. $\underline{\nabla} \wedge (\underline{v} \wedge \underline{Q}) = \underline{v}\, (\underline{\nabla} \bullet \underline{Q}) - (\underline{v} \bullet \underline{\nabla})\, \underline{Q}$
19. $\underline{\nabla} \wedge (\underline{a} \wedge \underline{\nabla}(1/x)) = \underline{\nabla}((\underline{a} \bullet \underline{x})/x^3)$ {if $x \neq 0$ and $\underline{a}$ = const}
20. $\underline{\nabla} \bullet (\underline{a} / x) = \underline{a} \bullet \underline{\nabla}(1/x) = -(\underline{a} \bullet \underline{x}) / x^3$
21. $\underline{\nabla} \wedge (\underline{a} \wedge \underline{Q}) = \underline{a}\, (\underline{\nabla} \bullet \underline{Q}) + \underline{a} \wedge (\underline{\nabla} \wedge \underline{Q}) - \underline{\nabla}(\underline{a} \bullet \underline{Q})$

10.2 NATURAL VECTOR IDENTITIES

These identities all use the basic conjugate multiplication form introduced in section 4.4 and 4.6; these formulae are evaluated using the 3D vector calculus results of section 10.1; again α and $\underline{Q}$ are scalar and vector functions.

1. $\nabla^* \mathbf{X} = -2\, \mathbf{I_0}$
2. $\nabla^* \mathbf{V} = -i\, \underline{\mathbf{I}} \bullet \partial_0 \underline{v}$
3. $\nabla^* \alpha = -i\, \mathbf{I_0}\, \partial_0\alpha + \underline{\mathbf{I}} \bullet \underline{\nabla}\alpha$
4. $\nabla^*(\mathbf{X}^* \mathbf{X}) = -2\, \mathbf{X}$
5. $\nabla^*(\mathbf{V}^* \mathbf{V}) = 2\, i\, (\underline{v} \bullet \partial_0\, \underline{v})\, \mathbf{I_0}$
6. $\nabla^*(\alpha\, \mathbf{Q}) = \alpha\, (\nabla^* \mathbf{Q}) + (\nabla^* \alpha)\mathbf{Q}$
7. $\nabla^* \nabla\alpha = \nabla\, \nabla^*\, \alpha = \mathbf{I_0}\, (\partial_0^2 - \underline{\nabla}^2)\alpha \equiv -\mathbf{I_0}\, \Box\alpha$
8. $\mathbf{V}^* \nabla^* \alpha = -\mathbf{I_0}\, (c\, \partial_0 + \underline{v} \bullet \underline{\nabla})\alpha - i\, \underline{\mathbf{I}} \bullet (c\, \underline{\nabla} + \underline{v}\, \partial_0)\alpha + \underline{\mathbf{I}} \bullet (\underline{v} \wedge \underline{\nabla}\alpha)$
9. $\mathbf{Q}\, \nabla^* \alpha = \mathbf{I_0}\, (Q_0\, \partial_0\alpha - \underline{Q} \bullet \underline{\nabla}\alpha) + i\, \underline{\mathbf{I}} \bullet (Q_0\, \underline{\nabla}\alpha - \underline{Q}\, \partial_0\alpha) + \underline{\mathbf{I}} \bullet (\underline{Q} \wedge \underline{\nabla}\alpha)$
10. $\nabla^*(\alpha\, \mathbf{V}) = \mathbf{I_0}\, (c\, \partial_0\alpha - \underline{v} \bullet \underline{\nabla}\alpha) + i\, \underline{\mathbf{I}} \bullet (c\, \underline{\nabla}\alpha - \underline{v}\, \partial_0\alpha - \alpha\, \partial_0\underline{v}) - \underline{\mathbf{I}} \bullet (\underline{v} \wedge \underline{\nabla}\alpha)$

A. REFERENCES

[1] Holton G, *Thematic Origins of Scientific Thought,* Cambridge (MA): Harvard University Press, 2nd ed. (1988)

[2] Minkowski H, Perrett W, Jeffery GB, *The Principle of Relativity,* NYC (NY): Dover Books (1953)

[3] Heaviside O, *Electromagnetic Theory,* reprinted NYC (NY): Dover (1950)

[4] Lange M, *Introduction to the Philosophy of Physics,* Malden (MA): Blackwell, p.224 (2002)

[5] Dirac PAM, *Directions in Physics*, NYC (NY): Wiley Interscience p.7, p.20, p. 36 (1978)

[6] Spencer HJ, *The CNV Theory of Electromagnetism*, (to be published)

[7] Hankins TL, *Sir William Rowan Hamilton,* Baltimore (MD): Johns Hopkins University Press (1980)

[8] Hamilton WR, *General Method in Dynamics,* Proc. Roy. Soc., **1**, p.95 (1835)

[9] Darbyshire J, *Unknown Quantity – a History of Algebra*, Washington (DC): John Henry Press (2006)

[10] Wilkins DR, *Hamilton's Research on Quaternions,* Trinity College, Dublin; available online at:

http://www.maths.tcd.ie/pub/HistMath/People/Hamilton/Quaternions.html

[11] Hamilton WR, *On a New Species of Imaginary Quantities,* Proc. Roy. Irish Acad., **2**, p.424 (1844)

[12] Hamilton WR, *On Quaternions,* Proc. Roy. Irish Acad., **3**, p.1 (1847)

[13] Goldstein H, *Classical Mechanics,* Reading (MA): Addison-Wesley (1959)

[14] Hamilton WR, *Researches respecting Quaternions,* Trans. Roy. Irish Acad., **21**, p.199 (1848)

[15] Hamilton WR, *On a New System of Imaginaries in Algebra,* Phil. Mag., **35**, p. 10 (1844) to **36**, p 305 (1850)

[16] Hamilton WR, *On Symbolical Geometry,* Cambridge & Dublin Math. J., **1**, p.45 (1846) to **4**, p.105 (1849)

[17] Hamilton WR, *Lectures on Quaternions,* Dublin: Hodges & Smith (1853)

[18] Hamilton WR, *Elements of Quaternions,* Dublin (1866); reprinted Chelsea Publishing, NY: (1969)

[19] *Quaternion,* Wikipedia, http://en.wikipedia.org/wiki/quaternion p. 15 (2007)

[20] Hestenes D, Sobczyk G, *Clifford Algebra to Geometric Calculus,* Boston (MA): Reidel Publ. (1984)

[21] Hestenes D, *Space-Time Algebra,* New York (NY): Gordon & Breach (1966)

[22] Clifford WK, Tucker ER, *Mathematical Papers,* (1882); reprinted Kessinger Publishing, NY: (2007)

[23] Gspooner A, Hurni JP, *Physical Heritage of W. R. Hamilton,* (2007); e-print arXiv:math-ph/0201058 v4

[24] Gspooner A, Hurni JP, *Quaternions in Mathematical Physics,* (2006); e-print arXiv:math-ph/0510059 v3

[25] Conway AW, *Applications of Quaternions to Electrical Theory,* Proc. Roy. Irish Acad., **29**, p.1 (1911)

[26] Silberstein L, *Quaternionic form of Relativity,* Phil. Mag., **23**, p.790 (1912)

[27] Rastall P, *Quaternions in Relativity,* Rev. Mod. Phys., **36**, p.820 (1964)

[28] Lanczos C, *Tensoranalytischen Beziehungen der Diracschen Gleichung,* Zeit. f. Phys., **57**, p.477 (1929)

[29] Conway AW, *Quaternion Treatment of the Relativistic Wave Equation,* Proc. Roy. Soc., A**162**, p.145 (1937)

[30] Davies AJ, *Quaternionic Dirac Equation,* Phys. Rev., D**41**, p.2628 (1990)

[31] Berkoff G, von Neumann J, *The Logic of Quantum Mechanics,* Ann. Math., **37**, p.823 (1936)

[32] Adler SL, *Quaternionic Quantum Mechanics & Quantum Fields,* London (UK): Oxford Univ. Press (1995)

[33] Finkelstein D, et al, *Foundations of Quaternion Quantum Mechanics,* J. Math. Phys., **2**, p.207 (1962)

[34] see Hankins, Ref. [7] p. 309

[35] see Goldstein, Ref. [13] p. 20

[36] Gluckman AG, *Coordinate Transformations of W. Voigt & Relativity*, Am. J. Phys., **36**, p.226 (1968)

[37] Lorenz LV, *On the Identity of the Vibrations of Light with Electric Currents*, Phil. Mag., **38**, p.287 (1867)

[38] Miller AI, *Albert Einstein's Special Theory of Relativity,* New York, NY: Springer-Verlag (1998)

[39] Planck M, *On the Dynamics of Moving Systems,* Berl. Ber., **13**, p.542 (1907)

[40] Kaufmann W, *Magnetic & Electric Deviations of Becquerel Rays,* Goett. Nachr., p.143 (1901)

[41] Newton I, *Philosophiae Naturalis Principia Mathematica,* (1687); reprinted as *The Principia*, translated by Mott A, Amherst (NY): Prometheus Books (1995)

[42] Miller AI, *Insights of Genius,* New York (NY): Springer-Verlag (1996)

[43] Whittaker ET, *The Hamiltonian Revival,* Math. Gazette, **24**, p.158 (1940)