Duplex Fraction Method to Compute the Determinant of a 4 × 4 Matrix

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Abstract. In this paper, we present a new method to compute the determinant of a real matrix of order 4.

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1. Introduction

In linear algebra and matrix theory, the determinant of a square matrix is very important. The basic formula to compute the determinant of a square matrix of order n, such as $A_n = [a_{ij}]_{n\times n}$, is equal to

$$D(A) = det(A) = |A| = \sum_{j_1 \dots j_n \in s_n} sgn(j_1 \dots j_n)\, a_{1j_1} \dots a_{nj_n} \,.$$

where $sgn(j_1 \dots j_n) = \begin{cases} +1 & , if\ j_1 \dots j_n\ is\ an\ even\ permutation \\ -1 & , if\ j_1 \dots j_n\ is\ an\ odd\ permutation \end{cases}$.

There are many methods to compute the determinant of a square matrix. C. L. Dodgson in 1866 [1], and A. Salihu in 2012 [3], presented two methods for compute the determinant of square matrices of order n that we will use of their methods to establish a new method in this article.

2. The main definitions and lemmas

First, we will establish a new definition related to the fraction of two matrices of order 2, that we have called this definition by the name of *duplex fraction* or *duplex division*.

Definition 2. 1. *Let* $A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ *and* $B_2 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ *are two real matrices of order 2. If* $det(B_2) \neq 0$ *and* $b_{ij} \neq 0$ $(\forall\, i, j = 1, 2)$, *then the duplex fraction (or duplex division) of the determinant of* A_2 *on* B_2 *is defined as follows*

$$\frac{|A|}{|B|} = \frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}{\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}} = \frac{\begin{vmatrix} \frac{a_{11}}{b_{11}} & \frac{a_{12}}{b_{12}} \\ \frac{a_{21}}{b_{21}} & \frac{a_{22}}{b_{22}} \end{vmatrix}}{\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}} \,. \qquad (1)$$

Definition 2. 2. *Let* $B_n = [b_{ij}]_{n\times n}$ *be a square real matrix. The Dodgson's condensation of matrix* B_n *is a* $(n-1)\times(n-1)$ *matrix that defined as follows*

$$DC(B_n) = \begin{bmatrix} \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} & \cdots & \begin{vmatrix} b_{1(n-1)} & b_{1n} \\ b_{2(n-1)} & b_{2n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} b_{(n-1)1} & b_{(n-1)2} \\ b_{n1} & b_{n2} \end{vmatrix} & \cdots & \begin{vmatrix} b_{(n-1)(n-1)} & b_{(n-1)n} \\ b_{n(n-1)} & b_{nn} \end{vmatrix} \end{bmatrix}_{(n-1)\times(n-1)} . \tag{2}$$

Definition 2. 3. *Let* $A_4 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}_{4\times4}$ *be a real matrix of order 4. If the Dodgson's condensation of the matrix* A_4 *be equal to*

$$DC(A_4) = \begin{bmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{bmatrix}_{3\times3} , \tag{3}$$

then, the Twice Dodgson's condensation of the matrix A_4 *is defined as follows*

$$TDC(A_4) = DC\big(DC(A_4)\big) = \begin{bmatrix} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \end{bmatrix}_{2\times2} . \tag{4}$$

The Dodgson's condensation for the first time used for compute the determinant of a $n \times n$ matrix by C. L. Dodgson in 1866 [1].

To prove the main theorem we need the following lemmas.

Lemma 2. 1 (Dodgson's method). *The determinant of matrix* $A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$ *, with assuming* $a_{22} \neq 0$ *, is equal to*

$$|A_3| = \frac{1}{a_{22}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} . \tag{5}$$

Proof. See Dodgson's condensation method [1, 2].

Lemma 2. 2 (Salihu's method). *The determinant of matrix* $A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}_{3\times 3}$ *is equal to*

$$|A| = \frac{1}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} \times \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} \end{vmatrix}. \qquad (6)$$

were $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \neq 0.$

Proof. See Salihu's method [3].

3. A new method

In the following theorem we present a new method, just to compute the determinant of a 4×4 matrix.

Theorem 3. 1. *Given the real matrix* $A_4 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}_{4\times 4}$ *and let* $a_{22}, a_{23}, a_{32}, a_{33} \neq 0$ *and also let* $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \neq 0$. *Then the determinant of matrix* A_4 *is equal to*

$$|A_4| = \frac{|\,TDC(A_4)\,|}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}. \qquad (7)$$

Proof. Since $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \neq 0$, so by Lemma 2. 2 we have

$$|A_4| = \frac{1}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} \times \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} \end{vmatrix},$$

Besides, if $a_{22}, a_{23}, a_{32}, a_{33} \neq 0$, then by Lemma 2. 1 we have

$$|A| = \frac{1}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} \times \begin{vmatrix} \frac{1}{a_{22}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{23}} \begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\ \frac{1}{a_{32}} \begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{33}} \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \end{vmatrix},$$

Using of the Definition 2. 1, we can write

$$|A_4| = \frac{\begin{vmatrix} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}$$

By the Definition 2. 3, we know that

$$|TDC(A_4)| = \begin{vmatrix} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \end{vmatrix}$$

Therefore, we have

$$|A_4| = \frac{|\,TDC(A_4)\,|}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}.$$

And proof is complete.

Example. The determinant of matrix $A_4 = \begin{bmatrix} 2 & 3 & 7 & 1 \\ 4 & 5 & 10 & 0 \\ 6 & 3 & 2 & 0 \\ 5 & 4 & 3 & 2 \end{bmatrix}_{4\times 4}$ is obtain as follows

$$\xrightarrow{DC(A_4)} \begin{bmatrix} -2 & -5 & -10 \\ -18 & -20 & 0 \\ 9 & 1 & 4 \end{bmatrix}_{3\times 3} \xrightarrow{TDC(A_4)} \begin{bmatrix} -50 & -200 \\ 162 & -80 \end{bmatrix}_{2\times 2};$$

$$|\mathrm{A}| = \frac{|\,TDC(A_4)\,|}{\begin{vmatrix} 5 & 10 \\ 3 & 2 \end{vmatrix}} = \frac{\begin{vmatrix} -50 & -200 \\ 162 & -80 \end{vmatrix}}{\begin{vmatrix} 5 & 10 \\ 3 & 2 \end{vmatrix}} = \frac{\begin{vmatrix} \frac{-50}{5} & \frac{-200}{10} \\ \frac{162}{3} & \frac{-80}{2} \end{vmatrix}}{\begin{vmatrix} 5 & 10 \\ 3 & 2 \end{vmatrix}} = -74.$$

References

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