Reflection of light from a moving mirror

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We demonstrate the fulfilment of the conservation laws with respect to fluxes of momentum, energy, spin, and photons when a plane circularly polarized electromagnetic wave reflects from a receding mirror at normal incidence.
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1. Introduction

The reflection of light from a moving mirror is essentially exhaustively investigated in a famous paper [1]. Nevertheless, it seems interesting to demonstrate the implementation of the conservation laws with respect to fluxes of momentum, energy, spin, and number of photons within such a reflection. The Maxwell tensor in Minkowski space [2],

$$T^{\alpha\beta} = g^{\alpha\mu}F_{\mu\lambda}F^{\lambda\beta} + g^{\alpha\beta}F_{\mu\lambda}F^{\mu\lambda}, \tag{1.1}$$

is used for the calculation of momentum and energy, and the canonical spin tensor [3,4],

$$\Upsilon^{\lambda\mu\nu} = -2A^{[\lambda}F^{\mu]\nu}, \tag{1.2}$$

is used for the calculation of spin. Here $F_{\mu\lambda}$ is the electromagnetic field tensor, and A^{λ} is the magnetic vector potential. The number of photons is calculated by means of a division of energy by $\hbar\omega$ or by means of a division of spin by $\hbar$. We confine ourselves by the normal incidence of light on a mirror and, for concreteness, we consider the case of a receding mirror. Due to the spin count, we consider the incident plane wave of circular polarization

$$\mathbf{E}_1 = E_1(\mathbf{x}+i\mathbf{y})\exp(ik_1 z - i\omega_1 t) \text{ [V/m]}, \quad \mathbf{H}_1 = -i\varepsilon_0 c\mathbf{E}_1 \text{ [A/m]}, \quad ck_1 = \omega_1 \tag{1.3}$$

and, respectively, the reflected wave

$$\mathbf{E}_2 = E_2(\mathbf{x}+i\mathbf{y})\exp(-ik_2 z - i\omega_2 t), \quad \mathbf{H}_2 = i\varepsilon_0 c\mathbf{E}_2. \quad ck_2 = \omega_2 \tag{1.4}$$

As is well known [1], the frequency ratio of the reflected and incident waves coincides with the ratio of the amplitudes of these waves and is given by the formula

$$\frac{\omega_2}{\omega_1} = \frac{E_2}{E_1} = \frac{1-\beta}{1+\beta}, \tag{1.5}$$

where $\beta = v/c$, and v is the speed of the mirror.

2. Momentum flux density, i.e. pressure $\mathscr{P}$

The wave, which impinges on the moving mirror, has the frequency related to the mirror, according to the Doppler effect [5, § 48],

$$\omega_0 = \omega_1\sqrt{\frac{1-\beta}{1+\beta}} \tag{2.1}$$

and, respectively, has the amplitude

$$E_0 = E_1\sqrt{\frac{1-\beta}{1+\beta}}. \tag{2.2}$$

We consider the mirror to be superconducting, thus the magnetic field doubles on the mirror under a zero electric field

$$\mathbf{H}_0 = 2\varepsilon_0 cE_0(\mathbf{x}+i\mathbf{y})\exp(-i\omega_0 t). \tag{2.3}$$

Therefore the pressure on the mirror is defined by the formula $\mathscr{P}_0 = \langle T^{zz}\rangle = \mu_0\langle H^2\rangle/2$ and turns out to be equal to

$$\mathcal{P}_0 = < T_0^{zz} > = \mu_0 \Re\{H_x \overline{H}_x + H_y \overline{H}_y\}/4 = 2\varepsilon_0 E_0^2 = 2\varepsilon_0 E_1^2 \frac{1-\beta}{1+\beta} \text{ [N/m}^2\text{]}. \quad (2.4)$$

In addition to the momentum flux, which gives pressure on the mirror, there is a filling of the space vacated by the moving mirror by momentum. The volume density of the filling, $G^z = < T_1^{zt} + T_2^{zt} >$, consists of two parts, belonging to the incident and to the reflected waves:

$$T_1^{zt} + T_2^{zt} = g^{zz}(F_{1zx}F_1^{xt} + F_{1zy}F_1^{yt} + F_{2zx}F_2^{xt} + F_{2zy}F_2^{yt}) \quad (2.5)$$

$$G^z = < T_1^{zt} + T_2^{zt} > = -\Re(-B_{1zx}\overline{D}_1^{xt} - B_{1zy}\overline{D}_1^{yt} - B_{2zx}\overline{D}_2^{xt} - B_{2zy}\overline{D}_2^{yt})/2$$

$$= \frac{\varepsilon_0}{c}(E_1^2 - E_2^2) = \frac{\varepsilon_0 E_1^2}{c}\left(1 - \frac{E_2^2}{E_1^2}\right) = \frac{\varepsilon_0 E_1^2 4\beta}{c(1+\beta)^2} \text{ [Ns/m}^3\text{]}. \quad (2.6)$$

This filling requires the momentum flux density $G^z v$, which we will call $\tilde{\mathcal{P}}$:

$$\tilde{\mathcal{P}} = G^z v = \frac{\varepsilon_0 E_1^2 4\beta^2}{(1+\beta)^2} \text{ [N/m}^2\text{]}. \quad (2.7)$$

The total flux density is equal to:

$$\mathcal{P} = \mathcal{P}_0 + \tilde{\mathcal{P}} = 2\varepsilon_0 E_1^2 \left[\frac{1-\beta}{1+\beta} + \frac{\varepsilon_0 E_1^2 2\beta^2}{(1+\beta)^2}\right] = 2\varepsilon_0 E_1^2 \frac{1+\beta^2}{(1+\beta)^2} \quad (2.8)$$

This total flux density is provided by the oncoming flux density $\mathcal{P} = < T_1^{zz} + T_2^{zz} >$. Really, in accordance with the formula (1.1), we have expressions such as

$$T^{zz} = g^{zz}(F_{zt}F^{tz} + F_{zx}F^{xz} + F_{zy}F^{yz} + F_{xt}F^{xt} + F_{yx}F^{yx} + F_{yt}F^{yt})/2$$

$$= -(B_{zx}H^{xz} + B_{zy}H^{yz} - E_x D^x - E_y D^y)/2, \quad (2.9)$$

$$< T^{zz} > = \mu_0(H_y^2 + H_x^2)/4 + \varepsilon_0(E_y^2 + E_x^2)/4 = \varepsilon_0 E^2 \quad (2.10)$$

for the incident or reflected waves. Thus the total momentum flux density,

$$\mathcal{P} = < T_1^{zz} + T_2^{zz} > = \varepsilon_0(E_1^2 + E_2^2) = \varepsilon_0 E_1^2\left(1 + \frac{E_2^2}{E_1^2}\right) = \varepsilon_0 E_1^2\left[1 + \frac{(1-\beta)^2}{(1+\beta)^2}\right] = 2\varepsilon_0 E_1^2 \frac{1+\beta^2}{(1+\beta)^2}, \quad (2.11)$$

coincides with expression (2.8).

3. Law of Conservation of Energy

The pressure on the mirror $\mathcal{P}_0$ (2.4) produces a work because of the movement of the mirror. The corresponding mass-energy flux density is equal to:

$$\Pi_0 = \frac{\mathcal{P}_0 v}{c^2} = \frac{2\varepsilon_0 E_1^2}{c}\frac{1-\beta}{1+\beta}\beta \left[\frac{kg}{m^2 s}\right] \quad (3.1)$$

In addition, there is a filling of the space vacated by the moving mirror by mass-energy. The volume density of this filling, $u = < T_1^{tt} + T_2^{tt} >$, consists of two parts, belonging to the incident and to the reflected waves. Taking into account formula (1.1), we have expressions such as

$$T^{tt} = g^{tt}(F_{tx}F^{xt} + F_{ty}F^{yt} + F_{tz}F^{zt} + F_{xy}F^{xy} + F_{xz}F^{xz} + F_{yz}F^{yz})/2$$

$$= (E_x D^x + E_y D^y + B_{xz}H^{xz} + B_{yz}H^{yz})/(2c^2), \quad (3.2)$$

$$< T^{tt} > = \varepsilon_0(E_x^2 + E_y^2)/(4c^2) + \mu_0(H_y^2 + H_x^2)/(4c^2) = \varepsilon_0 E^2/c^2 \text{ [kg/m}^3\text{]}. \quad (3.3)$$

for the incident or reflected waves. Thus the total volume mass-energy equals:

$$u = < T_1^{tt} + T_2^{tt} > = \varepsilon_0(E_1^2 + E_2^2)/c^2 = \frac{\varepsilon_0 E_1^2}{c^2}\left(1 + \frac{E_2^2}{E_1^2}\right) = \frac{\varepsilon_0 E_1^2}{c^2}\left[1 + \frac{(1-\beta)^2}{(1+\beta)^2}\right] = \frac{2\varepsilon_0 E_1^2}{c^2}\frac{1+\beta^2}{(1+\beta)^2}. \quad (3.4)$$

This filling requires the mass-energy flux density, which we call $\tilde{\Pi} = uv$,

$$\tilde{\Pi} = uv = \frac{2\varepsilon_0 E_1^2}{c}\frac{1+\beta^2}{(1+\beta)^2}\beta. \quad (3.5)$$

The total mass-energy flux density,

$$\Pi_0 + \tilde{\Pi} = \frac{2\varepsilon_0 E_1^2}{c}\left[\frac{1-\beta}{1+\beta} + \frac{1+\beta^2}{(1+\beta)^2}\right]\beta = \frac{4\varepsilon_0 E_1^2 \beta}{c(1+\beta)^2}\ \left[\frac{kg}{m^2 s}\right] \tag{3.6}$$

is provided by the Poynting vector $\Pi = < T_1^{tz} + T_2^{tz} >$. Really,

$$T_1^{tz} + T_2^{tz} = g^{zz}(F_{1zx}F_1^{xt} + F_{1zy}F_1^{yt} + F_{2zx}F_2^{xt} + F_{2zx}F_2^{xt}) = -(-B_{1zx}D_1^x - B_{1zy}D_1^y - B_{2zx}D_1^x - B_{2zy}D_1^y),$$

$$= \mu_0\varepsilon_0(H_{1y}E_{1x} - H_{1x}E_{1y} + H_{2y}E_{2x} - H_{2x}E_{2y}), \tag{3.7}$$

$$\Pi = < T_1^{tz} + T_2^{tz} > = \frac{\varepsilon_0}{c}(E_1^2 - E_2^2) = \frac{\varepsilon_0 E_1^2}{c}(1 - \frac{E_2^2}{E_1^2}) = \frac{\varepsilon_0 E_1^2}{c}\left[1 - \frac{(1-\beta)^2}{(1+\beta)^2}\right] = \frac{4\varepsilon_0 E_1^2 \beta}{c(1+\beta)^2}. \tag{3.8}$$

The value (3.8) coincides with (3.6).

4. Conservation of the number of photons

The volume density of photons, n, in the space, vacated by the moving mirror, is obtained by dividing the portions of the energy density (3.4) by the energy of a single photon, i.e. by $\hbar\omega_1$ or by $\hbar\omega_2$

$$n = \varepsilon_0(\frac{E_1^2}{\hbar\omega_1} + \frac{E_2^2}{\hbar\omega_2}) = \frac{\varepsilon_0 E_1^2}{\hbar\omega_1}(1 + \frac{\omega_2}{\omega_1}) = \frac{\varepsilon_0 E_1^2}{\hbar\omega_1}\left[1 + \frac{1-\beta}{1+\beta}\right] = \frac{2\varepsilon_0 E_1^2}{\hbar\omega_1(1+\beta)}\ [1/\mathrm{m}^3]. \tag{4.1}$$

Due to the motion of the mirror the number of the photons increases. This requires the photon number flux density

$$nv = \frac{2\varepsilon_0 E_1^2 v}{\hbar\omega_1(1+\beta)}\ [1/\mathrm{m}^2\mathrm{s}]. \tag{4.2}$$

This flux density is provided by the difference of Poynting vectors from formula (3.8)

$$< \frac{T_1^{tz}}{\hbar\omega_1} + \frac{T_1^{tz}}{\hbar\omega_2} > c^2 = \varepsilon_0 c(\frac{E_1^2}{\hbar\omega_1} - \frac{E_2^2}{\hbar\omega_2}) = \frac{\varepsilon_0 E_1^2 c}{\hbar\omega_1}(1 - \frac{\omega_2}{\omega_1}) = \frac{\varepsilon_0 E_1^2 c}{\hbar\omega_1}\left[1 - \frac{1-\beta}{1+\beta}\right] = \frac{2\varepsilon_0 E_1^2 v}{\hbar\omega_1(1+\beta)}. \tag{4.3}$$

Photon number flux density (4.3) coincides with flux density (4.2).

5. Conservation of spin

The number of photons can be calculated not only on the basis of wave energy, but also on the basis of wave spin. The volume density of wave spin is given by the component of the canonical spin tensor (1.2)

$$\Upsilon^{xyt} = -2A^{[x}F^{y]t} = -A_x D_y + A_y D_x\ [\mathrm{Js/m}^3], \tag{5.1}$$

and the spin flux density is given by the component

$$\Upsilon^{xyz} = -2A^{[x}F^{y]z} = A_x H_x + A_y H_y\ [\mathrm{J/m}^2]. \tag{5.2}$$

Note that the lowering of the spatial index of the vector potential is related to the change of the sign in the view of the metric signature $(+---)$.

Since for a monochromatic field $A_k = -\int E_k dt = -iE_k/\omega$, densities (5.1), (5.2) can be expressed through the electromagnetic field:

$$\Upsilon^{xyt} = (iE_x D_y - iE_y D_x)/\omega, \quad \Upsilon^{xyz} = (-iE_x H_x - iE_y H_y)/\omega. \tag{5.3}$$

In our case of reflection from a moving mirror (1.3), (1.4) volume density of the spin is equal to:

$$< \Upsilon^{xyt} > = \Re\{(iE_{1x}\overline{D}_{1y} - iE_{1y}\overline{D}_{1x})/\omega_1 + (iE_{2x}\overline{D}_{2y} - iE_{2y}\overline{D}_{2x})/\omega_2\}/2$$

$$= \varepsilon_0(\frac{E_1^2}{\omega_1} + \frac{E_2^2}{\omega_2}) = \frac{\varepsilon_0 E_1^2}{\omega_1}(1 + \frac{\omega_2}{\omega_1}) = \frac{\varepsilon_0 E_1^2}{\omega_1}\left[1 + \frac{1-\beta}{1+\beta}\right] = \frac{2\varepsilon_0 E_1^2}{\omega_1(1+\beta)}, \tag{5.4}$$

and the photon density is given through the division by $\hbar$ and coincides with the value (4.1).
The spin flux density is equal to:

$$< \Upsilon^{xyz} > = \Re\{(-iE_{1x}\overline{H}_{1x} - iE_{1y}\overline{H}_{1y})/\omega_1 + (-iE_{2x}\overline{H}_{2x} - iE_{2y}\overline{H}_{2y})/\omega_2\}/2$$

$$= \varepsilon_0 c\left(\frac{E_1^2}{\omega_1} - \frac{E_2^2}{\omega_2}\right) = \frac{\varepsilon_0 E_1^2 c}{\omega_1}\left(1 - \frac{\omega_2}{\omega_1}\right) = \frac{\varepsilon_0 E_1^2 c}{\omega_1}\left[1 - \frac{1-\beta}{1+\beta}\right] = \frac{2\varepsilon_0 E_1^2 v}{\omega_1 (1+\beta)}, \quad (5.5)$$

and the photon flux density is given through the division by $\hbar$ and coincides with the value (4.3). Naturally, the increase in the amount of spin is provided by the spin flux:

$$< \Upsilon^{xyt} > v =< \Upsilon^{xyz} > \quad (5.6)$$

6. Conclusion

The given calculations show that spin occurs to be the same natural property of a plane electromagnetic wave, as energy and momentum. Recognizing the existence of photons with momentum, energy and spin in a plane electromagnetic wave (1.3), it is strange to deny the existence of spin in such a wave, as is done in modern electrodynamics.

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See also "Note" in http://khrapkori.wmsite.ru/ftpgetfile.php?id=9&module=files

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