PROOF OF TWIN PRIME CONJECTURE

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ABSTRACT. In this paper we prove that there exist infinitely many twin prime numbers by studying n when $6n \pm 1$ are primes. By studying n we show that for every n that generates a twin prime number, there has to be $m > n$ that generates a twin prime number too.

1. INTRODUCTION

Considering that every twin prime can be written as $6n \pm 1$ except for 3 and 5. By studying the properties of n we make sure that there's $m > n$ that generates a twin prime which means $6m \pm 1$ are primes. First values of n that generate twin primes are $\{1, 2, 3, 5, 7, 10, 12, 17, 18, 23 ...\}$. Let's name our n a twin prime generator.

Theorem 1.1.

If $k_1 < k_2$ where $k_1 = p(n_1) + rem_1$ or $k_1 mod\ p = rem_1$, $k_2 = p(n_2) + rem_2$, $k_1 \neq k_2$ and $rem_1 \neq rem_2$ then $6k_1\ mod\ p \neq 6k_2\ mod\ p$.

Where $n \in \mathbb{N}$ and p is a prime number.

Proof 1.1.

Let $k_1 = pn_1 + rem_1$, and $k_2 = pn_2 + rem_2$

Let $rem_1 \neq rem_2$

$6k_1 = 6(pn_1 + rem_1)$

$6k_2 = 6(pn_2 + rem_2)$

$6k_1 = P(6n_1) + 6rem_1$ (1)

$6k_2 = P(6n_2) + 6rem_2$ (2)

If $6rem_1$ and $6rem_1$ are bigger than p, then we divide it to values $6rem_3 + p(L_1)$ where n can be zero if $6rem_1$ is not bigger than p.

$6k_1 = P(6n_1) + rem_3 + p(L_1)$ (3)

$6k_2 = P(6n_2) + rem_4 + p(L_2)$ (4)

$6k_1 = P(6n_1 + L_1) + rem_3$

$6k_2 = P(6n_2 + L_2) + rem_4$

Let $rem_3 = rem_4$

Then from 3 and 4, $6k_1 - P(6n_1 + L_1) = 6k_2 - P(6n_2 + L_2)$ (5)

From 1, $6rem_1 = 6k_1 - P(6n_1)$ (6)

From 2, $6rem_2 = 6k_2 - P(6n_2)$ (7)

From 5, 6, and 7, $6rem_1 + pL_1 = 6rem_2 + pL_2$

Since $6rem_1 = rem_3 + p(L_1)$ and $6rem_2 = rem_4 + p(L_2)$

$rem_3 + pL_1 + pL_1 = rem_4 + pL_2 + pL_2$

$L_1 = L_2$

From 5, $6k_1 - P(6n_1) = 6k_2 - P(6n_2)$

$rem_1 = rem_2$ which is a contradiction.

What we conclude from theorem 1 is that for every two numbers have not the same remainder from the division by a prime number, then after multiplying by 6 they can't have the same remainder too. In brief, every distinct remainder from the division by a prime number after multiplying by 6 will have a distinct remainder from the division by the same prime number.

Theorem 1.2.

If 6m $\pm$ 1 is divisible by $(6n+1)$ or $(6n-1)$ then $m\ mod\ (6n+1) = ((6n+1) \pm n)\ mod\ (6n+1)$ or $m\ mod\ (6n-1) = ((6n-1) \pm n)\ mod\ (6n-1)$, where $(6n+1)$ and $(6n-1)$ are primes.

Proof 1.2.

We know that $((6n+1)+n)\ mod\ (6n+1) = n$, $((6n+1)-n)\ mod\ (6n+1) = \mathbf{5}n+1$, $((6n-1)+n)\ mod\ (6n-1) = n$ and $((6n-1)-n)\ mod\ (6n-1) = \mathbf{5}n-1$

Let $x = k(6n+1)+n$, Then $6x = 36kn + 6k + 6n$

$6x + 1 = 36kn + 6k + 6n + 1$

$6x + 1 = 6k(6n+1) + (6n+1)$

$6x + 1 = (6k+1)(6n+1)$ which is divisible by $(6n+1)$.

$6x\ mod\ (6n+1) = 6n$

From theorem 1.1, the remainder n is the only reminder that can lead to the remainder $6n$ where the next number is divisible by $6n+1$ when it's multiplied by 6.

Let $x = k(6n+1) - n$ that $x\ mod\ (6n+1) = 5n+1$, Then $6x = 36kn + 6k - 6n$

$6x - 1 = 36kn + 6k - 6n - 1$

$6x - 1 = 6k(6n+1) - (6n+1)$

$6x - 1 = (6k-1)(6n+1)$ which is divisible by $(6n+1)$.

$6x\ mod\ (6n+1) = 1$

From theorem 1.1, the remainder $\mathbf{5}n+1$ is the only remainder that can lead to the remainder 1 where the behind number is divisible by $6n+1$ when it's multiplied by 6.

Let $x = k(6n-1)+n$, Then $6x = 36kn - 6k + 6n$

$6x - 1 = 36kn - 6k + 6n - 1$

$6x - 1 = 6k(6n-1) + (6n-1)$

$6x - 1 = (6k + 1)(6n - 1)$ which is divisible by $(6n - 1)$.

$6x \ mod\ (6n - 1) = 1$

From theorem 1.1, the remainder n is the only reminder that can lead to the remainder 1 where the behind number is divisible by $6n - 1$ when it's multiplied by 6.

Let $x = k(6n - 1) - n$ that $x \ mod\ (6n - 1) = \mathbf{5}n - 1$, Then $6x = 36kn - 6k - 6n$

$6x + 1 = 36kn - 6k - 6n + 1$

$6x + 1 = 6k(6n - 1) - (6n - 1)$

$6x + 1 = (6k - 1)(6n - 1)$ which is divisible by $(6n - 1)$.

$6x \ mod\ (6n - 1) = 6n$

From theorem 1.1, the remainder $\mathbf{5}n - 1$ is the only remainder that can lead to the remainder $6n$ where the next number is divisible by $6n - 1$ when it's multiplied by 6.

Lemma 1.1.

For m to be a twin prime generator, it has to fulfill the condition that $m \ mod\ (6n + 1) \neq ((6n + 1) \pm n) \ mod\ (6n + 1)$ and $m \ mod\ (6n - 1) \neq ((6n - 1) \pm n) \ mod\ (6n - 1)$, where $n \in \mathbb{N}$ and $n \neq 0$.

We know that m to be a twin prime generator, $6m \pm 1$ shouldn't be divisible by 5 or 7, $6m \pm 1$ shouldn't be divisible by 11 or 13, $6m \pm 1$ shouldn't be divisible by 17 or 19, $6m \pm 1$ shouldn't be divisible by 23 or 25, and so on.

From theorem 1.2, m to be a twin prime generator, $m \pm 1$ shouldn't be divisible by 5 or 7, $m \pm 2$ shouldn't be divisible by 11 or 13, $m \pm 3$ shouldn't be divisible by 17 or 19, $m \pm 4$ shouldn't be divisible by 23 or 25, and so on. From here we can say that m to be a twin prime generator $m \ mod\ (6n + 1) \neq ((6n + 1) \pm n) \ mod\ (6n + 1)$ and $m \ mod\ (6n - 1) \neq ((6n - 1) \pm n) \ mod\ (6n - 1)$.

Theorem 1.3.

The longest interval of integers covered by the union of $4n$ arithmetic progressions $\pm k \ mod(6k - 1)$ and $\pm k \ mod(6k + 1)$ is less than $4n^2$ where $1 < k \leq n$.

Proof 1.3.

The Integers of the interval $4n^2$that are covered at most by $-k \ mod(6k - 1)$ is $\left\lfloor \frac{4n^2}{6k-1} \right\rfloor + 1$ and by $+k \ mod(6k - 1)$ is $\left\lfloor \frac{4n^2}{6k-1} \right\rfloor + 1$. So, the integers that are covered by $\pm k \ mod(6k - 1)$ at most is $2\left\lfloor \frac{4n^2}{6k-1} \right\rfloor + 2$ and by $\pm k \ mod(6k + 1)$ is $2\left\lfloor \frac{4n^2}{6k+1} \right\rfloor + 2$..

So, the maximum integers covered, keeping in mind integers that get overlapped, by the $4n$ progressions equal $2 + 2\left(1 - \frac{2}{5}\right) + 2\left(1 - \frac{2}{5} - \frac{6}{35}\right) + \cdots + 4n^2\left(\frac{2}{5} + \frac{2}{7} - \frac{2}{7}\left(\frac{2}{5}\right) + \frac{2}{11} - \frac{2}{11}\left(\frac{2}{5}\right) - \frac{2}{11}\left(\frac{2}{7}\right) + \frac{2}{11}\left(\frac{2}{7}\right)\left(\frac{2}{5}\right) + \frac{2}{13} - \frac{2}{13}\left(\frac{2}{5}\right) - \frac{2}{13}\left(\frac{2}{7}\right) - \frac{2}{13}\left(\frac{2}{11}\right) + \cdots + \frac{2}{(6(n)+1)} - \frac{2}{(6(n)+1)}\left(\frac{2}{6(1)-1}\right) - \cdots\right) = 2 + 2\left(1 - \frac{2}{5}\right) + 2\left(1 - \frac{2}{5} - \frac{6}{35}\right) + \cdots + 4n^2\left(\frac{2}{5} + \frac{6}{35} + \frac{6}{77} + \frac{54}{1001} + \cdots + \frac{2}{(6(n)+1)}\left(1 - \frac{2}{5} + \frac{6}{35} + \frac{6}{77} + \frac{54}{1001} + \cdots\right)\right)$.

Integers that aren't covered by the $4n$ progressions equal $4n^2\left(1 - \frac{2}{5} + \frac{6}{35} + \frac{6}{77} + \frac{54}{1001} + \cdots\right) - \left(2 + 2\left(1 - \frac{2}{5}\right) + 2\left(1 - \frac{2}{5} - \frac{6}{35}\right) + \cdots\right)$. $4n^2\left(1 - \frac{2}{5} + \frac{2}{7} - \frac{2}{7}\left(\frac{2}{5}\right) + \cdots\right) - \left(2 + 2\left(1 - \frac{2}{5}\right) + 2\left(1 - \frac{2}{5} - \frac{6}{35}\right) + \right.$

$\cdots\Big) > 1$ when $n \geq 3$ which means that there're integers that are not covered by the $4n$ progregressions when $n \geq 3$.

Theorem 1.4.

Let m be a twin prime generator and x is the next twin prime generator where $x > m$ then $x < m + 4\left\lfloor\frac{m}{6}\right\rfloor^2 + 1$.

Proof 1.4.

From theorem 1.3 we know that the longest interval of integers covered by the union of $4n$ arithmetic progressions $\pm l \; mod \; (6(l) - 1), (6(l) + 1)$ where $l \leq n$ and $n = \left\lfloor\frac{m}{6}\right\rfloor$ is less than $4n^2$.

Let p be a prime number greater than m, then the integers covered by p equal $k_p = k(6\text{s} \pm 1) \pm s$ where $s > \lfloor\frac{m}{6}\rfloor$.

If $k < s$, then $k_p = k(6\text{s} \pm 1) \pm s = k6s \pm k \pm \text{s} = s(6k \pm 1) \pm k$ which means that they're already covered by primes less than m (by one of $\pm l \; mod \; (6(l) - 1), (6(l) + 1)$). We conclude from here that k has to be greater than or equal n to cover an integer that's not covered already.

$s(6s + 1) + s > s(6s + 1) - s > s(6s - 1) - s > s(6s - 1) - s > m + 4\left\lfloor\frac{m}{6}\right\rfloor^2 + 1$ which means that for primes greater than m, they can't cover integers between m and $m + 4\left\lfloor\frac{m}{6}\right\rfloor^2$. Thus, that leads to the fact that there has to be an integer that isn't covered (a twin prime generator) x where $m < x < m + 4\left\lfloor\frac{m}{6}\right\rfloor^2 + 1$.

2. NEXT TWIN PRIME

Definition 2.1.

Let $o_1 = (6n + 1) + n$, $o_2 = (6n + 1) - n$, $o_3 = (6n - 1) + n$, $o_4 = (6n - 1) - n$.

Let c be a twin prime generator.

Let $k_1 = c \; mod \; (6n + 1)$ and $k_2 = c \; mod \; (6n - 1)$.

Let $f_1 = |k_1 - o_1 - (6n + 1)| \; mod \; (6n + 1)$, $f_2 = |k_1 - o_2 - (6n + 1)| \; mod \; (6n + 1)$, , $f_3 = |\, k_2 - o_3 - (6n - 1)| \; mod \; (6n - 1)$, , $f_4 = |\, k_2 - o_4 - (6n - 1)| \; mod \; (6n - 1)$.

Let $F = \{x \in \mathbb{N}: x \leq \left(c + 4\left\lfloor\frac{c}{6}\right\rfloor^2 + 1\right) \; and \; x = f_1 \; or \; x = f_2 \; or \; x = f_3 \; or \; x = f_4\}$.

Let $T = \{x \in \mathbb{N}: x \leq \left(c + 4\left\lfloor\frac{c}{6}\right\rfloor^2 + 1\right) \; and \; x \notin F\}$.

Then $next \; twin \; prime \; generator = c + MIN[T]$

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Example 1.

We know that 12 is a twin prime generator, then

$c = 12$

The next twin prime generator is definitely within the next 7 numbers

We calculate just when $n \leq \left\lfloor \frac{c}{6} \right\rfloor$ or $n \leq 2$.

$f_1(1) = |k_1 - o_1 - (6(1)+1)|\ mod\ (6(1)+1) = |5 - 8 - 7|\ mod\ 7 = 10\ mod\ 7 = 3$

$f_1(2) = |k_1 - o_1 - (6(2)+1)|\ mod\ (6(2)+1) = |12 - 15 - 13|\ mod\ 13 = 16\ mod\ 13 = 3$

$f_2(1) = |k_1 - o_2 - (6(1)+1)|\ mod\ (6(1)+1) = |5 - 6 - 7|\ mod\ 7 = 8\ mod\ 7 = 1$

$f_2(2) = |k_1 - o_2 - (6(2)+1)|\ mod\ (6(2)+1) = |12 - 11 - 13|\ mod\ 13 = 12\ mod\ 13 = 12$

$f_3(1) = |\ k_2 - o_3 - (6(1)-1)|\ mod\ (6(1)-1) = |2 - 6 - 5|\ mod\ 5 = 9\ mod\ 5 = 4$

$f_3(2) = |\ k_2 - o_3 - (6(2)-1)|\ mod\ (6(2)-1) = |1 - 13 - 11|\ mod\ 11 = 23\ mod\ 11 = 1$

$f_4(1) = |\ k_2 - o_4 - (6(1)-1)|\ mod\ (6(1)-1) = |2 - 4 - 5|\ mod\ 5 = 7\ mod\ 5 = 2$

$F = x \in \mathbb{N}: x \leq \left(c + 4\left\lfloor \frac{c}{6} \right\rfloor^2 + 1\right)\ and\ x = f_1\ or\ x = f_2\ or\ x = f_3\ or\ x = f_4)\} = \{1,2,3,4,12\}$

$T == \{x \in \mathbb{N}: x \leq \left(c + 4\left\lfloor \frac{c}{6} \right\rfloor^2 + 1\right)\ and\ x \notin F\} = \{5,6,7,8,9,10,11,13,14,15,16,17,18,19,20,21,22\}$

$next\ twin\ prime\ generator = c + MIN[T] = 12 + MIN[5,6,7] = 12 + 5 = 17$

3. REFERENCES

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