Infinite Product Representations for Binomial Coefficient, Pochhammer's Symbol, Newton's Binomial and Exponential Function

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"It is the spirit that quickeneth; the flesh profiteth nothing: the words that I speak unto you, they are spirit, and they are life." - John 6:63.

Abstract. In this paper, I demonstrate one infinite product for binomial coefficient, Euler's and Weierstrass's infinite product for Pochhammer's symbol, limit formula for Pochhammer's symbol, limit formula for exponential function, Euler's and Weierstrass's infinite product for Newton's binomial and exponential function.

1. Introduction

In 1729, Leonhard Euler (1707-1783) gave the infinite product expansion for gamma function [1, p. 33]

$$\Gamma(z)=\frac{1}{z}\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{z}\left(1+\frac{z}{j}\right)^{-1} \tag{1}$$

which is valid in $\mathbb{C}$, except for $z\in\{0,-1,-2,...\}$.

In 1854, Karl Weierstrass (1815-1897) gave the infinte product expansion for gamma function [1, p. 34-35]

$$\Gamma(z)=ze^{\gamma z}\prod_{j=1}^{\infty}\left(1+\frac{z}{j}\right)e^{-z/j}, \tag{2}$$

which is valid for all $\mathbb{C}$.

The binomial coefficient may be defined by the finite product [2]

$$\binom{\ell}{n}=\prod_{r=1}^{n}\left(\frac{\ell+1-r}{r}\right), \tag{3}$$

which is valid for $\ell\in\mathbb{C}-\{0,-1,-2,...\}$ and $n\in\mathbb{N}$, and [3]

$$(\ell)_n=\prod_{r=0}^{n-1}(\ell+r), \tag{4}$$

which is valid for $\ell\in\mathbb{C}$ and $n\in\mathbb{N}_{\geqslant 1}$.

The beta function may be defined by [4]

$$B(a,b)\equiv\frac{(a-1)!(b-1)!}{(a+b-1)!}, \tag{5}$$

when a,b are positive integers.

In this paper, I prove the infinite product representation for binomial coefficient given by

$$\binom{\ell}{n}=\prod_{j=1}^{\infty}\left(1+\frac{n}{j}\right)\left(1-\frac{n}{j+\ell}\right),$$

the Pochhammer's symbol is given by

$$(\ell)_n=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{n}\left(1+\frac{n}{j+\ell-1}\right)^{-1}$$

and

$$\frac{1}{(\ell)_n} = e^{\gamma n} \prod_{j=1}^{\infty} \left(1 + \frac{n}{j+\ell-1}\right) e^{-n/j},$$

the Newton's binomial is given by

$$(a+b)^n = \prod_{j=1}^{\infty} \left(1+\frac{1}{j}\right)^{2n} \left[\left(1+\frac{b}{a+jb}\right)\left(1+\frac{1}{b+j-1}\right)\right]^{-n}$$

and

$$(a+b)^{-n} = e^{2\gamma n} \prod_{j=1}^{\infty} \left[\left(1+\frac{b}{a+jb}\right)\left(1+\frac{1}{b+j-1}\right)\right]^{n} e^{-2n/j},$$

the exponential function is given by

$$e^{\mathrm{i} n \tan^{-1}\left(\frac{y}{x}\right)} = \prod_{j=1}^{\infty} \left(1+\frac{1}{j}\right)^{n} \frac{\left[\left(1+\frac{x^2}{jx^2+y^2}\right)\left(1+\frac{1}{x^2+j-1}\right)\right]^{n/2}}{\left[\left(1+\frac{x}{jx+\mathrm{i}y}\right)\left(1+\frac{1}{x+j-1}\right)\right]^{n}}$$

and

$$e^{-\mathrm{i} n \tan^{-1}\left(\frac{y}{x}\right)} = e^{\gamma n} \prod_{j=1}^{\infty} \frac{\left[\left(1+\frac{x}{jx+\mathrm{i}y}\right)\left(1+\frac{1}{x+j-1}\right)\right]^{n}}{\left[\left(1+\frac{x^2}{jx^2+y^2}\right)\left(1+\frac{1}{x^2+j-1}\right)\right]^{n/2}} e^{-n/j},$$

among other things.

2. PRELIMINARY

Lemma 1. *If $a, b \in \mathbb{R}$ and $b \neq 0$, then*

$$\frac{a}{b} = \prod_{j=1}^{\infty} \frac{(a+j-1)(b+j)}{(a+j)(b+j-1)}.$$

Proof. I well-know the identity

$$\frac{a}{b} \cdot \frac{b!(a-1)!}{[b+(a-1)+1]!} = \frac{a!(b-1)!}{[a+(b-1)+1]!}, \tag{6}$$

using the definition for beta function (4), I obtain

$$\frac{a}{b} = \frac{B(a+1,b)}{B(a,b+1)}. \tag{7}$$

On the other hand, the beta function have the following infinite product representation [5, p. 899]

$$(a+b+1)B(a+1,b+1) = \prod_{j=1}^{\infty} \frac{j(a+b+j)}{(a+j)(b+j)}, \tag{8}$$

valid for $a, b \neq -1, -2, \ldots$ Setting $a \to a-1$ and $b \to b-1$, respectively, in both members of (6), I find

$$B(a,b+1) = \frac{1}{a+b} \prod_{j=1}^{\infty} \frac{j(a+b+j-1)}{(a+j-1)(b+j)}, \tag{9}$$

and

$$B(a+1,b) = \frac{1}{a+b} \prod_{j=1}^{\infty} \frac{j(a+b+j-1)}{(a+j)(b+j-1)}. \tag{10}$$

Substituting (8) and (9) into the right hand side of (6), I get

$$\frac{a}{b} = \frac{a+b}{a+b} \prod_{j=1}^{\infty} \frac{j(a+b+j-1)(a+j-1)(b+j)}{(a+j)(b+j-1)j(a+b+j-1)}.$$

Eliminate the same terms in the numerator and denominator of the above equation, and encounter

$$\frac{a}{b} = \prod_{j=1}^{\infty} \frac{(a+j-1)(b+j)}{(a+j)(b+j-1)},$$

which is the desired result. □

3. Binomial Coefficient: the Infinite Product

3.1. Infinite Product Representation for Binomial Coefficient.

Theorem 2. *If $\ell \in \mathbb{C} - \{-1, -2, ...\}$ and $n \in \mathbb{N}$, then*

$$\binom{\ell}{n} = \prod_{j=1}^{\infty} \left(1 + \frac{n}{j}\right)\left(1 - \frac{n}{j+\ell}\right),$$

where $\binom{\ell}{n}$ denotes the binomial coefficient.

Proof. Setting $a = \ell + 1 - r$ and $b = r$ in both members of the Lemma 1, I obtain

$$\frac{\ell+1-r}{r} = \prod_{j=1}^{\infty} \frac{(\ell-r+j)(r+j)}{(\ell-r+j+1)(r+j-1)}. \tag{11}$$

Substituting (11) into the right hand side of (3), I find

$$\begin{aligned}\binom{\ell}{n} &= \prod_{r=1}^{n} \prod_{j=1}^{\infty} \frac{(\ell-r+j)(r+j)}{(\ell-r+j+1)(r+j-1)} \\ &= \prod_{j=1}^{\infty} \prod_{r=1}^{n} \frac{(\ell-r+j)(r+j)}{(\ell-r+j+1)(r+j-1)} \\ &= \prod_{j=1}^{\infty} \frac{(j+n)(j+\ell-n)}{j(j+\ell)} = \prod_{j=1}^{\infty} \left(1 + \frac{n}{j}\right)\left(1 - \frac{n}{j+\ell}\right),\end{aligned}$$

which is the desired result. □

4. Euler's and Weierstrass's Infinite Product Representation for Pochhammer's Symbol

4.1. Euler's Infinite Product Representation for Pochhammer's Symbol.

Theorem 3. *If $\ell \in \mathbb{C} - \{0, -1, -2, ...\}$ and $n \in \mathbb{N}$, then*

$$(\ell)_n = \prod_{j=1}^{\infty} \left(1 + \frac{1}{j}\right)^n \left(1 + \frac{n}{j+\ell-1}\right)^{-1},$$

where $(\ell)_n$ denotes the Pochhammer's symbol.

Proof. Setting $a = \ell + r$ and $b = 1$ in both members of the Lemma 1, I obtain

$$\ell + r = \frac{\ell+r}{1} = \prod_{j=1}^{\infty} \frac{(\ell+r+j-1)(j+1)}{(\ell+r+j)j}. \tag{12}$$

Substituting (12) into the right hand side of (4), I find

$$
\begin{aligned}
(\ell)_n &= \prod_{r=0}^{n-1}\prod_{j=1}^{\infty}\frac{(\ell+r+j-1)(j+1)}{(\ell+r+j)j}\\
&=\prod_{j=1}^{\infty}\prod_{r=0}^{n-1}\frac{(\ell+r+j-1)(j+1)}{(\ell+r+j)j}\\
&=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^n\frac{j+\ell-1}{j+\ell+n-1}=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^n\left(1+\frac{n}{j+\ell-1}\right)^{-1},
\end{aligned}
$$

which is the desired result. □

4.2. Weierstrass's Infinite Product Representation for Pochhammer's Symbol.

Theorem 4. *If $\ell\in\mathbb{C}$ and $n\in\mathbb{N}_{>0}$, then*

$$
\frac{1}{(\ell)_n}=e^{\gamma n}\prod_{j=1}^{\infty}\left(1+\frac{n}{j+\ell-1}\right)e^{-n/j},
$$

where $(\ell)_n$ denotes the Pochhammer's symbol, e^x denotes the exponential function and γ denotes the Euler-Mascheroni constant.

Proof. The inverse of the Theorem 3, give me

$$
\begin{aligned}
\frac{1}{(\ell)_n}&=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{-n}\left(1+\frac{n}{j+\ell-1}\right)\\
&=\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^n\lim_{m\to\infty}\prod_{j=1}^{m}\left(1+\frac{1}{j}\right)^{-n}\left(1+\frac{n}{j+\ell-1}\right)\\
&=\lim_{m\to\infty}\left[\left(1+\frac{1}{m}\right)^n\prod_{j=1}^{m}\left(1+\frac{1}{j}\right)^{-n}\left(1+\frac{n}{j+\ell-1}\right)\right]\\
&=\lim_{m\to\infty}\left[\prod_{j=1}^{m-1}\left(1+\frac{1}{j}\right)^{-n}\prod_{j=1}^{m}\left(1+\frac{n}{j+\ell-1}\right)\right]\\
&=\lim_{m\to\infty}\left[m^{-n}\prod_{j=1}^{m}\left(1+\frac{n}{j+\ell-1}\right)\right]\\
&=\lim_{m\to\infty}\left[\exp\left(\left(1-1+\frac{1}{2}-\frac{1}{2}+\ldots+\frac{1}{m}-\frac{1}{m}-\ln m\right)n\right)\prod_{j=1}^{m}\left(1+\frac{n}{j+\ell-1}\right)\right]\\
&=\lim_{m\to\infty}\left[\exp\left(\left(1+\frac{1}{2}+\ldots+\frac{1}{m}-\ln m\right)n\right)\exp\left(\frac{-n}{1}+\frac{-n}{2}+\ldots+\frac{-n}{m}\right)\prod_{j=1}^{m}\left(1+\frac{n}{j+\ell-1}\right)\right]\\
&=\lim_{m\to\infty}\left[\exp\left(\left(1+\frac{1}{2}+\ldots+\frac{1}{m}-\ln m\right)n\right)\prod_{j=1}^{m}\left(1+\frac{n}{j+\ell-1}\right)e^{-n/j}\right]\\
&=\lim_{m\to\infty}\left(\exp\left(\left(1+\frac{1}{2}+\ldots+\frac{1}{m}-\ln m\right)n\right)\right)\lim_{m\to\infty}\left(\prod_{j=1}^{m}\left(1+\frac{n}{j+\ell-1}\right)e^{-n/j}\right)\\
&=e^{\gamma n}\prod_{j=1}^{\infty}\left(1+\frac{n}{j+\ell-1}\right)e^{-n/j},
\end{aligned}
$$

which is the desired result. □

5. Limit Formula for Pochhammer's Symbol and Exponential Function

5.1. Limit Formula for Pochhammer's symbol.

Lemma 5. *If $\ell \in \mathbb{R}_{>0}$ and n is a non-negative integer, then*

$$(\ell)_n = \lim_{m\to\infty} \frac{m^n (\ell)_m}{(\ell+n)_m},$$

where $(\ell)_n$ denotes the Pochhammer's symbol.

Proof. Consider the Euler's infinite product representation for Pochhammer's symbol and let as follows

$$\begin{aligned}(\ell)_n &= \lim_{m\to\infty} \prod_{j=1}^{m} \left(1+\frac{1}{j}\right)^n \left(1+\frac{n}{j+\ell-1}\right)^{-1}\\ &= \lim_{m\to\infty} \frac{(m+1)^n (\ell)_m}{(\ell+n)_m} = \lim_{m\to\infty} \frac{m^n (\ell)_m}{(\ell+n)_m},\end{aligned}$$

which is the desired result. □

Theorem 6. *If $|z|<1$, then*

$${}_1F_1\left(\begin{matrix} a \\ b \end{matrix} \middle| z \right) = \lim_{m\to\infty} {}_2F_1\left(\begin{matrix} a, b+m \\ b \end{matrix} \middle| \frac{z}{m} \right),$$

where ${}_1F_1\left(\begin{matrix} a \\ b \end{matrix} \middle| z \right)$ denotes the confluent hypergeometrid function of the first kind and ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z \right)$ denotes the Gaussian (or ordinary) hypergeometric function.

Proof. The definition of hypergeometric series ${}_1F_1$ is given by

$${}_1F_1\left(\begin{matrix} a \\ b \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}, \tag{13}$$

valid for $|z|<1$. Put $\ell = b$ in Lemma 5, and substitute in (13)

$$\begin{aligned}{}_1F_1\left(\begin{matrix} a \\ b \end{matrix} \middle| z \right) &= \lim_{m\to\infty} \frac{1}{(b)_m} \sum_{n=0}^{\infty} \frac{(a)_n (b+n)_m}{m^n} \frac{z^n}{n!}\\ &= \lim_{m\to\infty} \frac{1}{(b)_m} \left[(b)_m \cdot {}_2F_1\left(\begin{matrix} a, b+m \\ b \end{matrix} \middle| \frac{z}{m} \right) \right]\\ &= \lim_{m\to\infty} {}_2F_1\left(\begin{matrix} a, b+m \\ b \end{matrix} \middle| \frac{z}{m} \right).\end{aligned} \tag{14}$$

which is desired result. □

Corollary 7. *If $z \in \mathbb{C}$, then*

$$e^z = \lim_{m\to\infty} \left(1-\frac{z}{m}\right)^{-m},$$

where e^z denotes the exponential function.

Proof. I know [6] that

$$e^z = {}_1F_1\left(\begin{matrix} 1 \\ 1 \end{matrix} \middle| z \right), \tag{15}$$

Using the Theorem 6 in the right hand side of (15), it follows that

$$e^z = \lim_{m\to\infty} {}_2F_1\left(\begin{matrix} 1, 1+m \\ 1 \end{matrix} \middle| \frac{z}{m} \right) = \lim_{m\to\infty} \left(1-\frac{z}{m}\right)^{-(m+1)} = \lim_{m\to\infty} \left(1-\frac{z}{m}\right)^{-m},$$

which is the desired result. □

6. The Pochhammer's Symbol, Gamma, Sine and Cosine at Rational Argument as Finite Product of Gamma Functions

6.1. Pochhammer's Symbol at Rational Argument.

Theorem 8. *If p and q are positive integers and $p \leqslant q$, then*

$$\left(\frac{p}{q}\right)_n = \prod_{s=1}^{q} \left(\frac{\Gamma\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s+1}{q}\right)}\right)^n \left(\frac{\Gamma\left(\frac{p}{q^2} + \frac{n+s-1}{q}\right)}{\Gamma\left(\frac{p}{q^2} + \frac{s-1}{q}\right)}\right),$$

where $(\ell)_n$ denotes the Pochhammer's symbol and $\Gamma(\ell)$ denotes the gamma function.

Proof. Consider the Euler's infinite product representation for Pochhammer's symbol, let $\ell = p/q$, with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, and encounter

$$\begin{aligned}\left(\frac{p}{q}\right)_n &= \prod_{j=1}^{\infty} \left(1 + \frac{1}{j}\right)^n \left(1 + \frac{nq}{jq + p - q}\right)^{-1} \\ &= \prod_{k=0}^{\infty} \left(1 + \frac{1}{k+1}\right)^n \left(1 + \frac{nq}{(k+1)q + p - q}\right)^{-1} \\ &= \prod_{k=0}^{\infty} \left(1 + \frac{1}{k+1}\right)^n \left(1 + \frac{nq}{kq + p}\right)^{-1}.\end{aligned}$$

Now, notice that for any $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, there exists unique $c, d \in \mathbb{Z}$, such that $a = bc + d$ and $0 \leqslant d < b$ (division law in $\mathbb{Z}$, see [7, Lemma 7, p. 4]). Hither, this means that any ($k \in \mathbb{N}_0$, $q \in \mathbb{N}$) uniquely determine the integer r and s, such that $k = qr + s$, where $r = 0, 1, 2, \ldots$ and $s = 1, 2, \ldots, q - 1$. Thereupon, it follows (by uniform convergence) that

$$\begin{aligned}\left(\frac{p}{q}\right)_n &= \prod_{r=0}^{\infty} \prod_{s=0}^{q-1} \left(1 + \frac{1}{qr + s + 1}\right)^n \left(1 + \frac{nq}{q^2 r + qs + p}\right)^{-1} \\ &= \prod_{s=0}^{q-1} \prod_{r=0}^{\infty} \left(1 + \frac{1}{qr + s + 1}\right)^n \left(1 + \frac{nq}{q^2 r + qs + p}\right)^{-1} \\ &= \prod_{s=0}^{q-1} \left(1 + \frac{1}{s+1}\right)^n \left(\frac{\Gamma\left(1 + \frac{s+1}{q}\right)}{\Gamma\left(1 + \frac{s+2}{q}\right)}\right)^n \left(\frac{\Gamma\left(\frac{p + q(n+s)}{q^2}\right)}{\Gamma\left(\frac{p + qs}{q^2}\right)}\right) \\ &= \prod_{s=1}^{q} \left(1 + \frac{1}{s}\right)^n \left(\frac{\Gamma\left(1 + \frac{s}{q}\right)}{\Gamma\left(1 + \frac{s+1}{q}\right)}\right)^n \left(\frac{\Gamma\left(\frac{p + q(n+s-1)}{q^2}\right)}{\Gamma\left(\frac{p + q(s-1)}{q^2}\right)}\right),\end{aligned}$$

using the identity $\Gamma(1 + z) = z\Gamma(z)$ in previous equation, I have

$$\begin{aligned}\left(\frac{p}{q}\right)_n &= \prod_{s=1}^{q} \left(\frac{s+1}{s}\right)^n \left(\frac{\frac{s}{q}\Gamma\left(\frac{s}{q}\right)}{\frac{s+1}{q}\Gamma\left(\frac{s+1}{q}\right)}\right)^n \left(\frac{\Gamma\left(\frac{p + q(n+s-1)}{q^2}\right)}{\Gamma\left(\frac{p + q(s-1)}{q^2}\right)}\right) \\ &= \prod_{s=1}^{q} \left(\frac{\Gamma\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s+1}{q}\right)}\right)^n \left(\frac{\Gamma\left(\frac{p}{q^2} + \frac{n+s-1}{q}\right)}{\Gamma\left(\frac{p}{q^2} + \frac{s-1}{q}\right)}\right),\end{aligned}$$

which is the desired result. □

Example 9. Put $p=1$, $q=2$ in previous Theorem, and let n to be a non-negative integer, thus

$$\left(\frac{1}{2}\right)_n = \frac{2^n}{\pi\sqrt{2}}\Gamma\left(\frac{1}{4}+\frac{n}{2}\right)\Gamma\left(\frac{3}{4}+\frac{n}{2}\right)$$

or

$$\Gamma\left(\frac{1}{2}+n\right) = \frac{2^n}{\sqrt{2\pi}}\Gamma\left(\frac{1}{4}+\frac{n}{2}\right)\Gamma\left(\frac{3}{4}+\frac{n}{2}\right)$$

or

$$\frac{(2n)!}{2^{2n}n!} = \frac{2^n}{\pi\sqrt{2}}\Gamma\left(\frac{1}{4}+\frac{n}{2}\right)\Gamma\left(\frac{3}{4}+\frac{n}{2}\right).$$

Example 10. Put $p=1$, $q=3$ in previous Theorem, and let n to be a non-negative integer, thus

$$\left(\frac{1}{3}\right)_n = 3^n\,\frac{\Gamma\left(\frac{1}{9}+\frac{n}{3}\right)\Gamma\left(\frac{4}{9}+\frac{n}{3}\right)\Gamma\left(\frac{7}{9}+\frac{n}{3}\right)}{\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{7}{9}\right)}.$$

Example 11. Put $p=2$, $q=3$ in previous Theorem, and let n to be a non-negative integer, thus

$$\left(\frac{2}{3}\right)_n = 3^n\,\frac{\Gamma\left(\frac{2}{9}+\frac{n}{3}\right)\Gamma\left(\frac{5}{9}+\frac{n}{3}\right)\Gamma\left(\frac{8}{9}+\frac{n}{3}\right)}{\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{8}{9}\right)}.$$

Example 12. Put $p=1$, $q=4$ in previous Theorem, and let n to be a non-negative integer, thus

$$\left(\frac{1}{4}\right)_n = 4^n\,\frac{\Gamma\left(\frac{1}{16}+\frac{n}{4}\right)\Gamma\left(\frac{5}{16}+\frac{n}{4}\right)\Gamma\left(\frac{9}{16}+\frac{n}{4}\right)\Gamma\left(\frac{13}{16}+\frac{n}{4}\right)}{\Gamma\left(\frac{1}{16}\right)\Gamma\left(\frac{5}{16}\right)\Gamma\left(\frac{9}{16}\right)\Gamma\left(\frac{13}{16}\right)}.$$

Example 13. Put $p=2$, $q=4$ in previous Theorem, and let n to be a non-negative integer, thus

$$\left(\frac{1}{2}\right)_n = 4^n\,\frac{\Gamma\left(\frac{1}{8}+\frac{n}{4}\right)\Gamma\left(\frac{3}{8}+\frac{n}{4}\right)\Gamma\left(\frac{5}{8}+\frac{n}{4}\right)\Gamma\left(\frac{7}{8}+\frac{n}{4}\right)}{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)}$$

or, using the Euler's reflexion formula: $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$, I have

$$\left(\frac{1}{2}\right)_n = \frac{4^n}{2\pi^2\sqrt{2}}\Gamma\left(\frac{1}{8}+\frac{n}{4}\right)\Gamma\left(\frac{3}{8}+\frac{n}{4}\right)\Gamma\left(\frac{5}{8}+\frac{n}{4}\right)\Gamma\left(\frac{7}{8}+\frac{n}{4}\right)$$

Example 14. Put $p=3$, $q=4$ in previous Theorem, and let n to be a non-negative integer, thus

$$\left(\frac{3}{4}\right)_n = 4^n\,\frac{\Gamma\left(\frac{3}{16}+\frac{n}{4}\right)\Gamma\left(\frac{7}{16}+\frac{n}{4}\right)\Gamma\left(\frac{11}{16}+\frac{n}{4}\right)\Gamma\left(\frac{15}{16}+\frac{n}{4}\right)}{\Gamma\left(\frac{3}{16}\right)\Gamma\left(\frac{7}{16}\right)\Gamma\left(\frac{11}{16}\right)\Gamma\left(\frac{15}{16}\right)}.$$

6.2. Gamma Function at Rational Argument.

Theorem 15. *If p and q are positive integers and $p\leqslant q$, then*

$$\frac{p}{q}\Gamma\left(\frac{p}{q}\right) = \prod_{s=1}^{q}\left(\frac{\Gamma\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s+1}{q}\right)}\right)^{p/q}\left(\frac{\Gamma\left(\frac{s}{q}+\frac{p}{q^2}\right)}{\Gamma\left(\frac{s}{q}\right)}\right),$$

where $\Gamma(\ell)$ denotes the gamma function.

Proof. Consider the Euler's infinite product representation for gamma function, let $z = p / q$ in (1), with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, and encounter

$$\begin{aligned}\Gamma\left(\frac{p}{q}\right) &= \frac{q}{p}\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{\frac{p}{q}}\left(1+\frac{p}{jq}\right)^{-1}\\ &= \frac{q}{p}\prod_{k=0}^{\infty}\left(1+\frac{1}{k+1}\right)^{\frac{p}{q}}\left(1+\frac{p}{q(k+1)}\right)^{-1}.\end{aligned}$$

Now, notice that for any $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, there exists unique $c, d \in \mathbb{Z}$, such that $a = bc + d$ and $0 \leqslant d < b$ (division law in $\mathbb{Z}$, see [7, Lemma 7, p. 4]). Hither, this means that any $(k \in \mathbb{N}_0, q \in \mathbb{N})$ uniquely determine the integer r and s, such that $k = qr + s$, where $r = 0, 1, 2, \ldots$ and $s = 1, 2, \ldots, q - 1$. Thereupon, it follows (by uniform convergence) that

$$\begin{aligned}\Gamma\left(\frac{p}{q}\right) &= \frac{q}{p}\prod_{r=0}^{\infty}\prod_{s=0}^{q-1}\left(1+\frac{1}{qr+s+1}\right)^{\frac{p}{q}}\left(1+\frac{p}{q(qr+s+1)}\right)^{-1}\\ &= \frac{q}{p}\prod_{s=0}^{q-1}\prod_{r=0}^{\infty}\left(1+\frac{1}{qr+s+1}\right)^{\frac{p}{q}}\left(1+\frac{p}{q^2 r+qs+q}\right)^{-1}\\ &= \frac{q}{p}\prod_{s=0}^{q-1}\left(1+\frac{1}{s+1}\right)^{p/q}\left(\frac{\Gamma\left(1+\frac{s+1}{q}\right)}{\Gamma\left(1+\frac{s+2}{q}\right)}\right)^{p/q}\left(\frac{\Gamma\left(\frac{s+1}{q}+\frac{p}{q^2}\right)}{\Gamma\left(\frac{s+1}{q}\right)}\right)\\ &= \frac{q}{p}\prod_{s=1}^{q}\left(1+\frac{1}{s}\right)^{p/q}\left(\frac{\Gamma\left(1+\frac{s}{q}\right)}{\Gamma\left(1+\frac{s+1}{q}\right)}\right)^{p/q}\left(\frac{\Gamma\left(\frac{s}{q}+\frac{p}{q^2}\right)}{\Gamma\left(\frac{s}{q}\right)}\right),\end{aligned}$$

using the identity $\Gamma(1+z) = z\Gamma(z)$ in previous equation, I obtain

$$\begin{aligned}\Gamma\left(\frac{p}{q}\right) &= \frac{q}{p}\prod_{s=1}^{q}\left(\frac{s+1}{s}\right)^{p/q}\left(\frac{\frac{s}{q}\Gamma\left(\frac{s}{q}\right)}{\left(\frac{s+1}{q}\right)\Gamma\left(\frac{s+1}{q}\right)}\right)^{p/q}\left(\frac{\Gamma\left(\frac{s}{q}+\frac{p}{q^2}\right)}{\Gamma\left(\frac{s}{q}\right)}\right)\\ &= \frac{q}{p}\prod_{s=1}^{q}\left(\frac{\Gamma\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s+1}{q}\right)}\right)^{p/q}\left(\frac{\Gamma\left(\frac{s}{q}+\frac{p}{q^2}\right)}{\Gamma\left(\frac{s}{q}\right)}\right)\\ \Rightarrow \frac{p}{q}\Gamma\left(\frac{p}{q}\right) &= \prod_{s=1}^{q}\left(\frac{\Gamma\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s+1}{q}\right)}\right)^{p/q}\left(\frac{\Gamma\left(\frac{s}{q}+\frac{p}{q^2}\right)}{\Gamma\left(\frac{s}{q}\right)}\right)\end{aligned}$$

which is the desired result. □

Example 16. Put $p = 1$, $q = 3$ in previous Theorem, thus

$$2\pi\sqrt[6]{3} = \frac{\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{7}{9}\right)}{\Gamma\left(\frac{1}{3}\right)}.$$

Example 17. Put $p = 1$, $q = 4$ in previous Theorem, thus

$$\frac{4\pi}{9\sqrt[6]{3}} = \frac{\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{8}{9}\right)\Gamma\left(\frac{11}{9}\right)}{\Gamma\left(\frac{2}{3}\right)}$$

or, using the identity $\Gamma(z+1) = z\Gamma(z)$, I have

$$\frac{2\pi}{\sqrt[6]{3}} = \frac{\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{8}{9}\right)}{\Gamma\left(\frac{2}{3}\right)}.$$

Example 18. Put $p=1,\ q=4$ in previous Theorem, thus

$$\frac{\pi\sqrt{\pi}}{4}=\frac{\Gamma\left(\frac{5}{16}\right)\Gamma\left(\frac{9}{16}\right)\Gamma\left(\frac{13}{16}\right)\Gamma\left(\frac{17}{16}\right)}{\Gamma\left(\frac{1}{4}\right)}$$

or, using the identity $\Gamma(z+1)=z\Gamma(z)$, I have

$$4\pi\sqrt{\pi}=\frac{\Gamma\left(\frac{1}{16}\right)\Gamma\left(\frac{5}{16}\right)\Gamma\left(\frac{9}{16}\right)\Gamma\left(\frac{13}{16}\right)}{\Gamma\left(\frac{1}{4}\right)}.$$

Example 19. Put $p=3,\ q=4$ in previous Theorem, thus

$$\frac{3\pi\sqrt{\pi}}{8}=\frac{\Gamma\left(\frac{7}{16}\right)\Gamma\left(\frac{11}{16}\right)\Gamma\left(\frac{15}{16}\right)\Gamma\left(\frac{19}{16}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

or, using the identity $\Gamma(z+1)=z\Gamma(z)$, I have

$$6\pi\sqrt{\pi}=\frac{\Gamma\left(\frac{3}{16}\right)\Gamma\left(\frac{7}{16}\right)\Gamma\left(\frac{11}{16}\right)\Gamma\left(\frac{15}{16}\right)}{\Gamma\left(\frac{3}{4}\right)}.$$

Example 20. Put $p=1,\ q=5$ in previous Theorem, thus

$$\frac{4\pi^2}{5\sqrt[10]{5^7}}=\frac{\Gamma\left(\frac{6}{25}\right)\Gamma\left(\frac{11}{25}\right)\Gamma\left(\frac{16}{25}\right)\Gamma\left(\frac{21}{25}\right)\Gamma\left(\frac{26}{25}\right)}{\Gamma\left(\frac{1}{5}\right)}$$

or, using the identity $\Gamma(z+1)=z\Gamma(z)$, I have

$$\frac{20\pi^2}{\sqrt[10]{5^7}}=\frac{\Gamma\left(\frac{1}{25}\right)\Gamma\left(\frac{6}{25}\right)\Gamma\left(\frac{11}{25}\right)\Gamma\left(\frac{16}{25}\right)\Gamma\left(\frac{21}{25}\right)}{\Gamma\left(\frac{1}{5}\right)}.$$

6.3. Sine Function at Rational Argument.

Theorem 21. *If p and q are positive integers and $p\leqslant q$, then*

$$\frac{\sin\left(\frac{p\pi}{q}\right)}{pq\pi\Gamma^2(q)}=\prod_{s=1}^{q}\frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q}-\frac{p}{q^2}\right)\Gamma\left(\frac{s}{q}+\frac{p}{q^2}\right)},$$

where $\Gamma(\ell)$ denotes the gamma function and $\sin(z)$ denotes the sine function.

Proof. Consider the Euler's infinite product representation for sine function [8, p. 321]

$$\frac{\sin(\pi z)}{\pi z}=\prod_{j=1}^{\infty}\left(1-\frac{z^2}{j^2}\right). \tag{16}$$

Let $z=p/q$ in (16), with $p\in\mathbb{Z}$ and $q\in\mathbb{N}$, and encounter

$$\begin{aligned}\frac{q\sin\left(\frac{p\pi}{q}\right)}{p\pi}&=\prod_{j=1}^{\infty}\left(1-\frac{p^2}{j^2q^2}\right)\\&=\prod_{k=0}^{\infty}\left(1-\frac{p^2}{(k+1)^2q^2}\right)\end{aligned}$$

Now, notice that for any $a\in\mathbb{Z}$ and $b\in\mathbb{N}$, there exists unique $c,d\in\mathbb{Z}$, such that $a=bc+d$ and $0\leqslant d<b$ (division law in $\mathbb{Z}$, see [7, Lemma 7, p. 4]). Hither, this means that any $(k\in\mathbb{N}_0, q\in\mathbb{N})$ uniquely determine the integer r and s, such that $k=qr+s$, where $r=0,1,2,\ldots$ and $s=1,2,\ldots,$ $q-1$. Thereupon, it follows (by uniform convergence) that

$$\begin{aligned}\frac{q\sin\left(\frac{p\pi}{q}\right)}{p\pi} &= \prod_{r=0}^{\infty}\prod_{s=0}^{q-1}\left(1-\frac{p^2}{(qr+s+1)^2q^2}\right)\\ &= \prod_{s=0}^{q-1}\prod_{r=0}^{\infty}\left(1-\frac{p^2}{(qr+s+1)^2q^2}\right)\\ &= q^2\prod_{s=0}^{q-1}\frac{\Gamma^2\left(1+\frac{s+1}{q}\right)}{\Gamma\left(\frac{q(s+1)-p}{q^2}\right)\Gamma\left(\frac{q(s+1)+p}{q^2}\right)}\\ \Rightarrow\frac{\sin\left(\frac{p\pi}{q}\right)}{pq\pi} &= \prod_{s=0}^{q-1}\frac{\Gamma^2\left(1+\frac{s+1}{q}\right)}{\Gamma\left(\frac{s+1}{q}-\frac{p}{q^2}\right)\Gamma\left(\frac{s+1}{q}+\frac{p}{q^2}\right)}\\ &= \prod_{s=1}^{q}\frac{\Gamma^2\left(1+\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q}-\frac{p}{q^2}\right)\Gamma\left(\frac{s}{q}+\frac{p}{q^2}\right)},\end{aligned}$$

using the identity $\Gamma(1+z)=z\Gamma(z)$ in previous equation, I get

$$\begin{aligned}\frac{\sin\left(\frac{p\pi}{q}\right)}{pq\pi} &= \frac{1}{q^2}\prod_{s=1}^{q}\frac{s^2\,\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q}-\frac{p}{q^2}\right)\Gamma\left(\frac{s}{q}+\frac{p}{q^2}\right)}\\ \Rightarrow\sin\left(\frac{p\pi}{q}\right) &= \frac{p\pi\Gamma^2(q+1)}{q}\prod_{s=1}^{q}\frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q}-\frac{p}{q^2}\right)\Gamma\left(\frac{s}{q}+\frac{p}{q^2}\right)}\\ \Rightarrow\frac{\sin\left(\frac{p\pi}{q}\right)}{pq\pi\Gamma^2(q)} &= \prod_{s=1}^{q}\frac{\Gamma^2\left(\frac{s}{q}\right)}{\Gamma\left(\frac{s}{q}-\frac{p}{q^2}\right)\Gamma\left(\frac{s}{q}+\frac{p}{q^2}\right)},\end{aligned}$$

which is the desired result. □

6.4. Cosine Function at Rational Argument.

Theorem 22. *If p and q are positive integers and $p\leqslant q$, then*

$$\cos\left(\frac{p\pi}{q}\right)=\prod_{s=1}^{q}\frac{\Gamma^2\left(\frac{2s-1}{2q}\right)}{\Gamma\left(\frac{2s-1}{2q}+\frac{p}{q^2}\right)\Gamma\left(\frac{2s-1}{2q}-\frac{p}{q^2}\right)},$$

where $\Gamma(\ell)$ denotes the gamma function and $\sin(z)$ denotes the sine function.

Proof. Consider the Euler's infinite product representation for cosine function [8, p. 321]

$$\cos(\pi z)=\prod_{j=1}^{\infty}\left(1-\frac{4z^2}{(2j-1)^2}\right). \tag{17}$$

Let $z=p/q$ in (17), with $p\in\mathbb{Z}$ and $q\in\mathbb{N}$, and encounter

$$\begin{aligned}\cos\left(\frac{p\pi}{q}\right) &= \prod_{j=1}^{\infty}\left(1-\frac{4p^2}{(2j-1)^2q^2}\right)\\ &= \prod_{k=0}^{\infty}\left(1-\frac{4p^2}{(2k+1)^2q^2}\right)\end{aligned}$$

Now, notice that for any $a\in\mathbb{Z}$ and $b\in\mathbb{N}$, there exists unique $c,d\in\mathbb{Z}$, such that $a=bc+d$ and $0\leqslant d<b$ (division law in $\mathbb{Z}$, see [7, Lemma 7, p. 4]). Hither, this means that any $(k\in\mathbb{N}_0, q\in\mathbb{N})$ uniquely determine the integer r and s, such that $k=qr+s$, where $r=0,1,2,\ldots$ and $s=1,2,\ldots,q-1$. Thereupon, it follows (by uniform convergence) that

$$\begin{aligned}\cos\left(\frac{p\pi}{q}\right) &= \prod_{r=0}^{\infty}\prod_{s=0}^{q-1}\left(1-\frac{4p^2}{(2qr+2s+1)^2q^2}\right)\\ &= \prod_{s=0}^{q-1}\prod_{r=0}^{\infty}\left(1-\frac{4p^2}{(2qr+2s+1)^2q^2}\right)\\ &= \prod_{s=0}^{q-1}\frac{\left(1-\frac{4p^2}{(2s+1)^2q^2}\right)\Gamma^2\left(1+\frac{2s+1}{2q}\right)}{\Gamma\left(1+\frac{2s+1}{2q}+\frac{p}{q^2}\right)\Gamma\left(1+\frac{2s+1}{2q}-\frac{p}{q^2}\right)}\\ &= \prod_{s=1}^{q}\frac{\left(1-\frac{4p^2}{(2s-1)^2q^2}\right)\Gamma^2\left(1+\frac{2s-1}{2q}\right)}{\Gamma\left(1+\frac{2s-1}{2q}+\frac{p}{q^2}\right)\Gamma\left(1+\frac{2s-1}{2q}-\frac{p}{q^2}\right)},\end{aligned}$$

using the identity $\Gamma(1+z)=z\Gamma(z)$ in previous equation, I get

$$\begin{aligned}\cos\left(\frac{p\pi}{q}\right) &= \prod_{s=1}^{q}\frac{\left(1-\frac{4p^2}{(2s-1)^2q^2}\right)\left(\frac{2s-1}{2q}\right)^2\Gamma^2\left(\frac{2s-1}{2q}\right)}{\left(\frac{2s-1}{2q}+\frac{p}{q^2}\right)\left(\frac{2s-1}{2q}-\frac{p}{q^2}\right)\Gamma\left(\frac{2s-1}{2q}+\frac{p}{q^2}\right)\Gamma\left(\frac{2s-1}{2q}-\frac{p}{q^2}\right)}\\ &= \prod_{s=1}^{q}\frac{\Gamma^2\left(\frac{2s-1}{2q}\right)}{\Gamma\left(\frac{2s-1}{2q}+\frac{p}{q^2}\right)\Gamma\left(\frac{2s-1}{2q}-\frac{p}{q^2}\right)},\end{aligned}$$

which is the desired result. □

7. Euler's and Weierstrass's Infinite Product Representation for Newton's Binomial

7.1. Euler's Infinite Product Representation for Newton's Binomial.

Theorem 23. *If* $a,b\in\mathbb{C}-\{0,-1,-2,\ldots\}$ *and* $n\in\mathbb{N}$, *then*

$$(a+b)^n = \prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{2n}\left[\left(1+\frac{b}{a+jb}\right)\left(1+\frac{1}{b+j-1}\right)\right]^{-n}.$$

Proof. Consider the Newton's binomial

$$(1+x)^n = \left(\frac{1+x}{1}\right)^n. \tag{18}$$

Set $a=1+x$ and $b=1$ in both members of the Lemma 1

$$\frac{1+x}{1} = \prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)\left(1+\frac{1}{x+j}\right)^{-1}. \tag{19}$$

Substitute the right hand side of (19) into the right hand side of (18)

$$(1+x)^n = \prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^n\left(1+\frac{1}{x+j}\right)^{-n}. \tag{20}$$

Put $x = a/b$ in both members of (20)

$$(a+b)^n = b^n \prod_{j=1}^{\infty} \left(1+\frac{1}{j}\right)^n \left(1+\frac{b}{a+jb}\right)^{-n}. \tag{21}$$

On the other hand, I get $x = b-1$ in both members of (20)

$$b^n = \prod_{j=1}^{\infty} \left(1+\frac{1}{j}\right)^n \left(1+\frac{1}{b+j-1}\right)^{-n}. \tag{22}$$

From (21) and (22), it follows that

$$(a+b)^n = \prod_{j=1}^{\infty} \left(1+\frac{1}{j}\right)^{2n} \left[\left(1+\frac{b}{a+jb}\right)\left(1+\frac{1}{b+j-1}\right)\right]^{-n}, \tag{23}$$

which is the desired result. □

7.2. Weierstrass's Infinite Product Representation for Newton's Binomial.

Theorem 24. *If $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$, then*

$$(a+b)^{-n} = e^{2\gamma n} \prod_{j=1}^{\infty} \left[\left(1+\frac{b}{a+jb}\right)\left(1+\frac{1}{b+j-1}\right)\right]^n e^{-2n/j},$$

where e^x denotes the exponential function and γ denotes the Euler-Mascheroni constant.

Proof. The inverse of the Theorem 23 give me

$$\begin{aligned}
(a+b)^{-n} &= \prod_{j=1}^{\infty} \left(1+\frac{1}{j}\right)^{-2n} \left[\left(1+\frac{b}{a+jb}\right)\left(1+\frac{1}{b+j-1}\right)\right]^n \\
&= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{2n} \lim_{m\to\infty}\prod_{j=1}^{m} \left(1+\frac{1}{j}\right)^{-2n} \left[\left(1+\frac{b}{a+jb}\right)\left(1+\frac{1}{b+j-1}\right)\right]^n \\
&= \lim_{m\to\infty}\left\{\left(1+\frac{1}{m}\right)^{2n} \prod_{j=1}^{m} \left(1+\frac{1}{j}\right)^{-2n} \left[\left(1+\frac{b}{a+jb}\right)\left(1+\frac{1}{b+j-1}\right)\right]^n\right\} \\
&= \lim_{m\to\infty}\left\{\prod_{j=1}^{m-1} \left(1+\frac{1}{j}\right)^{-2n} \prod_{j=1}^{m} \left[\left(1+\frac{b}{a+jb}\right)\left(1+\frac{1}{b+j-1}\right)\right]^n\right\} \\
&= \lim_{m\to\infty}\left\{m^{-2n} \prod_{j=1}^{m} \left[\left(1+\frac{b}{a+jb}\right)\left(1+\frac{1}{b+j-1}\right)\right]^n\right\} \\
&= \lim_{m\to\infty}\left\{\exp\left(\left(1-1+\frac{1}{2}-\frac{1}{2}+\ldots+\frac{1}{m}-\frac{1}{m}-\ln m\right)2n\right) \prod_{j=1}^{m} \left[\left(1+\frac{b}{a+jb}\right)\left(1+\frac{1}{b+j-1}\right)\right]^n\right\} \\
&= \lim_{m\to\infty}\left\{\exp\left(\left(1+\frac{1}{2}+\ldots+\frac{1}{m}-\ln m\right)2n\right)\exp\left(\frac{-2n}{1}+\frac{-2n}{2}+\ldots+\frac{-2n}{m}\right)\times\right. \\
&\qquad \left.\times\prod_{j=1}^{m} \left[\left(1+\frac{b}{a+jb}\right)\left(1+\frac{1}{b+j-1}\right)\right]^n\right\} \\
&= \lim_{m\to\infty}\left\{\exp\left(\left(1+\frac{1}{2}+\ldots+\frac{1}{m}-\ln m\right)2n\right) \prod_{j=1}^{m} \left[\left(1+\frac{b}{a+jb}\right)\left(1+\frac{1}{b+j-1}\right)\right]^n e^{-2n/j}\right\} \\
&= \lim_{m\to\infty}\left(\exp\left(\left(1+\frac{1}{2}+\ldots+\frac{1}{m}-\ln m\right)2n\right)\right) \lim_{m\to\infty}\left(\prod_{j=1}^{m} \left[\left(1+\frac{b}{a+jb}\right)\left(1+\frac{1}{b+j-1}\right)\right]^n e^{-2n/j}\right) \\
&= e^{2\gamma n} \prod_{j=1}^{\infty} \left[\left(1+\frac{b}{a+jb}\right)\left(1+\frac{1}{b+j-1}\right)\right]^n e^{-2n/j},
\end{aligned}$$

which is the desired result. □

8. Euler's and Weierstrass's Infinite Product Representation for Exponential Function

8.1. Euler's Infinite Product Representation for Exponential Function.

Theorem 25. *If $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$, then*

$$e^{\mathrm{i}n\tan^{-1}\left(\frac{y}{x}\right)} = \prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{n}\frac{\left[\left(1+\frac{x^2}{jx^2+y^2}\right)\left(1+\frac{1}{x^2+j-1}\right)\right]^{n/2}}{\left[\left(1+\frac{x}{jx+\mathrm{i}y}\right)\left(1+\frac{1}{x+j-1}\right)\right]^{n}},$$

where $\mathrm{i}=\sqrt{-1}$ and e^z denotes the exponential function.

Proof. Let $a=\mathrm{i}y$ and $b=x$, in both members of the Theorem 23, and encounter

$$(x+\mathrm{i}y)^n = \prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{2n}\left[\left(1+\frac{x}{jx+\mathrm{i}y}\right)\left(1+\frac{1}{x+j-1}\right)\right]^{-n}. \tag{24}$$

On the other hand, I know that the complex exponentiation satisfies [9]

$$(x+\mathrm{i}y)^{a+\mathrm{i}b} = (x^2+y^2)^{(a+\mathrm{i}b)/2}\times e^{\mathrm{i}(a+\mathrm{i}b)\arg(x+\mathrm{i}y)}, \tag{25}$$

where $\arg(z)$ denotes the complex argument, which can be computed as $\arg(x+\mathrm{i}y)\equiv\tan^{-1}\left(\frac{y}{x}\right)$. Set $a=n$ and $b=0$ in both members of (25), and find

$$(x+\mathrm{i}y)^n = (x^2+y^2)^{n/2}\times e^{\mathrm{i}n\tan^{-1}\left(\frac{y}{x}\right)}. \tag{26}$$

Substitute the right hand side of (26) into the left hand side of (24), and obtain

$$e^{\mathrm{i}n\tan^{-1}\left(\frac{y}{x}\right)} = (x^2+y^2)^{-n/2}\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{2n}\left[\left(1+\frac{x}{jx+\mathrm{i}y}\right)\left(1+\frac{1}{x+j-1}\right)\right]^{-n}. \tag{27}$$

Put $a=y^2$, $b=x^2$ and $n\to -n/2$, in both members of the Theorem 23, and encounter

$$(x^2+y^2)^{-n/2} = \prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{-n}\left[\left(1+\frac{x^2}{jx^2+y^2}\right)\left(1+\frac{1}{x^2+j-1}\right)\right]^{n/2}. \tag{28}$$

From (27) and (28), it follows that

$$e^{\mathrm{i}n\tan^{-1}\left(\frac{y}{x}\right)} = \prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{n}\frac{\left[\left(1+\frac{x^2}{jx^2+y^2}\right)\left(1+\frac{1}{x^2+j-1}\right)\right]^{n/2}}{\left[\left(1+\frac{x}{jx+\mathrm{i}y}\right)\left(1+\frac{1}{x+j-1}\right)\right]^{n}}, \tag{29}$$

which is the desired result. □

8.2. Weierstrass's Infinite Product Representation for Exponential Function.

Theorem 26. *If $x, y \in \mathbb{C}$ and $n \in \mathbb{N}$, then*

$$e^{-\mathrm{i}n\tan^{-1}\left(\frac{y}{x}\right)} = e^{\gamma n}\prod_{j=1}^{\infty}\frac{\left[\left(1+\frac{x}{jx+\mathrm{i}y}\right)\left(1+\frac{1}{x+j-1}\right)\right]^{n}}{\left[\left(1+\frac{x^2}{jx^2+y^2}\right)\left(1+\frac{1}{x^2+j-1}\right)\right]^{n/2}}e^{-n/j},$$

where e^x denotes the exponential function and γ denotes the Euler-Mascheroni constant.

Proof. The inverse of the Theorem 25, give me

$$
\begin{aligned}
e^{-\mathrm{i}n\tan^{-1}\left(\frac{y}{x}\right)} &= \prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{-n}\frac{\left[\left(1+\frac{x}{jx+\mathrm{i}y}\right)\left(1+\frac{1}{x+j-1}\right)\right]^{n}}{\left[\left(1+\frac{x^2}{jx^2+y^2}\right)\left(1+\frac{1}{x^2+j-1}\right)\right]^{n/2}}\\
&= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{n}\lim_{m\to\infty}\prod_{j=1}^{m}\left(1+\frac{1}{j}\right)^{-n}\frac{\left[\left(1+\frac{x}{jx+\mathrm{i}y}\right)\left(1+\frac{1}{x+j-1}\right)\right]^{n}}{\left[\left(1+\frac{x^2}{jx^2+y^2}\right)\left(1+\frac{1}{x^2+j-1}\right)\right]^{n/2}}\\
&= \lim_{m\to\infty}\left\{\left(1+\frac{1}{m}\right)^{n}\prod_{j=1}^{m}\left(1+\frac{1}{j}\right)^{-n}\frac{\left[\left(1+\frac{x}{jx+\mathrm{i}y}\right)\left(1+\frac{1}{x+j-1}\right)\right]^{n}}{\left[\left(1+\frac{x^2}{jx^2+y^2}\right)\left(1+\frac{1}{x^2+j-1}\right)\right]^{n/2}}\right\}\\
&= \lim_{m\to\infty}\left\{\prod_{j=1}^{m-1}\left(1+\frac{1}{j}\right)^{-n}\prod_{j=1}^{m}\frac{\left[\left(1+\frac{x}{jx+\mathrm{i}y}\right)\left(1+\frac{1}{x+j-1}\right)\right]^{n}}{\left[\left(1+\frac{x^2}{jx^2+y^2}\right)\left(1+\frac{1}{x^2+j-1}\right)\right]^{n/2}}\right\}\\
&= \lim_{m\to\infty}\left\{m^{-n}\prod_{j=1}^{m}\frac{\left[\left(1+\frac{x}{jx+\mathrm{i}y}\right)\left(1+\frac{1}{x+j-1}\right)\right]^{n}}{\left[\left(1+\frac{x^2}{jx^2+y^2}\right)\left(1+\frac{1}{x^2+j-1}\right)\right]^{n/2}}\right\}\\
&= \lim_{m\to\infty}\left\{\exp\left(\left(1-1+\frac{1}{2}-\frac{1}{2}+\ldots+\frac{1}{m}-\frac{1}{m}-\ln m\right)n\right)\prod_{j=1}^{m}\frac{\left[\left(1+\frac{x}{jx+\mathrm{i}y}\right)\left(1+\frac{1}{x+j-1}\right)\right]^{n}}{\left[\left(1+\frac{x^2}{jx^2+y^2}\right)\left(1+\frac{1}{x^2+j-1}\right)\right]^{n/2}}\right\}\\
&= \lim_{m\to\infty}\left\{\exp\left(\left(1+\frac{1}{2}+\ldots+\frac{1}{m}-\ln m\right)n\right)\exp\left(\frac{-n}{1}+\frac{-n}{2}+\ldots+\frac{-n}{m}\right)\right.\\
&\qquad\left.\times\prod_{j=1}^{m}\frac{\left[\left(1+\frac{x}{jx+\mathrm{i}y}\right)\left(1+\frac{1}{x+j-1}\right)\right]^{n}}{\left[\left(1+\frac{x^2}{jx^2+y^2}\right)\left(1+\frac{1}{x^2+j-1}\right)\right]^{n/2}}\right\}\\
&= \lim_{m\to\infty}\left\{\exp\left(\left(1+\frac{1}{2}+\ldots+\frac{1}{m}-\ln m\right)n\right)\prod_{j=1}^{m}\frac{\left[\left(1+\frac{x}{jx+\mathrm{i}y}\right)\left(1+\frac{1}{x+j-1}\right)\right]^{n}}{\left[\left(1+\frac{x^2}{jx^2+y^2}\right)\left(1+\frac{1}{x^2+j-1}\right)\right]^{n/2}}e^{-n/j}\right\}\\
&= \lim_{m\to\infty}\left(\exp\left(\left(1+\frac{1}{2}+\ldots+\frac{1}{m}-\ln m\right)n\right)\right)\lim_{m\to\infty}\left\{\prod_{j=1}^{m}\frac{\left[\left(1+\frac{x}{jx+\mathrm{i}y}\right)\left(1+\frac{1}{x+j-1}\right)\right]^{n}}{\left[\left(1+\frac{x^2}{jx^2+y^2}\right)\left(1+\frac{1}{x^2+j-1}\right)\right]^{n/2}}e^{-n/j}\right\}\\
&= e^{\gamma n}\prod_{j=1}^{\infty}\frac{\left[\left(1+\frac{x}{jx+\mathrm{i}y}\right)\left(1+\frac{1}{x+j-1}\right)\right]^{n}}{\left[\left(1+\frac{x^2}{jx^2+y^2}\right)\left(1+\frac{1}{x^2+j-1}\right)\right]^{n/2}}e^{-n/j},
\end{aligned}
$$

which is the desired result. □

8.3. A Simple Infinite Product Representation for Exponential Function.

Theorem 27. *If $x\in\mathbb{C}$, then*

$$e^x=\prod_{j=1}^{\infty}e^{x/(j+j^2)},$$

where e^x denotes the exponential function.

Proof. Leonhard Euler (1707-1783) gave firstly the limit definition for exponential function as follows

$$e^x=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

In Theorem 23, I put $a=x$ and $b=n$ in both members and encounter

$$\left(1+\frac{x}{n}\right)^n=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^n\left(1+\frac{n}{x+jn}\right)^{-n}. \tag{30}$$

The letting n tends to infinity in both members of (30), give me

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \prod_{j=1}^{\infty}\lim_{n\to\infty}\left[\left(1+\frac{1}{j}\right)^n\left(1+\frac{n}{x+jn}\right)^{-n}\right]$$
$$\Rightarrow e^x = \prod_{j=1}^{\infty} e^{x/(j+j^2)},$$

which is the desired result. □

9. Infinite Product Representation for Kronecker Delta

9.1. A Simple Infinite Product Representation for Kronecker Delta.

I leave as easy exercise

Exercise 1. Prove that

$$\delta_{i\,j} = \prod_{k=1}^{\infty}\left(1-\frac{1}{k}\right)\left(1-\frac{1}{k+(i-j)^2}\right)^{-1},$$

where

$$\delta_{i\,j} = \begin{cases} 0, \text{if } i \neq j, \\ 1, \text{if } i = j, \end{cases}$$

denotes the Kronecker delta function.

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