Derivation of potentially important masses for physics and astrophysics by dimensional analysis

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Abstract

The Hubble constant has been added in addition to the three fundamental constants (speed of light, gravitational constant and Planck constant) used by Max Planck, for derivation of the Planck mass by dimensional analysis. As a result, a general solution is found for the mass dimension expression $m = \gamma^p m_P$, where m_P is the Planck mass, $\gamma \approx 1.23\times10^{-61}$ is a small dimensionless quantity and p is an arbitrary parameter in the interval [–1, 1]. The Planck mass $m_1 \equiv m_P$ = 2.17×10^{-8} *kg*, mass of the Hubble sphere m_2 ~ 10^{53} *kg*, minimum quantum of mass/energy m_3 = 2.68×10^{-69} *kg*, Weinberg mass m_5 = 1.08×10^{-28} *kg*, Eddington mass limit of stars M_3 = 6.6×10^{32} *kg*, mass of hypothetical quantum gravity atom M_2 = 3.8×10^{12} *kg* and some more masses potentially important for the physics and astrophysics represent particular solutions for values of p, expressed as fractions with small numerators and nominators.

Keywords: dimensional analysis, fundamental constants, mass of the Hubble sphere, Weinberg mass, Eddington mass limit

Introduction

The dimensional analysis is a conceptual tool often applied in physics and astrophysics to understand physical situations involving certain physical quantities [1-4]. It is routinely used to check the plausibility of the derived equations and computations. When it is known with which other determinative quantities a particular quantity would be connected, but the form of this connection is unknown, a dimensional equation $[q_0] \sim \prod_{i=1}^{n} [q_i]^{n_i}$ is composed for its finding. In the left side of the equation is placed the unit of this quantity q_0 with its dimensional exponent and in the right side of the equation is placed the product of units of the determinative quantities q_i raised to the unknown exponents n_i, where n is a positive integer and the exponents n_i are rational numbers. Most often, the dimensional analysis is applied in the mechanics, aerodynamics, astrophysics and other fields of the modern physics, where there are many problems with a few determinative quantities.

The Planck mass as defined by Planck [5] in terms of three fundamental constants, speed of light in vacuum (c), gravitational constant (G) and reduced Plank constant $\hbar$, is $m_P \sim \sqrt{\frac{\hbar c}{G}}$. Since the constants c, G and $\hbar$ represent three very basic aspects of the universe (i.e. the relativistic, gravitational and quantum phenomena), the Plank mass appears to a certain degree a unification of these phenomena. The Plank mass has many important theoretical ramifications in modern physics. One of them is that the energy equivalent of Planck mass $E_P = m_P c^2 \sim \sqrt{\frac{\hbar c^5}{G}} \sim 10^{19}\ GeV$ appears to be the unification energy of the fundamental interactions [6]. Additionally, the Planck mass can be derived approximately by defining it as a mass whose Compton wavelength and Schwartzchild radius are equal [7].

The Hubble constant H has been added to the set of constants c, G and $\hbar$, and thus a unique mass dimension quantity has been derived for every triad (c, G, H), (c, $\hbar$, H) and (G, $\hbar$, H) by dimensional analysis [8]. Thus, three new fundamental masses are found, i.e. $m_2 = \frac{c^3}{GH} \sim 10^{53} kg$, $m_3 = \frac{\hbar H}{c^2} = 2.68\times 10^{-69} kg$ and $m_4 = \sqrt[5]{\frac{H\hbar^3}{G^2}} = 1.43\times 10^{-20}\ kg$. The mass m_2 is identified with the mass of the Hubble sphere, m_3 with minimum quantum of mass/energy, and m_4 is conjectured to be the mass of a yet unknown super heavy particle or fundamental energetic scale. According to the recent cosmology, the Hubble constant slowly decreases with the age of the universe and there are indications that other constants, especially the gravitational constant G and fine structure constant α might also vary with comparable rate [9-11]. That is why the Hubble constant could deserve being treated on an equal level with the other three constants used by Planck.

In the present work we seek a mass dimension quantity that appears as a product of rational exponents of the four constants – c, G, $\hbar$ and H.

2. General solution of the problem for finding of a mass dimension quantity by means of fundamental constants c, G, $\hbar$ and H.

By dimensional analysis, we search for a mass dimension quantity m in the form of product of rational exponents n_1, n_2, n_3 and n_4 of the constants c, G, $\hbar$ and H:

$$m = kc^{n_1}G^{n_2}\hbar^{n_3}H^{n_4} \tag{1}$$

The exponents n_1, n_2, n_3 and n_4 are unknown quantities that can be found by matching dimensions on both sides of equation (1) and k is a dimensionless parameter (coefficient) on the order of unity.

Replacing dimensions of m, c, G, $\hbar$ and H in (1) we find the dimensional equation:

$$L^0T^0M^1 = L^{n_1+3n_2+2n_3}T^{-n_1-2n_2-n_3-n4}M^{-n_2+n_3} \tag{2}$$

From Equation (2) we find system of linear equations for unknown quantities n_1, n_2, n_3 and n_4:

$$\begin{aligned} & n_1 + 3n_2 + 2n_3 = 0 \\ & -n_1 - 2n_2 - n_3 - n_4 = 0 \\ & -n_2 + n_3 = 1 \end{aligned} \tag{3}$$

The rank of augmented matrix of the system $r = 3$ is equal to the rank of the coefficient matrix. Thus, in accordance with the Rouche-Capelli theorem the system is consistent and so must have at least one solution. The solution is unique if and only if the rank equals the number of variables. In the system (3) the number of variables $m = 4 > r = 3$, therefore the solution is not unique, but having infinitely many solutions. However, upon introducing the concept of a free parameter *p*, wherein it is accepted that $n_4 = p$, system (3) can be transform to:

$$
(4) \qquad
\begin{aligned}
& n_1 + 3n_2 + 2n_3 = 0 \\
& -n_1 - 2n_2 - n_3 = p \\
& -n_2 + n_3 = 1
\end{aligned}
$$

The determinant of system (4) is Δ = 2 ≠ 0 and the system has a solution that is dependent only upon the free parameter *p*. We find the solution of the system (4) by means of Cramer's rule:

$$(5) \qquad n_1 = (1-5p)/2, \; n_2 = (p-1)/2, \; n_3 = (p+1)/2, \; n_4 = p,$$

where *p* is the free parameter.

Replacing the solution (5) in Equation (1) we find Equation (6) for the mass *m*:

$$(6) \qquad m \sim c^{(1-5p)/2} G^{(p-1)/2} \hbar^{(p+1)/2} H^p$$

Obviously, the Equation (6) can be transformed in Equation (7):

$$(7) \qquad m \sim c^{\frac{1}{2}-\frac{5}{2}p} G^{-\frac{1}{2}+\frac{p}{2}} \hbar^{\frac{1}{2}+\frac{p}{2}} H^p = \sqrt{\frac{c\hbar}{G}} \left(\sqrt{\frac{G\hbar H^2}{c^5}} \right)^p$$

Therefore, we find the general solution (8):

$$(8) \qquad m \sim \gamma^p m_P,$$

where $\gamma = \sqrt{\frac{G\hbar H^2}{c^5}} \sim 10^{-61}$ is exceptionally small dimensionless quantity, $m_P = \sqrt{\frac{\hbar c}{G}}$ $= 2.17 \times 10^{-8}$ *kg* is the Planck mass and *p* is a free parameter.

Although the parameter *p* can take arbitrary values in the interval $(-\infty, +\infty)$, only solutions in the interval [–1, 1] could have physical meaning, because for limit values *p* = 1 and *p* = – 1 the resulting solutions are respectively the minimum measurable mass/energy in the universe $m_3 \sim \frac{\hbar H}{c^2} = 2.68 \times 10^{-69}$ kg $\sim 10^{-33}$ *eV* [12, 13] and the largest observable mass – mass of the Hubble sphere $m_2 \sim \frac{c^3}{GH} \sim 10^{53}$ *kg* [14, 15]. The exceptionally small mass m_3 seems close to the graviton mass m_G obtained by different methods [16-19].

According to Ockham's razor principle, all other things being equal, the simplest theory is the most likely to be true [20]. In science, this principle is used as a heuristic technique (discovery tool) to guide scientists in the development of theoretical models [21]. Therefore, in the following section, we consider particular solutions where the free parameter $|p| \le 1$ appears as a fraction with a small numerator and denominator, i.e.

$p = \pm\frac{1}{3}, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{5}, \pm\frac{2}{3}$. We will show that some such solutions result in mass formulas that could be interesting for contemporary particle physics and astrophysics.

3. Particular solutions potentially important for the particle physics or astrophysics where the free parameter $|p| \le 1$ appears as fractions having small numerators and denominators

From the general solution (8) we find the particular solution (9) at $p = \frac{1}{3}$:

$$m_5 = c^{-\frac{1}{3}} G^{-\frac{1}{3}} \hbar^{\frac{2}{3}} H^{\frac{1}{3}} = \sqrt[3]{\frac{H\hbar^2}{cG}} = 1.08\times10^{-28}\ kg \approx 60.8\ MeV \tag{9}$$

The Equation (9) represents the well known Weinberg mass formula [22]. The mass m_5 is about two times lighter than the pion mass $m_\pi \approx 2.48\times10^{-28} kg$. The physical meaning of the Weinberg mass was found from Sivaram [12]. He shows the Weinberg mass represents the lightest mass whose self-gravitational energy has measurable value for the time of existence of the universe $H^{-1} \sim 1.38\times10^{10}$ years.

From (8) we find the particular solution (10) at $p = -\frac{1}{3}$:

$$m_6 = c^{\frac{4}{3}} G^{-\frac{2}{3}} \hbar^{\frac{1}{3}} H^{-\frac{1}{3}} = c \cdot \sqrt[3]{\frac{c\hbar}{G^2 H}} = 4.36\times10^{12} kg \tag{10}$$

It has been shown [23] that the hypothetical 'Quantum gravity atom' built up from a central electro-neutral mass M_G around which orbits an electro-neutral particle of electron mass m_e at a distance equal to the Bohr radius $a_0 = \frac{\hbar^2}{m_e e^2}$ = 5.3×10^{-11} m, possesses gravitational potential $V = \frac{GM_G m_e}{a_0}$ equal to the electrostatic potential $V_E = \frac{e^2}{a_0}$. From here Forsythe finds the central mass M_G:

$$M_G = \frac{e^2}{Gm_e} = 3.8\times10^{12}\ kg \tag{11}$$

From the general solution (8) we find the particular solution (12) at $p = \frac{1}{2}$:

$$m_7 = c^{-\frac{3}{4}} G^{-\frac{1}{4}} \hbar^{\frac{3}{4}} H^{\frac{1}{2}} = \sqrt[4]{\frac{\hbar^3 H^2}{Gc^3}} = 7.64\times10^{-39} kg = 4.3\times10^{-3} eV \tag{12}$$

Obviously, the mass m_7, obtained from Equation (8) at $p=\frac{1}{2}$ is of the order of the neutrino rest mass [24].

From (8) we find the particular solution (13) at $p=-\frac{1}{2}$:

$$(13) \qquad m_8 = c^{\frac{7}{4}} G^{-\frac{3}{4}} \hbar^{\frac{1}{4}} H^{-\frac{1}{2}} = c \cdot \sqrt[4]{\frac{c^3 \hbar}{G^3 H^2}} = 6.18 \times 10^{22}\, kg$$

The mass m_8 appears approximately 1 % of the Earth mass $M_{\oplus}$ = 5.97×10^{24} *kg* and is close to the Moon mass M_{Moon} = 7.34×10^{22} *kg*, that appears typical satellite in the Solar system.

From the general solution (8) we find the particular solution (14) at $p=\frac{1}{4}$:

$$(14) \qquad m_9 = c^{-\frac{1}{8}} G^{-\frac{3}{8}} \hbar^{\frac{5}{8}} H^{\frac{1}{4}} = \sqrt[8]{\frac{\hbar^5 H^2}{c G^3}} = 1.29 \times 10^{-23}\, kg \ = 7.25\ TeV$$

This energy is typical for the energy of protons in Large Hadron Collider (*LHC*) and possibly is connected with mass of yet unobserved heavy particle or fundamental energetic scale.

From (8) we find the particular solution (15) at $p=-\frac{1}{4}$:

$$(15) \qquad m_{10} = c^{\frac{9}{8}} G^{-\frac{5}{8}} \hbar^{\frac{3}{8}} H^{-\frac{1}{4}} = c \cdot \sqrt[8]{\frac{c \hbar^3}{G^5 H^2}} = 3.67 \times 10^{7}\, kg$$

The mass m_{10} most probably has no reference to the fundamental physics.

From the general solution (8) we find the particular solution (16) at $p=-\frac{1}{5}$:

$$(16) \qquad m_{11} = c G^{-\frac{3}{5}} \hbar^{\frac{2}{5}} H^{-\frac{1}{5}} = c \cdot \sqrt[5]{\frac{\hbar^2}{G^3 H}} = 3.4 \times 10^{4}\, kg$$

The mass m_{11} hardly have some physical meaning.

The case $p=\frac{1}{5}$ uniquely yields the mass equation (17):

$$(17) \qquad m_4 = \sqrt[5]{\frac{H \hbar^3}{G^2}} = 1.43 \times 10^{-20}\, kg = 8.0 \times 10^{6}\ GeV$$

This mass also can't be identified but possibly could be considered a heuristic prediction of the suggested model for a super heavy unobserved particle or fundamental energetic scale intermediate for electroweak scale ~ 250 *GeV* and *GUT* scale ~ 10^{16} *GeV*.

From (8) we find the particular solution (18) at $p = \frac{2}{3}$:

$$(18) \quad m_{12} = c^{-\frac{7}{6}} G^{-\frac{1}{6}} \hbar^{\frac{5}{6}} H^{\frac{2}{3}} = \frac{1}{c} \cdot \sqrt[6]{\frac{\hbar^5 H^4}{cG}} = 5.39 \times 10^{-49} kg = 3.0 \times 10^{-13} eV$$

This mass is close to one of the seven fundamental equidistant masses found in [25], namely the mass $M_{(-1)} = 7.15 \times 10^{-49} kg$.

Finally, from the general solution (8) we find the particular solution (19) at $p = -\frac{2}{3}$:

$$(19) \quad m_{13} = c^{\frac{13}{6}} G^{-\frac{5}{6}} \hbar^{\frac{1}{6}} H^{-\frac{2}{3}} = c^2 \cdot \sqrt[6]{\frac{c\hbar}{G^5 H^4}} = 8.76 \times 10^{32} kg$$

The mass m_{13} has been identified in [25] with Eddington mass limit of the most massive stars $M_3 = 6.6\times10^{32}$ *kg*.

The above derived masses, whose free parameters are in the range $|p| \leq 1$ and appear in the general solution as fractions having small numerators and denominators, are presented in Table 1.

Table 1. Masses whose free parameters are in the range $|p| \leq 1$ and appear in the general solution as fractions having small numerators and denominators.

Parameter p	Mass corresponding to p	Identification of mass m_i
-1	$m_2 = \frac{c^3}{GH} = M = 1.76\times10^{53}$ *kg*	Mass of the Hubble sphere M_H
$-\frac{2}{3}$	$m_{13} = c^2 \cdot \sqrt[6]{\frac{c\hbar}{G^5 H^4}} = 8.76\times10^{32}$ *kg*	Eddington mass limit of stars
$-\frac{1}{2}$	$m_8 = c \cdot \sqrt[4]{\frac{c^3\hbar}{G^3 H^2}} = 6.18\times10^{22}$ *kg*	The Moon mass (Typical satellite in the Solar system)
$-\frac{1}{3}$	$m_6 = c \cdot \sqrt[3]{\frac{c\hbar}{G^2 H}} = 4.36\times10^{12}$ *kg*	Mass of 'Quantum Gravity Atom' M_G
$-\frac{1}{4}$	$m_{10} = c \cdot \sqrt[8]{\frac{c\hbar^3}{G^5 H^2}} = 3.67\times10^{7}$ *kg*	–
$-\frac{1}{5}$	$m_{11} = c \cdot \sqrt[5]{\frac{\hbar^2}{G^3 H}} = 3.40\times10^{4}$ *kg*	–
0	$m_1 = \sqrt{\frac{c\hbar}{G}} = 2.17\times10^{-8}$ *kg*	Planck mass m_P

$\frac{1}{5}$	$m_4 \sim \sqrt[5]{\frac{H\hbar^3}{G^2}}$ = 1.43×10^{-20} *kg*	Prediction for super massive unknown particle
$\frac{1}{4}$	$m_9 = \sqrt[8]{\frac{\hbar^5 H^2}{cG^3}}$ = 1.29×10^{-23} *kg*	Prediction for massive unknown particle
$\frac{1}{3}$	$m_5 = \sqrt[3]{\frac{H\hbar^2}{cG}}$ = 1.08×10^{-28} *kg*	Weinberg mass m_W
$\frac{1}{2}$	$m_7 = \sqrt[4]{\frac{\hbar^3 H^2}{Gc^3}}$ = 7.64×10^{-39} *kg*	Neutrino mass m_ν
$\frac{2}{3}$	$m_{12} = \frac{1}{c}\cdot\sqrt[6]{\frac{\hbar^5 H^4}{cG}}$ = 5.39×10^{-49} *kg*	Forsythe-Valev mass $M_{(-1)}$
1	$m_3 = \frac{\hbar H}{c^2}$ = 2.68×10^{-69} *kg*	Minimum quantum of mass/energy (Graviton mass m_G)

Probably, the general solution (8) includes additional masses interesting from the physical view point, but indefiniteness of the parameter *p* doesn't allow unambiguous finding of these masses.

4. Conclusions

The Hubble constant *H* has been added to the three fundamental constants (the speed of light in vacuum, Newtonian gravitational constant and reduced Planck constant) used from Max Planck for derivation of Planck mass by dimensional analysis.

We search by dimensional analysis a mass dimension quantity that appears a product of rational exponents of the four constants – *c*, *G*, $\hbar$ and *H*. In result, a general solution has been found of mass dimension quantity $m = \gamma^p m_P$, where $m_P = \sqrt{\frac{c\hbar}{G}}$ = 2.17×10^{-8} *kg* is the Planck mass, $\gamma = \sqrt{\frac{G\hbar H^2}{c^5}}$ = 1.23×10^{-61} is a small dimensionless quantity and *p* is an arbitrary parameter in the interval [–1, 1]. According to Ockham's razor principle, all other things being equal, the simplest theory is the most likely to be true. Therefore, we consider particular solutions where the free parameter $|p| \le 1$ appears as a fraction with a small numerator and denominator, i.e. $p = \pm\frac{1}{3}, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{5}, \pm\frac{2}{3}$.

In result, it has been found that the Planck mass $m_1 \equiv m_P = \sqrt{\frac{c\hbar}{G}}$ = 2.17×10^{-8} *kg*, mass of the Hubble sphere $m_2 \sim \frac{c^3}{GH}$ ~ 10^{53} *kg*, minimum quantum of mass/energy $m_3 \sim \frac{\hbar H}{c^2}$ = 2.68×10^{-69} *kg*, Weinberg mass $m_5 = \sqrt[3]{\frac{H\hbar^2}{cG}}$ = 1.08×10^{-28} *kg*, Eddington mass limit of stars M_3 = 6.6×10^{32} *kg*, mass of hypothetical quantum gravity atom M_2 = 3.8×10^{12} *kg* and some more masses potentially important for the physics and astrophysics represent particular solutions for values of *p*, expressed as fractions with small numerators and denominators. Likely, some of unidentified masses could have heuristic meaning for astrophysics and high energy physics.

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