Subnormal Distribution Derived from Evolving Networks with Variable Elements

Minyu Feng, Hong Qu *Member, IEEE*, Zhang Yi *Fellow, IEEE*, and Jürgen Kurths

***Abstract*—During the last decades, Power-law distributions played significant roles in analyzing the topology of scale-free (SF) networks. However, in the observation of degree distributions of practical networks and other unequal distributions such as wealth distribution, we uncover that, instead of monotonic decreasing, there exists a peak at the beginning of most real distributions, which cannot be accurately described by a Power-law. In this paper, in order to break the limitation of the Power-law distribution, we provide detailed derivations of a novel distribution called Subnormal distribution from evolving networks with variable elements and its concrete statistical properties. Additionally, simulations of fitting the subnormal distribution to the degree distribution of evolving networks, real social network, and personal wealth distribution are displayed to show the fitness of proposed distribution.**

***Index Terms*—Power-law Distribution, Degree Distribution, Probability Theory, Evolving Networks, Gibrat's law.**

I. Introduction

As well known, the Power-law distribution is a nonuniform distribution, in particular for networks, it appears that a majority of vertices hold a low number of links while a few vertices have many links. The history of the Power-law distribution starts from the Italian economist Pareto in the 19th century, who first put the "20-80" rule forward, i.e. 20% of a population possess 80% social welfare, apparently following a Power-law distribution. Bababási first employs the Power-law distribution to explain the degree distribution of SF networks and makes it gain considerable fame. In 1999, he revolutionarily evolved the network model into a scale-invariant state with the growing and preferential attachment character, and revealed that the degree distribution of evolving networks follows a Power-law distribution [1]. This discovery soon drew great attentions from many multidisciplinary researchers and brought a stirring of interest in SF network. It is well known that, in the real world, most practical networks such as web networks [2], interaction networks [3], sorting comparison network [4], social networks [5], [6], etc, all follow a Power-law distribution, which can be described as the "rich-get-richer" or Matthew Effect. The Power-law degree distribution therefore has shown significance in the study of complex systems and is the foundation of exploring the formation mechanism and organizational principle of SF networks. Inspired by SF networks, the discovery, analysis, and application of SF networks, i.e. pattern extraction [7], search [8] and synchronization [9] [10], now represent a "new science of networks". Then, many researchers devoted to the study of degree distribution of evolving networks. They proposed mean-field [11], master-equation [12] and Markov-chain approach [13] to mathematically solve the degree distribution. In addition, the logarithmic binning [14] and other algorithms [15], [16] were applied to obtain the statistical degree distribution of practical networks.

In a considerable time, the Power-law distribution holds its dominant position in network science, but some researchers doubt whether it fits in all practical networks [17], [18]. In actual, most practical SF networks are in accord with the rule "rich-get-richer", however, many of them are out of accord with "poor-get-poorer", indicating that the degree distribution of these networks is not simply Power-law. The movie actor collaboration network [19], for example, shows a lift instead of smoothly descending when the degree is low and can not completely fit in a Power-law distribution. Apart from network science, the social welfare in economics and the frequency of words in natural language also are different from Power-law distributions but show skew distributions. In this paper, we call this phenomenon "sub-normalization", since its curve seems to fall somewhere between the Power-law and normal distribution. In order to discover this phenomenon and put forward the novel distribution, we introduce certain variable elements to the modeling process of SF network and employ some common calculation methods to solve the distribution function of degree. This obtained distribution is called the "subnormal distribution" processing the properties of both Power-law and log-normal distribution. Through a mathematical analysis, we find out that this distribution is a joint probability density function produced by variable elements of networks, e.g, number of connections and selection of individuals. Furthermore, we study the statistical properties of this distribution. In simulations, we display the construction process of evolving networks by variable elements, and the similarity between the network distribution and the proposed distribution is compared. Besides, the distributions of social network degree and personal wealth are also compared with our distribution to show that it can be fit in with the practical. Finally, we try to find out the mechanism of SF networks

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and discuss the potential value of subnormal phenomenon and subnormal distribution in other fields such as economics.

The organization of this paper is as follows: A detailed presentation of the evolving networks with variable elements is provided in Section II. The derivations of subnormal distribution and its statistical properties are presented in section III. Simulations are carried out in Section IV to demonstrate the fitness to other distributions. Finally, some discussions, conclusions and outlooks are given in Section V.

II. Evolving Networks with Variable Elements

SF networks suggest the growth and preferential attachment and follow a perfect Power-law distribution. Though the BA network is closer to the real networks than other network model, the construction process is still ideal. For practical networks, it is impossible to introduce one vertex each time and connect it to m existing vertices. Contrarily, in practical situation, there exists lots of variable factors, such as the famous WWW that has a variable vertex growth rate and edge connection. To reveal these influences on the degree distribution of networks, we discuss the variable elements and show the construction process of a SF model with them in this section.

A. Variable Elements

First, the variable elements in the process of construction of a SF network are discussed.

The initial network is one of the most negligent issue which in fact is also ignored by the BA SF network. The variable elements of the initial network are the number of initial vertices and their connection rules. As we know, the number of initial vertices affects the final degree distribution if it is very huge. However, the initial network are always very small comparing to the final network, like ARPANET (Advanced Research Projects Agency Network), the origination of Internet, has only four host computers connecting to each other, and now Internet has billions of computer connections. Therefore, the number of an initial network is required to be small enough that it does not affect the final structure of the network. In addition, the small initial networks are always highly gathered, e.g, the beginning of a new journal network is cited by each other and its average short path distance is low. Considering both the small and low distance character, we suppose that a small-world network such as NW is appropriate to describe the initial network [20].

The other variable element is the arrival rate or interval time of vertices which is assumed as a constant by many theoretical models. BA networks, for example, suppose that one vertex is connected to the network each time. We suggest that the rates of vertices introduced to the network follow a certain rate, and in different period, the rate is varied, i.e. it is related to time. Specifically, the growth rate during the financial crisis is lower than during a boom time. Thus, a nonhomogeneous Poisson process can perfectly express the generation of vertices. In our last research article [21], we have proven that a nonhomogeneous Poisson process is irrelevant to the final degree distribution, and specifically, the size of an evolving networks and its relative rate of growth are independent, which is precisely consistent with the famous the Gibrat's law in economics [22]. In other words, this law is also valid for evolving networks.

The most significant variable in this paper is the connection of new arrival vertices which directly affects the degree distribution of the network. Accelerating growth network model [23] first notes that the connection is varied, and its connection follows a Power-law distribution. However, for this kind of network, a stationary distribution of the degree distribution does not exist, which disagrees with the practical networks. The stationary distribution for connections of a network is one of the goals of this paper. We mainly consider two kinds of distributions, uniform distributions and nonuniform distributions. The uniform distributions refer to those networks whose connections of new arrival vertices are homogeneous, hierarchical networks for example [24]. On the contrary, the most common connection rule follows a nonuniform distribution, since each new vertex has its own fitness to the network, to which the number of its connections to the existing vertices in the networks relates. Practically speaking, in the network of the research reference, once a high quality paper such as *Emergence of scaling in random networks* by Barabási is published, many related references will emerge in a short time, yet a low quality paper will not be cited in a long time. The universal expression for this connection is the Gaussian distribution, most connection numbers are a mean value, and the extremely high and low numbers are rare which is much more common in a steady network than the Power-law distribution. Furthermore, the Gaussian distribution is just an ideal situation. There are also many dilemmas by applying this distribution, e.g. the connection can not be negative. There is evidence that the most income is distributed log-normally, which can be also interpreted as the new added connection, the income of evolving networks, is also distributed log-normally [25]. Taking all factors into consideration, therefore, we employ the log-normal distribution to simulate the connection of evolving networks in this paper. The log-normal distribution has a significant influence on the degree distribution of complex networks, which makes it free of time, but disobey the traditional Power-law distribution and break the rule of "poor-get-poorer".

There are other variable elements, such as the connection rule and time, which are not topics of this paper.

B. Evolving Network Model based on Variable Elements

Next, we show a constructing process of evolving networks with variable elements. The constructing process mainly includes **Initialization**, **Growth**, **Connection** and **Termination**.

Initialization in this paper is a process of a small NW network. Assuming that the number of total vertices of the initial network is n, each of them links to k neighbors, and has the probability p to link to others. Self-loops are avoided.

Growth is the key step of evolving network, which consists of the vertex growing rate and then arrival vertex connection. For each time t, we add $\lambda(t)$ vertices, where $\lambda(t)$ is a continuous function for a nonhomogeneous Poisson process. In the interval $[t, t+\triangle t]$, the probability of number of new vertices is then

$$P\{N(t+\triangle t)-N(t)=k\}=\frac{[s(t+\triangle t)-s(t)]^k}{k!}e^{s(t+\triangle t)-s(t)}, k=0,1,\cdots \tag{1}$$

where $s(t)=\int_0^t \lambda(s)ds$, and the $s(t+\triangle t)-s(t)$ is the mean value of this process. Besides, for each vertex, we connect m edges to the m different vertices already present in the network, where m follows a log-normal distribution with the parameters μ and σ, i.e. the density of m is given by

$$f(m)=\begin{cases} \dfrac{1}{m\sqrt{2\pi}\sigma}e^{-\frac{(\ln m-\mu)^2}{2\sigma^2}}, x>0 \\ 0, \qquad\qquad x\le 0 \end{cases} \tag{2}$$

Connection is simply linearly dependent on the degree of the target vertex for the benefit of the following derivation, $\phi(i)$, the probability of a connection to a vertex i, is denoted as

$$\phi(i)=\frac{k_i}{\sum\limits_j k_j} \tag{3}$$

where k_i is the degree of the target vertex i.

Termination is controlled by time t, which directly affect the scale of networks.

C. Degree Distribution of Evolving Networks

For the proposed model, as $t\rightarrow\infty$, the small initial network has little effect on the degree distribution, therefore, in the derivation, the initial network is ignored.

Given that the input rate of vertices is $\lambda(t)$, each new vertex links to m edges, and the number of vertices $N(t)$ which is independent of connections m, then the expected value of the total degree is

$$\sum_j k_j = 2E[m]E[N(t)] = 2\mu s(t) \tag{4}$$

For a new vertex with m edges, one of which connects to the existing vertex i, the corresponding probability is

$$P=\binom{m}{1}[\phi(i)][1-\phi(i)]^{m-1}\approx\frac{mk_i}{2\mu s(t)} \tag{5}$$

where $\phi(i)$ relates to Eq. 3.

Obviously, for one unit time from t to $t+1$, the probability that the degree of vertex i increases one is approximately $\lambda(t)\frac{mk_i}{2\mu s(t)}$. Then, we assume that the degree distribution of the vertex i $P_{t+1}(k)$ follows a master degree,

$$P_k(t+1)\approx\frac{\lambda(t)m(k-1)}{2\mu s(t)}P_{k-1}(t)+(1-\frac{\lambda(t)mk}{2\mu s(t)})P_t(k), \tag{6}$$

then, we have

$$P_k(t+1)-P_k(t)=\frac{\lambda(t)m}{2\mu s(t)}[(k-1)P_{k-1}(t)-kP_t(k)]. \tag{7}$$

Note that the differences of t and k are both 1, and based on the definition of the partial derivative, we get

$$\frac{\partial P_k(t)}{\partial t}=\frac{-\lambda(t)m}{2\mu s(t)}\cdot\frac{\partial kP_k(t)}{\partial k}. \tag{8}$$

Then we multiply both sides of Eq. 8 by k, and integrate over k, that is

$$\int_0^\infty kdk\frac{\partial P_k(t)}{\partial t}=\frac{-\lambda(t)m}{2\mu s(t)}\int_0^\infty kdk\frac{\partial kP_k(t)}{\partial k}. \tag{9}$$

Consider that the definition of the expectant degree k_i and employ the integration by part, we can deduce that

$$\begin{aligned} k_i &\approx \int_0^\infty kP_k(t)dk \\ &= -\{[k^2P_k(i,t)]_0^\infty - \int_0^\infty kP_k(t)dk\} \\ &= -\int_0^\infty kd[kP_k(t)] = -\int_0^\infty kdk\frac{\partial[kP_k(t)]}{\partial k}. \end{aligned} \tag{10}$$

Thus, Eq. 9 equivalently denotes as

$$\frac{\partial k_i}{\partial t}=\frac{\lambda(t)m}{2\mu s(t)}\cdot k_i \tag{11}$$

the general solution is $k_i=C[s(t)]^{\frac{m}{2\mu}}$, where C is a constant.

Combining with the boundary condition $k_i(s^{-1}(i))=m$, where $s^{-1}(i)$ is the inverse function of $s(t)$ which indicates the generation time of vertex i, then we have the solution

$$k_i=m[\frac{s(t)}{i}]^{\frac{m}{2\mu}}. \tag{12}$$

After k_i is obtained, the degree distribution of the evolving network is in demand. As previously stated, the random variable m follows a log-normal distribution denoted as $f(m)$, while the vertex i is randomly selected from the vertices of the evolving network. This means that i follows a uniform distribution on $[0, s(t)]$, denoted as $f(i)$, and their joint distribution is denoted as $f(i,m)$. And obviously, $f(m)$ as well as $f(i)$ are the marginal probability densities for joint probability density $f(i,m)$, and the connection variable is mutually independent of the selected vertex, which means that $f(i,m)=f(i)f(m)$.

Then, based on the definition of the degree distribution function, the joint degree distribution $P\{k_i(t)<k\}$ is derived as

$$\begin{aligned} P\{k_i(t)<k\} &= \iint\limits_{k_i(t)<k} f(i,m)didm \\ &= \iint\limits_{i>s(t)(\frac{m}{k})^{\frac{2\mu}{m}}} f(i)f(m)didm \\ &= \int_0^k\int_{s(t)(\frac{m}{k})^{\frac{2\mu}{m}}}^{s(t)}\frac{1}{m\sqrt{2\pi}\sigma s(t)}e^{-\frac{(\ln m-\mu)^2}{2\sigma^2}}didm \\ &= \int_0^k\frac{1-(\frac{m}{k})^{\frac{2\mu}{m}}}{m\sqrt{2\pi}\sigma}e^{-\frac{(\ln m-\mu)^2}{2\sigma^2}}dm. \end{aligned} \tag{13}$$

For Eq. 13, to solve the derivative of the joint degree distribution, by applying the Leibniz integral rule, we yield

$$
\begin{aligned}
p(k) &= P'\{k_i(t) < k\} = \frac{d}{dk}\int_0^k \frac{1-(\frac{m}{k})^{\frac{2\mu}{m}}}{m\sqrt{2\pi}\sigma} e^{-\frac{(\ln m-\mu)^2}{2\sigma^2}} dm \\
&= \frac{1-(\frac{k}{k})^{\frac{2\mu}{m}}}{m\sqrt{2\pi}\sigma} e^{-\frac{(\ln m-\mu)^2}{2\sigma^2}} \cdot k' - \frac{1-(\frac{1}{k})^{\frac{2\mu}{m}}}{m\sqrt{2\pi}\sigma} e^{-\frac{(\ln m-\mu)^2}{2\sigma^2}} \cdot 0' \\
&\quad + \int_0^k \frac{d}{dk}\frac{1-(\frac{m}{k})^{\frac{2\mu}{m}}}{m\sqrt{2\pi}\sigma} e^{-\frac{(\ln m-\mu)^2}{2\sigma^2}} dm \\
&= \int_0^k \frac{-m^{\frac{2\mu}{m}}}{m\sqrt{2\pi}\sigma} e^{-\frac{(\ln m-\mu)^2}{2\sigma^2}} \frac{d}{dk} k^{-\frac{2\mu}{m}} dm \\
&= \frac{\frac{2\mu}{m}}{k^{\frac{2\mu}{m}+1}} \int_0^k \frac{m^{\frac{2\mu}{m}-1}}{\sqrt{2\pi}\sigma} e^{-\frac{(m-\mu)^2}{2\sigma^2}} dm.
\end{aligned} \tag{14}
$$

So far, the degree distribution is obtained. Obviously, if we let m in Eq. 14 be a constant, the distribution is reduced to a Power-law distribution, i.e. the Power-law distribution is a specific case of the this distribution and the internal mechanism for only those SF networks with constant connections. However, for most networks and other situations, variable elements always exist, i.e. the scope of applicability of the obtained distribution is much broader. Therefore, we suggest that this distribution is a very promising direction to study adaptive or evolving networks.

Moreover, from the mathematical derivation, we discover that the degree distribution for evolving networks is a 2-dimensional joint random variable consisting of the selection of vertices that follows a uniform distribution and the connection of new vertices that follows a log-normal distribution. The Matthew effect relates to the boundary of the joint probability density function. Consequently, the determinants of a degree distribution are the selection rule (whether it is selected randomly or certainly), the newly added links (whether they are constants or variables following such as a log-normal distribution, and the connection mechanism (e.g. the Matthew effect), whichever directly affects the final degree distribution. We reveal this direction is more significant than the traditional view that the connection mechanism is the crucial factor for evolving networks.

Additionally, we uncover that the exponential term of a degree distribution for an evolving network $\frac{2\mu}{m}$ is traced to that one link has two degrees, see Eq. 4. If we break the limit of the network, and regard the value of m as the increment or decrement of income in economics, then, one income can be spent on different places, which means the exponential term can be not only $\frac{2\mu}{m}$. In that sense, by setting the exponential term as a parameter γ, we then figure out the general form of this kind of distribution in next section.

III. Subnormal Distributions for Evolving Networks and Their Statistical Properties

In this section, we mainly focus on the definition of the subnormal distribution derived from an evolving network and displaying its statistical properties.

A. Definition and Derivation of a Subnormal Distribution

For Eqs. 13 and 14, theoretically, consider that the connection variable m as a pure log-normal distribution with the parameters λ and σ^2, while i as a uniform distribution, then the variable k referred to Eq. 12 is defined as a subnormal distribution is this paper. Above all, we present the detail definition of a general Subnormal Distribution by its probability density function and cumulative distribution function in mathematics.

Definition 1 *A continuous random variable X is said to have a subnormal distribution with the parameter $\gamma{>}0$, if its probability density function (PDF) is given by*

$$
f(x) = \begin{cases} \dfrac{\gamma}{x^{\gamma+1}} \displaystyle\int_0^x \frac{t^{\gamma-1}}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln t-\mu)^2}{2\sigma^2}} dt, x > 0 \\ 0, \qquad\qquad\qquad\qquad\qquad x \le 0 \end{cases} \tag{15}
$$

or, equivalently, if its cumulative distribution function (CDF) is given by

$$
F(x) = \begin{cases} \displaystyle\int_0^x \frac{1-(\frac{t}{x})^{\gamma}}{\sqrt{2\pi}\sigma t} e^{-\frac{(\ln t-\mu)^2}{2\sigma^2}} dt, x > 0 \\ 0, \qquad\qquad\qquad\qquad x \le 0 \end{cases} \tag{16}
$$

where t is a log-normal random variable with the parameters λ and σ^2.

At the first place, we prove that the function $f(x)$ indeed is a PDF.

Theorem 1 *$f(x)$ in Eq. 15 is a PDF having the properties that $f(x){\ge}0$ and $\int_{-\infty}^{+\infty} f(x){=}1$.*

Proof: Apparently, for $x{<}0$, $f(x){=}0$, otherwise, $f(x){>}0$. Overall, $f(x){\ge}0$.

For all t, having $0{<}t{<}x{<}{+}\infty$, we can exchange the order of integral, i.e.

$$
\begin{aligned}
\int_{-\infty}^{+\infty} f(x) &= \int_0^{+\infty}\int_0^x \frac{\frac{\gamma}{x^{\gamma+1}} t^{\gamma-1}}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln t-\mu)^2}{2\sigma^2}} dtdx \\
&= \int_0^{+\infty}\int_t^{+\infty} \frac{\frac{\gamma}{x^{\gamma+1}} t^{\gamma-1}}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln t-\mu)^2}{2\sigma^2}} dxdt \\
&= \int_0^{+\infty} [-\frac{1}{\gamma}x^{-\gamma}]_t^{+\infty} \frac{t^{\gamma-1}}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln t-\mu)^2}{2\sigma^2}} dt \\
&= \int_0^{+\infty} \frac{1}{t\sqrt{2\pi}\sigma} e^{-\frac{(\ln t-\mu)^2}{2\sigma^2}} dt,
\end{aligned} \tag{17}
$$

let $y{=}\frac{\ln t-\mu}{\sigma}$, that is $dy{=}\frac{1}{\sigma t}dt$, then by substitution y and dy into Eq. 17, we have

$$
\int_{-\infty}^{+\infty} f(x) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = 1. \tag{18}
$$

The results follow. ■

As stated above, a subnormal variable is a joint probability density of a uniform variable and a log-normal distribution, thus we have Th. 2.

Theorem 2 *Given random variables X and Y are mutually independent, and X follows a uniform distribution on [0,a], Y follows a log-normal distribution with the parameters μ and σ, if*

$$
Z = Y(\frac{a}{X})^{\gamma}, \tag{19}
$$

then, the random variable Z follows a subnormal distribution with the parameter γ.

Proof: For $z{\le}0$,

$$F_Z(z) = P\{Y(\frac{a}{X})^\gamma \le z\} = 0. \tag{20}$$

Otherwise, for $z{>}0$, since X and Y are mutually independent, the joint probability density $f_Z(z)$ is the product of marginal probability densities $f_X(x)$ and $f_Y(y)$, then

$$\begin{aligned} F_Z(z) &= P\{Y(\frac{a}{X})^\gamma \le z\} = \iint\limits_{y(\frac{a}{x})^\gamma \le z} f_Z(z)dxdy \\ &= \iint\limits_{y(\frac{a}{x})^\gamma \le z} f(x,y)dxdy = \iint\limits_{x>a(\frac{y}{z})^\gamma} f_X(x)f_Y(y)dxdy \\ &= \int_0^z \int_{a(\frac{y}{z})^\gamma}^a \frac{1}{\sqrt{2\pi}\sigma a x} e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}} dxdy \\ &= \int_0^z \frac{1-(\frac{x}{z})^\gamma}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} dx. \end{aligned} \tag{21}$$

Consider both Eqs. 20 and 21, we have

$$F_Z(z) = \begin{cases} \int_0^z \frac{1-(\frac{x}{z})^\gamma}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} dz, & z > 0 \\ 0, & z \le 0 \end{cases} \tag{22}$$

obviously, $F_Z(z)$ follows the CDF of a subnormal distribution.

Further,

$$f_Z(z) = F'_Z(z) = \begin{cases} \frac{\gamma}{z^{\gamma+1}} \int_0^z \frac{x^{\gamma-1}}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} dx, & z > 0 \\ 0, & z \le 0 \end{cases} \tag{23}$$

which indicates the PDF of a subnormal distribution.

In summary, the results follow. ∎

Additionally, the integration of Def. 1 is difficult to calculate in practical situation. To address this issue, we can also use the error function, also called Gaussian error function, to denote Eq. 15 and 16 in Def. 1, which is presented in Th. 3.

Theorem 3 *The PDF of a subnormal distribution is given by*

$$f(x) = \begin{cases} \frac{\gamma}{2x^{\gamma+1}} e^{\mu+\frac{1}{2}\gamma^2\sigma^2} [1 + \text{erf}(\frac{\ln x - \mu - \gamma\sigma^2}{\sqrt{2}\sigma})], \\ \qquad\qquad x > 0 \\ 0, \qquad\qquad x \le 0 \end{cases} \tag{24}$$

and its CDF is given by

$$F(x) = \begin{cases} \frac{1}{2}\{\text{erf}(\frac{\ln x - \mu}{\sqrt{2}\sigma}) - \text{erf}[\frac{\ln x - \mu - (\gamma+1)\sigma^2}{\sqrt{2}\sigma}]\}, \\ \qquad\qquad x > 0 \\ 0, \qquad\qquad x \le 0 \end{cases} \tag{25}$$

where t is a log-normal random variable with the parameters λ and σ^2, and $\text{erf}(x)$ is the Gaussian error function.

Proof: Consider Def. 1, for $x{\le}0$, both $f(x)$ and $F(x)$ are equal to 0.

Otherwise, for $x{>}0$, we set $y{=}\frac{\ln t-\mu-\gamma\sigma^2}{\sqrt{2}\sigma}$, and $dy{=}\frac{1}{\sqrt{2}\sigma t}dt$, notice the integral range, then PDF can be expressed as

$$\begin{aligned} f(x) &= \frac{\gamma}{x^{\gamma+1}} \int_{-\infty}^{\frac{\ln x-\mu-\gamma\sigma^2}{\sqrt{2}\sigma}} \frac{e^{\gamma(\sqrt{2}\sigma y+\gamma\sigma^2+\mu)}}{\sqrt{\pi}\sigma} e^{-\frac{(\sqrt{2}y+\gamma\sigma)^2}{2}} dy \\ &= \frac{\gamma}{x^{\gamma+1}} e^{\mu+\frac{1}{2}\gamma^2\sigma^2} \int_{-\infty}^{\frac{\ln x-\mu-\gamma\sigma^2}{\sqrt{2}\sigma}} \frac{1}{\sqrt{\pi}} e^{-y^2} dy \\ &= \frac{\gamma}{x^{\gamma+1}} e^{\mu+\frac{1}{2}\gamma^2\sigma^2} (1 + \int_0^{\frac{\ln x-\mu-\gamma\sigma^2}{\sqrt{2}\sigma}} \frac{1}{\sqrt{\pi}} e^{-y^2} dy) \\ &= \frac{\gamma}{2x^{\gamma+1}} e^{\mu+\frac{1}{2}\gamma^2\sigma^2} [1 + \text{erf}(\frac{\ln x - \mu - \gamma\sigma^2}{\sqrt{2}\sigma})] \end{aligned} \tag{26}$$

where $\text{erf}(x){=}\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$.

Applying the same method, for $x{>}0$, we solve CDF

$$F(x) = \frac{1}{2}\{\text{erf}(\frac{\ln x - \mu}{\sqrt{2}\sigma}) - \text{erf}[\frac{\ln x - \mu - (\gamma+1)\sigma^2}{\sqrt{2}\sigma}]\}. \tag{27}$$

The results follow. ∎

Remark 1 *The PDF can also expressed as the complementary error function and standard normal cumulative distribution function*

$$\begin{aligned} f(x) &= \frac{\gamma}{x^{\gamma+1}} e^{\mu+\frac{1}{2}\gamma^2\sigma^2} \int_{\frac{\mu+\gamma\sigma^2-\ln x}{\sqrt{2}\sigma}}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} dy \\ &= \frac{\gamma}{2x^{\gamma+1}} e^{\mu+\frac{1}{2}\gamma^2\sigma^2} \text{erfc}(\frac{\mu+\gamma\sigma^2-\ln x}{\sqrt{2}\sigma}) \\ &= \frac{\gamma}{x^{\gamma+1}} e^{\mu+\frac{1}{2}\gamma^2\sigma^2} \Phi(\frac{\ln x - \mu - \gamma\sigma^2}{\sigma}). \end{aligned} \tag{28}$$

where $\text{erfc}(x){=}\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$, and $\Phi(x){=}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{t^2}{2}}dt$.

Analogously, CDF can also be denoted as

$$\begin{aligned} F(x) &= \frac{1}{2}\{\text{erfc}(\frac{\mu - \ln x}{\sqrt{2}\sigma}) - \text{erfc}[\frac{\mu + (\gamma+1)\sigma^2 - \ln x}{\sqrt{2}\sigma}]\} \\ &= \Phi(\frac{\ln x - \mu}{\sigma}) - \Phi(\frac{\ln x - \mu - (\gamma+1)\sigma^2}{\sigma}). \end{aligned} \tag{29}$$

B. Some Statistical Properties of a Subnormal Distribution

In practical situations, a distribution function is of the nonessential; instead some special properties are more useful. In this subsection, we provide some common statistical properties in numerals such as the expected value, variance, etc, and display their solving processes. The default of γ is non-zero in all derivations.

1) Expectation Value: In probability theory, the expectation value of a random variable is intuitively the long-run average value of repetitions of the experiment it represents. It is the weighted average of all possible values. Practically, if $Z{=}G(x,y)$ is a continuous random variable having a joint probability density function $f(x,y)$, and $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|G(x,y)|f(x,y)dxdy{<}{+\infty}$ then the expectation value of Z is given by

$$E(Z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(x,y)f(x,y)dxdy. \tag{30}$$

Then, we have

Theorem 4 *The expectation value of a general subnormal variable with the parameter* $\gamma\neq1$ *is* $\frac{\gamma}{\gamma-1}e^{\mu+\frac{\sigma^2}{2}}$.

Proof: A subnormal variable Z is jointed by mutually independent random variables, a uniform variable X distributed on $[0, a]$ and a log-normal variable Y distributed distributed on $(0, +\infty)$, according to Th. 2, and can be expressed as

$$z = y(\frac{a}{x})^{\frac{1}{\gamma}}. \tag{31}$$

For $\gamma\neq1$, we have

$$\begin{aligned} E(Z) &= \int_0^{+\infty}\int_0^a y(\frac{a}{x})^{\frac{1}{\gamma}} f(x,y)dxdy \\ &= \int_0^{+\infty}\int_0^a y(\frac{a}{x})^{\frac{1}{\gamma}} f(x)f(y)dxdy \\ &= \int_0^{+\infty}\int_0^a y(\frac{a}{x})^{\frac{1}{\gamma}} \frac{1}{ay\sqrt{2\pi}\sigma}e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}dxdy \\ &= [\frac{\gamma}{\gamma-1}x^{1-\frac{1}{\gamma}}]_0^a \cdot \int_0^{+\infty}\frac{a^{\frac{1}{\gamma}-1}}{\sqrt{2\pi}\sigma}e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}dy, \end{aligned} \tag{32}$$

let $t{=}\frac{\ln y-\mu}{\sigma}$, then

$$E(Z) = \frac{\gamma}{\gamma-1}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}+\sigma y+\mu}dt = \frac{\gamma}{\gamma-1}e^{\mu+\frac{\sigma^2}{2}}. \tag{33}$$

The result follows. ■

2) Variance: Additionally, we employ the variance to display the dispersion degree measuring how far a set of numbers is spread out. Specifically, for a variable X with the expectation value $E(X)$, then variance is given by $Var(X){=}E([X-E(X)]^2)$.

Then, we carry out the variance of a subnormal distribution.

Theorem 5 *The variance of a general subnormal variable with the parameter* $\gamma\neq1$ *and* $\gamma\neq2$ *is* $e^{2\mu+\sigma^2}[\frac{\gamma}{\gamma-2}e^{\sigma^2}-\frac{\gamma^2}{(\gamma-1)^2}]$.

Proof: The variance of a subnormal variable Z can be expressed as

$$Var(Z) = E(Z^2) - [E(Z)]^2 \tag{34}$$

where, by utilizing Eq. 31,

$$\begin{aligned} E(Z^2) &= \int_0^{+\infty}\int_0^a y^2(\frac{a}{x})^{\frac{2}{\gamma}}\frac{1}{ay\sqrt{2\pi}\sigma}e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}dxdy \\ &= \frac{\gamma}{\gamma-2}[x^{1-\frac{2}{\gamma}}]_0^a \cdot \int_0^{+\infty}\frac{ya^{\frac{2}{\gamma}-1}}{\sqrt{2\pi}\sigma}e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}dy, \end{aligned} \tag{35}$$

here, we render $t{=}\frac{\ln y-\mu}{\sigma}$,

$$\begin{aligned} E(Z^2) &= \frac{\gamma}{\gamma-2}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}+2\sigma t+2\mu}dt \\ &= \frac{\gamma}{\gamma-2}e^{2\mu+2\sigma^2}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(t-2\sigma)^2}{2}}dt \\ &= \frac{\gamma}{\gamma-2}e^{2\mu+2\sigma^2}, \end{aligned} \tag{36}$$

and note that Th. 4

$$[E(Z)]^2 = \frac{\gamma^2}{(\gamma-1)^2}e^{2\mu+\sigma^2}. \tag{37}$$

Then, Eq. 36 and 37 are substituted into 34, we have

$$Var(Z) = e^{2\mu+\sigma^2}[\frac{\gamma}{\gamma-2}e^{\sigma^2}-\frac{\gamma^2}{(\gamma-1)^2}]. \tag{38}$$

The result follows. ■

Remark 2 *Apparently, if* $\gamma{=}1$ *or* $\gamma{=}2$*, the nonexistence of the variance follows. Furthermore, the variance should be nonnegative, that is*

$$e^{2\mu+\sigma^2}[\frac{\gamma}{\gamma-2}e^{\sigma^2}-\frac{\gamma^2}{(\gamma-1)^2}] \geq 0. \tag{39}$$

For $\gamma\neq1$ *or* 2

$$e^{\sigma} \geq \frac{\gamma(\gamma-2)}{(\gamma-1)^2}. \tag{40}$$

Analogously,

$$\sigma \geq \ln\frac{\gamma(\gamma-2)}{(\gamma-1)^2}. \tag{41}$$

Finally, we conclude that only if $\gamma\neq1$ *or* 2 *and* $\sigma\geq\ln\frac{\gamma(\gamma-2)}{(\gamma-1)^2}$*, the variance of the subnormal distribution will exist.*

Remark 3 *From Th. 4 and 5, we obtain the relationship between the parameters* μ*,* σ*,* γ *and the expectation value* $E(X)$ *and the variance* $Var(X)$*. Specifically,* μ *is denoted as*

$$\begin{aligned} \mu &= \ln[\frac{\gamma}{\gamma-1}E(X)] - \frac{1}{2}\ln\{\frac{\gamma(\gamma-2)}{(\gamma-1)^2}(1+\frac{Var(X)}{[E(X)]^2})\} \\ &= \ln[\frac{\gamma}{\gamma-1}E(X)] - \frac{1}{2}\sigma^2, \end{aligned} \tag{42}$$

and σ *is*

$$\sigma = \sqrt{\ln\{\frac{\gamma(\gamma-2)}{(\gamma-1)^2}+\frac{\gamma(\gamma-2)}{(\gamma-1)^2}\cdot\frac{Var(X)}{[E(X)]^2}\}}. \tag{43}$$

From the derivation, we can see that the variance is possibly nonexistent. And if $\gamma\to\infty$, the variance numerically equals to $e^{2\mu+\sigma^2}[e^{\sigma^2}-1]$, equivalent to the log-normal distribution. And we can learn from the Eq. 42 and 43 that the values of both μ and σ are relatively low, directing the variance to a low value, which agrees with the assumption in the process of derivation of a subnormal distribution.

3) Other Statistical Properties: For any real number k, the kth moment variable X is given by $E(X^k)$, thus we have

Theorem 6 *The* kth *moment of a general subnormal variable with the parameter* $\gamma\neq k$ *is* $\frac{\gamma}{\gamma-k}e^{k\mu+\frac{1}{2}k^2\sigma^2}$.

Proof:

$$\begin{aligned} E(Z^k) &= \int_0^{+\infty}\int_0^a y^k(\frac{a}{x})^{\frac{k}{\gamma}}\frac{1}{ay\sqrt{2\pi}\sigma}e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}dxdy \\ &= \frac{\gamma}{\gamma-k}[x^{1-\frac{k}{\gamma}}]_0^a \cdot \int_0^{+\infty}\frac{y^{k-1}a^{\frac{k}{\gamma}-1}}{\sqrt{2\pi}\sigma}e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}dy, \end{aligned} \tag{44}$$

let $t{=}\frac{\ln y-\mu}{\sigma}$, then

$$\begin{aligned} E(Z^k) &= \frac{\gamma}{\gamma-k}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}+k\sigma t+k\mu}dt \\ &= \frac{\gamma}{\gamma-k}e^{k\mu+\frac{1}{2}k^2\sigma^2}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(t-k\sigma)^2}{2}}dt \\ &= \frac{\gamma}{\gamma-k}e^{k\mu+\frac{1}{2}k^2\sigma^2}. \end{aligned} \tag{45}$$

The result follows. ∎

The arithmetic coefficient of variation $CV(X)$ is the ratio $\frac{SD(X)}{E(X)}$, where $SD(X)=\sqrt{Var(X)}$.

Theorem 7 *The $CV(X)$ of a general subnormal variable with the parameter $\gamma\neq 1$ and $\gamma\neq 2$ is $\frac{\gamma}{\gamma-k}e^{k\mu+\frac{1}{2}k^2\sigma^2}$.*

Proof:

$$\begin{aligned} CV(X) &= \frac{SD(X)}{E(X)} = \frac{\sqrt{e^{2\mu+\sigma^2}[\frac{\gamma}{\gamma-2}e^{\sigma^2}-\frac{\gamma^2}{(\gamma-1)^2}]}}{\frac{\gamma}{\gamma-1}e^{\mu+\frac{\sigma^2}{2}}} \\ &= \sqrt{\frac{(r-1)^2}{r(r-2)}e^{\sigma^2}-1}. \end{aligned} \tag{46}$$

∎

The partial expectation (PE) value of a random variable X with respect to a threshold ξ is denoted as $PE(\xi) = \int_{\xi}^{\infty} xf(x)dx$.

Theorem 8 *The partial expectation value of variation $PE(X)$ of a general subnormal variable with the parameter $\gamma\neq 1$ is $\frac{\gamma}{\gamma-1}e^{\mu+\frac{\sigma^2}{2}}\Phi(\sigma-\frac{\ln\xi-\mu}{\sigma})$.*

Proof:

$$\begin{aligned} PE(X) &= \int_{\xi}^{\infty} xf(x)dx \\ &= \int_{\xi}^{+\infty}\int_{0}^{x}\frac{\frac{\gamma}{x^\gamma}t^{\gamma-1}}{\sqrt{2\pi}\sigma}e^{-\frac{(\ln t-\mu)^2}{2\sigma^2}}dtdx \\ &= \frac{\gamma}{\gamma-1}\int_{\xi}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(\ln t-\mu)^2}{2\sigma^2}}dt, \end{aligned} \tag{47}$$

let $y=\frac{\ln t-\mu}{\sigma}-\sigma$, we have

$$\begin{aligned} PE(X) &= \frac{\gamma}{\gamma-1}\int_{\frac{\ln\xi-\mu}{\sigma}-\sigma}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}+\mu+\frac{\sigma^2}{2}}dy \\ &= \frac{\gamma}{\gamma-1}e^{\mu+\frac{\sigma^2}{2}}\int_{\frac{\ln\xi-\mu}{\sigma}-\sigma}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \\ &= \frac{\gamma}{\gamma-1}e^{\mu+\frac{\sigma^2}{2}}\Phi(\sigma-\frac{\ln\xi-\mu}{\sigma}). \end{aligned} \tag{48}$$

where Φ is the standard normal cumulative distribution function.

The result follows. ∎

In addition, many other statistical properties, such as the characteristic function $E[e^{itX}]$, the moment generating function $E[e^{tX}]$ (which are easily proved divergent), mode, peak (which are without analytic solutions), etc, are omitted in this paper.

IV. Simulation and Analysis

In this section, we first provide the analysis of the influence of the different parameters γ, μ, and σ of Def. 1 on the curve of the subnormal distribution. Then, we carry out some simulations of fitting the subnormal distribution to the the degree distribution of evolving network in theory and social network in practical, as well as the wealth distribution in economics.

A. Parameter Analysis

To explore the influence of the parameters γ, μ, and σ on the curve of the subnormal distribution of Eq. 15, we let two of them be constants, and the other one deals with three distinct values, then the corresponding plots are drawn. The results are illustrated in Fig. 1.

For the parameter γ, as shown in Fig. 1(a), the higher value makes the curve taller and thinner, which means the peak grows higher. However, a change from 2 to 10 is much more obvious than from 10 to 40. And the influence of γ on the mode (x of the peak) is inconspicuous. As a result, γ speeds up ascent rate before mode, and descent rate after the mode. This character is very similar to the exponent index of the Power-law distribution denoting the slope in logarithmic coordinates.

We can see from Fig. 1(b), with the rise of the parameter μ, the curve becomes shorter and fatter, and the change is very apparent, even μ rarely increases by 1. That is to say, μ is positively related to the mode, but has a visibly negative effect on the peak.

As the last parameter σ illustrated in Fig. 1(c), the higher value makes the curve taller and thinner. For the peak, the influence of σ is similar to γ, in other words, with the increase of σ, the peak rises, but very slow. Different from γ, σ is negatively related to the mode in the exponential, the higher value makes the mode much lower. Fig. 1(c) displays that the corresponding x move leftward.

In summary, γ and σ positively affects the peak of a subnormal distribution indicating the height of the curve, while μ does it negatively. Besides, μ has a positive influence on the mode indicating the location of the peak. Contrarily, σ has a negative influence, and γ has an inapparent influence on the mode. Furthermore, γ and σ have a positive relationship with the rate of rise and fall, the latter is more severe, otherwise, μ has an obviously inverse relationship. With these relationships, we can determine the approximate curve shape of the required subnormal distribution.

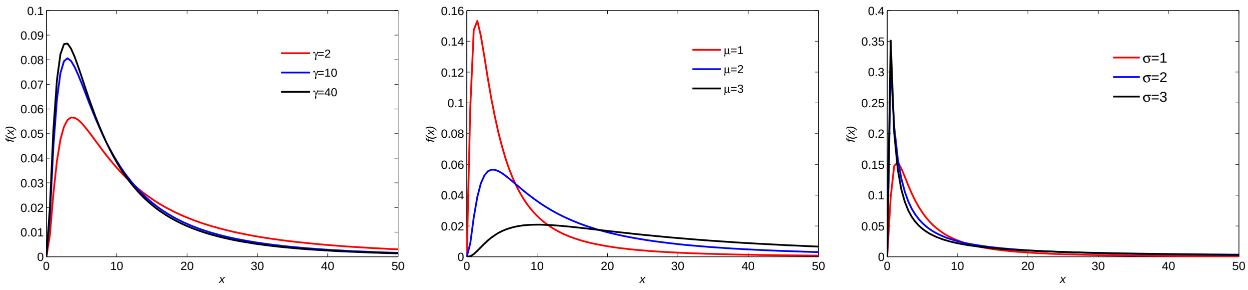
B. Fitting Subnormal Distribution to Other Distributions

To fit these distributions, the subnormal distribution is required to be discretized, in other words, x can only be integers. To clearly compare the fitness of two distributions, we apply the Pearson product-moment correlation coefficient. Specifically, for vectors of subnormal variables X and other variables, such as the degree distribution of evolving network Y, the correlation coefficient is denoted as

$$\rho_{X,Y} = \frac{E[(X-E(X))(Y-E(Y))]}{SD(X)SD(Y)} \tag{49}$$

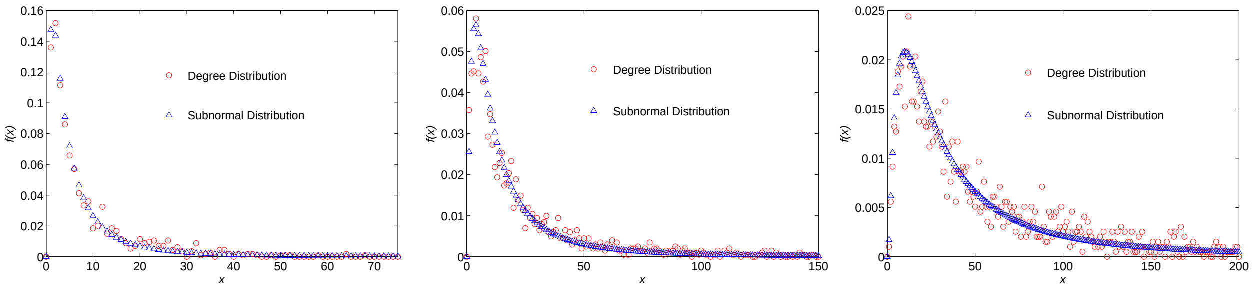
1) Fitting the Degree Distribution to an Evolving Network: The modeling of the evolving network is referred to Section $II.B$.

In the initialization, we build a NW small-world network with 20 vertices as the initial network, each connects to 2 neighbors and has a 50% chance to add a link to others. In the evolving process, for simplicity, we use a homogeneous Poisson process instead of a nonhomogeneous one since the growing is essentially independent of the degree distribution. Specifically,



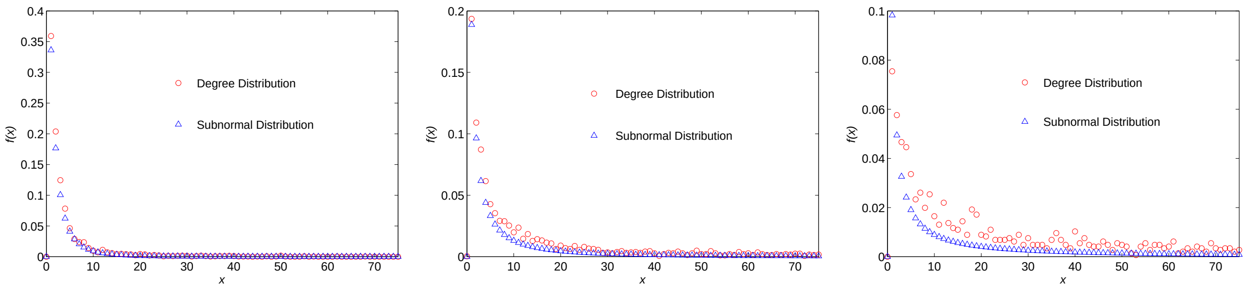
(a) The subnormal curve with different γs. (b) The subnormal curve with different μs. (c) The subnormal curve with different σs.

Figure 1. The influence of different values of parameters on the curve of the subnormal distribution. The constant arguments are set as (a) $\mu == 2$ and $\sigma == 1$, (b)$\gamma == 2$ and $\sigma == 1$, and (c)$\gamma == 2$ and $\mu == 1$.



(a) μ=1. (b) μ=2. (c) μ=3.

Figure 2. The comparison of the degree distributions of evolving networks marked with red circles and their corresponding subnormal distributions marked with blue circles, all have that σ=1 and γ=2, but different μs.



(a) σ=1. (b) σ=2. (c) σ=3.

Figure 3. The comparison of the degree distributions of evolving networks marked by red circles and their corresponding subnormal distributions marked by blue circles, all have that μ=0 and γ=2, but different σs.

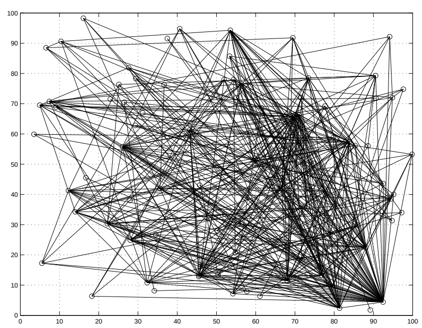


Figure 4. An illustration of an evolving network produced by λ=1, t=100, and μ=1, σ=1.

$\diamond$ Set the values of input rate λ and termination time t;
$\diamond$ Generate exponential distribution random values with λ, denoted as t_i, i=$\{1, 2, 3, \cdots\}$;
$\diamond$ If the cumulative time $T_i \leq t$, let $T_i = T_i + t_i$, else stop and output the temporal series.

Then we have the temporal series of arrival vertices. In the process of connection, we employ the $lognrnd$ function in Matlab to produce the number of connections, the result is rounded by $round$ function, and the roulette algorithm is applied to simulate Eq. 3. As a result, a relatively sparse sample of a evolving network produced by λ=1, t=100, and μ=1, σ=1 of log-normal is demonstrated in Fig. 4.

After an evolving network is obtained, we utilize the association matrix to record its degrees, and plot the corresponding degree distribution. Three evolving networks with different μs and σs are recorded. Then, with the same parameters, we also use Def. 1 to draw the distributions of the discrete subnormal variables in the same coordinate. The results are shown in Figs. 2 and 3. Since the distributions are obtained from networks, we let $\gamma \approx 2$.

Table I
THE CORRELATION COEFFICIENT OF THE DEGREE DISTRIBUTIONS OF EVOLVING NETWORKS AND THEIR CORRESPONDING SUBNORMAL DISTRIBUTIONS.

Parameter \ Value	1	2	3
μ	0.9937	0.9810	0.9639
σ	0.9985	0.9908	0.9207

In Fig. 2, different μs of the degree distributions and subnormal distributions are compared. To reduce the interruption of σ, we let it be the smallest integer 1. Since the tails of both distributions are extremely close, we take 150 , 200, and 250 values of x for μ=1, 2, and 3 for illustration, respectively. By Eq. 49, the similarity of both distributions is calculated and listed in Tab. I, (first row). From the results, we see that the correlation coefficient of two distributions are very high, all above 95%, implying that the degree distribution of evolving networks is highly similar to the subnormal distribution with the same parameters. And we also observe that the higher μ the lower the correlation coefficient, the deviation degree of both distributions becomes more obvious.

In Fig. 3, different σs of the degree distributions and the subnormal distributions are compared. To reduce the interruption of μ, we let it be the possibly smallest integer 0. 75, 100, and 125 values of x for σ=1, 2, and 3 are illustrated. By Eq. 49, the similarity of both distributions is calculated and listed in Tab. I, (second row). The results also show a good evaluation of similarity of both distributions, all above 90%. However, the higher σ will rise the variance of connections, and consequently, the deviation degree is more apparent, leading to a lower correlation coefficient.

Above all, the subnormal distribution well fits the degree distribution of evolving networks, and the result is better for relatively low values of μ and σ.

2) Fitting Distribution of Real Networks: First, a collaboration network of Arxiv Astro Physics from the e-print arXiv which covers scientific collaborations between authors of papers submitted to Astro Physics category is fitted to the subnormal distribution [26].

For the collaboration network, if an author i co-authored a paper with author j, the network contains a undirected edge from i to j. The data covers papers in the period from January 1993 to April 2003. It begins within a few months of the inception of the arXiv, and thus represents essentially the complete history of its ASTRO-PH section. The number of nodes is $18,772$, and of edges is $39,6160$. From this network, we can obtain its related association matrix, and calculate the degree distribution, shown in Fig. 5 as red scatters.

From the data, we can estimate the expectation value of degree of collaboration network, which is $E(cn)$=21.1038≈21. Assuming that one subnormal distribution fits this network degree, as we know, the γ for networks approximatively equals to 2, then by employing Th. 4, the relationship of the parameters μ and σ can be denoted as $\mu+\frac{\sigma^2}{2}=\frac{\log[E(cn)]}{2}\approx1.522$. Thus, we let μ=1.15, σ=0.86. The illustration of blue scatters of subnormal distribution is shown in Fig. 5.

Because in previous studies, the Power-law distribution is most commonly utilized to describe social networks, we also introduce a Power-law distribution for comparison which is denoted as $f(x)=\gamma m^\gamma x^{-\gamma-1}$ $(x>\mu)$, where γ=2. The expectation value of Power-law distribution, if the distribution fits the network degree, can be denoted as $2m=E(cn)$. Then, to simulate the collaboration network, we set the parameter $m\approx11$. Its black scatters are shown in Fig. 5.

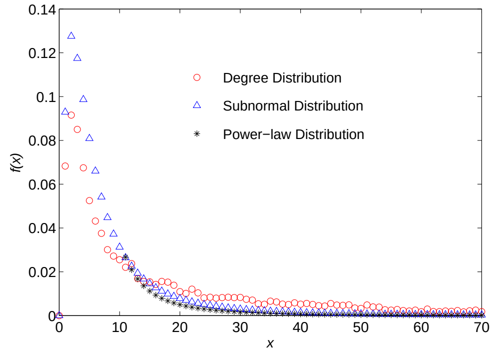


Figure 5. The comparison of the degree distributions of collaboration network marked by red circles and its corresponding subnormal distributions marked by blue circles, Power-law distribution marked by black circles.

Again, the Eq. 49 is employed to calculate the correspondence of the subnormal, Power-law distribution with the degree distribution of the collaboration network. The first 200 values are taken into calculation, but for a clear displaying, only the first 70 values are shown in Fig. 5. The correlation coefficient with the subnormal distribution is 98.68%, while that with the Power-law distribution is 76.31% only (97.45% and 94.45% if only the values beyond 11 are compared).

Obviously, the subnormal distribution fits the collaboration network degree distribution much better in tendency as well as correlation. Actually, the Power-law distribution only describes the tails, i.e. those authors having many collaborations, but ignores those ones having a few only. From our perspective, the lower values are much more to be considered, since they represent the majority. Specifically, the highest collaboration number is 3 in the collaboration network, and can be described as the most probable value of authors. Additionally, the Power-law is monotonously decreasing, but social networks usually grow with a peak, indicating that the poorest are not the most, and the subnormal distribution perfectly fits that character.

3) Fitting the Wealth Distribution: As mentioned before, the subnormal can also describe the inequality distribution in economy. Therefore, we try to fit the subnormal distribution to the personal wealth distribution.

However, wealth distribution is not easy to measure, since people avoid reporting their total wealth routinely. And the statistical data are often quartered or more, which makes it difficult to seek precise values of each wealth level. But when a person dies, all assets must be reported for the purpose of inheritance tax. Using these data and an adjustment procedure, the British tax agency, the Inland Revenue (IR), reconstructed wealth distribution of the whole UK population. We mainly employ the 2008 to 2010 data of total gross capital value obtained from their Web site [27]. These data divide the people into five levels of net estate, $\pounds$ 0 to $50,000$, $50,000$ to $100,000$, $100,000$ to $200,000$, $200,000$ to $300,000$, $300,000$ to $500,000$, $500,000$ to $1,000,000$, $1,000,000$ to $2,000,000$, and over $2,000,000$, we average the level as $\{0.25, 0.75, 1.5, 2.5, 4, 7.5, 10, 25\}*10^5$. And the number of people for each level are $\{3053, 2382, 4207, 2515, 1682, 889, 224, 98\}$. From these data, we obtain the wealth distribution. Since the scatters are sparse, we connect them as a plot, see the red plot in Fig. 6.

As the data are too poor to calculate the expectation value, so we can not employ the previous method to evaluate the parameters of the fit subnormal distribution. Lots of parameters are tested to present better results, and one of them is that $\gamma=1.9$, μ=0.6, and σ=0.5, see the blue plot of subnormal distribution in Fig. 6.

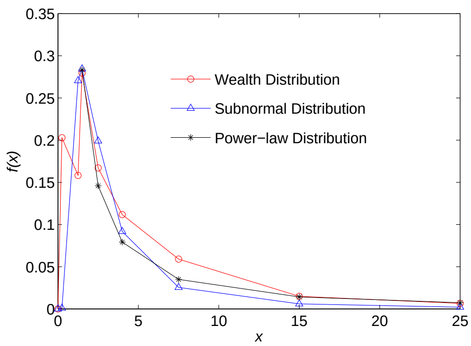


Figure 6. The comparison of the degree distributions of collaboration network marked by red circles and its corresponding subnormal distributions marked by blue circles, Power-law distribution marked by black circles.

For the Power-law distribution, except the expectation value, we can also calculate the slope of the date in logarithm to obtain γ, and further m. The result is that γ=0.3 and $m\approx$0.34. Then, the Power-law distribution in black plot is illustrated in Fig. 6.

By Eq. 49, the result of correlation coefficient of the subnormal distribution with wealth distribution is 80.53%, while the Power-law distribution is 68.35% only, (99.24% and 97.55% if only the values beyond 2.5 are compared). Once again, we display that the Power-law only fits the tail, but the subnormal approximatively fits the whole plot. For the wealth, the head of the distribution represent the lower-middle-classes, which play significant role in the social wealth and stability and should not be ignored. Therefore, the fit of a subnormal distribution to wealth distribution is worth exploring. However, we observe that at the point of 0.25 to 0.75, the tendency of plot declines, the reason is that negative asset owners are included in 0 to 0.5 making the value at 0.25 higher, and for both subnormal and Power-law distribution negative values are ignored.

V. Discussion and Outlook

In this paper, we have provided a new distribution called *subnormal distribution* to simulate the distributions of the degree of evolving networks such as SF networks, real networks such as social networks, economic distribution, and other uneven distributions. Essentially, from the derivation, we discover that the degree distribution is a 2-dimensional joint probability density consisting of the selection of vertices that follows a uniform distribution and the connection of new vertices that particular follows a log-normal distribution in here, while the inequality of the Matthew effect relates to the boundary of the joint probability density function. Actually, the connection may also be another distribution like the uniform distribution in some special cases, but in this paper we employ the log-normal distribution and obtain the subnormal distribution. In further work, we may continue the study to other joint probability density functions as well.

We find that the subnormal distribution can also describe the wealth distribution, which can be explained by network theory. The income is regarded as the new coming vertex, and the arrangement or consumption for this income is its connections to the network. The whole network is the total wealth, each one vertex is the wealth of the individual. Apparently, the inequality of the Matthew effect influences the consumption of individual, for example, people are more likely to buy goods with famous brand, and these firms are getting richer, which is highly similar with the connection process of evolving networks. Therefore, beyond the evolving networks, we speculate that the subnormal distribution can be universally employed to describe the distribution with inequality and growing which requires further studies.

Additionally, as Gibrat's law describes, the size of a firm and its relative rate of growth are independent, which is also available for evolving networks. One result of Gibrat's law is that processes characterized by Gibrat's law converge to a limiting distribution, which may be log-normal or power law, depending on more specific assumptions about the stochastic growth process. Furthermore, we precisely deduce that this kind of distribution based on the unequal growth follows that the poor tends to be poorer; while the rich richer is subnormal in this paper, which holds both characters of log-normal and Power-law. For the special situation that the connection or income is constant, the subnormal distribution reduces to a Power-law, and if the individual is constant and non-random, the distribution reduces to a log-normal one. In that sense, we can also argue that the subnormal distribution is a combination of log-normal and Power-law with the peak of the former and the tail of the latter. Above all, we agree with Gibrat's law that the income/connection is log-normally distributed, while the final wealth/degree follows a subnormal distribution.

However, we also have a dilemma on how to confirm the parameters of subnormal distribution. In this paper, we can only roughly decide the influence of γ, μ, and σ on the tendency of a subnormal curve, but not accurately deduce their attributes for a subnormal function. One possible solution is to solve the mode and median of the probability density function, which greatly contributes to determine the function curve and is our goal for the next stage.

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