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TOPSIS for Solving Multi-Attribute Decision Making Problems under Bi-Polar Neutrosophic Environment

Abstract

The paper investigates a technique for order preference by similarity to ideal solution (TOPSIS) method to solve multi-attribute decision making problems with bipolar neutrosophic information. We define Hamming distance function and Euclidean distance function to determine the distance between bipolar neutrosophic numbers. In the decision making situation, the rating of performance values of the alternatives with respect to the attributes are provided by the decision maker in terms of bipolar neutrosophic numbers. The weights of the attributes are determined using maximizing deviation method. We define bipolar neutrosophic relative positive ideal solution (BNRPIS) and bipolar neutrosophic relative negative ideal solution (BNRNIS). Then, the ranking order of the alternatives is obtained by TOPSIS method and most desirable alternative is selected. Finally, a numerical example for car selection is solved to demonstrate the applicability and effectiveness of the proposed approach and comparison with other existing method is also provided.

Keywords

Single valued neutrosophic sets; bipolar neutrosophic sets; TOPSIS; multi-attribute decision making.

1. Introduction

Zadeh [1] introduced the concept of fuzzy set to deal with problems with imprecise information in 1965. However, Zadeh [1] considers one single value to express the grade of membership of the fuzzy set defined in a universe. But, it is not always possible to represent the grade of membership value by a single point. In order to overcome the difficulty, Turksen [2] incorporated interval valued fuzzy sets. In 1986, Atanassov [3] extended the concept of fuzzy sets [1] and defined intuitionistic fuzzy sets which are characterized by grade of membership and non-membership functions. Later, Lee [4, 5] introduced the notion of bipolar fuzzy sets by extending the concept of fuzzy sets where the degree of membership is expanded from [0, 1] to [-1, 1]. In a bipolar fuzzy set, if the degree of membership is zero then we say the element is unrelated to the corresponding property, the membership degree (0, 1] of an element specifies that the element somewhat satisfies the property, and the membership degree $[-1, 0)$ of an element implies that the element somewhat satisfies the implicit counter-property [6]. Zhou and Li [7] incorporated the notion of bipolar fuzzy semirings and investigated relative properties using positive t- cut, negative s- cut and equivalence relation. Smarandache [8, 9, 10, 11] incorporated indeterminacy membership function as independent component and defined neutrosophic set on three components truth, indeterminacy and falsehood. However, from practical point of view, Wang et al. [12] defined single valued neutrosophic sets (SVNSs) where degree of truth membership, indeterminacy membership and falsity membership $\in [0, 1]$. Deli et al. [13] introduced the notion of bipolar neutrosophic sets (BNSs) which is a generalization of the fuzzy sets, bipolar fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets. Pramanik and Mondal defined rough bipolar neutrosophic set [14].

Zhang and Wu [15] presented a TOPSIS [16] method for solving single valued neutrosophic multi-criteria decision making with incomplete weight information. Chi and Liu [17] proposed an extended TOPSIS method for MADM problems where the attribute weights are unknown and the attribute values are expressed in terms of interval neutrosophic numbers. Biswas et al. [18] developed a new TOPSIS based approach for solving multi-attribute group decision making problem with simplified neutrosophic information. Broumi et al. [19] extended TOPSIS method for multiple attribute decision making based on interval neutrosophic uncertain linguistic variables. In neutrosophic hybrid environment, Pramanik et al. [20] extended TOPSIS method for singled valued soft expert set based multi-attribute decision making problems. Dey et al. [21] presented TOPSIS method for generalized neutrosophic soft multi-attribute group decision making. Mondal et al. [22] presented TOPSIS in rough neutrosophic environment and provided illustrative example.

Deli et al. [13] investigated a bipolar neutrosophic multi-criteria decision making approach based on bipolar neutrosophic weighted average and geometric operators and the score, certainty and accuracy functions. Uluçay et al. [23] studied similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making. Literature review suggests that TOPSIS method in bipolar neutrosophic environment is yet to appear. Therefore this issue needs to be addressed.

In this paper, we define Hamming distances and Euclidean distances between two BNSs and develop a new TOPSIS based method for solving MADM problems under bipolar neutrosophic assessments.

The content of the paper is organized as follows. Section 2 presents some basic definitions concerning neutrosophic sets, SVNSs, BNSs which are helpful for the construction of the paper. Hamming and Euclidean distances between two bipolar neutrosophic numbers (BNNs) are also defined in the Section 2. Section 3 is devoted to present TOPSIS method for MADM problems under bipolar neutrosophic environment. A car selection problem is solved in Section 4 to illustrate the applicability of the proposed method. Sectin 5 presents conclusion.

2. Preliminaries

In this Section, we provide basic definitions regarding neutrosophic sets, SVNSs, BNSs.

2.1 Neutrosophic Sets [8, 9, 10, 11]

Consider U be a space of objects with a generic element of U denoted by x. Then, a neutrosophic set N on U is defined as follows:

$$N = \{x, \langle \mathrm{T}_N(x), \mathrm{I}_N(x), \mathrm{F}_N(x) \rangle \mid x \in U\}$$

where, $\mathrm{T}_N(x)$, $\mathrm{I}_N(x)$, $\mathrm{F}_N(x)$: $U \rightarrow\ ]^-0,\ 1^+[$ represent respectively the degrees of truth-membership, indeterminacy-membership, and falsity-membership of a point $x \in U$ to the set N with the condition $^-0 \leq \mathrm{T}_N(x) + \mathrm{I}_N(x) + \mathrm{F}_N(x) \leq 3^+$.

2.2 Single valued neutrosophic Sets [12]

Let U be a universal space of points with a generic element of X denoted by x, then a SVNS S is presented as follows:

$$S = \{x, \langle \mathrm{T}_S(x), \mathrm{I}_S(x), \mathrm{F}_S(x) \rangle \mid x \in U\}$$

where, $\mathrm{T}_S(x)$, $\mathrm{I}_S(x)$, $\mathrm{F}_S(x)$: $U \rightarrow [0, 1]$ and $0 \leq \mathrm{T}_S(x) + \mathrm{I}_S(x) + \mathrm{F}_S(x) \leq 3$ for each point $x \in U$.

2.3 Bipolar Neutrosophic Set [13]

Definition 1. Let U be a universal space of points, then a BNS B in U is defined as follows

$$B = \{x, \langle \mathrm{T}_B^+(x), \mathrm{I}_B^+(x), \mathrm{F}_B^+(x), \mathrm{T}_B^-(x), \mathrm{I}_B^-(x), \mathrm{F}_B^-(x) \rangle \mid x \in U\},$$

where $\mathrm{T}_B^+(x)$, $\mathrm{I}_B^+(x)$, $\mathrm{F}_B^+(x)$: $U \rightarrow [0, 1]$ and $\mathrm{T}_B^-(x)$, $\mathrm{I}_B^-(x)$, $\mathrm{F}_B^-(x)$: $U \rightarrow [-1, 0]$.

The positive membership degrees $\mathrm{T}_B^+(x)$, $\mathrm{I}_B^+(x)$, and $\mathrm{F}_B^+(x)$ represent the truth membership, indeterminate membership, and false membership of an element $x \in U$ corresponding to a bipolar neutrosophic set B and the negative membership degrees $\mathrm{T}_B^-(x)$, $\mathrm{I}_B^-(x)$, and $\mathrm{F}_B^-(x)$ represent the truth membership, indeterminate membership, and false membership of an element $x \in U$ to some implicit counter property corresponding to a bipolar neutrosophic set B. For convenience, a bipolar neutrosophic number is represented by $\tilde{b} = <\mathrm{T}_B^+, \mathrm{I}_B^+, \mathrm{F}_B^+, \mathrm{T}_B^-, \mathrm{I}_B^-, \mathrm{F}_B^->$.

Example: Consider $U = \{u_1, u_2, u_3, u_4\}$. Then

B = {< u_1, 0.6, 0.2, 0.1, -0.7, -0.1, -0.04>; < u_2, 0.4, 0.3, 0.1, -0.5, -0.09, -0.4>; < u_3, 0.8, 0.5, 0.4, -0.3, -0.01, -0.5>; < u_4, 0.3, 0.6, 0.7, -0.2, -0.3, -0.7>]

is a bipolar neutrosophic subset of U.

Definition 2. Let, $B_1 = \{x, \langle \mathrm{T}_{B_1}^+(x), \mathrm{I}_{B_1}^+(x), \mathrm{F}_{B_1}^+(x), \mathrm{T}_{B_1}^-(x), \mathrm{I}_{B_1}^-(x), \mathrm{F}_{B_1}^-(x) \rangle \mid x \in U\}$ and $B_2 = \{x, \langle \mathrm{T}_{B_2}^+(x), \mathrm{I}_{B_2}^+(x), \mathrm{F}_{B_2}^+(x), \mathrm{T}_{B_2}^-(x), \mathrm{I}_{B_2}^-(x), \mathrm{F}_{B_2}^-(x) \rangle \mid x \in U\}$ be two BNSs. Then $B_1 \subseteq B_2$ if and only if

$\mathrm{T}_{B_1}^+(x) \leq \mathrm{T}_{B_2}^+(x)$, $\mathrm{I}_{B_1}^+(x) \leq \mathrm{I}_{B_2}^+(x)$, $\mathrm{F}_{B_1}^+(x) \geq \mathrm{F}_{B_2}^+(x)$; $\mathrm{T}_{B_1}^-(x) \geq \mathrm{T}_{B_2}^-(x)$, $\mathrm{I}_{B_1}^-(x) \geq \mathrm{I}_{B_2}^-(x)$, $\mathrm{F}_{B_1}^-(x) \leq \mathrm{F}_{B_2}^-(x)$ for all $x \in U$.

Definition 3. Consider, $B_1 = \{x, \langle T^{+}_{B_1}(x), I^{+}_{B_1}(x), F^{+}_{B_1}(x), T^{-}_{B_1}(x), I^{-}_{B_1}(x), F^{-}_{B_1}(x) \rangle \mid x \in U\}$ and $B_2 = \{x, \langle T^{+}_{B_2}(x), I^{+}_{B_2}(x), F^{+}_{B_2}(x), T^{-}_{B_2}(x), I^{-}_{B_2}(x), F^{-}_{B_2}(x) \rangle \mid x \in U\}$ be two BNSs. Then $B_1 = B_2$ if and only if

$T^{+}_{B_1}(x) = T^{+}_{B_2}(x)$, $I^{+}_{B_1}(x) = I^{+}_{B_2}(x)$, $F^{+}_{B_1}(x) = F^{+}_{B_2}(x)$; $T^{-}_{B_1}(x) = T^{-}_{B_2}(x)$, $I^{-}_{B_1}(x) = I^{-}_{B_2}(x)$, $F^{-}_{B_1}(x) = F^{-}_{B_2}(x)$ for all $x \in U$.

Definition 4. Consider, $B = \{x, \langle T^{+}_{B}(x), I^{+}_{B}(x), F^{+}_{B}(x), T^{-}_{B}(x), I^{-}_{B}(x), F^{-}_{B}(x) \rangle \mid x \in U\}$ be a BNS. The complement of B is denoted by B^{c} and is defined by

$T^{+}_{B^c}(x) = \{1^{+}\} - T^{+}_{B}(x)$, $I^{+}_{B^c}(x) = \{1^{+}\} - I^{+}_{B}(x)$, $F^{+}_{B^c}(x) = \{1^{+}\} - F^{+}_{B}(x)$;

$T^{-}_{B^c}(x) = \{1^{-}\} - T^{-}_{B}(x)$, $I^{-}_{B^c}(x) = \{1^{-}\} - I^{-}_{B}(x)$, $F^{-}_{B^c}(x) = \{1^{-}\} - F^{-}_{B}(x)$ for all $x \in U$.

Definition 5. Consider, $B_1 = \{x, \langle T^{+}_{B_1}(x), I^{+}_{B_1}(x), F^{+}_{B_1}(x), T^{-}_{B_1}(x), I^{-}_{B_1}(x), F^{-}_{B_1}(x) \rangle \mid x \in U\}$ and $B_2 = \{x, \langle T^{+}_{B_2}(x), I^{+}_{B_2}(x), F^{+}_{B_2}(x), T^{-}_{B_2}(x), I^{-}_{B_2}(x), F^{-}_{B_2}(x) \rangle \mid x \in U\}$ be two BNSs. Then their union $B_1 \cup B_2$ is defined as follows:

$$B_1 \cup B_2 = \{\mathrm{Max}\,(T^{+}_{B_1}(x), T^{+}_{B_2}(x)), \frac{I^{+}_{B_1}(x) + I^{+}_{B_2}(x)}{2}, \mathrm{Min}\,(F^{+}_{B_1}(x), F^{+}_{B_2}(x)), \mathrm{Min}\,(T^{-}_{B_1}(x), T^{-}_{B_2}(x)), \frac{I^{-}_{B_1}(x) + I^{-}_{B_2}(x)}{2}, \mathrm{Max}\,(F^{-}_{B_1}(x), F^{-}_{B_2}(x))\} \text{ for all } x \in U.$$

Definition 6. Consider, $B_1 = \{x, \langle T^{+}_{B_1}(x), I^{+}_{B_1}(x), F^{+}_{B_1}(x), T^{-}_{B_1}(x), I^{-}_{B_1}(x), F^{-}_{B_1}(x) \rangle \mid x \in U\}$ and $B_2 = \{x, \langle T^{+}_{B_2}(x), I^{+}_{B_2}(x), F^{+}_{B_2}(x), T^{-}_{B_2}(x), I^{-}_{B_2}(x), F^{-}_{B_2}(x) \rangle \mid x \in U\}$ be two BNSs. Then their intersection $B_1 \cap B_2$ is defined as follows:

$$B_1 \cap B_2 = \{\mathrm{Min}\,(T^{+}_{B_1}(x), T^{+}_{B_2}(x)), \frac{I^{+}_{B_1}(x) + I^{+}_{B_2}(x)}{2}, \mathrm{Max}\,(F^{+}_{B_1}(x), F^{+}_{B_2}(x)), \mathrm{Max}\,(T^{-}_{B_1}(x), T^{-}_{B_2}(x)), \frac{I^{-}_{B_1}(x) + I^{-}_{B_2}(x)}{2}, \mathrm{Min}\,(F^{-}_{B_1}(x), F^{-}_{B_2}(x))\} \text{ for all } x \in U.$$

Definition 7. Suppose $\widetilde{b}_1 = < T^{+}_{B_1}, I^{+}_{B_1}, F^{+}_{B_1}, T^{-}_{B_1}, I^{-}_{B_1}, F^{-}_{B_1} >$ and $\widetilde{b}_2 = < T^{+}_{B_2}, I^{+}_{B_2}, F^{+}_{B_2}, T^{-}_{B_2}, I^{-}_{B_2}, F^{-}_{B_2} >$ are two BNNs, then

i. $\alpha . \widetilde{b}_1 = <1 - (1 - T^{+}_{B_1})^{\alpha}, (I^{+}_{B_1})^{\alpha}, (F^{+}_{B_1})^{\alpha}, -(-T^{-}_{B_1})^{\alpha}, -(-I^{-}_{B_1})^{\alpha}, -(1 - (1 - (-F^{-}_{B_1}))^{\alpha})>$;

ii. $(\widetilde{b}_1)^{\alpha} = < (T^{+}_{B_1})^{\alpha}, 1 - (1 - I^{+}_{B_1})^{\alpha}, 1 - (1 - F^{+}_{B_1})^{\alpha}, -(1 - (1 - (-T^{-}_{B_1}))^{\alpha}), -(-I^{-}_{B_1})^{\alpha}, -(-F^{-}_{B_1}))^{\alpha}>$;

iii. $\widetilde{b}_1 + \widetilde{b}_2 = < T^{+}_{B_1} + T^{+}_{B_2} - T^{+}_{B_1} . T^{+}_{B_2}, I^{+}_{B_1} . I^{+}_{B_2}, F^{+}_{B_1} . F^{+}_{B_2}, -T^{-}_{B_1} . T^{-}_{B_2}, -(-I^{-}_{B_1} - I^{-}_{B_2} - I^{-}_{B_1} . I^{-}_{B_2}), -(-F^{-}_{B_1} - F^{-}_{B_2} - F^{-}_{B_1} . F^{-}_{B_2})>$;

iv. $\widetilde{b}_1 . \widetilde{b}_2 = <T^+_{B_1} . T^+_{B_2}, I^+_{B_1} + I^+_{B_2} - I^+_{B_1} . I^+_{B_2}, F^+_{B_1} + F^+_{B_2} - F^+_{B_1} . F^+_{B_2}, (-T^-_{B_1} - T^-_{B_2} - T^-_{B_1} . T^-_{B_2}), -I^-_{B_1} . I^-_{B_2}, -F^-_{B_1} . F^-_{B_2}>$,

where $\alpha > 0$.

2.4. The distance between two BNNs

In this sub-section, we propose the distance between two BNNs.

Consider $B_1 = \sum_{i=1}^{m} (x_i, <T^+_{B_1}(x_i), I^+_{B_1}(x_i), F^+_{B_1}(x_i), T^-_{B_1}(x_i), I^-_{B_1}(x_i), F^-_{B_1}(x_i)>)$, $B_2 = \sum_{i=1}^{m} (x_i, <T^+_{B_2}(x_i), I^+_{B_2}(x_i), F^+_{B_2}(x_i), T^-_{B_2}(x_i), I^-_{B_2}(x_i), F^-_{B_2}(x_i)>)$ be two BNNs then,

(1). The Hamming distance between two BNNs is defined as follows:

$$D_H(B_1, B_2) = \sum_{i=1}^{m} \{|(T^+_{B_1}(x_i) - T^+_{B_2}(x_i))| + |(I^+_{B_1}(x_i) - I^+_{B_2}(x_i))| + |(F^+_{B_1}(x_i) - F^+_{B_2}(x_i))| + |(T^-_{B_1}(x_i) - T^-_{B_2}(x_i))| + |(I^-_{B_1}(x_i) - I^-_{B_2}(x_i))| + |(F^-_{B_1}(x_i) - F^-_{B_2}(x_i))|\} \quad (1)$$

(2). The normalized Hamming distance between two BNNs is defined as follows:

$$^N D_H(B_1, B_2) = \frac{1}{6m} \sum_{i=1}^{m} \{|(T^+_{B_i}(x_i) - T^+_{B_2}(x_i))| + |(I^+_{B_1}(x_i) - I^+_{B_2}(x_i))| + |(F^+_{B_1}(x_i) - F^+_{B_2}(x_i))| + |(T^-_{B_1}(x_i) - T^-_{B_2}(x_i))| + |(I^-_{B_1}(x_i) - I^-_{B_2}(x_i))| + |(F^-_{B_1}(x_i) - F^-_{B_2}(x_i))|\} \quad (2)$$

(3). The Euclidean distance between two BNNs is defined as follows:

$$E_H(B_1, B_2) = \sqrt{\sum_{i=1}^{m} \left\{ \begin{array}{l} (T^+_{B_1}(x_i) - T^+_{B_2}(x_i))^2 + (I^+_{B_1}(x_i) - I^+_{B_2}(x_i))^2 + (F^+_{B_1}(x_i) - F^+_{B_2}(x_i))^2 + \\ (T^-_{B_1}(x_i) - T^-_{B_2}(x_i))^2 + (I^-_{B_1}(x_i) - I^-_{B_2}(x_i))^2 + (F^-_{B_1}(x_i) - F^-_{B_2}(x_i))^2 \end{array} \right\}} \quad (3)$$

(4). The normalized Euclidean distance between two BNNs is defined as follows:

$$^N E_H(B_1, B_2) = \sqrt{\frac{1}{6m} \sum_{i=1}^{m} \left\{ \begin{array}{l} (T^+_{B_1}(x_i) - T^+_{B_2}(x_i))^2 + (I^+_{B_1}(x_i) - I^+_{B_2}(x_i))^2 + (F^+_{B_1}(x_i) - F^+_{B_2}(x_i))^2 + \\ (T^-_{B_1}(x_i) - T^-_{B_2}(x_i))^2 + (I^-_{B_1}(x_i) - I^-_{B_2}(x_i))^2 + (F^-_{B_1}(x_i) - F^-_{B_2}(x_i))^2 \end{array} \right\}} \quad (4)$$

with the following properties:

(1). $0 \le D_H(B_1, B_2) \le 6m$

(2). $0 \le {}^N D_H(B_1, B_2) \le 1$

(3). $0 \le E_H(B_1, B_2) \le \sqrt{6m}$

(4). $0 \le {}^N E_H(B_1, B_2) \le 1$.

3. TOPSIS method for MADM with bipolar neutrosophic information

In this Section, we present an approach based on TOPSIS method to deal with MADM problems under bipolar neutrosophic environment.

Let $A = \{A_1, A_2, \ldots, A_m\}$, ($m \geq 2$) be a discrete set of m feasible alternatives, $C = \{C_1, C_2, \ldots, C_n\}$, ($n \geq 2$) be a set of attributes under consideration and $w = (w_1, w_2, \ldots, w_n)^T$ be the unknown weight vector of the attributes with $0 \leq w_j \leq 1$ and $\sum_{j=1}^{n} w_j = 1$. The rating of performance value of alternative A_i, ($i = 1, 2, \ldots, m$) with respect to the predefined attribute C_j, ($j = 1, 2, \ldots, n$) is presented by the decision maker (DM) and they can be expressed by BNNs. Therefore, the proposed approach is presented using the following steps:

Step 1. Construction of decision matrix with BNNs

The rating of performance value of alternative A_i ($i = 1, 2, \ldots, m$) with respect to the attribute C_j, ($j = 1, 2, \ldots, n$) is expressed by BNNs and they can be presented in the decision matrix as follows:

$$\left\langle \tilde{r}_{ij} \right\rangle_{m\times n} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix}$$

Here, we have $r_{ij} = (T^{+}_{ij}, I^{+}_{ij}, F^{+}_{ij}, T^{-}_{ij}, I^{-}_{ij}, F^{-}_{ij})$ with $T^{+}_{ij}, I^{+}_{ij}, F^{+}_{ij}, -T^{-}_{ij}, -I^{-}_{ij}, -F^{-}_{ij} \in [0, 1]$ and $0 \leq T^{+}_{ij} + I^{+}_{ij} + F^{+}_{ij} - T^{-}_{ij} - I^{-}_{ij} - F^{-}_{ij} \leq 6$ for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

Step 2. Determination of weights of the attributes

We assume that the weights of the attributes are not equal and they are fully unknown to the DM. Therefore, in this paper, maximizing deviation method [24] is used to find the unknown weights. The main idea of maximizing deviation method can be expressed as follows. If the attribute values r_{ij} ($j = 1, 2, \ldots, n$) in the attribute C_j have small differences between the alternatives, then C_j has a small significance in ranking of all alternatives and a small weight is assigned for the attribute. If the attribute values r_{ij} ($j = 1, 2, \ldots, n$) in the attribute C_j are same, then C_j has no effect in the ranking results and zero is assigned to the weight of the attribute. However, if the attribute values r_{ij} ($j = 1, 2, \ldots, n$) over the attribute C_j have big differences, then C_j will play a key role in ranking of all alternatives and we will allocate a big weight for the attribute. The deviation values of alternative A_i ($i = 1, 2, \ldots, m$) to all other alternatives under the attribute C_j ($j = 1, 2, \ldots, n$) can be defined as $Z_{ij}(w_j) = \sum_{k=1}^{m} z(r_{ij}, r_{kj}) w_j$, then $Z_j(w_j) = \sum_{i=1}^{m} Z_{ij} w_j = \sum_{i=1}^{m}\sum_{k=1}^{m} z(r_{ij}, r_{kj}) w_j$ presents the total deviation values of all alternatives to the other alternatives for the attribute C_j ($j = 1, 2, \ldots, n$). Now $Z(w_j) = \sum_{j=1}^{n} Z_j(w_j) = \sum_{j=1}^{n} \sum_{i=1}^{m}\sum_{k=1}^{m} z(r_{ij}, r_{kj}) w_j$ presents the total deviation of all attributes to the other alternatives with respect to all alternatives. Now we construct the non-linear optimizing model based on above analysis to obtain unknown attribute weight w_j as follows:

$$\text{Max } Z(w_j) = \sum_{j=1}^{n} \sum_{i=1}^{m}\sum_{k=1}^{m} z(r_{ij}, r_{kj}) w_j \tag{5}$$

Subject to $\sum_{j=1}^{n} w_j^2 = 1$, $w_j \geq 0$, $j = 1, 2, \ldots, n$.

We now formulate the Lagrange multiplier function, and obtain

$$L(w_j, \rho) = \sum_{j=1}^{n} \sum_{i=1}^{m}\sum_{k=1}^{m} z(r_{ij}, r_{kj}) w_j + \rho \ (\sum_{j=1}^{q} w_j^2 - 1)$$

where ρ is the Lagrange multiplier.

Then, we calculate the partial derivatives of L with respect to w_j and ρ respectively as follows:

$$\frac{\partial L(w_j, \rho)}{\partial w_j} = \sum_{i=1}^{m}\sum_{k=1}^{m} z(r_{ij}, r_{kj}) w_j + \rho \ (\sum_{j=1}^{n} w_j^2 - 1) = 0,$$

$$\frac{\partial L(w_j, \rho)}{\partial \rho} = \sum_{j=1}^{n} w_j^2 - 1 = 0.$$

Therefore, the weight of the attribute C_j is obtained as

$$w_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m} z(r_{ij}, r_{kj})}{\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m} z(r_{ij}, r_{kj})\right)^2}} \tag{6}$$

and the normalized weight of the attribute C_j is given by

$$w_j^* = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m} z(r_{ij}, r_{kj})}{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m} z(r_{ij}, r_{kj})\right)}. \tag{7}$$

Step 3. Construction of weighted decision matrix

We find aggregated weighted decision matrix by multiplying weights [25] of the attributes and the aggregated decision matrix $\left\langle r_{ij}^{w_j} \right\rangle_{m\times n}$ is constructed as follows:

$$\left\langle r_{ij} \right\rangle_{m\times n} \otimes w_j = \left\langle r_{ij}^{w_j} \right\rangle_{m\times n} = \begin{bmatrix} r_{11}^{w_1} & r_{12}^{w_2} & \dots & r_{1n}^{w_n} \\ r_{21}^{w_1} & r_{22}^{w_2} & \dots & r_{2n}^{w_n} \\ . & . & \dots & . \\ . & . & \dots & . \\ r_{m1}^{w_1} & r_{m2}^{w_2} & \dots & r_{mn}^{w_n} \end{bmatrix}$$

$r_{ij}^{w_j} = (T_{ij}^{w_j+}, I_{ij}^{w_j+}, F_{ij}^{w_j+}, T_{ij}^{w_j-}, I_{ij}^{w_j-}, F_{ij}^{w_j-})$ with $T_{ij}^{w+}, I_{ij}^{w+}, F_{ij}^{w+}, T_{ij}^{w-}, I_{ij}^{w-}, F_{ij}^{w-} \in [0, 1]$ and $0 \le T_{ij}^{w+} + I_{ij}^{w+} + F_{ij}^{w+} - T_{ij}^{w-} - I_{ij}^{w-} - F_{ij}^{w-} \le 6$ for i = 1, 2, …, m; j = 1, 2, …, n.

Step 4. Identify the bipolar neutrosophic relative positive ideal solution (BNRPIS) and bipolar neutrosophic relative negative ideal solution (BNRNIS)

In real life decision making, we confront two types of attributes namely, benefit type attributes (β_1) and cost type attributes (β_2). In bipolar neutrosophic environment, assume that Q_{BNRPIS}^{w+} and

Q^{w-}_{BNRNIS} be the bipolar neutrosophic relative positive ideal solution (BNRPIS) and bipolar neutrosophic relative negative ideal solution (BNRNIS). Then, Q^{w+}_{BNRPIS} and Q^{w-}_{BNRNIS} are defined as follows:

$$Q^{w+}_{BNRPIS} = \left(\left\langle {}^{+}T_1^{w_1+}, {}^{+}I_1^{w_1+}, {}^{+}F_1^{w_1+}, {}^{+}T_1^{w_1-}, {}^{+}I_1^{w_1-}, {}^{+}F_1^{w_1-}, \right\rangle, \left\langle {}^{+}T_2^{w_2+}, {}^{+}I_2^{w_2+}, {}^{+}F_2^{w_2+}, {}^{+}T_2^{w_2-}, {}^{+}I_2^{w_2-}, {}^{+}F_2^{w_2-} \right\rangle, \ldots, \left\langle {}^{+}T_n^{w_n+}, {}^{+}I_n^{w_n+}, {}^{+}F_n^{w_n+}, {}^{+}T_n^{w_n-}, {}^{+}I_n^{w_n-}, {}^{+}F_n^{w_n-} \right\rangle \right) \tag{8}$$

$$Q^{w-}_{BNRNIS} = \left(\left\langle {}^{-}T_1^{w_1+}, {}^{-}I_1^{w_1+}, \text{-}F_1^{w_1+}, {}^{-}T_1^{w_1-}, {}^{-}I_1^{w_1-}, {}^{-}F_1^{w_1-}, \right\rangle, \left\langle {}^{-}T_2^{w_2+}, {}^{-}I_2^{w_2+}, {}^{-}F_2^{w_2+}, {}^{-}T_2^{w_2-}, {}^{-}I_2^{w_2-}, {}^{-}F_2^{w_2-} \right\rangle, \ldots, \left\langle {}^{-}T_n^{w_n+}, {}^{-}I_n^{w_n+}, {}^{-}F_n^{w_n+}, {}^{-}T_n^{w_n-}, {}^{-}I_n^{w_n-}, {}^{-}F_n^{w_n-} \right\rangle \right) \tag{9}$$

where

$\left\langle {}^{+}T_j^{w_j+}, {}^{+}I_j^{w_j+}, {}^{+}F_j^{w_j+}, {}^{+}T_j^{w_j-}, {}^{+}I_j^{w_j-}, {}^{+}F_j^{w_j-} \right\rangle$ = <[{ $\underset{i}{Max}(T_{ij}^{w_j+}) \mid j \in \beta_1$ }; { $\underset{i}{Min}(T_{ij}^{w_j+}) \mid j \in \beta_2$ }], [{ $\underset{i}{Min}(I_{ij}^{w_j+}) \mid j \in \beta_1$ }; { $\underset{i}{Max}(I_{ij}^{w_j+}) \mid j \in \beta_2$ }], [{ $\underset{i}{Min}(F_{ij}^{w_j+}) \mid j \in \beta_1$ }; { $\underset{i}{Max}(F_{ij}^{w_j+}) \mid j \in \beta_2$ }], [{ $\underset{i}{Min}(T_{ij}^{w_j-}) \mid j \in \beta_1$ }; { $\underset{i}{Max}(T_{ij}^{w_j-}) \mid j \in \beta_2$ }], [{ $\underset{i}{Max}(I_{ij}^{w_j-}) \mid j \in \beta_1$ }; { $\underset{i}{Min}(I_{ij}^{w_j-}) \mid j \in \beta_2$ }],[{ $\underset{i}{Max}(F_{ij}^{w_j-}) \mid j \in \beta_1$ }; { $\underset{i}{Min}(F_{ij}^{w_j-}) \mid j \in \beta_2$ }]>, j = 1, 2, …, n;

$\left\langle {}^{-}T_j^{w_j+}, {}^{-}I_j^{w_j+}, {}^{-}F_j^{w_j+}, {}^{-}T_j^{w_j-}, {}^{-}I_j^{w_j-}, {}^{-}F_j^{w_j-} \right\rangle$ = < [{ $\underset{i}{Min}(T_{ij}^{w_j+}) \mid j \in \beta_1$ }; { $\underset{i}{Max}(T_{ij}^{w_j+}) \mid j \in \beta_2$ }], [{ $\underset{i}{Max}(I_{ij}^{w_j+}) \mid j \in \beta_1$ }; { $\underset{i}{Min}(I_{ij}^{w_j+}) \mid j \in \beta_2$ }], [{ $\underset{i}{Max}(F_{ij}^{w_j+}) \mid j \in \beta_1$ }; { $\underset{i}{Min}(F_{ij}^{w_j+}) \mid j \in \beta_2$ }], [{ $\underset{i}{Max}(T_{ij}^{w_j-}) \mid j \in \beta_1$ }; { $\underset{i}{Min}(T_{ij}^{w_j-}) \mid j \in \beta_2$ }], [{ $\underset{i}{Min}(I_{ij}^{w_j-}) \mid j \in \beta_1$ }; { $\underset{i}{Max}(I_{ij}^{w_j-}) \mid j \in \beta_2$ }], [{ $\underset{i}{Min}(F_{ij}^{w_j-}) \mid j \in \beta_1$ }; { $\underset{i}{Max}(F_{ij}^{w_j-}) \mid j \in \beta_2$ }] >, j = 1, 2, …, n.

Step 5. Calculation of distance of each alternative from BNRPIS and BNRNIS

The normalized Euclidean distance of each alternative $\left\langle T_{ij}^{w_j+}, I_{ij}^{w_j+}, F_{ij}^{w_j+}, T_{ij}^{w_j-}, I_{ij}^{w_j-}, F_{ij}^{w_j-} \right\rangle$ from the BNRPIS $\left\langle {}^{+}T_j^{w_j+}, {}^{+}I_j^{w_j+}, {}^{+}F_j^{w_j+}, {}^{+}T_j^{w_j-}, {}^{+}I_j^{w_j-}, {}^{+}F_j^{w_j-} \right\rangle$ for i = 1, 2, …, m; j = 1, 2, …., n can be defined as follows:

$$Euc_N^{i+} = \sqrt{\frac{1}{6n} \sum_{j=1}^{n} \begin{Bmatrix} (T_{ij}^{w_j+} - {}^{+}T_{ij}^{w_j+})^2 + (I_{ij}^{w_j+} - {}^{+}I_{ij}^{w_j+})^2 + (F_{ij}^{w_j+} - {}^{+}F_{ij}^{w_j+})^2 + \\ T_{ij}^{w_j-} - {}^{+}T_{ij}^{w_j-})^2 + (I_{ij}^{w_j-} - {}^{+}I_{ij}^{w_j-})^2 + (F_{ij}^{w_j-} - {}^{+}F_{ij}^{w_j-})^2 \end{Bmatrix}} \tag{10}$$

Similarly, normalized Euclidean distance of each alternative $\left\langle T_{ij}^{w_j+}, I_{ij}^{w_j+}, F_{ij}^{w_j+}, T_{ij}^{w_j-}, I_{ij}^{w_j-}, F_{ij}^{w_j-} \right\rangle$ from the BNRNIS $\left\langle {}^{-}T_j^{w_j+}, {}^{-}I_j^{w_j+}, {}^{-}F_j^{w_j+}, {}^{-}T_j^{w_j-}, {}^{-}I_j^{w_j-}, {}^{-}F_j^{w_j-} \right\rangle$ for i = 1, 2, …, m; j = 1, 2, …., n can be written as follows:

$$\mathrm{Euc}_{N}^{i-} = \sqrt{\frac{1}{6n}\sum_{j=1}^{n}\left\{\begin{matrix}(T_{ij}^{w_j+} - {}^{-}T_{ij}^{w_j+})^2 + (I_{ij}^{w_j+} - {}^{-}I_{ij}^{w_j+})^2 + (F_{ij}^{w_j+} - {}^{-}F_{ij}^{w_j+})^2 + \\ T_{ij}^{w_j-} - {}^{-}T_{ij}^{w_j-})^2 + (I_{ij}^{w_j-} - {}^{-}I_{ij}^{w_j-})^2 + (F_{ij}^{w_j-} - {}^{-}F_{ij}^{w_j-})^2\end{matrix}\right\}} \quad (11)$$

Step 6. Evaluate the relative closeness co-efficient

The relative closeness co-efficient of each alternative A_i, (i = 1, 2, …, m) with respect to the BNRPIS Q_{BNRPIS}^{w+} is defined as follows:

$$cc_i^{*} = \frac{\mathrm{Euc}_{N}^{i-}}{\mathrm{Euc}_{N}^{i+} + \mathrm{Euc}_{N}^{i-}} \quad (12)$$

where, $0 \le cc_i^{*} \le 1$, i = 1, 2, …, m.

Step 7. Rank the alternatives

Rank the alternatives according to the descending order of the alternatives and select the best alternative with maximum value of cc_i^{*}.

4. A numerical example

We consider the problem [13] where a customer wants to buy a car. There are four types cars (alternatives) A_i, i = 1, 2, 3, 4 are available. The customer considers four attributes namely Fuel economy (C_1), Aerod (C_2), Comfort (C_3), Safety C_4 to assess the alternatives. Now we solve the problem with bipolar neutrosophic information based on TOPSIS method to select most desirable car for the customer. Then, the proposed TOPSIS approach for solving the problem is presented in the following steps:

Step 1: Formulation of decision matrix

We construct the decision matrix with bipolar neutrosophic information presented by the DM as given below (see Table 1).

Table 1. *The decision matrix provided by the DM*

	C_1	C_2	C_3	C_4
A_1	(0.5, 0.7, 0.2, -0.7, -0.3, -0.6)	(0.4, 0.4, 0.5, -0.7, -0.8, -0.4)	(0.7, 0.7, 0.5, -0.8, -0.7, -0.6)	(0.1, 0.5, 0.7, -0.5, -0.2, -0.8)
A_2	(0.9, 0.7, 0.5, -0.7, -0.7, -0.1)	(0.7, 0.6, 0.8, -0.7, -0.5, -0.1)	(0.9, 0.4, 0.6, -0.1, -0.7, -0.5)	(0.5, 0.2, 0.7, -0.5, -0.1, -0.9)
A_3	(0.3, 0.4, 0.2, -0.6, -0.3, -0.7)	(0.2, 0.2, 0.2, -0.4, -0.7, -0.4)	(0.9, 0.5, 0.5, -0.6, -0.5, -0.2)	(0.7, 0.5, 0.3, -0.4, -0.2, -0.2)
A_4	(0.9, 0.7, 0.2, -0.8, -0.6, -0.1)	(0.3, 0.5, 0.2, -0.5, -0.5, -0.2)	(0.5, 0.4, 0.5, -0.1, -0.7, -0.2)	(0.4, 0.2, 0.8, -0.5, -0.5, -0.6)

Step 2. Calculation of the weights of the attributes

We use normalized Hamming distance and obtain the weights of the attributes by maximizing deviation method as follows:

$w_1 = 0.2585$, $w_2 = 0.2552$, $w_3 = 0.2278$, $w_4 = 0.2585$, where $\sum_{j=1}^{4} w_j = 1$.

Step 3. Construction of weighted decision matrix

The weighted decision matrix is obtained by multiplying weights to decision matrix as given below (see Table 2)

Table 2. *The weighted decision matrix*

	C_1	C_2	C_3	C_4
A_1	(0.164, 0.912, 0.66, -0.912, -0.732, -0.211)	(0.122, 0.791, 0.838, -0.913, -0.945, -0.122)	(0.24, 0.922, 0.854, -0.95, -0.922, -0.208)	(0.027, 0.836, 0.912, -0.836, -0.66, -0.337)
A_2	(0.488, 0.912, 0.836, -0.912, -0.912, -0.027)	(0.264, 0.874, 0.945, -0.913, -0.838, -0.026)	(0.408, 0.812, 0.89, -0.592, -0.922, -0.162)	(0.164, 0.66, 0.912, -0.836, -0.551, -0.444)
A_3	(0.088, 0.789, 0.66, -0.876, -0.732, -0.267)	(0.055, 0.663, 0.663, -0.791, -0.913, -0.122)	(0.408, 0.854, 0.854, -0.89, -0.854, -0.055)	(0.267, 0.836, 0.088, -0.789, -0.66, -0.055)
A_4	(0.448, 0.912, 0.66, -0.944, -0.876, -0.027)	(0.087, 0.838, 0.663, -0.838, -0.838, -0.055)	(0.146, 0.812, 0.854, -0.592, -0.922, -0.055)	(0.124, 0.66, 0.944, -0.836, -0.836, -0.208)

Step 4. Recognize the BNRPIS and BNRNIS

The BNRPIS (R^{w+}_{BPRPIS}) and BNRNIS (R^{w-}_{BPRNIS}) are obtained from the weighted decision matrix as follows:

R^{w+}_{BPRPIS}= < (0.448, 0.789, 0.66, -0.944, -0.732, -0.027); (0.264, 0.663, 0.663, -0.913, -0.838, -0.026); (0.408, 0.812, 0.854, -0.89, -0.854, -0.055); (0.267, 0.66, 0.88, -0.836, -0.551, -0.055) >;

R^{w-}_{BPRNIS}= < (0.088, 0.912, 0.836, -0.876, -0.912, -0.267); (0.055, 0.878, 0.945, -0.791, -0.945, -0.122); (0.146, 0.922, 0.89, -0.592, -0.922, -0.208); (0.027, 0.836, 0.912, -0.789, -0.836, -0.444) >.

Step 5. Distance measures of each alternative from the BNRPISs and BNRNISs

The normalized Euclidean distances of each alternative from the BNRPISs are computed as follows:

$\mathrm{Euc}_N^{1+} = 0.0479$, $\mathrm{Euc}_N^{2+} = 0.0161$, $\mathrm{Euc}_N^{3+} = 0.013$, $\mathrm{Euc}_N^{4+} = 0.0469$.

Similarly, the normalized Euclidean distances of each alternative from the BNRNISs are computed as follows:

$\mathrm{Euc}_N^{1-} = 0.0123$, $\mathrm{Euc}_N^{2-} = 0.0247$, $\mathrm{Euc}_N^{3-} = 0.0548$, $\mathrm{Euc}_N^{4-} = 0.0192$.

Step 6. Calculation of the relative closeness coefficient

We determine the relative closeness co-efficient cc_i^*, (i = 1, 2, 3, 4) using Eq. (12).

$cc_1^* = 0.2043$, $cc_2^* = 0.6054$, $cc_3^* = 0.8082$, $cc_4^* = 0.2905$.

Step 7. Rank the alternatives

The ranking order of the cars is presented according to the relative closeness coefficient as given below.

$A_3 \succ A_2 \succ A_4 \succ A_1$

Consequently, A_3 is the most preferable alternative.

Note 1: Deli et al. [13] consider the weight vector of the attributes as $w = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})$ for car selection. However, if we take weight vector of the attributes as $w = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})$, then relative closeness co-efficient cc_i^*, (i = 1, 2, 3, 4) are computed as given below.

$cc_1^* = 0.3746$, $cc_2^* = 0.5761$, $cc_3^* = 0.4716$, $cc_4^* = 0.6944$.

Therefore, the ranking order of the cars can be represented as follows:

$A_4 \succ A_2 \succ A_3 \succ A_1$

So, A_4 would be the most suitable alternative.

5. Conclusion

In this paper, we present a TOPSIS method for solving MADM problem with bipolar neutrosophic information. We define Hamming distance function and Euclidean distance function to determine the distance between BNNs. In the decision making situation, the rating of performance values of the alternatives with respect to the attributes are provided by the DM in terms of BNNs. The weights of the attributes are obtained by maximizing deviation method and we construct the weighted decision matrix. We also define BNRPIS and BNRNIS. Euclidean distance measure is employed to compute the distances of each alternative from BNRPISs as well as BNRNISs. Relative closeness coefficients are calculated to rank the alternative and to obtain the best alternative. Finally, the proposed method is applied to solve a car selection problem to verify the applicability of the proposed method and comparison with other existing method is also provided.

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