On the Lorentz transformation and time dilation

Per Hokstad

Trondheim, Norway;

phokstad@gmail.com

Abstract. In the special theory of relativity there is a time dilation between two reference frames moving relative to each other at a constant speed. The Lorentz transformation provides the magnitude of this time dilation. The present work focuses on the fact that the times observed on the 'other' system will depend on the location of the clocks used for time comparisons, and we refer to positional (location specific) time. The paper points to the various 'observational principles', *i.e.* the specification of which clocks to apply, and we present a unified framework for these principles. It is argued that the total picture of the observed time dilations is more informative than the usual approach of focusing on one specific expression for time dilation, apparently being based on a somewhat arbitrary definition of simultaneity. The motivation of the paper is to challenge the current narrative regarding time dilation.

Key words: Lorentz transformation; time dilation; special theory of relativity; symmetry; length contraction; positional time.

1. Introduction

The Lorentz transformation provides the mathematical description of space-time for two reference systems moving relative to each other at constant speed; *i.e.* the situation described in the special theory of relativity (STR). This paper strives to explore the full potential of the Lorentz transformation, and thereby the interpretation of time dilation.

We specify the various expressions for the time dilation following from the Lorentz transformation. In doing so, we introduce the concept of 'observational principle'; that is, the specification of which clocks to use for the required time comparisons. A unified framework for these observational principles is given, stressing that time measurements for 'the other system' is given by the location where the time reading/comparison is carried out. Thus, we will refer to positional (location specific) time.

So rather than specifying *one* 'generic' time dilation formula – typically being based on a somewhat arbitrary definition of simultaneity – we look at the total picture of all expressions for time dilation.

Regarding *simultaneity,* we will in this paper restrict to consider events which occur at the same location *and* time. We will assume that each reference frame has a set of calibrated clocks, located at virtually any position. So in principle we can at any position compare the clocks of the two reference frames. Thus, any convention to define simultaneity across reference frames by use of light rays is not part of the considerations in the present paper.

A vast literature exists on the special theory of relativity (STR). As a basis for our discussions we will in Chapter 2 review and shortly comment on a few aspects from a small sample of these references. Next we present a list of assumptions for the further discussion. In particular we assume a strict symmetry between the two reference frames.

The various 'observational principles' based on the Lorentz transformation are thoroughly discussed in chapters 3, 4. The provided results are well-known, but the presentation is in some respect believed to be original. Next, the concluding chapters 5-6 present a common, consistent framework for all the time dilation expressions. In an Annex we includes a simple derivation of the Lorentz transformation and finally provide a discussion of length contraction. Throughout we restrict to consider *one* space co-ordinate (x).

Obviously, a substantial part of the present work presents standard results, for most readers providing trivial material. However, the main objective is not tutorial but to arrive a consistent description of time dilation, as derived from the Lorentz transformation. We arrive at views that in some respect seem to challenge the prevailing understanding of time dilation. We note that the present work is a report of an observer from outside the physical community. The considerations are essentially mathematical, but with necessity we also touch upon the physical interpretation of time dilation.

2. The Lorentz transformation

This chapter provides a background for the discussions in subsequent chapters. First we review a small sample of the literature on time dilation. Next, we specify basic assumptions, and also present the Lorentz transformation (for one space coordinate). Finally, we comment on the concept of simultaneity.

2.1 Review of some literature

The basis for the discussions is the standard theoretical experiment, with two co-ordinate systems (reference frames), K and K' moving relative to each other at speed, v. We investigate the relation between space and time parameters, (x, t) on system K and the corresponding parameters (x', t') on system K'. The relation is provided by the Lorentz transformation, (*e.g.* [1] - [4]).

A vast amount of literature exists on this topic. As a background we consider a small sample, authored by experienced scientists in the field: First two older books ('classics'), Einstein's introduction to the STR, [1], and Chapters 3 and 4 of the Feynman lectures, [2]. Further, the more recent and insightful books by Giulini, [3] and Mermin, [4], which are frequently referred on the topic. Finally we consider some web pages: that of Andrew Hamilton, [5]; Pössel, ('Einstein Online'), [6]. These references mainly address non-experts: But it is of interest to see how the main ideas of the STR are presented, and we shortly review a few aspects of these works, being relevant for the subsequent discussions.

Definition of *simultaneity* becomes crucial as clocks are moving relative to each other. It seems to be a common understanding that no unique definition of simultaneity exists *across systems* (reference frames); these definitions being based on utilizing light rays. The arguments on time dilation are frequently based on such a definition of simultaneity. This tends to introduce some asymmetry between the two systems, (see Chapter 4, below); resulting in an asymmetric solution to a symmetric situation.

The question of *symmetry* is interesting. The STR essentially describes a symmetric situation for the two systems/observers moving relative to each other. And for instance the reference [5] specifies an experiment of complete symmetry, referring to two spaceships moving relative to each other. Other references, however, are not found that explicit. Some will for instance include examples, like the 'travelling twin', *e.g.* [4], which clearly involves asymmetry. But claiming that the slower aging of the 'travelling twin' is not restricted to the acceleration/deceleration periods; this seems to be taken as an example of time dilation occurring under the conditions of the STR. The present paper, however, restrict to consider absolute symmetry.

Regarding the basic concept of *time dilation*, I firstly miss a more precise discussion of the multitude of (time) solutions offered by the Lorentz transformation. It is treated by some authors, *e.g.* in [4], but in my opinion not in sufficient depth. More generally, it is to me not completely clear that the sources are fully consistent; *i.e.* telling 'the same story' regarding time dilation. In particular, to what extent do they agree regarding the physical reality of time dilation?

So, how should we interpret the common statement: 'Moving clock goes slower'? Many authors apply the expression 'as seen' by the observer on the other reference system, indicating that it is an apparent effect, not a physical reality, without elaborating on the interpretation of 'as seen'. However, others stress that 'everything goes slower' on the 'moving system', not only the clocks; truly stating the time dilation represents a physical reality also under the conditions of STR, (no gravitation *etc.*).

Giulini [3] is relatively clear (see Section 3.3 of his book) by stating: 'Moving clocks slow down' is 'potentially misleading and should not be taken too literally'. However, he does not follow up on this and explicitly state whether time dilation is just an apparent effect without physical reality. I do not find his expression 'not to be taken too literally' to be a very strong (or precise) statement.

As pointed out *e.g.* by Pössel [6] the phenomenon of time dilation stems from the fact that clocks of the two systems have to be compared at least twice, so it cannot be the same two clocks being compared. Since movement is relative, however, an interesting question is how to decide which system (clock) is moving. Mermin [4] states that what 'moves' is decided by which clocks are chosen to be synchronized. This seems to be is in line with the views of the present paper: the procedure of clock synchronization and clock comparison decides which reference system has the time moving faster/slower. But this might also imply that time dilation does not represent a physical reality (in this situation).

In conclusion, I find the sources somewhat ambiguous regarding the very interpretation of time dilation. What is actually the message of the physical community? In what sense – and under which precise conditions – is time dilation to be considered a true physical phenomenon? In the completely symmetric situation – *e.g.* the case of two free floating spaceships moving relative to each other – one would hardly consider time dilation as a 'physical reality', even if it is an 'observational reality'. So, will the phenomenon described by the Lorentz transformation with necessity imply that time dilation is a true physical phenomenon? In other words: when do we have a 'true time dilation'.

2.2 Basic assumptions

The main focus of this paper is the Lorentz transformation, describing two reference frames, K and K' passing each other at a relative speed, v. We consider just one space co-ordinate, (x-axis), and all observations, (space, time), represent *differences,* relative to an arbitrarily chosen origin, (time = 0, space coordinate = 0), being identical for the two systems. This represents the definite starting point, from which all events are measured. So this *point of initiation* is chosen freely, but will then play a special role in the description of subsequent events. In total the discussions of the present paper will be based on the following assumptions regarding this idealized experiment:

- *Speed of light* will be measured to be constant in both directions and equal to c, independent both of the speed of the observer and speed of the light source.
- *Length contraction*. There is observed a length contraction, k_x along the x- axis of 'the other' reference frame. When we from a specific location on K observe the passing of a measure stick of length, x', (as measured on K'), then the time observed between the passing of its two endpoints, will correspond to the stick (apparently) having a length $k_x x'$; (*cf.* Annex B).
- There is a complete *symmetry* between the two co-ordinate systems, K and K'. This symmetry will include the past history of the systems; (how they came into this state of relative movement). Thus, in our idealized experiment we consider the systems to be identical in all respects.
- On both reference frames we can apply an arbitrary number of identical, synchronized clocks, located at various positions where it is required to measure time. All discussions on simultaneity 'across systems' will relate to readings of clocks being at the 'same location at the same time', *cf.* Section 2.4.
- In order to observe a time dilation, we must somehow distinguish between the two systems. Thus, we will choose the *perspective* of one of the systems (typically K), and refer to this as the *primary* system. This will simply mean that at any instant, the time on this system for any position, x is given as a constant, $t(x) \equiv t$; and consequently the observed time t' on the 'other' system will depend on the location, x. We might think of a 'primary' observer located on the primary system, making observations (of clocks) on the 'other' system. What this actually means is that the time comparisons are performed at instances when the clocks on K give identical time readings.
- Further, we apply 'Newtonian/classical' arguments for events relative to a specific system. We also assume a completely *idealized situation*: systems are 'free floating', moving without disturbance, neglecting gravitational forces; all data can be gathered precisely, without measurement errors; *etc.*

2.3 Formulation of the Lorentz transformation

The Lorentz transformation represents the fundament for our discussion of time dilation (*cf.* Annex A). We consider the 'standard situation': The reference frame K' moves relative to K with the velocity, v along the x-axis. At any instant, the position, x on K is at the same location as x' on K', and at this position time equals t on K and t' on K'. Initially at time $t = t' = 0$ the origins 0 on K and 0' on K' have the same location, ('point of initiation'). We also introduce the so-called length contraction,

$$k_x = \sqrt{1 - (\frac{v}{c})^2} \tag{1}$$

being the inverse of the so-called Lorentz factor. The Lorentz transformation for one space co-ordinate is now given by

$$x' = \frac{x - vt}{\sqrt{1-(\frac{v}{c})^2}} \tag{2}$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1-(\frac{v}{c})^2}} \tag{3}$$

So this relates simultaneous time readings, t and t' performed at identical locations x and x'.

2.4 The concept of simultaneity

Before starting to apply the Lorentz transformation we remark that, within one reference frame the concept of simultaneity does not represent any problem. We may locate stationary clocks 'all over' the system (at any location we want), synchronize these in a standard way, and simultaneity may then be confirmed by comparing clock readings.

But when we now consider events involving more than one reference system, we will restrict to consider simultaneity of events occurring at the same time *and* at the same location. So the events are actually identical (but measured in different bodies of reference). At these identical points there will be a difference in the time readings of the clock located on K and the clock on K'. However, the time readings as such are objective, and the observers on both reference frames agree on the observed times of the two clocks at the specified location.

This direct coupling to the same location *and* time is also what we find in the Lorentz transformation. Here (x, t) on K corresponds to (x', t') on K', which means that when a clock on K, located at x shows time t, then relative to K' this point has location x', and the clock in this position shows time t'. In short:

i. Observers (observational equipment) at different locations on K will all agree regarding the current time on K, (synchronization), but they will disagree on the time at K', *cf.* Lorentz transformation, (2)-(3).
ii. Similarly all observers (clocks) on K' agree regarding the time on K', but will disagree on the current time on K.
iii. An observer at K and an observer at K', which at an instant in time are at the same location - actually passing each other at the moment in question - will (usually) observe $t \neq t'$, but they will agree both on the time t at K and on the time t' at K'; these observed values being specified by the Lorentz transformation.

3 Two observational principles regarding time dilation

We now point out some direct consequences of the Lorentz transformation presented in Section 2.3. Due to the relative movement of the two reference frames, one should specify exactly how the comparisons of time and length are carried out. So now we present two different observational principles for observing time on 'the other system' moving relative to the observer. We choose the *perspective* of system K; just meaning that clock comparisons at various locations are carried out at instances when all clocks on K show the same time; (*cf.* Section 2.2).

- *Principle A*: Following a fixed clock on ‘the other system’, K'. A set of clocks and observational instruments are located along the x-axis, allowing the observer on K to follow a fixed clock on K'; thus, observing both the time t' on K' and t on K at the moment when this clock passes. In particular, considering a clock at position, $x' \equiv 0$ on K', this corresponds to specifying the location $x = vt$ at time t on K, and the Lorentz transformation, (3) directly gives the relation

$$t' = t\sqrt{1 - (\frac{v}{c})^2} \tag{4}$$

  which equals the 'standard' time dilation formula; e.g. see [1].

- *Principle B*: Observing various clocks on ‘the other system’ from a fixed location. The observer on K has one clock at a fixed location on the x-axis, and from this position makes registrations of clocks on K' as they pass along. For example choosing the location $x \equiv 0$ on K, the Lorentz transformation gives that at time t on K the observed time on K' at this location equals

$$t' = \frac{1}{\sqrt{1-(\frac{v}{c})^2}}t \tag{5}$$

In conclusion, considering observations carried out when time equals t *all over* K, both $t' = t\sqrt{1 - (\frac{v}{c})^2}$ and $t' = t/\sqrt{1 - (\frac{v}{c})^2}$ are valid results for the ‘simultaneous’ time observed on K'. The first expression being valid at position, $x = vt$, ($x' = 0$); the second for position $x = 0$, ($x' = -\frac{vt}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = -vt'$). As observed, all results being a direct consequences of the Lorentz transformation.

Of course we achieve the identical result considering the perspective of system K'. It is still the case that if we choose to follow the clock on K which at time 0 is located at position 0, we conclude that the relation (5) holds between the times on K and K'. If, however, we choose to follow the clock located at 0' on K' at time 0 we arrive at the relation (4) for describing relative time.

So, actually, it is not the perspective (location of ‘primary observer’) that matters, but the specification of which clocks are used for the comparisons. The question is: Which reference system applies a single clock, and which reference system utilizes at least two clocks for the time comparisons. Thus, assuming both observers utilize the same clocks (both on K and K') for the comparisons, it could be more informative to summarize the above results as follows:

Let A be the reference frame where there are used at least two clocks for time comparison, letting x_A and t_A be the position and time measurements on this system. Thus, there at least two clocks on A, located at $x_A = 0$ and $x_A = vt_A$, respectively.

Let B be the reference frame with just one clock, letting x_B and t_B be the length and time measurements on this system. Thus, system B has one clock located in $x_B = 0$.

Then, identical time measurements, (*i.e.* clock readings at same location, same time), will be related by the formula:

$$t_B = t_A\sqrt{1 - (\frac{v}{c})^2} \tag{6}$$

Given this observational set-up, it is essential to point out that observers on both reference frames agree on this. Thus, I find it utterly misleading in this situation to use the phrase 'as seen'[1] (by the observer on the other system); an expression used by several authors. Time readings are objective, and all observers on the location 'see' the same thing. The point is that observers (observational equipment) at different locations will disagree.

Again, the result could be summarized as follows: An observer at rest at one location, seeing clocks passing by, will see these clocks going slower than his own set of synchronized clocks. So, of course, this confirms-– in a rather narrow sense – the standard phrase: 'moving clock goes slower'. However, in such a narrow sense, I find the statement of limited interest. It can hardly follow from the choice of such an observational set-up, (which may also be interchanged at random), will represent a 'true' time dilation; *i.e.* affect the relative *time* as such on the two systems, (including aging).

In the next chapter we explore further implications of the Lorentz transformation; still providing well-known results, but within the framework suggested by the above discussion.

4 Time dilation and light rays

We now consider the utilization of light rays to provide time measurements at the reference frames.

4.1 Unidirectional flashes. A third observational principle based on light rays

In Chapter 3 we compared time measurements when the clock on one of the reference frames had a fixed location, and this clock was compared with time readings of (at least two) clocks on the other reference frame passing by. So no clock/object was moving with respect to both reference frames, and no light flashes were involved.

Now consider the result obtained by inserting $x = ct$, and thus also $x' = ct'$ in (3). Thus, at time $t = t' = 0$ there is emitted a flash of light at location $x = x' = 0$ along the positive x-axis, and we compare the times at the location of this flash at a later time, t on K. The Lorentz transformation directly gives the following relation between the clock readings at $x = ct$, and $x' = ct'$:

$$t' = \frac{1-v/c}{\sqrt{1-(\frac{v}{c})^2}}t = \frac{\sqrt{1-v/c}}{\sqrt{1+v/c}}t; \quad (x = ct) \tag{7}$$

So here we apply (at least) two clocks on both system: one at $x = 0$ and $x = ct$ on K, and similarly, one at $x' = 0$ and $x' = ct'$ at K'. We may refer to this approach for time measurement/comparison as observational principle C. So it utilizes the constancy of speed of light to give the relation between time, t' on K' and time, t on K at a specific position along the positive x-axis, (*i.e.* at locations, $x = ct$).

So eq. (7) is valid when the light ray is emitted in the positive direction ($x > 0$; *i.e.* c having the same direction as the velocity v, as seen from K). In the negative direction, ($x = - ct$) we similarly get

$$t' = \frac{1+v/c}{\sqrt{1-(\frac{v}{c})^2}}t = \frac{\sqrt{1+v/c}}{\sqrt{1-v/c}}t; \quad (x = -ct) \tag{8}$$

We refer to this result as being obtained using observational principle C*. So the difference between these two observational principles is not so much the different use of clocks, rather the direction of the light flash. Also the results (7) and (8) are of course well-known, *e.g.* see [3], [4]. Again these relations demonstrate that we may observe both $t' < t$, and $t' > t$; now depending on whether we observe a light flash having the same or the opposite direction of the relative movement, v.

[1] Here we talk about clock readings. Length contraction, on the other side, implies that observers on different systems will 'see' (*i.e.* observe) different lengths of the same object, *cf.* Annex B.

Note that the time measurements of (7)-(8) are based on unidirectional flashes of light. In Section, 4.3 we will also consider taking averages of these time measurements; thus, applying bidirectional rays. However, we first present a generalization of observational principles, A, B and C.

4.2 A generalized observational principle

We have seen that different observational principles give different relations between t and t'. The case is that when the time all over K is measured to equal t, one will at different locations on K observe different times, t' on K'. For observational principle B we choose the fixed position, $x = 0$, which implies $x' = -\frac{vt}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = -vt'$. Applying principle A, we have a moving position, $x = vt$, corresponding to $x' = 0$, and for principle C, we have $x = ct$ and $x' = ct'$.

Now consider a generalization of these principles A, B and C. In general let $x' = ut'$, and then specify a w so that at any instant, the position, $x' = ut'$ corresponds to (has same location as) $x = wt$. Then we find the relation between t and t' at this position. Thus, we consider an 'object' that starts out from origin, 0 = 0′ at time 0 and moves on with speed u relative to K', along the positive x'-axis. First by inserting $x = wt$ in (3) we get

$$t' = \frac{1-\frac{vw}{c^2}}{\sqrt{1-\left(\frac{v}{c}\right)^2}}t \tag{9}$$

By also inserting $x' = ut'$, in (2) we get

$$t' = \frac{\sqrt{1-\left(\frac{v}{c}\right)^2}}{1+\frac{uv}{c^2}}t \tag{10}$$

These are seen as two fundamental relations regarding time dilation. As a by-product we first use (9) and (10) to obtain the well-known result regarding the speed of the 'moving object relative to K:

$$w = \frac{x}{t} = \frac{u+v}{1+\frac{uv}{c^2}} \tag{11}$$

But more fundamentally, we directly see that the results for the observational principles A, B and C directly follows from (9) by choosing $w = 0$, $w = v$ and $w = c$, respectively. And they follow from (10) by choosing $u = -v$, $u = 0$ and $u = c$, respectively.

Thus, equations (9) and (10) provide the relation between clock readings at identical locations if the location at K is given by $x = wt$ and the location at K' is given by $x' = ut'$. So in general at least two clocks are required on both reference frames; (principles A and B representing exceptions). Further, *eq.* (9) for instance tells that, given time t on K, the time t' on K' is a linear, decreasing function in w. *Eq.* (10) similarly tells that time t' is a decreasing, nonlinear function in u.

In addition, now having these general expressions, (9), (10), we could ask which value of u and w would results in $t = t'$. It is easily derived that this equality is obtained by choosing

$$w = -u = w_0 = \frac{c^2}{v}\left(1 - \sqrt{1-\left(\frac{v}{c}\right)^2}\right) = \frac{v}{1+\sqrt{1-\left(\frac{v}{c}\right)^2}} \tag{12}$$

This means that if we consistently consider the positions where simultaneously $x = w_0t$ and $x' = -w_0t'$, then no time dilation will be observed in these positions. Thus, at such locations we have $t' = t$ and $x' = -x$, providing a nice symmetry. Note that we also obtain (12) directly from the Lorentz transformation, by requiring $t = t'$.

We add this choice, $x = w_0t$ as an observational principle D. As stated, this gives $t' = t$ and will in all respects maintain the symmetry between the two reference systems.

Again we note that the above approach is based on applying direct clock comparisons 'on location'. Actually, it now seems possible to define simultaneity for events at moving reference frames without the use of light rays. At location $x = w_0t$ on K and $x' = -w_0t'$ on K' there is actually a perfect simultaneity across the reference systems. The two clocks at this position show same time, $t = t'$; further, all clocks on K show the same time, t; and all clocks on K' show the same time t'. So the described symmetry of the situation might actually suggest $t = t'$ as a candidate for defining a kind of time simultaneity.

4.3 Bidirectional rays. The atomic clock

Now return to the investigation of time determined by light rays, representing a common approach for investigating time dilation. Section 4.1 presented the approach of light flashes emitted along the *x*-axis. However, it was observed that the result depended on the direction of the flashes. So when comparing time measurements on K and K' it is common to consider a ‘round trip’; *i.e.* a flash going from one location, then being reflected, and finally returning to the ‘same’ location. Such a light flash can also be seen to represent an atomic clock, the time of one round trip representing the time unit.

So let a ray be emitted from the origin, 0 = 0' at time $t = t' = 0$. Further, the ray is reflected back in opposite direction simultaneously at locations, say D and D′ at time t_1 and t_1', measured in K and K', respectively. Finally they return (simultaneously) to the location of emission at times t_2 and t_2', respectively. Obviously the point of return must be specified, as location 0 no longer coincides with 0'.

If the flash starts in the positive direction, eq. (7) is valid until the reflection occurs, thereafter eq. (8). Thus, we have

$$t_1' = \frac{\sqrt{1-\frac{v}{c}}}{\sqrt{1+\frac{v}{c}}}t_1 = \frac{1-v/c}{\sqrt{1-(\frac{v}{c})^2}}t_1 \tag{13a}$$

$$t_2' - t_1' = \frac{\sqrt{1+\frac{v}{c}}}{\sqrt{1-\frac{v}{c}}}(t_2 - t_1) = \frac{1+v/c}{\sqrt{1-(\frac{v}{c})^2}}(t_2 - t_1) \tag{13b}$$

So, to decide on the time dilation it is actually common to compare the times of return (total time of a round trip), that is t_2' and t_2. But at this point, we have to decide on the ‘point of return'. Is it 0 (located on K or 0' located of K'. At the time of emission these points where located at the same place. But by the return they have moved relative to each other. If we choose 0 as the point of return, the distance in negative direction becomes longer than if we chose 0', and so the weighting of equations (7) and (8); *i.e.* (13a) and (13b) will differ.

First, if we follow the ray on K', (return to 0′). Then the light will on K' passes the distance x' in both directions, and obviously $t_2' - t_1' = t_1'$. Then equations (13a), (13b) directly give

$$t_2 = (t_2 - t_1) + t_1 = (\frac{\sqrt{1-v/c}}{\sqrt{1+v/c}} + \frac{\sqrt{1+v/c}}{\sqrt{1-v/c}})\, t_1'$$

By also using $t_2' = 2t_1'$ it directly follows that

$$t_2 = \frac{1}{\sqrt{1-(\frac{v}{c})^2}}t_2' \tag{14}$$

This means that if we consider times of return only, then the relation between time measured on K', (t') and time measured on K, (t) is given by, (14), being equivalent to (4). Now using subscript AvL to indicate Average Low, we will in this case get

$$t'_{AvL} = t\sqrt{1-(\frac{v}{c})^2} \tag{15}$$

Secondly, if we choose to follow the ray on K, that is return to 0, we have $t_2 - t_1 = t_1$, and (13a) and (13b) directly give

$$t_2{}' = \frac{1}{\sqrt{1-(\frac{v}{c})^2}}\, t_2 \tag{16}$$

So rather than (15) and (4) we now get the 'opposite' result, (16), being identical to (5). Letting subscript AvH indicate Average High we can write this as:

$$t'_{AvH} = t/\sqrt{1-\left(\frac{v}{c}\right)^2} \tag{17}$$

In conclusion, we have that the two formulas (4) and (5) of Chapter 3 can also be interpreted as the times for a round trip, *i.e.* time measurement when we restrict to record the time to the instants of the return of 'round trip' flashes.

The two solutions (15) and (17) demonstrate the problem of utilizing a specific definition of simultaneity (using light rays) to derive time dilation. The convention of current literature leads to (15) as *the* time dilation formula, even if one seems to agree that the definition of simultaneity is rather arbitrarily chosen.

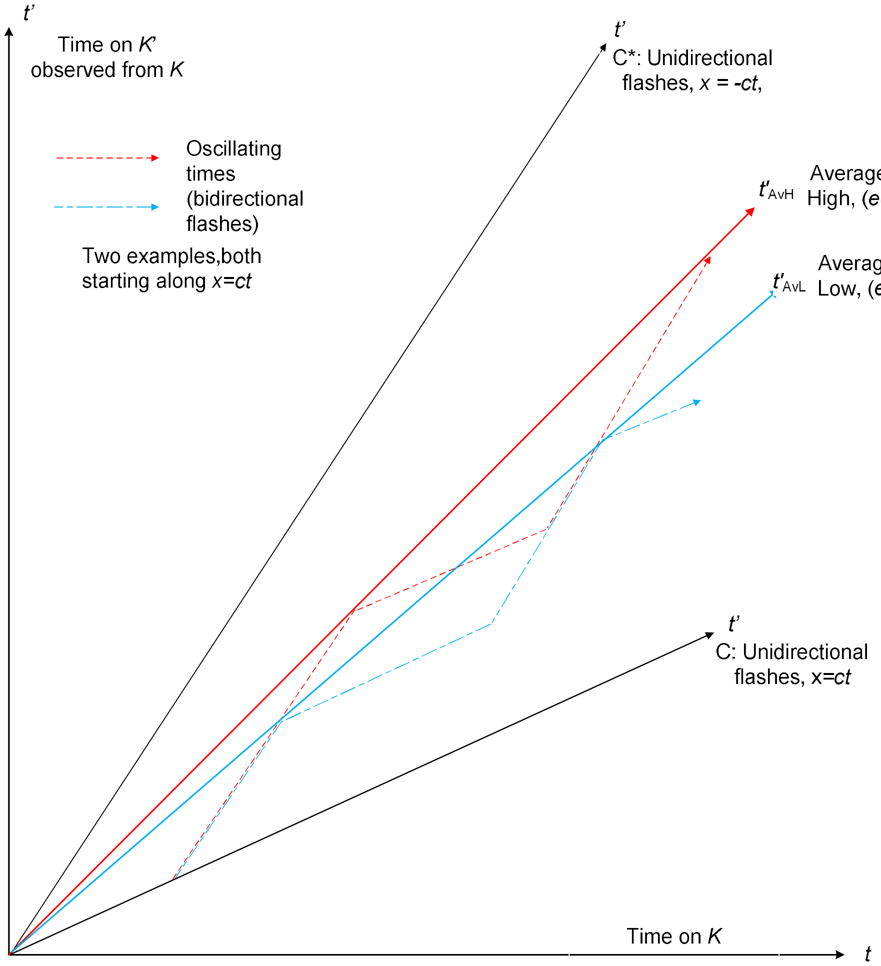

Figure 1 Illustration of time readings, utilizing light rays for time comparison.

Figure 1 provides an illustration. It gives the relation between t' and t when using unidirectional light rays, *i.e.* principle C (*eq*. (7)) and principle C* (*eq*. (8)); *cf.* the solid lines. The dotted lines represent examples of time for bidirectional (reflected) light flashes. These are alternately parallel with the C and C* line, and are here referred to as oscillating times. By considering only the points of return, we arrive at the time measurements $t'_{AvL} = t\sqrt{1-(\frac{v}{c})^2}$ and $t'_{AvH} = t/\sqrt{1-\left(\frac{v}{c}\right)^2}$, respectively. Which result is obtained will depend on whether we follow a light ray returning to *K'*, or a light ray returning to *K*.

So in case we do not restrict to consider the points of return, we get the oscillating time measurements, where *eqs*. (13a) and (13b) apply alternatingly; see dotted lines in figure. Note that we in the figure have exaggerated the length of the flashes before reflection. Usually the round trips are assumed very short, and the oscillating times are oscillating very closely around t'_{AvL}, (or possibly t'_{AvH}). The figure does, however, illustrate that having these long round trips is indeed a possibility.

This oscillating time is the more general time readings based on light flashes. It also depends on the length of the reflection distance, L. When $L \rightarrow 0$, we get the average time(s), (15), (17), and when $L \rightarrow \pm \infty$ we get the directional times, (*i.e.* principles C and C*, respectively).

We should, however, realize that applying light flashes is just one approach for obtaining expressions for time dilation, (applying a specific type of clocks). As seen in Chapter 3, we can in general just refer to the clock readings, as given by the relevant version of the Lorentz transformation.

For completeness, we also mention another possible approach for use of bidirectional rays. We could have an observer located at $x = 0$, also having two observers (equipment) located along the x-axis; one at location, x, and another at location $-x$. Emitting a ray in both directions from $x = 0$, the equipment at both positions observe the clock reading on K' when the ray arrives (after time $t = x/c$). These clock readings are given by (7) and (8), respectively, giving the arithmetic mean, t'_{AvH}, and geometric mean, t, (neither being equal to the standard time dilation expression); again demonstrating that it is hard to point to one specific time dilation formula.

5 Summing up on time dilation. Discussion.

We now sum up the main findings of the previous chapters regarding time dilation.

5.1 Time dilation and observational principles

A number of equations have been obtained, all expressing the time dilation between the two systems K and K'. This could seem confusing, but the different results have been related to various observational principles, hopefully contributing to a clearer picture. Table 1 summarizes these principles, providing reference to the corresponding equation relating t' and t. In particular it points out which expression for x (x') we insert in the Lorentz transformation in order to obtain this particular result for t'.

Table 1 Review of observational principles; perspective of K; (initially, $t = t' = 0$ at $x = 0$)

Principle	Expression for t'	Description
A	*Eq*. (4)	Observation on K follows fixed point on K'; $x' = 0$ and $x=vt$
B	*Eq*. (5)	Observation on K from fixed location; $x = 0$
C	*Eq*. (7)	Observation follows light ray on both K and K'; $x = ct$ and $x' = ct'$
C*	*Eq*. (8)	Observation follows light ray; $x = - ct$ and $x' = - ct'$
D	$t' = t$	Observation on K obtained at $x = w_0 t$ and $x' = -w_0 t'$; w_0 given by *eq*. (12)

So again we stress that the Lorentz transformation does not provide one unique answer regarding the time dilation; it rather provides a multitude of answers. Also recall that when we refer to time, we actually refer to time differences, that is the time elapsed since origins $x = 0$ and $x' = 0'$ were at the same location at time $t = t' = 0$. Further, all clocks on K (distributed along the x-axis) are synchronized, and all show the same time, t on K. The same hold for the clocks on K'. However, the time t' observed on K' will depend on the observational principle, *i.e.* the location at time, t, where the observation is carried out; (and of course vice versa).

The various results, summarized in Table 1, are also illustrated in Figure 2; giving the observed time, t' at various positions on K, when time on K equals t. The figure demonstrates that the observational principles A, B, C, C* and D are just special cases of a general 'principle', saying that at time t we observe the time t' on K' from position $x = wt$. The possible values of w are ranging from $-c$ to c.

The principles C and C*, based on light rays, represent the extremes, providing the total range of t' values, given as $(\frac{\sqrt{1-v/c}}{\sqrt{1+v/c}}t,\ \frac{\sqrt{1+v/c}}{\sqrt{1-v/c}}t)$. Observational principles A and B represent a considerable narrowing of the range of t' values: $(t\sqrt{1-(\frac{v}{c})^2}, t/\sqrt{1-\left(\frac{v}{c}\right)^2})$. We note that principle A, which provides the lower limit of this interval, represents the 'standard' time dilation formula. However, it is worth mentioning that the geometric mean of the endpoints of both these two intervals equals t. This result, $t = t'$, is also obtained by principle D.

Figure 2 summarizes all possible observational principles*, i.e.* gives all possible t'-values that can be observed, when time on K is equal to t. Actually, this is just a presentation of eq. (3), giving t' as a linear, decreasing function of x, when t is fixed. It presents a total picture of the relation between t and t', as obtained by the Lorentz transformation, without utilizing *e.g.* any definition of simultaneity. From this total picture, it is far from obvious that observational principle A gives a more 'natural' time dilation formula than for instance principles B or D.

An essential feature is that time, t' is uniquely given by the position, x on K where it is observed. At time t a comparisons of t and t' can be carried out at any position, satisfying $-ct < x < ct$. Each t'-value is the one directly being observed at the specified position, x on K, and we might thus refer to *positional* (*i.e.* location specific) time. The figure directly illustrates that at positions $x > w_0t$ we observe time, $t' < t$; while when $x < w_0t$, then we observe time, $t' > t$ on K'. Thus, the dilation factor is indeed varying all along the x-axis, taking values both >1 and <1.

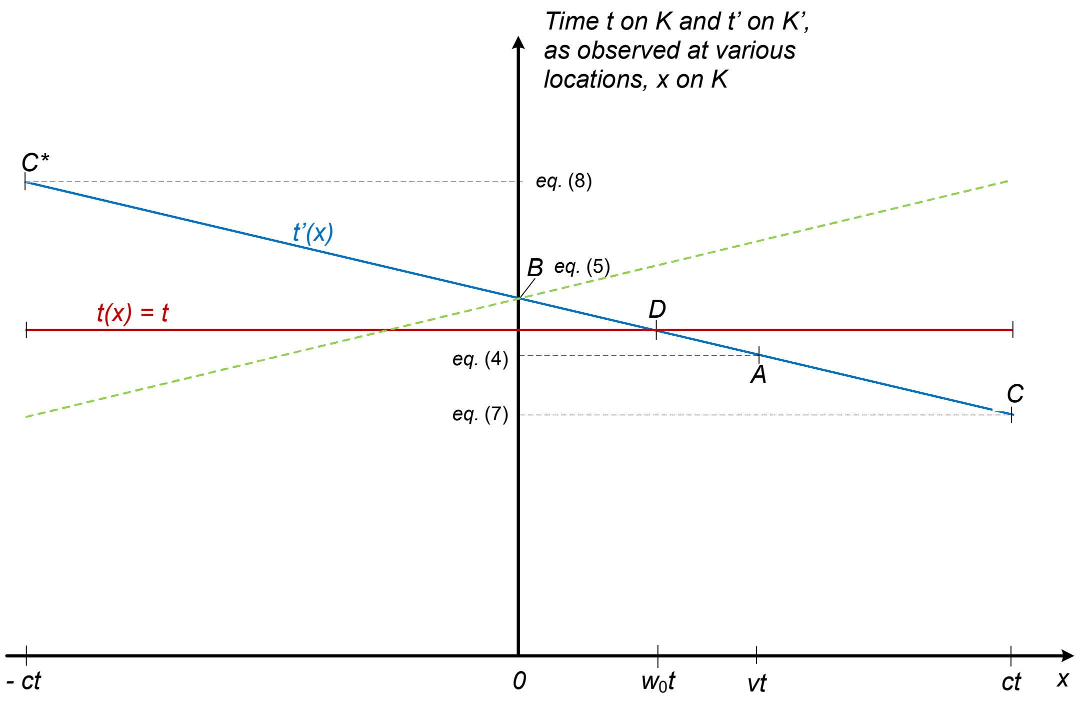

Figure 2 Positional time: Time, $t' = t'(x)$, (blue) on K' at different locations on K, when the time on K equals t, (red). Various observational principles specified along the line, $t'(x)$. The dotted green line gives values of, $t' = t'(x)$, when v is replaced by $-v$.

As a further illustration we have in Figure 2 also included a dotted green line, representing t', as measured from position x on K, on a system K', moving at speed $-v$. We see that the two lines corresponding to v and $-v$, respectively, forms a 'bow tie' with a knot in observational principle B, *i.e.* it is 'shifted upwards', relative to the line, t, with a factor $1/\sqrt{1-\left(\frac{v}{c}\right)^2}$. If – from this figure – we should suggest an 'overall time dilation' for the 'moving system, then this factor could seem quite appropriate; which would actually tell that 'moving clock goes faster'!

Next a small comment regarding the notation: We could introduce $t_v(x)$ to represent t' as given by eq. (3). Then $t_0(x) = t$ represents the time on K, (red line), and the blue line and the dotted green line represent $t_v(x)$ and $t_{-v}(x)$, respectively.

We consider the relations presented in Figure 2 to be rather fundamental for the interpretation of relative time. Being a direct consequence of the Lorentz transformation, it is of course well-known; being pointed out *e.g.* in Feynmann, [2], *p* 175. Further, Mermin [4] gives a thorough discussion on this relation, focusing on how the exact expression for $t_2' - t_1'$ depends on $x_2 - x_1$. In spite of this, also these authors seem to accept the common formulation that the time dilation is basically given by equation (4); representing a rather narrow description of the phenomenon of time dilation.

An alternative illustration of the various relations between t and t' is given in Figure 3. The lines of this figure present the relation between t and t', given specific observational principles; that is, the time comparisons are carried out at positions satisfying $x = wt$, with w being constant; thus, the oscillating times of Figure 1 are not included.

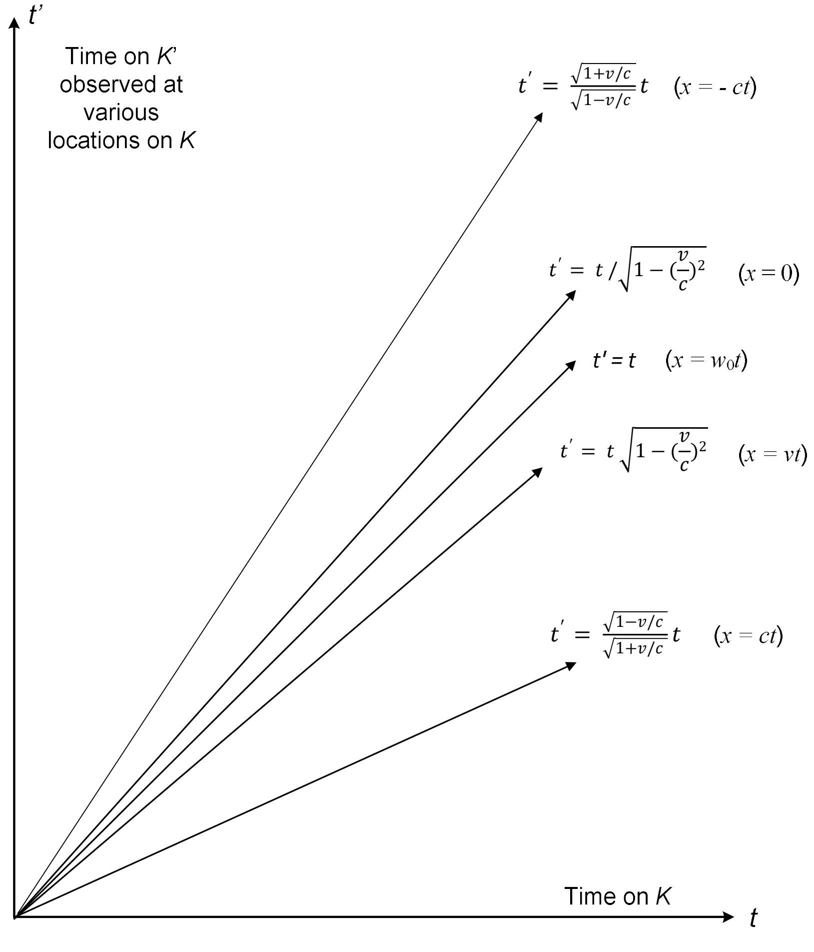

Figure 3 Values for time t' on K' as a function of time t on K. Various observational principles, (*i.e.* locations).

5.2 Observing time by use of bidirectional light rays

As stated, Figure 2 presents the total picture regarding time dilation. It tells the reading of a clock on K' at a given position on K, and given that time on K equals t. But as discussed in Section 4.3, it is in the literature also rather common to consider light flashes being reflected, and then comparing the time at the instances of return. At these instances one gets 'average times', see *eqs*. (15) and (17). These average times are here denoted t_{AvL}' (Average Low) and t_{AvH}' (Average High), respectively, and both represent averages of observational principles C and C*; also see Figure 1.

However, this actually just corresponds to specifying the type of clocks to be used in the time comparisons. And these clocks - constructed by light rays performing roundtrips – are assumed to have fixed location. Thus, we actually apply observational principle A, (*eq*. (4); $x = vt$), and principle B, (*eq*. (5); $x = 0$), respectively. In this sense the results of Section 4.3 are special cases of the general result given in Figure 2, and provide no essential new information.

However, if we do not restrict to define time (clock reading) by the instants of flash return, we could in principle also define an 'oscillating time'. These 'oscillating' lines of Figure 1 present generalizations not included in Figure 3. Thus, as Figure 2 gives the complete picture regarding time comparisons at time t, Figure 3 only presents those cases where $x = wt$; and not the situations where w is changing with time, (as in the 'oscillating' case illustrated in Figure 1).

6 Conclusions

The main objective of this paper is to provide an overview and hopefully a deeper understanding of the time dilation occurring between two reference frame moving relative to each other under the conditions of the special theory of relativity (STR); including a strict symmetry between the two reference systems. The richness of the Lorentz transformation is utilized to explore the phenomenon of time dilation, and a number of –essentially well-known – results are presented within a structured framework.

The overall conclusion is that there is a multitude of time dilation formulas to be obtained by a systematic use of the Lorentz transformation. In particular, time on K' - as observed from K - can certainly be seen to go slower than time on K itself. But – by following the Lorentz transformation- it is equally certain that it can also be seen to go faster. As the situation is here assumed to be completely symmetric, the results are (of course) also completely symmetric.

We introduce the concept of observational principle to determine the time t' on (a clock at) K' corresponding to time t on K. In particular, the two relations $t' = t\sqrt{1 - (\frac{v}{c})^2}$ and $t' = t/\sqrt{1 - \left(\frac{v}{c}\right)^2}$ appear, corresponding to what we here denote observational principle A and B, respectively. So the choice of clocks used for time comparisons is crucial.

This leads to the formulation of a framework for all possible observational principles, and the concept of positional time; *i.e*. stressing the fact that at a given time, t on K, the time t' observed on K' will depend on the position, x of the observer. We argue that the choice between these options (observational principles) could rather be seen as a practical decision on how we choose to observe and compare clock readings, not necessarily implying that 'time' on one reference frame is faster than 'time' of the other frame. We also stress that observers (observational equipment) on both reference frames agree on all the time readings (clock comparisons); as they are carried out 'on location', without utilizing any definition of simultaneity.

Thus, by considering the total picture, it is hard to claim that one specific expression for time dilation should be considered to be *the* correct one, and in that case there is no unique value for time dilation. Thus, relying on just one observational principle may not give a trustworthy result. Therefore I am sceptical to reducing the Lorentz transformation to the simplified expression, $t' = t\sqrt{1 - (\frac{v}{c})^2}$, (‘moving clock goes faster’), and argue that one should look at the total picture, taking all information into account.

So, could the conclusion be that the time on the 'other' system, K' is undetermined, since it seems unreasonable to favor one of the numerous time expression over the others? This is a possibility. The lack of one single solution relating t and t' seems to give a rather strong signal to be careful before adopting one specific time dilation expression as the 'truth'.

However, the strict symmetry assumed here, after all points at the relation $t' = t$ as the most plausible for relating the 'overall' time readings of the two systems. After all, we observed that if also the observational principle is symmetric (*i.e.* applying principle D: $x = w_0 t$ and $x' = -w_0 t'$,), then we obtain the relation $t = t'$. So deviations from this identity is caused by applying an *asymmetric observational principle*, (as everything else is symmetric!). In my view this further suggests that time dilation *under the stated symmetry conditions* does not represent a physical reality. It is rather describing a deficiency of observations when the observer is moving relative to the phenomenon in question. It further points to the important question on which departure(s) from this complete symmetry would result in a ‘true’ time dilation to occur. To my knowledge this does not seem to be fully explored.

An additional conclusion might be that an observer moving relative to the reference frame where the event occurs, is a rather unreliable observer regarding time. The different observational principles will give different results. So one should be careful to let such an observer define the phenomenon. Even if a phenomenon appears in a particular way for this observer, it does not need to be the ‘correct’ answer; one should rather realize the imperfection of an observer to comprehend in full depth a phenomenon occurring on another reference frame. This comment could be relevant when discussing ‘time dilation phenomena’, like the ‘travelling twin’ and μ-meson; *cf. e.g.* [4].

Thus, the paper presents reservations to what seems to be the prevailing view regarding time dilation (under the conditions of STR). In addition to the complete symmetry between the two reference frames, the framework suggested here, is characterized by the following features:

- We do not utilize any definition of *simultaneity* across systems. The approach restricts to explore direct comparisons of clocks being at the same location at the same time.
- We do not use the expression ‘*as seen’* (from the other reference frame). Observers on both frames will see the same time measurements; when being on the same location, *i.e*. comparing the same clocks.
- We look at the *total picture* regarding time dilation; *i.e*. the overall solutions, as given by the Lorentz transformation. Such a holistic view is suggested to give a deeper understanding.
- In particular, we specify how observed time, t' (on the 'other' system) depends on the position, x on the ‘primary’ reference frame; leading to the concept of *positional time*.
- We specify the applied *observational principle*. The approach includes a general framework for all observational principles.
- We specify the *perspective* of one of the reference frames (the ‘primary’); otherwise this perspective will be implicitly (and perhaps somewhat arbitrarily) chosen. To take the perspective of a specific reference system, means that time on this system is given as $t(x) \equiv t$, all x.

In summary, the present work utilizes the Lorentz transformation to present a narrative on time dilation, which seems to deviate somewhat from the presentations of current literature. Overall it is found that a truly symmetric situation also has a symmetric solution. This might suggest that when we discuss cases of time dilation, we should identify any asymmetry between the two reference systems, and specify the effects of this asymmetry: could it possibly result in a ‘true’ time dilation? Thus, in further discussions on the topic, it could be an interesting task to identify precise conditions - in particular departures from symmetry - which could cause time dilation to represent a physical reality also for systems at constant speed.

References

[1] Einstein, Albert, *Relativity. The Special and the General Theory*. Authorised Translation by Robert W. Lawson. Reprint with Introduction by Roger Penrose. The Folio Society, 2004.

[2] Feynman, Richard P., *Easy & Not-so-Easy Pieces*, (Chapter 3, The Special Theory of Relativity. Chapter 4: Relativistic Energy and Momentum). Reprint with Introduction by Roger Penrose. The Folio Society, 2008.

[3] Giulini, Domenico, Special Relativity, A First Encounter, Oxford University Press, 2005.

[4] Mermin. N. David, It's About Time. Understanding Einstein's Relativity. Princeton University Press. 2005.

[5] Hamilton, Andrew, Hamilton's Homepage, http://casa.colorado.edu/~ajsh/sr/sr.shtml

[6] Pössel, Markus, Special Relativity - Einstein online, http://www.einstein-online.info/elementary/specialRT

Annex A. A derivation of the Lorentz transformation

Various derivations of the Lorentz transformation exist. The following - for one space co-ordinate (x-axis) - is derived from the following three assumptions, *cf.* the three first assumptions of Section 2.2.

1. *Speed of light* will be measured to be constant in both directions and equal to c, independent both of the speed of the observer and speed of the light source.
2. *Length contraction*. There is observed a length contraction, k_x along the x- axis of 'the other' reference frame. When we from a specific location on K observe the passing of a measure stick of length, x', (as measured on K'), then the time observed between the passing of its two endpoints, will correspond to the stick (apparently) having a length $k_x x'$; *cf* Annex B below.
3. There is a complete *symmetry* between the two co-ordinate systems, K and K', and we consider the systems to be identical in all respects.

The first assumption, (speed of light being constant) is the fundamental one, as length contraction can be seen as a consequence of this.

We consider the 'standard situation': The reference frame K' moves relative to K with the velocity, v. Initially at time $t = t' = 0$, the origins $x = 0$ on K and $x' = 0'$ on K' have the same location. At any later instant, any location, x on K is positioned at the same location as x' on K', and at this position time is measured to equal t on K and t' on K'.

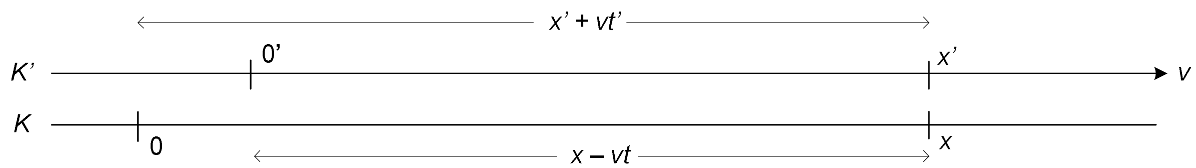

Figure A1 Identical positions, x and x' at time t on K and time t' on K', (measured at this location).

Observed on K the origin 0' has at time t moved a distance vt, (all clocks on K are synchronized). Thus $x - vt$ corresponds to the length x' on K', (Figure A1). Utilizing the assumption of contraction, we will from K observe the distance, x' to have length $k_x \cdot x'$. Thus, as measured on K:

$$x - vt = k_x \cdot x' \tag{A1}$$

In exactly the same way we have (*cf* symmetry) that observed from K':

$$x' + vt' = k_x \cdot x \tag{A2}$$

Next, we consider the case that a flash of light is emitted from the origin at time $t = t' = 0$. We now utilize the constancy of speed of light; *i.e.* if $x = ct$ then also $x' = ct'$, and vice versa. That is

$$(x = ct) \Longleftrightarrow (x' = ct') \tag{A3}$$

So as a special case we insert $x = ct$ and $x' = ct'$ in (A1) and (A2) and get, respectively

$$t' = \frac{1-\frac{v}{c}}{k_x} t \tag{A4a}$$

$$t' = \frac{k_x}{1+\frac{v}{c}} t \tag{A4b}$$

By combining these two expressions we determine the length contraction:

$$k_x = \sqrt{1 - (\frac{v}{c})^2} \tag{A5}$$

Thus, the requirement $x=ct$ iff $x'=ct'$ is sufficient to determine k_x. Next inserting the result, (A5) into (A1) and (A2), we easily obtain the Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1-(\frac{v}{c})^2}} \tag{A6}$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1-(\frac{v}{c})^2}} \tag{A7}$$

Thus, the Lorentz transformation, (A6) - (A7) in a very simple way follows from the three basic relations (A1) - (A3). We observe that the relations (A4) apply for the special case, $x = ct$, $x' = ct'$; while (A6), (A7) are general relations of simultaneous time readings, t and t' performed at identical locations x and x'.

The expressions (A6), (A7) are also valid for negative x and x'. They are, however, derived under the assumptions that K' moves in the direction of the positive x-axis, as seen from K. By changing the direction of the relative movement, one should either let v be negative (replace v by $-v$), or, alternatively, interchange x and x', and t and t' in the formulas.

Annex B. Length contraction

We include a short discussion on length contraction. The common approach is to place a rod of length x_0 along the x'-axis of K'. We locate one end in the origin, O', and the other in a point, C' along the negative x'-axis. At time $t = t' = 0$, we have that the location of the origin, O', ($x' = 0'$) on K' coincides with the origin, O, ($x = 0$) on K. The other end of the rod, located in C' with coordinate $x' = - x_0$, corresponds to a point C on K. According to *eq* (2) this coordinate equals $x = x'\sqrt{1 - (\frac{v}{c})^2} = -x_0\sqrt{1 - (\frac{v}{c})^2}$. This directly gives the distance x_0 as measured from K. We simply observe that at time $t = 0$ the distance OC equals $x = - x_0\sqrt{1 - (\frac{v}{c})^2}$. So since OC at this instant 'corresponds to' $O'C'$, and $O'C'$ has length x_0, it will of course follow that the length contraction is given by *eq* (1).

So when the length of a rod on 'the other' reference frame (K') is measured by performing simultaneous position measurements within your own reference frame, (K), then we will observe the specified length contraction. We could, however, ask whether there exist observational principles, possibly giving other results. Above the observer utilized two observational positions (O and C), in this respect applying an approach similar to observational principle A of Chapter 3. Alternatively, we could apply just one observational position (say in $x = 0$), and observe the rod as it passes along. This would correspond to observational principle B. We note that in order to apply this principle for lengths it is required that we already has measured the velocity, v between the two reference frames. (Both observers agree on this velocity, obtained by measuring the time it takes for a single point on the other system to pass a certain distance on his own reference frame.)

Now using this second principle, we will at time $t = 0$ again let O' be positioned at O; thus, $x = x' = 0$, also giving $t' = 0$. So we have the same starting position as above. Now, however, time t is given as the time when C' is positioned in O. So this t is defined by having $x = 0$ and $x' = -x_0$, resulting in $t = t' \sqrt{1 - (\frac{v}{c})^2}$ and $x' = -vt'$. Utilizing both these results, the length of $O'C'$ observed from K will equal $x = vt = vt' \sqrt{1 - (\frac{v}{c})^2} = -x' \sqrt{1 - (\frac{v}{c})^2}$. Thus, we have the same result as obtained by the first observational principle.

In conclusion, the Lorentz transformation gives that a rod, located parallel to the movement of the reference frames, will be observed to have a length contraction, irrespective of observational principle applied. In a way this differs from the phenomenon of time dilation.