Integer, Fractional, and Anomalous Quantum Hall Effect Explained with Eyring's Rate Process Theory and Free Volume Concept

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Abstract

The Hall effect, especially integer, fractional and anomalous quantum Hall effect, has been addressed with the Eyring's rate process theory and free volume concept. The basic assumptions are that the conduction process is a common rate controlled "reaction" process that can be described with Eyring's absolute rate process theory; the mobility of electrons should be dependent on the free volume available for conduction electrons. The obtained Hall conductivity is clearly quantized as e^2/h with prefactors related to both the magnetic flux quantum number and the magnetic quantum number via azimuthal quantum number, with and without an externally applied magnetic field. This article focuses on two dimensional (2D) systems, but the approaches developed in this article can be extended to 3D systems.

I. INTRODUCTION

The Hall effect, the current induced by a magnetic field perpendicular to an initial electric field, is the most amazing and rich phenomena in condensed matter physics, in addition to the superconductivity. There are so many successful theories generated to explain the beautiful integer and fractional quantum Hall effect.[1–7] No matter which theory one may prefer, the conductivity is generally considered to be induced from the movements of charge carriers like electrons and ions; The inter-electron interferences cannot be overlooked and the resistivity comes from not only electron-lattice interactions, but also electron-electron interactions; At low temperatures close to zero Kelvin, electrons may tend to condensate, forming pair structures as suggested in BCS theory[8] and even crystal lattice structures predicted by Wigner.[9] Recent experimental evidences confirm the existences of cooper pairs,[10] the body-centered cubic (bcc) structures in 3D (three dimension),[11] and triangular lattice structures in 2D (two dimension) systems.[12]

Like other elementary particles, electrons should bear particle and wave duality, resulting in classical and quantum mechanical treatments in dealing with the rich conductivity phenomena. Both approaches show advantages and limitations, and the general consensus is that the classical or semiclassical approaches provide relatively simple and easy interpretations on Hall effect.[13] Complications come from interactions among electrons and between electrons and phonon, a tough many-body problem and hard to be resolved satisfactorily. Theories of Hall effect thus remains highly controversial and need to be refined from a brand new stand of points.

Free volume theory,[14–17] originated from molecular systems, is a most successful mean field theory in dealing with many-body problems. All different kinds of interactions among entities are factored into a single term, the free volume available in the systems. At extremely low temperatures, electrons may show high probability of particle rather than wave behaviors, if electron condensation truly happened. Therefore, one may reasonably assume that there should be a free volume unoccupied by electrons available in low temperature systems, as Hao[18] introduced previously to treat superconductivity phenomena. In this article, we borrow the free volume concept again to treat the behaviors of electrons, with the aid of the Eyring's rate process theory that successfully describes chemical reactions, diffusions, viscosity, electrical conductivity processes.[19] Attempt is made to explain the fascinating

normal and anomalous integer and fractional quantum Hall effect in a much simpler way, avoiding the complicated quantum mechanical calculations.

II. THEORY

Let's consider 2D systems under an electric field E and a perpendicular magnetic field B, a typical Hall experimental set up. In such a case, the conductivity of this system should be a tensor and may be expressed as:

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{yx} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix} \tag{1}$$

where σ_{xx} is the conductivity along the applied electric field direction, and σ_{yx} is the Hall conductance. The relationship between these two parameters may be expressed as[20,21]:

$$\begin{aligned} \sigma_{yx} &= \sigma_{xx}\omega_c\tau \\ \sigma_{xx} &= \frac{\sigma_0}{1+\omega_c^2\tau^2} \end{aligned} \tag{2}$$

where σ_0 is the zero magnetic field conductivity, ω_c is the cyclotron frequency, $\omega_c = eB/m^*$, m^* is the effective mass of electron, e is the elementary charge, B is the magnetic field strength, and τ is the relaxation time related parameter. The conductivity of a material in general without a magnetic field may be expressed as[20,21]:

$$\sigma_0 = eN_c\frac{v_d}{E} \tag{3}$$

where N_c is the number of conduction electrons, v_d is the drift velocity of conduction electrons, and E is the applied electric field strength. Combining Eq. 2 and Eq. 3 leads to:

$$\sigma_{yx} = \frac{eB\tau}{m*}\frac{\sigma_0}{[1+(\frac{eB\tau}{m*})^2]} = \frac{e^2B\tau}{Em*}\frac{N_cv_d}{[1+(\frac{eB\tau}{m*})^2]} \tag{4}$$

For a material with area A, Eq. 4 may be re-written as:

$$\sigma_{yx} = \frac{e^2BA\tau}{AEm*}\frac{N_cv_d}{[1+(\frac{eB\tau}{m*})^2]} \tag{5}$$

The term BA in the equation above is the magnetic flux and should obey the Dirac quantization condition at extremely low temperature[22]:

$$BA = n\frac{h}{e} \tag{6}$$

where n is an integer number and h is the Planck constant. Therefore Eq. 5 may be further re-written as:

$$A\sigma_{yx} = \frac{nehτ}{AEm*} \frac{N_c v_d}{[1 + (\frac{nhτ}{Am*})^2]} \tag{7}$$

where $\frac{N_c}{A}$ is the conduction electron concentration in 2D systems and has been worked out through the zero order incomplete Fermi-Dirac integral as[23,24]:

$$\frac{N_c}{A} = \frac{4\pi m^* k_B T}{h^2} \ln[1 + \exp \frac{(E_f - E_c)}{k_B T}] \tag{8}$$

where k_B is the Boltzmann constant, T is the temperature, E_f is the Fermi energy, and E_c is the conduction band edge energy. Substituting Eq. 8 into Eq. 7 leads to:

$$\sigma_{yx} = 4\pi ne \frac{k_B T}{Eh} \frac{\tau v_d}{[1 + (\frac{nhτ}{Am*})^2]} \ln[1 + \exp \frac{(E_f - E_c)}{k_B T}] \tag{9}$$

Based on Eyring's rate process theory and free volume concept, Hao[18] has recently derived conductivity equations that seem to work for many conductivity phenomena observed so far, especially for superconductivity. The obtained drift velocity of electrons, v_d, can be expressed as[18]:

$$v_d = K^+ \lambda [\exp \frac{\alpha e \lambda E}{k_B T} - \exp \frac{-(1-\alpha) e \lambda E}{k_B T}] \tag{10}$$

where K^+ is the specific "reaction" rate in any direction for undisturbed systems, λ is the distance between the initial equilibrium position and the final position, similar to mean free path of electrons, and α is directly related to the coordinate number, c_n, of an electron with the relationship, $\alpha = 1/c_n$. The term $[\exp \frac{\alpha e \lambda E}{k_B T} - \exp \frac{-(1-\alpha) e \lambda E}{k_B T}]$ in Eq.10 can be mathematically written in a more concise form:

$$[\exp \frac{\alpha e \lambda E}{k_B T} - \exp \frac{-(1-\alpha) e \lambda E}{k_B T}] = G \frac{e \lambda E}{k_B T} \tag{11}$$

where the parameter G is an exponential function. Replacing v_d with Eq.11 in Eq. 9, one may obtain:

$$\sigma_{yx} = 4\pi n \frac{e^2}{h} \frac{\tau K^+ \lambda^2 G}{1 + (\frac{nhτ}{Am*})^2} \ln[1 + \exp \frac{E_f - E_c}{k_B T}] \tag{12}$$

Now we need to determine the relaxation time parameter τ and the specific reaction rate K^+. One may consider the conduction process analogously as a rate controlled "reaction" process, adopting the same ideas from the theory of rate process proposed by Eyring.[19] As we know, there are a huge amount of electrons in the systems, assuming its effective concentration is N_{e1}. The effective concentration has a chemical name dubbed "activity"

and both two terms will be used interchangeably. Some small portion of these electrons are "activated" into the conduction band under an external electric field to participate the conduction process with effective concentration or called activity, a_c, while the big portion of electrons still remains uninvolved, with the effective concentration of N_{e2}. This process can be simply described as below:

$$N_{e1} \overset{E}{\rightleftharpoons} a_c + N_{e2}$$

Since the concentration of conduction electrons should be much smaller than the total amount of electrons, N_{e1} should be almost identical to N_{e2}. So the equilibrium constant of this process, K_c, can be written as:

$$K_c = \frac{a_c N_{e2}}{N_{e1}} = a_c \tag{13}$$

It is worth emphasizing that in the equilibrium constant calculation shown in Eq.13, the activity or effective concentrations of each "reactants" should be used, instead of the regular concentrations. This is why we need to figure out what is the effective concentrations of conduction electrons. As one may already realize, electrons move in spiral cyclo-circles under both an electric and magnetic fields that are perpendicular with each others.[25,26] So the effective free area that electrons can freely travel could be larger than the sample area A in 2D systems. Since electrons can only spiral in one direction, the perimeter instead of the linear distance should be used to calculate the area. Therefore, the active/effective area thus should be $2\pi(L/2)W = \pi A$, where L and W are the length and width of the sample, under the assumption that electrons spiral along the horizontal length direction, L. So the activity or the effective concentration of conduction electrons, should be the "effective" number of conduction electrons divided by the active free area (corresponding to the free volume in 3D). The occupied area isn't available for conduction electrons and shouldn't be counted in the activity calculation. As Hao already formulated,[18] the specific free volume, or the free volume per unit volume, of an electron, may be expressed as:

$$f_v = (1 - \frac{4}{9\pi N_v}) \tag{14}$$

where N_v is the number of valence electrons per unit cell. So for 2D systems with an effective area πA, the active free area that electrons may travel, A_a, is:

$$A_a = \pi A(1 - \frac{4}{9\pi N_v}) \tag{15}$$

Eq. 15 tells the effective free area available for conduction electrons. We still need to know the effective number of conduction electrons in order to calculate the effective concentration of conduction electrons. From quantum mechanics, a set of four quantum numbers is used to describe the unique states of electrons; Conduction electrons usually reside the outer layer of orbitals and the energy levels are mainly determined by the principal and Azimuthal quantum numbers. However, in a magnetic field the energy levels are further split into sub-levels, and the number of states, the magnetic quantum numbers m_l, has a relationship with the Azimuthal quantum number, ℓ, $m_l = 2\ell + 1$. This is why the Zeeman effect occurs. As a general rule, two electrons with opposite spin directions would reside at each state and the inner orbitals will be filled up first; the most outer orbital could be occupied by just one electron or simply unoccupied, depending on how many conduction electrons are available. So in the end the energy states that conduction electrons may occupy is $2\ell \pm 1$, and the number of electrons, n_{ce}, that may occupy these states at these three cases, may be expressed as:

$$n_{ce} = \begin{cases} 2 \times 2\ell + 1 = 4\ell + 1 & \text{odd number of conduction electrons,} \\ 2 \times (2\ell + 1) = 4\ell + 2 & \text{even number of conduction electrons,} \\ 2 \times 2\ell = 4\ell & \text{the most outer obital unoccupied.} \end{cases} \tag{16}$$

Note that electrons occupying the same states are indistinguishable and can be considered as "same" entities. If the total number of conduction electrons is N_c, then the effective number of conduction electrons, N_{ce}, with distinguishable energy states could be expressed as:

$$N_{ce} = \frac{N_c}{n_{ce}} = \begin{cases} \frac{N_c}{4\ell+1} & \text{odd number of conduction electrons,} \\ \frac{N_c}{4\ell+2} & \text{even number of conduction electrons,} \\ \frac{N_c}{4\ell} & \text{the most outer obital unoccupied.} \end{cases} \tag{17}$$

Since electrons can only move freely in the free area, the effective concentration of conduction electrons, a_c, thus should be written as:

$$a_c = \frac{N_{ce}}{A_a} = \begin{cases} \frac{N_c}{A\pi(4\ell+1)(1-\frac{4}{9\pi N_v})} & \text{odd number of conduction electrons,} \\ \frac{N_c}{A\pi(4\ell+2)(1-\frac{4}{9\pi N_v})} & \text{even number of conduction electrons,} \\ \frac{N_c}{(4\ell A\pi)(1-\frac{4}{9\pi N_v})} & \text{the most outer obital unoccupied.} \end{cases} \tag{18}$$

Once we know the activity of the conduction electrons in the systems as shown in Eq. 18, one may easily obtain the equilibrium constant of the conduction process from Eq. 13, $K_c = a_c$.

For rate controlled activation processes, according to Eyring, the relaxation time τ is equal to the average time of crossing the activation energy barrier that can be expressed as[19]:

$$\tau = \delta(\frac{2\pi m^*}{k_B T})^{1/2} \tag{19}$$

and the specific rate can be written as[19]:

$$K^+ = K_c \frac{1}{\delta}(\frac{k_B T}{2\pi m^*})^{1/2} \tag{20}$$

where δ is the length of top activation barrier. From Eq. 19, 20, and Eq. 13, one may easily obtain the term τK^+ shown below:

$$\tau K^+ = K_c = a_c \tag{21}$$

Substituting Eq. 21 into Eq. 12 leads to:

$$\sigma_{yx} = \begin{cases} \frac{4n}{4\ell+1}\frac{e^2}{h}\frac{N_c}{A(1-\frac{4}{9\pi N_v})}\frac{\lambda^2 G}{1+(\frac{nh\tau}{Am*})^2}\ln[1+\exp\frac{(E_f-E_c)}{k_B T}] & \text{odd number of conduction electrons,} \\ \frac{4n}{4\ell+2}\frac{e^2}{h}\frac{N_c}{A(1-\frac{4}{9\pi N_v})}\frac{\lambda^2 G}{1+(\frac{nh\tau}{Am*})^2}\ln[1+\exp\frac{(E_f-E_c)}{k_B T}] & \text{even number of conduction electrons,} \\ \frac{n}{\ell}\frac{e^2}{h}\frac{N_c}{A(1-\frac{4}{9\pi N_v})}\frac{\lambda^2 G}{1+(\frac{nh\tau}{Am*})^2}\ln[1+\exp\frac{(E_f-E_c)}{k_B T}] & \text{the most outer obital unoccupied.} \end{cases} \tag{22}$$

According to definition, λ^2 simply represents the free area that an electron may have in 2D geometry and should be equal to the total free area divided by the number of conduction electrons:

$$\lambda^2 = \frac{A(1-\frac{4}{9\pi N_v})}{N_c} \tag{23}$$

therefore, Eq. 22 can be re-written as:

$$\sigma_{yx} = \begin{cases} \frac{4n}{4\ell+1}\frac{e^2}{h}\frac{G}{1+(\frac{nh\tau}{Am*})^2}\ln[1+\exp\frac{(E_f-E_c)}{k_B T}] & \text{odd number of conduction electrons,} \\ \frac{4n}{4\ell+2}\frac{e^2}{h}\frac{G}{1+(\frac{nh\tau}{Am*})^2}\ln[1+\exp\frac{(E_f-E_c)}{k_B T}] & \text{even number of conduction electrons,} \\ \frac{n}{\ell}\frac{e^2}{h}\frac{G}{1+(\frac{nh\tau}{Am*})^2}\ln[1+\exp\frac{(E_f-E_c)}{k_B T}] & \text{the most outer obital unoccupied.} \end{cases} \tag{24}$$

Using Eq. 19 to replace τ and Eq. 8 to replace $m*$, one may reach:

$$\sigma_{yx} = \begin{cases} \frac{4n}{4\ell+1}\frac{e^2}{h}\frac{G\ln[1+\exp\frac{E_f-E_c}{k_BT}]}{1+\frac{8\pi^2n^2\delta^2}{AN_c}\ln[1+\exp\frac{E_f-E_c}{k_BT}]} & \text{odd number of conduction electrons,} \\ \frac{4n}{4\ell+2}\frac{e^2}{h}\frac{G\ln[1+\exp\frac{E_f-E_c}{k_BT}]}{1+\frac{8\pi^2n^2\delta^2}{AN_c}\ln[1+\exp\frac{E_f-E_c}{k_BT}]} & \text{even number of conduction electrons,} \\ \frac{n}{\ell}\frac{e^2}{h}\frac{G\ln[1+\exp\frac{E_f-E_c}{k_BT}]}{1+\frac{8\pi^2n^2\delta^2}{AN_c}\ln[1+\exp\frac{E_f-E_c}{k_BT}]} & \text{the most outer obital unoccupied.} \end{cases} \quad (25)$$

Since δ is a very small distance of nanometer scales and N_c is very large number, one may reasonably assume that the term $\frac{8\pi^2n^2\delta^2}{AN_c}\ln[1+\exp\frac{(E_f-E_c)}{k_BT}]$ is much smaller than 1, thus $1+\frac{8\pi^2n^2\delta^2}{AN_c}\ln[1+\exp\frac{(E_f-E_c)}{k_BT}] \cong 1$. This approximation should be true, as the applied magnetic field, as well as the corresponded parameter n, cannot be huge enough to overtake the term δ^2/N_c. Thus Eq. 25 can be simplified as:

$$\sigma_{yx} = \begin{cases} \frac{4n}{4\ell+1}\frac{e^2}{h}G\ln[1+\exp\frac{E_f-E_c}{k_BT}] & \text{odd number of conduction electrons,} \\ \frac{4n}{4\ell+2}\frac{e^2}{h}G\ln[1+\exp\frac{E_f-E_c}{k_BT}] & \text{even number of conduction electrons,} \\ \frac{n}{\ell}\frac{e^2}{h}G\ln[1+\exp\frac{E_f-E_c}{k_BT}] & \text{the most outer obital unoccupied.} \end{cases} \quad (26)$$

Let's evaluate the conduction equations, Eq. 26, in two extreme limits, non-degenerate limit when $E_f - E_c \ll 0$, and degenerate limit when $E_f - E_c \gg 0$, defined according to the literature.[23,24] When $E_f - E_c \ll 0$, the term $\ln[1+\exp\frac{E_f-E_c}{k_BT}]$ approaches to zero, so the conductivity becomes zero, too. In other words, Hall resistance should be huge. When $E_f - E_c \gg 0$, the situation becomes complicated and can be resolved with Wigner's idea that electrons may condensate at extremely low temperature and form Wigner crystal structures,[9,27] which was adopted in my previous publication to successfully treat the conductivity.[18] If electrons form pair structures like Cooper pairs, then the coordinate number $c_n = 1$, and thus $\alpha = 1/c_n = 1$. Eq. 11 will becomes:

$$[\exp\frac{e\lambda E}{k_BT} - 1] = G\frac{e\lambda E}{k_BT} \quad (27)$$

Note that we are dealing with the degenerate case when $E_f - E_c \gg 0$. One may reasonably assume that $\frac{E_f-E_c}{k_BT} = -\frac{e\lambda E}{k_BT} = x$. i.e., the work required for electrons to move a distance λ under an electric field E, within the framework of the Eyring's rate controlled conduction process, is equal to the energy gap. The negative sign is used due to the negative charge of electrons. Replacing the parameter G with Eq. 27 in Eq. 26 leads to:

$$\sigma_{yx} = \begin{cases} \frac{4n}{4\ell+1}\frac{e^2}{h}\frac{\exp(-x)-1}{-x}\ln[1+\exp x] & \text{odd number of conduction electrons,} \\ \frac{4n}{4\ell+2}\frac{e^2}{h}\frac{\exp(-x)-1}{-x}\ln[1+\exp x] & \text{even number of conduction electrons,} \\ \frac{n}{\ell}\frac{e^2}{h}\frac{\exp(-x)-1}{-x}\ln[1+\exp x] & \text{the most outer obital unoccupied.} \end{cases} \quad (28)$$

Eq. 28 is the Hall conductivity under the assumption that electrons form pair structures and the applied magnetic field is not unreasonably strong. Again, n and ℓ are integers, originated from the quantization of magnetic flux assumption for n, and the magnetic quantum number via Azimuthal quantum number or called the orbital angular momentum quantum number, ℓ. The term $\frac{\exp(-x)-1}{-x}\ln[1+\exp x]$ tends out to be 1 when $x \gg 0$, which is the case at extremely low temperature. Fig. 1 show the vales of this term at wide x ranges, plotted as normalized Hall conductivity $\sigma_{yx}/(\frac{n}{\ell}\frac{e^2}{h})$ vs. $x(= -e\lambda E/k_BT)$.

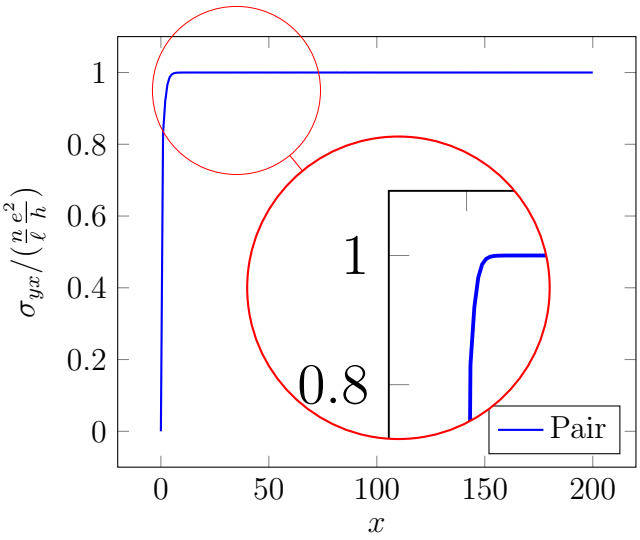

FIG. 1: Normalized Hall conductivity $\sigma_{yx}/(\frac{n}{\ell}\frac{e^2}{h})$ vs. $x(= -e\lambda E/k_BT)$ when the electrons form pair structures. The circle shows the magnified area at the left top region.

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Therefore, the Hall conductivity equations, Eq. 28, may be written as:

$$\sigma_{yx} = \begin{cases} \frac{4n}{4\ell+1}\frac{e^2}{h} & \text{odd number of conduction electrons,} \\ \frac{4n}{4\ell+2}\frac{e^2}{h} & \text{even number of conduction electrons,} \\ \frac{n}{\ell}\frac{e^2}{h} & \text{the most outer obital unoccupied.} \end{cases} \quad (29)$$

These are the Hall conductivities at three different cases in term of the number of conduction electrons. Note that both n and ℓ are integer and have physical meanings, too. Obviously, these equations are consistent with experimental observations on both integer and fractional quantum Hall effects. The Hall coefficient, ν, the prefactor in front of the term $\frac{e^2}{h}$, is experimentally found to be integers of either a general expression $\nu = p/(2p \pm 1)$ where p is another integer, or $5/2$.[5,7,28,29] With Eq. 29, one can easily verify these experimental results. The integer quantum Hall effect corresponds to the equation $\sigma_{yx} = \frac{n}{\ell}\frac{e^2}{h}$ with $\ell = 1$ and n takes integer numbers, while the fractional quantum Hall effect corresponds to the remainder two equations with $\ell = n/2 \pm 1$.

Utilizing the same procedures above, one may easily obtained the Hall conductivity equations when electrons form Wigner crystals of the coordinate number 2, 3, 4, 8, and so on, which correspond to α =1/2, 1/3, 1/4, and 1/8..., respectively. The obtained general equations can be expressed as below:

$$\sigma_{yx} = \begin{cases} \frac{4n}{4\ell+1}\frac{e^2}{h}\frac{\exp(-\alpha x)-\exp(1-\alpha)x}{-x}\ln[1+\exp x] & \text{odd number of conduction electrons,} \\ \frac{4n}{4\ell+2}\frac{e^2}{h}\frac{\exp(-\alpha x)-\exp(1-\alpha)x}{-x}\ln[1+\exp x] & \text{even number of conduction electrons,} \\ \frac{n}{\ell}\frac{e^2}{h}\frac{\exp(-\alpha x)-\exp(1-\alpha)x}{-x}\ln[1+\exp x] & \text{the most outer obital unoccupied.} \end{cases} \quad (30)$$

Since the term $\frac{\exp(-\alpha x)-\exp(1-\alpha)x}{-x}\ln[1+\exp x]$ isn't equal to 1 in these cases, thus Eq. 30 cannot predict integer or fractional quantum Hall effects. Using the simple form equation $\sigma_{yx} = \frac{n}{\ell}\frac{e^2}{h}\frac{\exp(-\alpha x)-\exp(1-\alpha)x}{-x}\ln[1+\exp x]$, the normalized conductivity $\sigma_{yx}/(\frac{n}{\ell}\frac{e^2}{h})$ is plotted against $x(= -e\lambda E/k_BT)$ in Fig.2 with three different Wigner crystal structures: honeycomb for $\alpha = 1/3$, tetrahedral for $\alpha = 1/4$, and body-centered cubic (bcc) for $\alpha = 1/8$. When many more electrons coordinate together, the predicted Hall conductivity becomes even large. Note that when temperature approaches to zero, the parameter x approaches to infinity and the Hall conductivity reaches the infinity, too; at high temperatures, x approaches to zero, so does the Hall conductivity, no matter which crystal structures are going to be formed. Comparing with Fig. 1, one may easily reaches the conclusion that the integer and fractional quantum Hall effect can only happen when electrons form pair structures. As we know, the electron pair structures provide the possibility for the materials to reach superconductivity state where the magnetic flux is always quantized. This coincidence is consistent with one of the early assumptions that the magnetic flux, the parameter BA, is quantized, as shown in Eq. 6. This is the power and beauty of treating electrical conduction

process with Eyring's rate process theory and free volume concept: many amazing phenomena like Kondo insulators and Anderson transitions etc. can be easily understood, without introducing complicated quantum mechanical treatments. The detailed discussions on how electrical conductivity even superconductivity is treated with the Eyring's rate process theory and the free volume concept, can be found in my previous article.[18] The current article simply extends the previous same approaches and/or ideas to Hall effect.

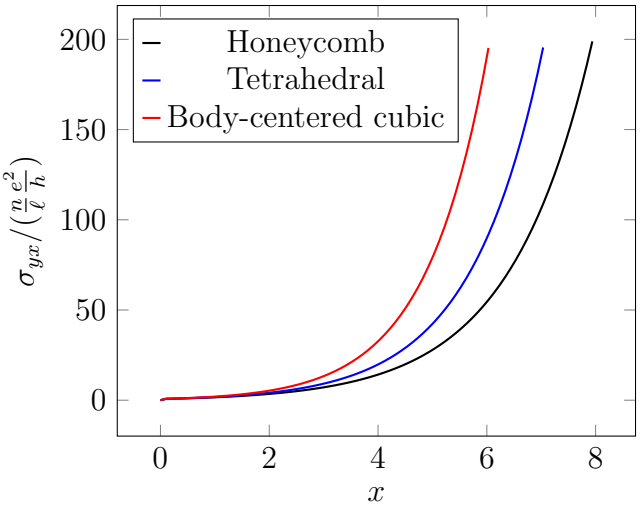

FIG. 2: Normalized Hall conductivity $\sigma_{yx}/(\frac{n}{\ell}\frac{e^2}{h})$ vs. $x(=-e\lambda E/k_BT)$ when the electrons form honeycomb, tetrahedral, and body-centered cubic structures

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Note that the derivations above only take use of two approximations: the number of conduction electrons is much smaller than the number of total electrons in the systems; the applied magnetic is not unreasonably strong, thus $1+\frac{8\pi^2n^2\delta^2}{AN_c}\ln[1+\exp\frac{E_f-E_c}{k_BT}]\approxeq 1$. The first approximation should be always true, and the second one may not hold all the time if the applied magnetic field is exceptionally strong and the temperature is extremely low. Let's now consider the extremity, when $\omega_c^2\tau^2 \gg 1$, corresponding to the very high magnetic field (large $\omega_c, \omega_c = eB/m*$) and extremely low temperature (large τ, see Eq.19). Thus Eq. 2 will becomes:

$$\sigma_{yx} = \frac{\sigma_0}{\omega_c\tau} = \frac{\sigma_0 m*}{eB\tau} \tag{31}$$

Using the same procedures for deriving Eq. 26, one may obtain the Hall conductivity

equations as:

$$\sigma_{yx} = \begin{cases} \frac{e^2}{h}\frac{N_c AG}{2\pi^2\delta^2 n(4\ell+1)} & \text{odd number of conduction electrons,} \\ \frac{e^2}{h}\frac{N_c AG}{2\pi^2\delta^2 n(4\ell+2)} & \text{even number of conduction electrons,} \\ \frac{e^2}{h}\frac{N_c AG}{8\pi^2\delta^2 n\ell} & \text{the most outer obital unoccupied.} \end{cases} \tag{32}$$

Again δ is the length of top activation barrier, which, theoretically, should vary with the applied magnetic field, becoming smaller, i.e., more easy for electrons to transport, at higher magnetic field strengths. Assuming that the parameter δ is equal to:

$$\delta^2 = \begin{cases} \frac{1}{b}\frac{N_c AG}{2\pi^2(4\ell+1)^2} & \text{odd number of conduction electrons,} \\ \frac{1}{b}\frac{N_c AG}{2\pi^2(4\ell+2)^2} & \text{even number of conduction electrons,} \\ \frac{1}{b}\frac{N_c AG}{2\pi^2\ell^2} & \text{the most outer obital unoccupied.} \end{cases} \tag{33}$$

where b is a magnetic field flux related constant, increasing with the magnetic field strength. Then Eq. 32 becomes:

$$\sigma_{yx} = \begin{cases} \frac{e^2}{h}\frac{b(4\ell+1)}{n} & \text{odd number of conduction electrons,} \\ \frac{e^2}{h}\frac{b(4\ell+2)}{n} & \text{even number of conduction electrons,} \\ \frac{e^2}{h}\frac{b\ell}{4n} & \text{the most outer obital unoccupied.} \end{cases} \tag{34}$$

the parameter b could be assumed as an integer, due to the quantization nature of magnetic flux at low temperatures. Comparing Eq. 29 with Eq. 34, one may realize immediately, no matter which approximations are taken, $\omega_c^2\tau^2 \gg 1$ or $\frac{8\pi^2 n^2\delta^2}{AN_c}\ln[1+\exp\frac{E_f-E_c}{k_BT}] \ll 1$, integer and fractional quantum Hall effect will be achieved in the end. Interestingly, Eq. 33 may indicate that the top activation barrier length δ is actually related to the number of distinguishable conduction electrons per unit area on one direction reflected with the term $(\frac{N_c}{A})^{1/2}/(4\ell+2)$, times the square area in term of the radius of the sample area (A/π), a measure on how crowded the electrons are in the conduction (electric field) direction with a factor of $(G/2b)^{1/2}$ in the unit of the radius of the sample area. This seems to be very intuitive and reasonable, as more conduction electrons on the conduction direction per unit length means more hard for electrons to move due to the crowdedness, i.e., a lengthy barrier; Obviously, it should be dependent on the applied magnetic field strength as well. For graphene the Hall conductance follows $\sigma_{yx} = \pm 4(p+1/2)\frac{e^2}{h}$.[28] This actually corresponds to the equation $\sigma_{yx} = \frac{e^2}{h}\frac{b(4\ell+2)}{n}$ with $b/n = 1$. By the definitions, the parameters b and n will increase when the applied magnetic field increases, always shifting to a same direction.

The anomalous quantum Hall effect,[5,7,29] happened without an externally applied magnetic field, could be explained with the same approach shown above. Without an externally applied magnetic field, the parameter B in Eq. 2 cannot be assumed as zero, since an "intrinsic" magnetic field may exist, resulting from the ferromagnetic properties of the material or spins of electrons. In this case, the parameter B should be very small, therefore one may reasonably assume $\omega_c^2\tau^2 \ll 1$, i.e.,$1+\omega_c^2\tau^2 \cong 1$. Eq. 2 should be re-written as:

$$\sigma_{yx} = \sigma_{xx}\omega_c\tau \tag{35}$$

$$\sigma_{xx} = \sigma_0 \tag{36}$$

Following the same procedures in deriving Eq. 29, one may easily obtain the anomalous Hall conductivity equations:

$$\sigma_{yx} = \begin{cases} \frac{n}{4\ell+1}\frac{e^2}{h} & \text{odd number of conduction electrons,} \\ \frac{n}{4\ell+2}\frac{e^2}{h} & \text{even number of conduction electrons,} \\ \frac{n}{4\ell}\frac{e^2}{h} & \text{the most outer obital unoccupied.} \end{cases} \tag{37}$$

under the assumption that $\frac{E_f-E_c}{k_BT} = -\frac{e\lambda E}{k_BT}$, which requires the electrons to form pair structures. Comparing Eq. 37 with Eq. 29, one may notice that these two equations are very similar, but the anomalous quantum Hall conductivities are 4 times smaller than the normal Hall conductivity predicted with Eq. 29, the prefactors have a difference of "4".

III. DISCUSSION

Note that Eq. 29 is obtained under the approximation $\frac{8\pi^2n^2\delta^2}{AN_c}\ln[1+\exp\frac{(E_f-E_c)}{k_BT}] \ll 1$; at such conditions, electrons are required to form pair structures in order to show the integer and fractional quantum Hall effect; Other Wigner crystal structures cannot generate integer and fractional quantum Hall effect. While in contrast, Eq. 34 is derived under the condition $\omega_c^2\tau^2 \gg 1$. There is no requirement for electrons to form pair structures and nothing restricted on how low the temperature and how high the magnetic field should be. Replacing ω_c with $\omega_c = eB/m*$, $m*$ with Eq. 8, and τ with Eq. 19, one may obtain:

$$\omega_c^2\tau^2 = \frac{8\pi^2e^2B^2\delta^2A}{N_ch^2}\ln[1+\exp\frac{(E_f-E_c)}{k_BT}] \tag{38}$$

when temperature is very high like room temperature, $T = 300K$, $e^{(E_f-E_c)/k_BT} \approx 1$, thus $\ln[1+\exp\frac{(E_f-E_c)}{k_BT}] \approx \ln 2$. The condition $\omega_c^2\tau^2 \gg 1$ may require:

$$B \gg (\frac{N_c h^2 \ln 2}{8\pi^2 e^2 \delta^2 A})^{1/2} \tag{39}$$

Taking graphene as an example, $N_c/A = 10^{16}\text{m}^{-2}$ from the literature,[30] one may reach:

$$B \gg \frac{3.89 \times 10^{-8}}{\delta} \quad \text{or,} \quad \delta \gg \frac{3.89 \times 10^{-8}}{B} \tag{40}$$

If the applied magnetic field $B = 38.9$ T, then $\delta \gg 10^{-9}$m, 1 nm, which can be achieved easily for electrons under an electric field, as electron mean free path usually is double digits nanometers.[31] This means that integer and fractional quantum Hall effect could potentially happen even at room temperature if the applied magnetic field is strong enough, which has been evidenced experimentally.[30]

It is worth mentioning that both the integer and fractional quantum Hall effect at condition $\frac{8\pi^2 n^2 \delta^2}{AN_c} \ln[1 + \exp\frac{(E_f - E_c)}{k_B T}] \ll 1$ and the anomalous quantum Hall effect at condition $\omega_c^2 \tau^2 \ll 1$, require that electrons form pair structures, where the superconductivity becomes possible. It is thus not surprising to see the anomalous quantum Hall effect is observed in magnetic topological insulators,[29] as the topological insulators potentially are superconductors.[18] Although several 2D systems like graphene mentioned earlier are found to be consistent with our predictions, our theory is more generic and unnecessary to bind with any particular systems. The Hall conductivity equations obtained in this article clearly show that the Hall conductivity is strongly related to the energy gap: the larger the gap is, the stronger the Hall conductivity effect. The first-principal calculations using the density function theory have recently demonstrated to successfully predict the large gap quantum spin Hall 2D topological insulators.[32–35] Combining our theory and the density function calculations, one may potentially discover more large gap topological insulators of strong quantum Hall effect. For example, Eq. 25 may indicate that a material of large N_c may show a strong Hall effect.

The approach developed in this article is quite different, simple, and easy comprehensive, especially without the needs of going through the complicated wave functions. Objections are expected from peoples who are familiar and stuck with quantum mechanical approach, as my approach seems to be a classic one. However, there really is no a conflict between these two approaches, as Eyring's rate process theory has the quantum mechanical origins and electronic quantum energy states are directly utilized in my derivations.

IV. CONCLUSIONS

In conclusions, Eyring's rate process theory and free volume concept is successfully employed to treat the Hall effect in 2D systems. Conductivity equations with and without an external magnetic fields are derived, and both the normal and anomalous quantum Hall effects are predicted; The Hall prefactors are revealed to be related to the magnetic flux integer n and magnetic quantum numbers m_l via the Azimuthal quantum number ℓ. The quantum Hall effect can be realized when the conduction electrons form the pair structures under the approximation $\frac{8\pi^2 n^2 \delta^2}{AN_c} \ln[1 + \exp \frac{(E_f - E_c)}{k_B T}] \ll 1$ for normal or $\omega_c^2 \tau^2 \ll 1$ for anomalous quantum hall effect; Under these two conditions the quantum Hall effect wouldn't occur if electrons form Wigner's crystal structures other than pairs. When the applied magnetic field strength is very high or the temperature is very low that satisfies $\omega_c^2 \tau^2 \gg 1$, the integer and fractional quantum Hall effect doesn't require electrons to form pair structures and could even happen at room temperature if the applied magnetic field is strong enough. Many other mystical puzzles like 5/2 filling factor can be easily understood with the equations developed in this article. Our approaches are very simple and don't involve the complicated wavefunctions, providing a new way to look at the important and interesting Hall effect. Our approach sheds impactful new physical chemistry insights on the important quantum Hall effects, implying that Hall conductivity process can be analogously treated with a common rate controlled process and electrons can be treated as particles in low temperatures. In the meanwhile, our approach broadens the application of Eyrings rate process and free volume theories.

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