Electromagnetic-Power-based Characteristic Mode Theory for Material Bodies

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Abstract—**In this paper, an ElectroMagnetic-Power-based Characteristic Mode Theory (CMT) for Material bodies (Mat-EMP-CMT) is provided. The Mat-EMP-CMT is valid for the inhomogeneous and lossy material bodies, and it is applicable to the bodies which are placed in complex electromagnetic environments.**

Under the Mat-EMP-CMT framework, a series of power-based Characteristic Mode (CM) sets are constructed, and they have abilities to depict the inherent power characteristics of material bodies from different aspects. All power-based CM sets are independent of the external electromagnetic environment and excitation.

Among the various power-based CM sets constructed in Mat-EMP-CMT, only the Input power CM (InpCM) set has the same physical essence as the CM set constructed in Mat-VIE-CMT (the Volume Integral Equation based CMT for Material bodies), and the other CM sets are completely new. However, the power characteristic of the InpCM set is more physically reasonable than the CM set derived from Mat-VIE-CMT.

In addition, not only radiative CMs and real characteristic currents but also non-radiative CMs and complex characteristic currents can be constructed under the Mat-EMP-CMT framework; the traditional characteristic quantity, Modal Significance (MS), is generalized, and some new characteristic and non-characteristic quantities are introduced to depict the modal characteristics from different aspects; a variational formulation for the scattering problem of material scatterer is established based on the conservation law of energy.

***Index Terms*—Characteristic Mode (CM), Electromagnetic Power, Input Power, Material Body, Modal Expansion, Modal Significance (MS), Output Power.**

I. INTRODUCTION

THE Theory of Characteristic Mode (TCM), or equivalently called as Characteristic Mode Theory (CMT), was firstly introduced by R. J. Garbacz [1], and subsequently refined by R. F. Harrington and J. R. Mautz under the MoM framework. In 1971, Harrington and Mautz built their CMT for PEC systems based on the Surface EFIE-based MoM (PEC-SEFIE-CMT) [2]. Afterwards, some variants for the PEC-SEFIE-CMT were introduced one after another under the MoM framework, such as the Volume Integral Equation-based CMT for Material bodies (Mat-VIE-CMT) [3] and the Surface Integral Equation-based CMT for Material bodies (Mat-SIE-CMT) [4] etc. Recently, the PEC-SEFIE-CMT is re-derived from complex Poynting's theorem in [5], and an alternative surface formulation for the Mat-SIE-CMT is provided in [6]. In [5]-[6], the power characteristics of the Characteristic Mode (CM) sets derived from the PEC-SEFIE-CMT and Mat-SIE-CMT are analyzed, such that the physical pictures of the PEC-SEFIE-CMT and Mat-SIE-CMT become clearer. In fact, to analyze the power characteristic of the CM set derived from Mat-VIE-CMT is also valuable for both theoretical research and engineering application, and it is done in this paper.

In this paper, an ElectroMagnetic-Power-based CMT for Material bodies (Mat-EMP-CMT) is built. The Mat-EMP-CMT is valid for the inhomogeneous and lossy material bodies, and it is applicable to the bodies which are placed in complex electromagnetic environments. Under the Mat-EMP-CMT framework, a series of power-based CM sets are constructed, and the various CM sets have abilities to depict the inherent power characteristics of the objective material body from different aspects. All power-based CM sets are independent of the external electromagnetic environment and excitation.

Except the Input power CM (InpCM) set, all power-based CM sets constructed in this paper are completely new. The InpCM set has the same physical essence as the CM set constructed in Mat-VIE-CMT, but the former is more advantageous than the latter in the following aspects.

1) The InpCM set has a more reasonable power characteristic.

2) The applicable range of the InpCM set is wider. For example, the InpCM set not only includes the real characteristic currents and radiative CMs, but also includes the complex characteristic currents and non-radiative CMs. In fact, both the complex characteristic currents and non-radiative CMs are valuable for electromagnetic engineering, because:

(2.1) although the real characteristic currents are more suitable for depicting the resonant material antennas [7], the complex characteristic currents are more suitable for depicting the travelling wave material antennas [7];

(2.2) although the radiative CMs are more suitable for characterizing the material antennas [8], the non-radiative CMs are more suitable for characterizing the material resonators [9].

In addition, based on a new normalization way for various electromagnetic quantities, the traditional characteristic quantity Modal Significance (MS) is generalized, and some new characteristic and non-characteristic quantities are

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introduced in this paper. Various characteristic and non-characteristic quantities have abilities to depict the modal characteristics from different aspects. A functional variation formulation for the scattering problem of material scatterer is established in this paper, based on the conservation law of energy [10].

This paper is organized as follows. Sections II-VII give the principles and formulations of Mat-EMP-CMT, and then some necessary discussions related to the Mat-EMP-CMT are provided in Sec. VIII. Section IX concludes this paper. In what follows, the $e^{j\omega t}$ convention is used throughout.

II. Source-Field Relationships and Normalization

The material body, which is treated as a whole object, can be placed either in vacuum or in an arbitrary time-harmonic environment, and the material body is simply called as scatterer. When an external source is impressed, there exist three kinds of fields in whole space $\mathbb{R}^3$, that are the $\bar{F}^{im}$ generated by impressed source, the $\bar{F}^{en}$ generated by external environment, and the $\bar{F}^{sca}$ generated by the scattering sources on scatterer V, here $F = E,H$. The term "time-harmonic environment" means that the $\bar{F}^{en}$ operates at the same frequency as $\bar{F}^{im}$. Various sources and fields are illustrated in Fig. 1. Based on the linear superposition principle [10], the $\bar{F}^{sca}$ is considered as the scattering field excited by incident field $\bar{F}^{inc} \triangleq \bar{F}^{im} + \bar{F}^{en}$, because the scatterer is regarded as a whole object in this paper. The summation of $\bar{F}^{inc}$ and $\bar{F}^{sca}$ is the total field, and it is denoted as $\bar{F}^{tot}$, i.e., $\bar{F}^{tot} = \bar{F}^{inc} + \bar{F}^{sca}$.

A. *Source-field relationships.*

When the conductivity of scatterer is not infinity, the scattering sources include the volume ohmic electric current $\bar{J}^{vo}$ and the related electric charges $\{\rho^{vo},\rho^{so}\}$ due to the conduction phenomenon, the volume polarized electric current $\bar{J}^{vp}$ and the related electric charges $\{\rho^{vp},\rho^{sp}\}$ due to the polarization phenomenon, and the volume magnetic current $\bar{M}^{vm}$ and the related magnetic charges $\{\rho_m^{vm},\rho_m^{sm}\}$ due to the magnetization phenomenon [11]-[13]. The $\{\rho^{vo},\rho^{vp},\rho_m^{vm}\}$ are the volume charges, and the $\{\rho^{so},\rho^{sp},\rho_m^{sm}\}$ are the surface charges on the boundary of scatterer. The various charges are related to the corresponding currents by current continuity equations, so it is sufficient to only use the scattering currents to determine the scattering field [11]-[13]. In this paper, the various scattering currents are expressed as the linear functions about the total field $\bar{F}^{tot}$ in scatterer as below, and the reason will be explained in Sec. VIII-D.

The Maxwell's equations for the scattering fields $\{\bar{E}^{sca},\bar{H}^{sca}\}$ are as follows [11]-[12]

$$\nabla\times\bar{H}^{sca}(\bar{r}) = \bar{J}^{vop} + j\omega\varepsilon_0\bar{E}^{sca}(\bar{r}) \quad , \quad (\bar{r}\in\mathbb{R}^3) \quad (1.1)$$

$$\nabla\times\bar{E}^{sca}(\bar{r}) = -\bar{M}^{vm} - j\omega\mu_0\bar{H}^{sca}(\bar{r}) \quad , \quad (\bar{r}\in\mathbb{R}^3) \quad (1.2)$$

here

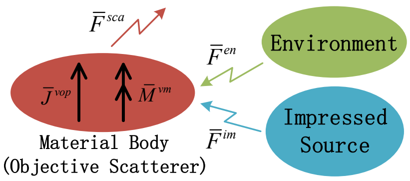

Fig. 1. Various fields generated by various sources.

$$\bar{J}^{vop}(\bar{r}) = \bar{J}^{vo}(\bar{r}) + \bar{J}^{vp}(\bar{r}) \quad , \quad (\bar{r}\in V) \quad (2.1)$$

$$\bar{M}^{vm}(\bar{r}) = j\omega\Delta\mu\bar{H}^{tot}(\bar{r}) \quad , \quad (\bar{r}\in V) \quad (2.2)$$

in which $\bar{J}^{vo}(\bar{r}) = \sigma\bar{E}^{tot}(\bar{r})$, and $\bar{J}^{vp}(\bar{r}) = j\omega\Delta\varepsilon\bar{E}^{tot}(\bar{r})$, so $\bar{J}^{vop}(\bar{r}) = j\omega\Delta\varepsilon_c\bar{E}^{tot}(\bar{r})$. In (1) and (2), $\Delta\mu = \mu - \mu_0$, $\Delta\varepsilon = \varepsilon - \varepsilon_0$, and $\Delta\varepsilon_c = \varepsilon_c - \varepsilon_0$; the $\varepsilon_c = \varepsilon + \sigma/j\omega$ is complex permittivity; the ε and ε_0 are the permitivities in scatterer and vacuum; the μ and μ_0 are the permeabilities in scatterer and vacuum; the σ is the electric conductivity in scatterer, and its vacuum version is zero. All these material parameters can be the functions about spatial position, except the ε_0 and μ_0. The $\omega = 2\pi f$ is angle frequency, and the f is frequency.

If the source of $\bar{F}^{inc}$ doesn't distribute on scatterer, the $\bar{F}^{tot}$ on scatterer V satisfies following Maxwell's equations [13].

$$\left.\begin{aligned}\nabla\times\bar{H}^{tot}(\bar{r}) &= j\omega\varepsilon_c\bar{E}^{tot}(\bar{r})\\ \nabla\times\bar{E}^{tot}(\bar{r}) &= -j\omega\mu\bar{H}^{tot}(\bar{r})\end{aligned}\right\} \quad , \quad (\bar{r}\in V) \quad (3)$$

so the $\bar{E}^{tot}$ and $\bar{H}^{tot}$ on V can be expressed by each other as

$$\left.\begin{aligned}\bar{E}^{tot}(\bar{r}) &= (1/j\omega\varepsilon_c)\nabla\times\bar{H}^{tot}(\bar{r})\\ \bar{H}^{tot}(\bar{r}) &= -(1/j\omega\mu)\nabla\times\bar{E}^{tot}(\bar{r})\end{aligned}\right\} \quad , \quad (\bar{r}\in V) \quad (4)$$

Based on the (2) and (4) and that the scattering field is the one generated by scattering sources in vacuum [11]-[13], the fields $\{\bar{E}^{inc},\bar{H}^{inc}\}$ and $\{\bar{E}^{tot},\bar{H}^{tot}\}$ on scatterer V, the scattering fields $\{\bar{E}^{sca},\bar{H}^{sca}\}$ on whole space $\mathbb{R}^3$, and the various scattering currents $\{\bar{J}^{vo},\bar{J}^{vp},\bar{J}^{vop},\bar{M}^{vm}\}$ on scatterer V can be related to the any one of the $\bar{E}^{tot}$ and $\bar{H}^{tot}$ on scatterer V, and they can be simply expressed as the following linear operator forms.

$$\left.\begin{aligned}\bar{E}_F^X(\bar{r}) &= \bar{E}^X(\bar{F}^{tot};\bar{r})\\ \bar{H}_F^X(\bar{r}) &= \bar{H}^X(\bar{F}^{tot};\bar{r})\end{aligned}\right\} \quad , \quad (\bar{r}\in V) \quad (5)$$

$$\left.\begin{aligned}\bar{E}_F^{sca}(\bar{r}) &= \bar{E}^{sca}(\bar{F}^{tot};\bar{r})\\ \bar{H}_F^{sca}(\bar{r}) &= \bar{H}^{sca}(\bar{F}^{tot};\bar{r})\end{aligned}\right\} \quad , \quad (\bar{r}\in\mathbb{R}^3) \quad (6)$$

$$\left.\begin{aligned}\bar{J}_F^Y(\bar{r}) &= \bar{J}^Y(\bar{F}^{tot};\bar{r})\\ \bar{M}_F^{vm}(\bar{r}) &= \bar{M}^{vm}(\bar{F}^{tot};\bar{r})\end{aligned}\right\} \quad , \quad (\bar{r}\in V) \quad (7)$$

here $X = inc,tot$, and $Y = vo,vp,vop$. In this paper, the $\bar{F}^{tot}$ in (5)-(7) is called as *basic variable*.

The subscripts "F" in the left-hand sides of (5)-(7) are to emphasize that the basic variable is $\bar{F}^{tot}$; that the subscript "F" doesn't appear in the right-hand sides of (5)-(7) is due to that the basic variable $\bar{F}^{tot}$ has appeared in brackets. In the (5)-(7), $F = E$ or H, and it depends on that various currents and fields are expressed as the functions of whom. For example:

1) The $\bar{E}^{inc}$ on scatterer V can be expressed as $\bar{E}_E^{inc}(\bar{r}) = \bar{E}^{inc}(\bar{E}^{tot};\bar{r}) = \bar{E}^{tot}(\bar{r}) - \bar{E}(\bar{J}^{vop},\bar{M}^{vm};\bar{r})$, here the $\bar{E}(\bar{J}^{vop},\bar{M}^{vm};\bar{r})$ is the electric field generated by $\bar{J}^{vop}$ and $\bar{M}^{vm}$ in vacuum, and its mathematical expression can be found in [11]-[12], and the $\bar{J}^{vop}$ is expressed as (2.1), and $\bar{M}^{vm} = -(\Delta\mu/\mu)\nabla\times\bar{E}^{tot}$.

2) The $\bar{E}^{inc}$ on scatterer V can also be expressed as $\bar{E}_H^{inc}(\bar{r}) = \bar{E}^{inc}(\bar{H}^{tot};\bar{r}) = (1/j\omega\varepsilon_c)\nabla\times\bar{H}^{tot}(\bar{r}) - \bar{E}(\bar{J}^{vop},\bar{M}^{vm};\bar{r})$, here the $\bar{M}^{vm}$ is expressed as (2.2), and $\bar{J}^{vop} = (\Delta\varepsilon_c/\varepsilon_c)\nabla\times\bar{H}^{tot}$.

To simplify the symbolic system of this paper, the subscripts "F" in the left-hand sides of (5)-(7) are omitted in the following sections, and it will not lead to any difficulty for understanding the Mat-EMP-CMT.

B. Normalization.

Following the normalization way introduced in [14], the basic variable $\bar{F}^{tot}$ is normalized as follows

$$\tilde{\bar{F}}^{tot}(\bar{r}) \triangleq \frac{\bar{F}^{tot}(\bar{r})}{\left(1/\sqrt{2}\right)\left\langle \bar{F}^{tot},\bar{F}^{tot}\right\rangle_V^{1/2}} \quad , \quad (\bar{r}\in V) \tag{8}$$

and then the incident fields $\{\bar{E}^{inc},\bar{H}^{inc}\}$ on V, the scattering fields $\{\bar{E}^{sca},\bar{H}^{sca}\}$ on $\mathbb{R}^3$, and the various currents $\{\bar{J}^{vo},\bar{J}^{vp},\bar{J}^{vop},\bar{M}^{vm}\}$ on V are automatically normalized as follows

$$\begin{aligned} \tilde{\bar{E}}^{inc}(\bar{r}) &= \bar{E}^{inc}(\bar{r})\Big/\left(1/\sqrt{2}\right)\left\langle \bar{F}^{tot},\bar{F}^{tot}\right\rangle_V^{1/2} \\ \tilde{\bar{H}}^{inc}(\bar{r}) &= \bar{H}^{inc}(\bar{r})\Big/\left(1/\sqrt{2}\right)\left\langle \bar{F}^{tot},\bar{F}^{tot}\right\rangle_V^{1/2} \end{aligned} \quad , \quad (\bar{r}\in V) \tag{9}$$

and

$$\begin{aligned} \tilde{\bar{E}}^{sca}(\bar{r}) &= \bar{E}^{sca}(\bar{r})\Big/\left(1/\sqrt{2}\right)\left\langle \bar{F}^{tot},\bar{F}^{tot}\right\rangle_V^{1/2} \\ \tilde{\bar{H}}^{sca}(\bar{r}) &= \bar{H}^{sca}(\bar{r})\Big/\left(1/\sqrt{2}\right)\left\langle \bar{F}^{tot},\bar{F}^{tot}\right\rangle_V^{1/2} \end{aligned} \quad , \quad (\bar{r}\in\mathbb{R}^3) \tag{10}$$

and

$$\begin{aligned} \tilde{\bar{J}}^{Y}(\bar{r}) &= \bar{J}^{Y}(\bar{r})\Big/\left(1/\sqrt{2}\right)\left\langle \bar{F}^{tot},\bar{F}^{tot}\right\rangle_V^{1/2} \\ \tilde{\bar{M}}^{vm}(\bar{r}) &= \bar{M}^{vm}(\bar{r})\Big/\left(1/\sqrt{2}\right)\left\langle \bar{F}^{tot},\bar{F}^{tot}\right\rangle_V^{1/2} \end{aligned} \quad , \quad (\bar{r}\in V) \tag{11}$$

here $Y = vo, vp, vop$; the superscript "$1/2$" represents square root. The inner products in (8)-(11) are defined as $\langle \bar{g},\bar{h}\rangle_\Omega \triangleq \int_\Omega \bar{g}^*\cdot\bar{h}\, d\Omega$, here the symbol "$*$" denotes the complex conjugate of relevant quantity, and the symbol "$\cdot$" is the scalar product for field vectors.

III. Various Electromagnetic Powers

The destination of Mat-EMP-CMT is to optimize the various electromagnetic powers related to the objective material scatterer, and the various powers and their normalized versions are discussed in this section.

The power done by $\{\bar{E}^{inc},\bar{H}^{inc}\}$ on $\{\bar{J}^{vop},\bar{M}^{vm}\}$ is the *input power* P^{inp} from external sources to scatterer, and it is expressed as follows

$$\begin{aligned} P^{inp} &= (1/2)\left\langle \bar{J}^{vop},\bar{E}^{inc}\right\rangle_V + (1/2)\left\langle \bar{H}^{inc},\bar{M}^{vm}\right\rangle_V \\ &= (1/2)\left\langle \bar{J}^{vop},\bar{E}^{tot}\right\rangle_V + (1/2)\left\langle \bar{H}^{tot},\bar{M}^{vm}\right\rangle_V \\ &\quad - (1/2)\left\langle \bar{J}^{vop},\bar{E}^{sca}\right\rangle_V - (1/2)\left\langle \bar{H}^{sca},\bar{M}^{vm}\right\rangle_V \end{aligned} \tag{12}$$

The reason why the $(1/2)\langle \bar{H}^{inc},\bar{M}^{vm}\rangle_V$ instead of the $(1/2)\langle \bar{M}^{vm},\bar{H}^{inc}\rangle_V$ appears in the first equality of (12) will be explained in Sec. VIII. The second equality in (12) is due to that $\bar{F}^{inc} = \bar{F}^{tot} - \bar{F}^{sca}$.

Multiplying the complex conjugate of (1.1) with $\bar{E}^{sca}$ and doing some necessary simplifications, the following Poynting's theorem for the scattering field is obtained [12].

$$\begin{aligned} P^{sca,vac} &= P^{sca,rad} + j\, P^{sca,react,vac} \\ &= P^{sca,rad} + j\, 2\omega\left(W_m^{sca,vac} - W_e^{sca,vac}\right) \end{aligned} \tag{13}$$

here the superscripts "*sca*" represent that the relevant quantities only correspond to the scattering field instead of the total field or incident field; the $P^{sca,rad}$ is the radiated power carried by scattering field, and the $W_m^{sca,vac}$ and $W_e^{sca,vac}$ are respectively the magnetically and electrically stored energies in scattering field, and their mathematical expressions are as follows [12]

$$P^{sca,vac} = -(1/2)\left\langle \bar{J}^{vop},\bar{E}^{sca}\right\rangle_V - (1/2)\left\langle \bar{H}^{sca},\bar{M}^{vm}\right\rangle_V \tag{14.1}$$

$$P^{sca,rad} = (1/2)\oint\!\!\!\oint_{S_\infty}\left[\bar{E}^{sca}\times\left(\bar{H}^{sca}\right)^*\right]\cdot d\bar{S} \tag{14.2}$$

$$W_m^{sca,vac} = (1/4)\left\langle \bar{H}^{sca},\mu_0\bar{H}^{sca}\right\rangle_{\mathbb{R}^3} \tag{14.3}$$

$$W_e^{sca,vac} = (1/4)\left\langle \varepsilon_0\bar{E}^{sca},\bar{E}^{sca}\right\rangle_{\mathbb{R}^3} \tag{14.4}$$

here the S_∞ is a closed spherical surface at infinity.

Considering of (2) and (13)-(14), the P^{inp} in (12) can be rewritten as

$$\begin{aligned} P^{inp} &= P^{inp,act} + j\, P^{inp,react} \\ &= P^{sca,rad} + P^{tot,loss} + j\left(P^{sca,react,vac} + P^{tot,react,mat}\right) \\ &= P^{sca,rad} + P^{tot,loss} + j\left[P^{sca,react,vac} + 2\omega\left(W_m^{tot,mat} - W_e^{tot,mat}\right)\right] \end{aligned} \tag{15}$$

here the $P^{tot,loss}$ is the total ohmic loss due to the interaction between the total electric field $\bar{E}^{tot}$ and scatterer, and the $W_m^{tot,mat}$ and $W_e^{tot,mat}$ are respectively the total magnetized and polarized energies stored in matter due to the interaction between the total fields $\{\bar{E}^{tot},\bar{H}^{tot}\}$ and scatterer, and [15]

$$P^{tot,loss} = (1/2)\left\langle \sigma\bar{E}^{tot},\bar{E}^{tot}\right\rangle_V \tag{16.1}$$

$$W_m^{tot,mat} = (1/4)\left\langle \bar{H}^{tot},\Delta\mu\bar{H}^{tot}\right\rangle_V \tag{16.2}$$

$$W_e^{tot,mat} = (1/4)\left\langle \Delta\varepsilon\bar{E}^{tot},\bar{E}^{tot}\right\rangle_V \tag{16.3}$$

Besides the above-mentioned powers P^{inp} and $P^{sca,vac}$, there also exist many other kinds of powers which can be selected as the objective powers to be optimized by Mat-EMP-CMT, such as the following $P^{inp,part,rad}$, P^{sca}, and $P^{sca,part,rad}$.

$$P^{inp,part,rad} \triangleq P^{sca,rad} + j\,P^{inp,react} \tag{17}$$

$$\begin{aligned} P^{sca} &= P^{sca,act} + j\,P^{sca,react} \\ &= P^{sca,rad} + P^{sca,loss} + j\left(P^{sca,react,vac} + P^{sca,react,mat}\right) \\ &= P^{sca,rad} + P^{sca,loss} \\ &\quad + j\left[P^{sca,react,vac} + 2\omega\left(W_m^{sca,mat} - W_e^{sca,mat}\right)\right] \end{aligned} \tag{18}$$

$$P^{sca,part,rad} \triangleq P^{sca,rad} + j\,P^{sca,react} \tag{19}$$

The mathematical expressions for the $P^{sca,loss}$, $W_m^{sca,mat}$, and $W_e^{sca,mat}$ in (18) are as follows

$$P^{sca,loss} = (1/2)\left\langle \sigma\bar{E}^{sca}, \bar{E}^{sca}\right\rangle_V \tag{20.1}$$

$$W_m^{sca,mat} = (1/4)\left\langle \bar{H}^{sca}, \Delta\mu\bar{H}^{sca}\right\rangle_V \tag{20.2}$$

$$W_e^{sca,mat} = (1/4)\left\langle \Delta\varepsilon\bar{E}^{sca}, \bar{E}^{sca}\right\rangle_V \tag{20.3}$$

The superscript "$part$" on $P^{inp,part,rad}$ is to emphasize that the $P^{inp,part,rad}$ is a part of P^{inp}, and the superscript "rad" on $P^{inp,part,rad}$ is to emphasize that the CMs constructed by orthogonalizing $P^{inp,part,rad}$ have the orthogonal radiation patterns as illustrated in Sec. V-C. The superscripts on $P^{sca,part,rad}$ can be similarly explained. Obviously, when the scatterer is lossless, the $P^{inp,part,rad}$ and $P^{sca,part,rad}$ are respectively the same as the P^{inp} and P^{sca}. The symbols "≙" in (17) and (19) represent that these powers are artificially defined for various practical destinations, and the practical value to introduce $P^{inp,part,rad}$ is specifically discussed in Sec. VIII-B. The reason why the symbol "≙" doesn't appear in the (18) will be explained in Sec. VIII-C.

For the convenience of following discussions, the relations among various powers are specifically given in (21), in which $P^{inc,loss} = (1/2)\left\langle \sigma\bar{E}^{inc}, \bar{E}^{inc}\right\rangle_V$, and $P^{coup,loss} = (1/2)\left\langle \sigma\bar{E}^{sca}, \bar{E}^{inc}\right\rangle_V + (1/2)\left\langle \sigma\bar{E}^{inc}, \bar{E}^{sca}\right\rangle_V$, and $P^{inc,react,mat} = 2\omega\left(W_m^{inc,mat} - W_e^{inc,mat}\right)$ (here $W_m^{inc,mat} = (1/4)\left\langle \bar{H}^{inc}, \Delta\mu\bar{H}^{inc}\right\rangle_V$ and $W_e^{inc,mat} = (1/4)\left\langle \Delta\varepsilon\bar{E}^{inc}, \bar{E}^{inc}\right\rangle_V$), and $P^{coup,react,mat} = 2\omega\left(W_m^{coup,mat} - W_e^{coup,mat}\right)$ (here $W_m^{coup,mat} = (1/4)\left\langle \bar{H}^{sca}, \Delta\mu\bar{H}^{inc}\right\rangle_V + (1/4)\left\langle \bar{H}^{inc}, \Delta\mu\bar{H}^{sca}\right\rangle_V$ and $W_e^{coup,mat} = (1/4)\left\langle \Delta\varepsilon\bar{E}^{sca}, \bar{E}^{inc}\right\rangle_V + (1/4)\left\langle \Delta\varepsilon\bar{E}^{inc}, \bar{E}^{sca}\right\rangle_V$).

The normalized versions of various powers appearing in (21) are as follows

$$\tilde{P}\left(\bar{F}^{tot}\right) = P\left(\tilde{\bar{F}}^{tot}\right) = \frac{P\left(\bar{F}^{tot}\right)}{(1/2)\left\langle \bar{F}^{tot}, \bar{F}^{tot}\right\rangle_V} \tag{22}$$

here the $\bar{F}^{tot}$ is the basic variable, and the symbol $P\left(\bar{F}^{tot}\right)$ is the operator form of related power.

IV. The Matrix Forms for various Powers

In this section, the matrix forms for various powers are provided. The basic variable $\bar{F}^{tot}$ is expanded in terms of the basis function set $\{\bar{b}_\xi(\bar{r})\}_{\xi=1}^{\Xi}$ as follows

$$\bar{F}^{tot}(\bar{r}) = \sum\nolimits_{\xi=1}^{\Xi} a_\xi \bar{b}_\xi(\bar{r}) = \bar{B}\cdot\bar{a} \quad , \quad (\bar{r}\in V) \tag{23}$$

here $\bar{B} = \left[\bar{b}_1(\bar{r}), \bar{b}_2(\bar{r}), \cdots, \bar{b}_\Xi(\bar{r})\right]$, and $\bar{a} = [a_1, a_2, \cdots, a_\Xi]^T$, and the superscript "$T$" represents matrix transposition. The symbol "$\cdot$" in (23) represents matrix multiplication.

Inserting the (5), (6), and (23) into the powers $P^{inc,loss}$, $P^{sca,loss}$, $P^{tot,loss}$, $P^{inc,react,mat}$, $P^{sca,react,mat}$, and $P^{tot,react,mat}$, their matrix forms and normalized versions can be written as follows

$$P^{Z,loss}(\bar{a}) = \bar{a}^H \cdot \overline{\overline{P}}^{Z,loss} \cdot \bar{a} \tag{24.1}$$

$$P^{Z,react,mat}(\bar{a}) = \bar{a}^H \cdot \overline{\overline{P}}^{Z,react,mat} \cdot \bar{a} \tag{24.2}$$

and

$$\tilde{P}^{Z,loss}(\bar{a}) = \bar{a}^H \cdot \overline{\overline{P}}^{Z,loss} \cdot \bar{a} \big/ \bar{a}^H \cdot \overline{\overline{F}}^{tot} \cdot \bar{a} \tag{25.1}$$

$$\tilde{P}^{Z,react,mat}(\bar{a}) = \bar{a}^H \cdot \overline{\overline{P}}^{Z,react,mat} \cdot \bar{a} \big/ \bar{a}^H \cdot \overline{\overline{F}}^{tot} \cdot \bar{a} \tag{25.2}$$

in which $Z = inc, sca, tot$, and the superscript "H" represents the transpose conjugate of matrix. $\overline{\overline{P}}^{Z,loss} = \left[p_{\xi\zeta}^{Z,loss}\right]_{\Xi\times\Xi}$, $\overline{\overline{P}}^{Z,react,mat} = \left[p_{\xi\zeta}^{Z,react,mat}\right]_{\Xi\times\Xi}$, and $\overline{\overline{F}}^{tot} = \left[f_{\xi\zeta}^{tot}\right]_{\Xi\times\Xi}$, here

$$p_{\xi\zeta}^{Z,loss} = \frac{1}{2}\left\langle \sigma\bar{E}^Z(\bar{b}_\xi), \bar{E}^Z(\bar{b}_\zeta)\right\rangle_V \tag{26.1}$$

$$p_{\xi\zeta}^{Z,react,mat} = 2\omega\left[\frac{1}{4}\left\langle \bar{H}^Z(\bar{b}_\xi), \Delta\mu\bar{H}^Z(\bar{b}_\zeta)\right\rangle_V - \frac{1}{4}\left\langle \Delta\varepsilon\bar{E}^Z(\bar{b}_\zeta), \bar{E}^Z(\bar{b}_\zeta)\right\rangle_V\right] \tag{26.2}$$

$$f_{\xi\zeta}^{tot} = \frac{1}{2}\left\langle \bar{b}_\xi, \bar{b}_\zeta\right\rangle_V \tag{26.3}$$

Obviously, the matrices $\overline{\overline{P}}^{Z,loss}$, $\overline{\overline{P}}^{Z,react,mat}$, and $\overline{\overline{F}}^{tot}$ are Hermitian. The matrix $\overline{\overline{P}}^{Z,loss}$ is positive definite, if $\sigma\neq 0$; the $\overline{\overline{P}}^{Z,loss}$ is zero, if $\sigma = 0$.

Inserting (6), (7), and (23) into (14.1), the $P^{sca,vac}$ can be written as the following matrix form.

$$P^{sca,vac}(\bar{a}) = \bar{a}^H \cdot \overline{\overline{P}}^{sca,vac} \cdot \bar{a} \tag{27.1}$$

here $\overline{\overline{P}}^{sca,vac} = \left[p_{\xi\zeta}^{sca,vac}\right]_{\Xi\times\Xi}$, and

$$p_{\xi\zeta}^{sca,vac} = -\frac{1}{2}\left\langle \bar{J}^{vop}(\bar{b}_\xi), \bar{E}^{sca}(\bar{b}_\zeta)\right\rangle_V - \frac{1}{2}\left\langle \bar{H}^{sca}(\bar{b}_\xi), \bar{M}^{vm}(\bar{b}_\zeta)\right\rangle_V \tag{27.2}$$

The matrix $\overline{\overline{P}}^{sca,vac}$ can be decomposed as follows

$$\overline{\overline{P}}^{sca,vac} = \overline{\overline{P}}^{sca,rad} + j\,\overline{\overline{P}}^{sca,react,vac} \tag{28.1}$$

$$\begin{aligned} P^{inp} &= \overbrace{P^{inc,loss} + P^{coup,loss} + P^{sca,loss} + P^{sca,rad}}^{P^{inp,act}} + j\overbrace{\left(P^{sca,react,vac} + P^{sca,react,mat} + P^{coup,react,mat} + P^{inc,react,mat}\right)}^{P^{inp,react}} \\ &= \overbrace{P^{inc,loss} + P^{coup,loss} + P^{sca,loss}}^{P^{tot,loss}} + \overbrace{P^{sca,rad} + j\,P^{sca,react,vac}}^{P^{sac,vac}} + j\overbrace{\left(P^{sca,react,mat} + P^{coup,react,mat} + P^{inc,react,mat}\right)}^{P^{tot,react,mat}} \\ &= P^{inc,loss} + P^{coup,loss} + \underbrace{\overbrace{P^{sca,loss} + P^{sca,rad}}^{P^{sca,act}} + j\overbrace{\left(P^{sca,react,vac} + P^{sca,react,mat}\right)}^{P^{sca,react}}}_{P^{sac}} + j\left(P^{coup,react,mat} + P^{inc,react,mat}\right) \\ &= P^{inc,loss} + P^{coup,loss} + P^{sca,loss} + \underbrace{\overbrace{P^{sca,rad} + j\left(P^{sca,react,vac} + P^{sca,react,mat}\right)}^{P^{sac,part,rad}} + j\left(P^{coup,react,mat} + P^{inc,react,mat}\right)}_{P^{inp,part,rad}} \end{aligned} \tag{21}$$

here

$$\begin{aligned}\overline{\overline{P}}^{sca,rad} &= \frac{1}{2}\left[\overline{\overline{P}}^{sca,vac}+\left(\overline{\overline{P}}^{sca,vac}\right)^H\right]\\ \overline{\overline{P}}^{sca,react,vac} &= \frac{1}{2j}\left[\overline{\overline{P}}^{sca,vac}-\left(\overline{\overline{P}}^{sca,vac}\right)^H\right]\end{aligned} \tag{28.2}$$

Obviously, the matrices $\overline{\overline{P}}^{sca,rad}$ and $\overline{\overline{P}}^{sca,react,vac}$ are Hermitian, so the $\overline{a}^H\cdot\overline{\overline{P}}^{sca,rad}\cdot\overline{a}$ and $\overline{a}^H\cdot\overline{\overline{P}}^{sca,react,vac}\cdot\overline{a}$ are always real numbers for any vector $\overline{a}$ [16], and then

$$P^{sca,rad}(\overline{a}) = \overline{a}^H\cdot\overline{\overline{P}}^{sca,rad}\cdot\overline{a} \tag{29.1}$$

$$P^{sca,react,vac}(\overline{a}) = \overline{a}^H\cdot\overline{\overline{P}}^{sca,react,vac}\cdot\overline{a} \tag{29.2}$$

and

$$\tilde{P}^{sca,rad}(\overline{a}) = \overline{a}^H\cdot\overline{\overline{P}}^{sca,rad}\cdot\overline{a}\Big/\overline{a}^H\cdot\overline{\overline{F}}^{tot}\cdot\overline{a} \tag{30.1}$$

$$\tilde{P}^{sca,react,vac}(\overline{a}) = \overline{a}^H\cdot\overline{\overline{P}}^{sca,react,vac}\cdot\overline{a}\Big/\overline{a}^H\cdot\overline{\overline{F}}^{tot}\cdot\overline{a} \tag{30.2}$$

Based on the above discussions, the following relations are derived.

$$P^{inp,act}(\overline{a}) = \overline{a}^H\cdot\overline{\overline{P}}^{inp,act}\cdot\overline{a} \tag{31.1}$$

$$P^{inp,react}(\overline{a}) = \overline{a}^H\cdot\overline{\overline{P}}^{inp,react}\cdot\overline{a} \tag{31.2}$$

$$P^{sca,act}(\overline{a}) = \overline{a}^H\cdot\overline{\overline{P}}^{sca,act}\cdot\overline{a} \tag{31.3}$$

$$P^{sca,react}(\overline{a}) = \overline{a}^H\cdot\overline{\overline{P}}^{sca,react}\cdot\overline{a} \tag{31.4}$$

and

$$\tilde{P}^{inp,act}(\overline{a}) = \overline{a}^H\cdot\overline{\overline{P}}^{inp,act}\cdot\overline{a}\Big/\overline{a}^H\cdot\overline{\overline{F}}^{tot}\cdot\overline{a} \tag{32.1}$$

$$\tilde{P}^{inp,react}(\overline{a}) = \overline{a}^H\cdot\overline{\overline{P}}^{inp,react}\cdot\overline{a}\Big/\overline{a}^H\cdot\overline{\overline{F}}^{tot}\cdot\overline{a} \tag{32.2}$$

$$\tilde{P}^{sca,act}(\overline{a}) = \overline{a}^H\cdot\overline{\overline{P}}^{sca,act}\cdot\overline{a}\Big/\overline{a}^H\cdot\overline{\overline{F}}^{tot}\cdot\overline{a} \tag{32.3}$$

$$\tilde{P}^{sca,react}(\overline{a}) = \overline{a}^H\cdot\overline{\overline{P}}^{sca,react}\cdot\overline{a}\Big/\overline{a}^H\cdot\overline{\overline{F}}^{tot}\cdot\overline{a} \tag{32.4}$$

and

$$\begin{aligned}P^{inp}(\overline{a}) &= \overline{a}^H\cdot\overline{\overline{P}}^{inp}\cdot\overline{a}\\ &= \overline{a}^H\cdot\left(\overline{\overline{P}}^{inp,act}+j\,\overline{\overline{P}}^{inp,react}\right)\cdot\overline{a}\end{aligned} \tag{33.1}$$

$$\begin{aligned}P^{inp,part,rad}(\overline{a}) &= \overline{a}^H\cdot\overline{\overline{P}}^{inp,part,rad}\cdot\overline{a}\\ &= \overline{a}^H\cdot\left(\overline{\overline{P}}^{sca,rad}+j\,\overline{\overline{P}}^{inp,react}\right)\cdot\overline{a}\end{aligned} \tag{33.2}$$

$$\begin{aligned}P^{sca}(\overline{a}) &= \overline{a}^H\cdot\overline{\overline{P}}^{sca}\cdot\overline{a}\\ &= \overline{a}^H\cdot\left(\overline{\overline{P}}^{sca,act}+j\,\overline{\overline{P}}^{sca,react}\right)\cdot\overline{a}\end{aligned} \tag{33.3}$$

$$\begin{aligned}P^{sca,part,rad}(\overline{a}) &= \overline{a}^H\cdot\overline{\overline{P}}^{sca,part,rad}\cdot\overline{a}\\ &= \overline{a}^H\cdot\left(\overline{\overline{P}}^{sca,rad}+j\,\overline{\overline{P}}^{sca,react}\right)\cdot\overline{a}\end{aligned} \tag{33.4}$$

here

$$\overline{\overline{P}}^{inp} = \overline{\overline{P}}^{inp,act} + j\,\overline{\overline{P}}^{inp,react} \tag{34.1}$$

$$\overline{\overline{P}}^{inp,part,rad} = \overline{\overline{P}}^{sca,rad} + j\,\overline{\overline{P}}^{inp,react} \tag{34.2}$$

$$\overline{\overline{P}}^{sca} = \overline{\overline{P}}^{sca,act} + j\,\overline{\overline{P}}^{sca,react} \tag{34.3}$$

$$\overline{\overline{P}}^{sca,part,rad} = \overline{\overline{P}}^{sca,rad} + j\,\overline{\overline{P}}^{sca,react} \tag{34.4}$$

and

$$\overline{\overline{P}}^{inp,act} = \overline{\overline{P}}^{sca,rad} + \overline{\overline{P}}^{tot,loss} \tag{35.1}$$

$$\overline{\overline{P}}^{inp,react} = \overline{\overline{P}}^{sca,react,vac} + \overline{\overline{P}}^{tot,react,mat} \tag{35.2}$$

$$\overline{\overline{P}}^{sca,act} = \overline{\overline{P}}^{sca,rad} + \overline{\overline{P}}^{sca,loss} \tag{35.3}$$

$$\overline{\overline{P}}^{sca,react} = \overline{\overline{P}}^{sca,react,vac} + \overline{\overline{P}}^{sca,react,mat} \tag{35.4}$$

Because the powers P^{inp}, $P^{inp,part,rad}$, P^{sca}, and $P^{sca,part,rad}$ are complex numbers, the Mat-EMP-CMT developed in this paper doesn't want to maximize and minimize them, so their normalized versions are not specifically listed here.

V. MAT-EMP-CMT

The main destination of Mat-EMP-CMT is to optimize the interesting powers discussed in Secs. III and IV. As the typical examples, the $\tilde{P}^{sca,rad}$, P^{inp}, and $P^{inp,part,rad}$ are respectively optimized in this section, and the procedures to optimize other powers are not provided, because their procedures are similar. Only some important formulations and conclusions are simply provided here, but the detailed procedures are not given, because a similar and detailed procedure can be found in [14].

A. *To maximize and minimize* $\tilde{P}^{sca,rad}$.

The matrices $\overline{\overline{P}}^{sca,rad}$ and $\overline{\overline{F}}^{tot}$ are Hermitian, and the $\overline{\overline{F}}^{tot}$ is positive definite, so the necessary condition to maximize and minimize power $\tilde{P}^{sca,rad}$ is the following generalized characteristic equation [16].

$$\overline{\overline{P}}^{sca,rad}\cdot\overline{a}_\xi = \lambda_\xi\,\overline{\overline{F}}^{tot}\cdot\overline{a}_\xi \tag{36}$$

here $\xi=1,2,\cdots,\Xi$, and the characteristic value λ_ξ is real [16].

The modal incident and total fields on scatterer, the modal scattering field on whole space, and the modal scattering currents on scatterer are respectively as follows

$$\left.\begin{aligned}\overline{E}_\xi^X(\overline{r}) &= \overline{E}^X\left(\overline{F}_\xi^{tot};\overline{r}\right)\\ \overline{H}_\xi^X(\overline{r}) &= \overline{H}^X\left(\overline{F}_\xi^{tot};\overline{r}\right)\end{aligned}\right.,\quad(\overline{r}\in V) \tag{37.1}$$

$$\left.\begin{aligned}\overline{E}_\xi^{sca}(\overline{r}) &= \overline{E}^{sca}\left(\overline{F}_\xi^{tot};\overline{r}\right)\\ \overline{H}_\xi^{sca}(\overline{r}) &= \overline{H}^{sca}\left(\overline{F}_\xi^{tot};\overline{r}\right)\end{aligned}\right.,\quad(\overline{r}\in\mathbb{R}^3) \tag{37.2}$$

$$\left.\begin{aligned}\overline{J}_\xi^Y(\overline{r}) &= \overline{J}^Y\left(\overline{F}_\xi^{tot};\overline{r}\right)\\ \overline{M}_\xi^{vm}(\overline{r}) &= \overline{M}^{vm}\left(\overline{F}_\xi^{tot};\overline{r}\right)\end{aligned}\right.,\quad(\overline{r}\in V) \tag{37.3}$$

here $X=inc,tot$, and $Y=vo,vp,vop$, and $\overline{F}_\xi^{tot}=\overline{\overline{B}}\cdot\overline{a}_\xi$. The relevant operators in (37) are defined in (5)-(7).

By doing some necessary orthogonalizations for the characteristic vectors corresponding to the same characteristic values (i.e., the degenerate modes) [16], the coupling modal powers satisfy following orthogonality for any $\xi,\zeta=1,2,\cdots,\Xi$.

$$P_{\xi}^{sca,rad}\delta_{\xi\zeta} = \bar{a}_{\xi}^{H}\cdot\bar{\bar{P}}^{sca,rad}\cdot\bar{a}_{\zeta} = \frac{1}{2}\oiint_{S_{\infty}}\left[\bar{E}_{\zeta}^{sca}\times\left(\bar{H}_{\xi}^{sca}\right)^{*}\right]\cdot d\bar{S} \tag{38}$$

here $\delta_{\xi\zeta}$ is the Kronecker delta symbol. In particular, when $\xi=\zeta$, the modal powers and their normalized versions are as follows

$$P_{\xi}^{sca,rad} = (1/2)\oiint_{S_{\infty}}\left[\bar{E}_{\xi}^{sca}\times\left(\bar{H}_{\xi}^{sca}\right)^{*}\right]\cdot d\bar{S} \tag{39}$$

and

$$\tilde{P}_{\xi}^{sca,rad} = \frac{P_{\xi}^{sca,rad}}{(1/2)\left\langle \bar{F}_{\xi}^{tot},\bar{F}_{\xi}^{tot}\right\rangle_{V}} \tag{40}$$

The proof for the orthogonality relation in (38) is similar to [14].

The modal currents and the modal fields in (37) can be normalized by using the method in (8)-(11). The CMs derived above are collectively referred to as Radiated power CMs (RaCMs) due to their ability to optimize system radiation [14].

B. To orthogonalize the power P^{inp}.

In this subsection, the lossless and lossy cases are separately discussed, because the matrix $\bar{\bar{P}}^{inp,act}$ can be either positive definite or positive semi-definite for lossless case, however the matrix $\bar{\bar{P}}^{inp,act}$ must be positive definite for lossy case.

1) Lossless Case

When the scatterer is lossless, $\bar{\bar{P}}^{tot,loss}=0$, and then $\bar{\bar{P}}^{inp,act}=\bar{\bar{P}}^{sca,rad}$. When the $\bar{\bar{P}}^{sca,rad}$ is positive definite at frequency f, the CM set which orthogonalizes $P^{inp}(\bar{a})$ can be obtained by solving the following generalized characteristic equation [16].

$$\bar{\bar{P}}^{inp,react}(f)\cdot\bar{a}_{\xi}(f) = \lambda_{\xi}(f)\,\bar{\bar{P}}^{sca,rad}(f)\cdot\bar{a}_{\xi}(f) \tag{41}$$

here $\xi=1,2,\cdots,\Xi$, and all $\lambda_{\xi}(f)$ are real [16]. When the $\bar{\bar{P}}^{sca,rad}$ is positive semi-definite at frequency f_0, the frequency f_0 can be determined by using the method given in [14]. Once the frequency f_0 is determined, the characteristic vector set $\{\bar{a}_{\xi}(f_0)\}_{\xi=1}^{\Xi}$ can be obtained by using the following limitation as explained in [14].

$$\bar{a}_{\xi}(f_0) = \lim\nolimits_{f\to f_0}\bar{a}_{\xi}(f) \tag{42}$$

At any frequency, the modal currents and modal fields can be similarly obtained as (37), and they satisfy the following orthogonalities.

$$P_{\xi}^{inp}\delta_{\xi\zeta} = \frac{1}{2}\left\langle \bar{J}_{\xi}^{vp},\bar{E}_{\zeta}^{inc}\right\rangle_{V}+\frac{1}{2}\left\langle \bar{H}_{\xi}^{inc},\bar{M}_{\zeta}^{vm}\right\rangle_{V} \tag{43.1}$$

$$\begin{aligned}P_{\xi;\,act}^{inp}\delta_{\xi\zeta} &= \mathrm{Re}\{P_{\xi}^{inp}\}\delta_{\xi\zeta} = P_{\xi;\,sca,rad}^{inp}\delta_{\xi\zeta}\\ &= \frac{1}{2}\oiint_{S_{\infty}}\left[\bar{E}_{\zeta}^{sca}\times\left(\bar{H}_{\xi}^{sca}\right)^{*}\right]\cdot d\bar{S}\end{aligned} \tag{43.2}$$

$$\begin{aligned}P_{\xi,\,react}^{inp}\delta_{\xi\zeta} &= \mathrm{Im}\{P_{\xi}^{inp}\}\delta_{\xi\zeta}\\ &= 2\omega\left[\frac{1}{4}\left\langle \bar{H}_{\xi}^{sca},\mu_0\bar{H}_{\zeta}^{sca}\right\rangle_{\mathbb{R}^3}-\frac{1}{4}\left\langle \varepsilon_0\bar{E}_{\xi}^{sca},\bar{E}_{\zeta}^{sca}\right\rangle_{\mathbb{R}^3}\right]\\ &\quad+2\omega\left[\frac{1}{4}\left\langle \bar{H}_{\xi}^{tot},\Delta\mu\bar{H}_{\zeta}^{tot}\right\rangle_{V}-\frac{1}{4}\left\langle \Delta\varepsilon\bar{E}_{\xi}^{tot},\bar{E}_{\zeta}^{tot}\right\rangle_{V}\right]\end{aligned} \tag{43.3}$$

and the modal input power P_{ξ}^{inp} is as follows

$$\begin{aligned}P_{\xi}^{inp} &= \begin{cases}P_{\xi;\,act}^{inp}+j\,P_{\xi;\,react}^{inp}, & (\xi=r_1,r_2,\cdots,r_R)\\ j\,P_{\xi;\,react}^{inp}, & (\xi=n_1,n_2,\cdots,n_N)\end{cases}\\ &= \begin{cases}P_{\xi;\,sca,rad}^{inp}+j\,P_{\xi;\,react}^{inp}, & (\xi=r_1,r_2,\cdots,r_R)\\ j\,P_{\xi;\,react}^{inp}, & (\xi=n_1,n_2,\cdots,n_N)\end{cases}\end{aligned} \tag{44}$$

here $\{r_1,r_2,\cdots,r_R\}\cap\{n_1,n_2,\cdots,n_N\}=\varnothing$, and $\{r_1,r_2,\cdots,r_R\}\cup\{n_1,n_2,\cdots,n_N\}=\{1,2,\cdots,\Xi\}$; the subscript "$act$" in $P_{\xi;\,act}^{inp}$ represents that the power $P_{\xi;\,act}^{inp}$ is the active part of modal input power P_{ξ}^{inp}; the subscript "sca,rad" in $P_{\xi;\,sca,rad}^{inp}$ represents that the power $P_{\xi;\,sca,rad}^{inp}$ is the radiated power carried by modal scattering field $\bar{F}_{\xi}^{sca}$; the other subscripts can be similarly explained. The proofs for the orthogonality relations in (43) are similar to [14].

The modes corresponding to $\xi=r_1,r_2,\cdots,r_R$ and $\xi=n_1,n_2,\cdots,n_N$ are respectively the radiative and non-radiative modes. The modes corresponding to $P_{\xi;\,react}^{inp}<0$, $P_{\xi;\,react}^{inp}=0$, and $P_{\xi;\,react}^{inp}>0$ are respectively the capacitive, resonant, and inductive modes. All these modes are collectively referred to as Input power CMs (InpCMs) to be distinguished from the CMs constructed in Secs. V-A and V-C. In fact, the InpCM set has the same physical essence as the Output power CM (OutCM) set constructed in [14]. To emphasize the equivalence between the OutCM set and InpCM set, the terms "OutCM" and "InpCM" are respectively used in [14] and this paper.

In addition, for the radiative modes, their characteristic values λ_{ξ} satisfy the following relation.

$$\lambda_{\xi} = P_{\xi;\,react}^{inp}\big/P_{\xi;\,sca,rad}^{inp} \tag{45}$$

2) Lossy Case

When $\sigma\neq 0$, the matrix $\bar{\bar{P}}^{tot,loss}$ is positive definite, so the matrix $\bar{\bar{P}}^{inp,act}=\bar{\bar{P}}^{sca,rad}+\bar{\bar{P}}^{tot,loss}$ must be positive definite [16]. The CM set which orthogonalizes $P^{inp}(\bar{a})$ can be obtained by solving the following generalized characteristic equation for any $\xi=1,2,\cdots,\Xi$ [16].

$$\bar{\bar{P}}^{inp,react}\cdot\bar{a}_{\xi} = \lambda_{\xi}\,\bar{\bar{P}}^{inp,act}\cdot\bar{a}_{\xi} \tag{46}$$

The modal currents and modal fields can be similarly obtained as the formulations in (37), and they satisfy the following orthogonalities.

$$P_{\xi}^{inp}\delta_{\xi\zeta} = \frac{1}{2}\left\langle \bar{J}_{\xi}^{vop},\bar{E}_{\zeta}^{inc}\right\rangle_{V}+\frac{1}{2}\left\langle \bar{H}_{\xi}^{inc},\bar{M}_{\zeta}^{vm}\right\rangle_{V} \tag{47.1}$$

$$\begin{aligned} P_{\xi;act}^{inp}\delta_{\xi\zeta} &= \operatorname{Re}\left\{P_{\xi}^{inp}\right\}\delta_{\xi\zeta} \\ &= \left(P_{\xi;sca,rad}^{inp}+P_{\xi;tot,loss}^{inp}\right)\delta_{\xi\zeta} \\ &= \frac{1}{2}\oint\!\!\!\oint_{S_\infty}\left[\bar{E}_{\zeta}^{sca}\times\left(\bar{H}_{\xi}^{sca}\right)^{*}\right]\cdot d\bar{S}+\frac{1}{2}\left\langle\sigma\bar{E}_{\xi}^{tot},\bar{E}_{\zeta}^{tot}\right\rangle_{V} \end{aligned} \tag{47.2}$$

$$\begin{aligned} P_{\xi;react}^{inp}\delta_{\xi\zeta} &= \operatorname{Im}\left\{P_{\xi}^{inp}\right\}\delta_{\xi\zeta} \\ &= \left(P_{\xi;sca,react,vac}^{inp}+P_{\xi;tot,react,mat}^{inp}\right)\delta_{\xi\zeta} \\ &= 2\omega\left[\frac{1}{4}\left\langle\bar{H}_{\xi}^{sca},\mu_0\bar{H}_{\zeta}^{sca}\right\rangle_{\mathbb{R}^3}-\frac{1}{4}\left\langle\varepsilon_0\bar{E}_{\xi}^{sca},\bar{E}_{\zeta}^{sca}\right\rangle_{\mathbb{R}^3}\right] \\ &\quad+2\omega\left[\frac{1}{4}\left\langle\bar{H}_{\xi}^{tot},\Delta\mu\bar{H}_{\zeta}^{tot}\right\rangle_{V}-\frac{1}{4}\left\langle\Delta\varepsilon\bar{E}_{\xi}^{tot},\bar{E}_{\zeta}^{tot}\right\rangle_{V}\right] \end{aligned} \tag{47.3}$$

and the modal power is as follows

$$\begin{aligned} P_{\xi}^{inp} &= \operatorname{Re}\left\{P_{\xi}^{inp}\right\} + j\operatorname{Im}\left\{P_{\xi}^{inp}\right\} \\ &= P_{\xi;act}^{inp} + j\,P_{\xi;react}^{inp} \\ &= P_{\xi;sca,rad}^{inp}+P_{\xi;tot,loss}^{inp}+j\left(P_{\xi;sca,react,vac}^{inp}+P_{\xi;tot,react,mat}^{inp}\right) \end{aligned} \tag{48}$$

The proofs for the orthogonality relations in (47) are similar to [14]. In addition, the characteristic values λ_ξ derived from (46) satisfy the following relation for any $\xi=1,2,\cdots,\Xi$.

$$\lambda_\xi = P_{\xi;react}^{inp}\Big/P_{\xi;act}^{inp} \tag{49}$$

It must be clearly pointed out here that the radiated power orthogonality like (38) and (43.2) cannot be guaranteed in this case. In fact, this is just the main reason to introduce the CM set given in the following Sec. V-C.

C. To orthogonalize the $P^{inp,\,part,\,rad}$ for lossy material bodies.

Because $P^{inp,\,part,\,rad}=P^{inp}$ for lossless material body case, only the lossy case is discussed in this subsection.

When the $\bar{\bar{P}}^{sca,\,rad}$ is positive definite at frequency f, the CM set which orthogonalizes $P^{inp,\,part,\,rad}(\bar{a})$ can be obtained by solving the following equation for any $\xi=1,2,\cdots,\Xi$ [16].

$$\bar{\bar{P}}^{inp,\,react}(f)\cdot\bar{a}_\xi(f) = \lambda_\xi(f)\,\bar{\bar{P}}^{sca,\,rad}(f)\cdot\bar{a}_\xi(f) \tag{50}$$

When the $\bar{\bar{P}}^{sca,\,rad}$ is positive semi-definite at frequency f_0, the frequency f_0 can be determined by using the method provided in [14], and then the $\bar{a}(f_0)$ at frequency f_0 can be obtained as the following (51) for any $\xi=1,2,\cdots,\Xi$ [14].

$$\bar{a}_\xi(f_0) = \lim_{f\to f_0}\bar{a}_\xi(f) \tag{51}$$

The modal currents and modal fields can be similarly obtained as (37), and they satisfy the following orthogonalities.

$$\begin{aligned} P_{\xi}^{inp,\,part,\,rad}\delta_{\xi\zeta} &= \frac{1}{2}\left\langle\bar{J}_{\xi}^{vop},\bar{E}_{\zeta}^{inc}\right\rangle_{V}+\frac{1}{2}\left\langle\bar{H}_{\xi}^{inc},\bar{M}_{\zeta}^{vm}\right\rangle_{V} \\ &\quad-\frac{1}{2}\left\langle\sigma\bar{E}_{\xi}^{tot},\bar{E}_{\zeta}^{tot}\right\rangle_{V} \end{aligned} \tag{52.1}$$

$$\operatorname{Re}\left\{P_{\xi}^{inp,\,part,\,rad}\right\}\delta_{\xi\zeta} = P_{\xi;sca,rad}^{inp,\,part,\,rad}\delta_{\xi\zeta} = \frac{1}{2}\oint\!\!\!\oint_{S_\infty}\left[\bar{E}_{\zeta}^{sca}\times\left(\bar{H}_{\xi}^{sca}\right)^{*}\right]\cdot d\bar{S} \tag{52.2}$$

$$\begin{aligned} \operatorname{Im}\left\{P_{\xi}^{inp,\,part,\,rad}\right\}\delta_{\xi\zeta} &= \left(P_{\xi;sca,react,vac}^{inp,\,part,\,rad}+P_{\xi;tot,react,mat}^{inp,\,part,\,rad}\right)\delta_{\xi\zeta} \\ &= 2\omega\left[\frac{1}{4}\left\langle\bar{H}_{\xi}^{sca},\mu_0\bar{H}_{\zeta}^{sca}\right\rangle_{\mathbb{R}^3}-\frac{1}{4}\left\langle\varepsilon_0\bar{E}_{\xi}^{sca},\bar{E}_{\zeta}^{sca}\right\rangle_{\mathbb{R}^3}\right] \\ &\quad+2\omega\left[\frac{1}{4}\left\langle\bar{H}_{\xi}^{tot},\Delta\mu\bar{H}_{\zeta}^{tot}\right\rangle_{V}-\frac{1}{4}\left\langle\Delta\varepsilon\bar{E}_{\xi}^{tot},\bar{E}_{\zeta}^{tot}\right\rangle_{V}\right] \end{aligned} \tag{52.3}$$

The modal power is as follows

$$\begin{aligned} P_{\xi}^{inp,\,part,\,rad} &= \begin{cases} \operatorname{Re}\left\{P_{\xi}^{inp,\,part,\,rad}\right\}+j\operatorname{Im}\left\{P_{\xi}^{inp,\,part,\,rad}\right\}, & (\xi=r_1,\cdots,r_R) \\ j\operatorname{Im}\left\{P_{\xi}^{inp,\,part,\,rad}\right\}, & (\xi=n_1,\cdots,n_N) \end{cases} \\ &= \begin{cases} P_{\xi;sca,rad}^{inp,\,part,\,rad}+j\left(P_{\xi;sca,react,vac}^{inp,\,part,\,rad}+P_{\xi;tot,react,mat}^{inp,\,part,\,rad}\right), & (\xi=r_1,\cdots,r_R) \\ j\left(P_{\xi;sca,react,vac}^{inp,\,part,\,rad}+P_{\xi;tot,react,mat}^{inp,\,part,\,rad}\right), & (\xi=n_1,\cdots,n_N) \end{cases} \end{aligned} \tag{53}$$

The proofs for the orthogonality relations in (52) are similar to [14]. In addition, for the radiative modes, their characteristic values λ_ξ satisfy the following relation (54).

$$\lambda_\xi = \operatorname{Im}\left\{P_{\xi}^{inp,\,part,\,rad}\right\}\Big/\operatorname{Re}\left\{P_{\xi}^{inp,\,part,\,rad}\right\} \tag{54}$$

It must be clearly pointed out that the orthogonality for the total active power like (47.2) cannot be guaranteed in this case, though the radiation pattern orthogonality (52.2) exists.

D. To optimize the other powers.

The fundamental principles and procedures to optimize the normalized powers $\tilde{P}^{inc,\,loss}$, $\tilde{P}^{sca,\,loss}$, $\tilde{P}^{tot,\,loss}$, $\tilde{P}^{inc,\,react,\,mat}$, $\tilde{P}^{sca,\,react,\,mat}$, $\tilde{P}^{tot,\,react,\,mat}$, $\tilde{P}^{sca,\,act}$, $\tilde{P}^{sca,\,react,\,vac}$, $\tilde{P}^{sca,\,react}$, $\tilde{P}^{inp,\,act}$, and $\tilde{P}^{inp,\,react}$ are completely similar to the $\tilde{P}^{sca,\,rad}$ provided in Sec. V-A; the fundamental principles and procedures to orthogonalize P^{sca} and $P^{sca,\,vac}$ are completely similar to the P^{inp} provided in Sec. V-B; the fundamental principles and procedures to orthogonalize $P^{sca,\,part,\,rad}$ are completely similar to the $P^{inp,\,part,\,rad}$ provided in Sec. V-C.

Various CM sets can be efficiently distinguished from each other based on their different superscripts, as illustrated in the following Sec. VI.

VI. Modal Expansion

In this section, the method to expand the fields, currents, and powers in terms of various CM sets is provided, and it is particularly called as the CM-based modal expansion method to be distinguished from the other kinds of expansion methods, such as the eigen-mode-based expansion method [17]-[18] and the special function-based expansion method [19].

A. The modal expansions for fields and currents.

Due to the completeness of any kind of CM set, the expansion vector $\bar{a}$, the field $\bar{F}^{X}$ on scatterer, the field $\bar{F}^{sca}$ on whole space, and the various currents $\left\{\bar{J}^{vp},\bar{J}^{vo},\bar{J}^{vop},\bar{M}^{vm}\right\}$ can be expanded in terms of the any kind of CM set as follows

$$\bar{a} = \sum\nolimits_{\xi=1}^{\Xi}c_\xi\bar{a}_\xi \tag{55}$$

and

$$\begin{aligned} \bar{E}^X(r) &= \sum_{\xi=1}^{\Xi} c_\xi \bar{E}_\xi^X(r) \\ \bar{H}^X(r) &= \sum_{\xi=1}^{\Xi} c_\xi \bar{H}_\xi^X(r) \end{aligned} \quad , \quad (\bar{r} \in V) \tag{56.1}$$

$$\begin{aligned} \bar{E}^{sca}(r) &= \sum_{\xi=1}^{\Xi} c_\xi \bar{E}_\xi^{sca}(r) \\ \bar{H}^{sca}(r) &= \sum_{\xi=1}^{\Xi} c_\xi \bar{H}_\xi^{sca}(r) \end{aligned} \quad , \quad (\bar{r} \in \mathbb{R}^3) \tag{56.2}$$

$$\begin{aligned} \bar{J}^Y(\bar{r}) &= \sum_{\xi=1}^{\Xi} c_\xi \bar{J}_\xi^Y(\bar{r}) \\ \bar{M}^{vm}(\bar{r}) &= \sum_{\xi=1}^{\Xi} c_\xi \bar{M}_\xi^{vm}(\bar{r}) \end{aligned} \quad , \quad (\bar{r} \in V) \tag{56.3}$$

here $X = inc, tot$, and $Y = vo, vp, vop$.

B. The modal expansions for various powers.

Based on the power orthogonalities of the CM sets derived in Sec. V, various system powers can be expanded as follows

$$P^{inp} = \sum_{\xi=1}^{\Xi} \left|c_\xi^{inp}\right|^2 P_\xi^{inp} \tag{57.1}$$

$$P^{inp,part,rad} = \sum_{\xi=1}^{\Xi} \left|c_\xi^{inp,part,rad}\right|^2 P_\xi^{inp,part,rad} \tag{57.2}$$

$$P^{sca} = \sum_{\xi=1}^{\Xi} \left|c_\xi^{sca}\right|^2 P_\xi^{sca} \tag{57.3}$$

$$P^{sca,vac} = \sum_{\xi=1}^{\Xi} \left|c_\xi^{sca,vac}\right|^2 P_\xi^{sca,vac} \tag{57.4}$$

$$P^{sca,part,rad} = \sum_{\xi=1}^{\Xi} \left|c_\xi^{sca,part,rad}\right|^2 P_\xi^{sca,part,rad} \tag{57.5}$$

and

$$\begin{aligned} P^{sca,rad} &= \sum_{\xi=1}^{\Xi} \left|c_\xi^{inp,part,rad}\right|^2 \mathrm{Re}\left\{P_\xi^{inp,part,rad}\right\} \\ &= \sum_{\xi=1}^{\Xi} \left|c_\xi^{sca,part,rad}\right|^2 \mathrm{Re}\left\{P_\xi^{sca,part,rad}\right\} \\ &= \sum_{\xi=1}^{\Xi} \left|c_\xi^{sca,vac}\right|^2 \mathrm{Re}\left\{P_\xi^{sca,vac}\right\} \\ &= \sum_{\xi=1}^{\Xi} \left|c_\xi^{sca,rad}\right|^2 P_\xi^{sca,rad} \end{aligned} \tag{58.1}$$

$$\begin{aligned} P^{sca,act} &= \sum_{\xi=1}^{\Xi} \left|c_\xi^{sca}\right|^2 \mathrm{Re}\left\{P_\xi^{sca}\right\} \\ &= \sum_{\xi=1}^{\Xi} \left|c_\xi^{sca,act}\right|^2 P_\xi^{sca,act} \end{aligned} \tag{58.2}$$

$$\begin{aligned} P^{sca,react,vac} &= \sum_{\xi=1}^{\Xi} \left|c_\xi^{sca,vac}\right|^2 \mathrm{Im}\left\{P_\xi^{sca,vac}\right\} \\ &= \sum_{\xi=1}^{\Xi} \left|c_\xi^{sca,react,vac}\right|^2 P_\xi^{sca,react,vac} \end{aligned} \tag{58.3}$$

$$\begin{aligned} P^{sca,react} &= \sum_{\xi=1}^{\Xi} \left|c_\xi^{sca,part,rad}\right|^2 \mathrm{Im}\left\{P_\xi^{sca,part,rad}\right\} \\ &= \sum_{\xi=1}^{\Xi} \left|c_\xi^{sca}\right|^2 \mathrm{Im}\left\{P_\xi^{sca}\right\} \\ &= \sum_{\xi=1}^{\Xi} \left|c_\xi^{sca,react}\right|^2 P_\xi^{sca,react} \end{aligned} \tag{58.4}$$

$$\begin{aligned} P^{inp,act} &= \sum_{\xi=1}^{\Xi} \left|c_\xi^{inp}\right|^2 \mathrm{Re}\left\{P_\xi^{inp}\right\} \\ &= \sum_{\xi=1}^{\Xi} \left|c_\xi^{inp,act}\right|^2 P_\xi^{inp,act} \end{aligned} \tag{58.5}$$

$$\begin{aligned} P^{inp,react} &= \sum_{\xi=1}^{\Xi} \left|c_\xi^{inp,part,rad}\right|^2 \mathrm{Im}\left\{P_\xi^{inp,part,rad}\right\} \\ &= \sum_{\xi=1}^{\Xi} \left|c_\xi^{inp}\right|^2 \mathrm{Im}\left\{P_\xi^{inp}\right\} \\ &= \sum_{\xi=1}^{\Xi} \left|c_\xi^{inp,react}\right|^2 P_\xi^{inp,react} \end{aligned} \tag{58.6}$$

here the superscripts in expansion coefficients are to emphasize that different CM-based expansions have different coefficients.

C. The variational formulation to determine the expansion coefficients.

When the scatterer is excited by external field $\bar{F}^{inc}$, the one and only one total field $\bar{F}^{tot}$ on scatterer is resulted, here $\bar{F}^{tot}$ is the basic variable. Based on the discussions in Sec. III, the system input power P^{inp} can be equivalently determined as the following two ways.

$$P^{inp} = \frac{1}{2}\left\langle \bar{J}^{vop}\left(\bar{F}^{tot}\right), \bar{E}^{inc} \right\rangle_V + \frac{1}{2}\left\langle \bar{H}^{inc}, \bar{M}^{vm}\left(\bar{F}^{tot}\right) \right\rangle_V \tag{59.1}$$

$$P^{inp} = \frac{1}{2}\left\langle \bar{J}^{vop}\left(\bar{F}^{tot}\right), \bar{E}^{inc}\left(\bar{F}^{tot}\right) \right\rangle_V + \frac{1}{2}\left\langle \bar{H}^{inc}\left(\bar{F}^{tot}\right), \bar{M}^{vm}\left(\bar{F}^{tot}\right) \right\rangle_V \tag{59.2}$$

The $\bar{E}^{inc}$ and $\bar{H}^{inc}$ in (59.1) are known. However, the $\bar{E}^{inc}\left(\bar{F}^{tot}\right)$ and $\bar{H}^{inc}\left(\bar{F}^{tot}\right)$ in (59.2) are expressed as the functions of $\bar{F}^{tot}$ as (5) regardless of whether $\bar{E}^{inc}$ and $\bar{H}^{inc}$ are known or not.

The real $\bar{F}^{tot}$ on scatterer will make the following functional be zero and stationary.

$$\begin{aligned} F\left(\bar{F}^{tot}\right) = &\frac{1}{2}\left\langle \bar{J}^{vop}\left(\bar{F}^{tot}\right), \bar{E}^{inc} \right\rangle_V + \frac{1}{2}\left\langle \bar{H}^{inc}, \bar{M}^{vm}\left(\bar{F}^{tot}\right) \right\rangle_V \\ &- \frac{1}{2}\left\langle \bar{J}^{vop}\left(\bar{F}^{tot}\right), \bar{E}^{inc}\left(\bar{F}^{tot}\right) \right\rangle_V - \frac{1}{2}\left\langle \bar{H}^{inc}\left(\bar{F}^{tot}\right), \bar{M}^{vm}\left(\bar{F}^{tot}\right) \right\rangle_V \end{aligned} \tag{60}$$

Inserting (56) into (60) and employing the Ritz's procedure [20], the following simultaneous equations for the expansion coefficients $\{c_\xi\}_{\xi=1}^{\Xi}$ are derived for any $\zeta = 1, 2, \cdots, \Xi$.

$$\begin{aligned} &\frac{1}{2}\left\langle \bar{J}_\zeta^{vop}, \bar{E}^{inc} \right\rangle_V + \frac{1}{2}\left\langle \bar{H}^{inc}, \bar{M}_\zeta^{vm} \right\rangle_V \\ &= \frac{1}{2}\left\langle \bar{J}_\zeta^{vop}, \sum_{\xi=1}^{\Xi} c_\xi \bar{E}_\xi^{inc} \right\rangle_V + \frac{1}{2}\left\langle \bar{H}_\zeta^{inc}, \sum_{\xi=1}^{\Xi} c_\xi \bar{M}_\xi^{vm} \right\rangle_V \\ &\quad + \frac{1}{2}\left\langle \sum_{\xi=1}^{\Xi} c_\xi \bar{J}_\xi^{vop}, \bar{E}_\zeta^{inc} \right\rangle_V + \frac{1}{2}\left\langle \sum_{\xi=1}^{\Xi} c_\xi \bar{H}_\xi^{inc}, \bar{M}_\zeta^{vm} \right\rangle_V \end{aligned} \tag{61.1}$$

and

$$\begin{aligned} &-\frac{1}{2}\left\langle \bar{J}_\zeta^{vop}, \bar{E}^{inc} \right\rangle_V + \frac{1}{2}\left\langle \bar{H}^{inc}, \bar{M}_\zeta^{vm} \right\rangle_V \\ &= -\frac{1}{2}\left\langle \bar{J}_\zeta^{vop}, \sum_{\xi=1}^{\Xi} c_\xi \bar{E}_\xi^{inc} \right\rangle_V - \frac{1}{2}\left\langle \bar{H}_\zeta^{inc}, \sum_{\xi=1}^{\Xi} c_\xi \bar{M}_\xi^{vm} \right\rangle_V \\ &\quad + \frac{1}{2}\left\langle \sum_{\xi=1}^{\Xi} c_\xi \bar{J}_\xi^{vop}, \bar{E}_\zeta^{inc} \right\rangle_V + \frac{1}{2}\left\langle \sum_{\xi=1}^{\Xi} c_\xi \bar{H}_\xi^{inc}, \bar{M}_\zeta^{vm} \right\rangle_V \end{aligned} \tag{61.2}$$

By solving the above (61), the $\{c_\xi\}_{\xi=1}^{\Xi}$ can be determined.

D. The expansion coefficients of the InpCM-based expansion.

If the CM set is derived by orthogonalizing P^{inp} , the coefficients $\{c_\xi^{inp}\}_{\xi=1}^{\Xi}$ in (61) can be easily solved and concisely expressed as following (62) based on the orthogonality (47).

$$c_\xi^{inp} = \begin{cases} \dfrac{1}{P_\xi^{inp}} \cdot \dfrac{1}{2}\left\langle \bar{J}_\xi^{vop}, \bar{E}^{inc} \right\rangle_V & , \ (\Delta\mu = 0, \Delta\varepsilon_c \neq 0) \\ \left[\dfrac{1}{P_\xi^{inp}} \cdot \dfrac{1}{2}\left\langle \bar{H}^{inc}, \bar{M}_\xi^{vm} \right\rangle_V\right]^* & , \ (\Delta\mu \neq 0, \Delta\varepsilon_c = 0) \\ \dfrac{1}{P_\xi^{inp}} \cdot \dfrac{1}{2}\left\langle \bar{J}_\xi^{vop}, \bar{E}^{inc} \right\rangle_V = \left[\dfrac{1}{P_\xi^{inp}} \cdot \dfrac{1}{2}\left\langle \bar{H}^{inc}, \bar{M}_\xi^{vm} \right\rangle_V\right]^* & , \ (\Delta\mu, \Delta\varepsilon_c \neq 0) \\ 0 & , \ (\Delta\mu, \Delta\varepsilon_c = 0) \end{cases} \tag{62}$$

However, it must be pointed out that the (62) is not valid for the non-radiative resonant InpCMs corresponding to $P_\xi^{inp} = 0$.

VII. The Modal Quantities for InpCMs

In (55)-(58), the system electromagnetic quantities (such as $\bar{J}^{vop}$, $\bar{F}^{sca}$, and P^{out}) are expanded in terms of series of modal components (such as $c_\xi \bar{J}_\xi^{vop}$, $c_\xi \bar{F}_\xi^{sca}$, and $|c_\xi|^2 P_\xi^{out}$), and to consider of the weight of every modal component in whole expansion formulation is important. However, the components are either the complex vectors (such as $c_\xi \bar{J}_\xi^{vop}$ and $c_\xi \bar{F}_\xi^{sca}$) or the complex scalars (such as $|c_\xi|^2 P_\xi^{out}$), so the modal weights cannot be directly derived from the modal components themselves, and then to establish an appropriate mapping from the modal index set $\{\xi\}_{\xi=1}^{\Xi}$ to a real number set is necessary.

In this section, the InpCM-based expansion method is considered, and it is assumed that $\Delta\varepsilon_c \neq 0$, and then the $c_\xi^{inp} = (1/2P_\xi^{inp})\langle \bar{J}_\xi^{vop}, \bar{E}^{inc}\rangle_V$ in (62) is used throughout. In fact, the case $(\Delta\mu \neq 0, \Delta\varepsilon_c = 0)$ can be similarly discussed, and it will not be repeated in this paper.

A. An illuminating example.

Let us consider the following example at first.

$$20\,\hat{x} + 20\,\hat{y} = \left[1\cdot(10\,\hat{x}+10\,\hat{y})\right] + \left[10\cdot(1\,\hat{x}+1\,\hat{y})\right] \tag{63}$$

Obviously, using the following mapping (64.1) to depict the weights of term $1\cdot(10\,\hat{x}+10\,\hat{y})$ and term $10\cdot(1\,\hat{x}+1\,\hat{y})$ is unreasonable.

$$\{1\cdot(10\,\hat{x}+10\,\hat{y})\ ,\ 10\cdot(1\,\hat{x}+1\,\hat{y})\} \leftrightarrow \{1\ ,\ 10\} \tag{64.1}$$

here the symbol $\{a,b\}$ represents the ordered array of numbers a and b. In fact, a reasonable mapping is as follows

$$\{1\cdot(10\,\hat{x}+10\,\hat{y})\ ,\ 10\cdot(1\,\hat{x}+1\,\hat{y})\} \leftrightarrow \left\{1\cdot\sqrt{10^2+10^2}\ ,\ 10\cdot\sqrt{1^2+1^2}\right\} \tag{64.2}$$

because the (63) can be equivalently rewritten as follows

$$20\,\hat{x} + 20\,\hat{y} = \left[1\cdot\sqrt{10^2+10^2}\cdot\frac{(10\,\hat{x}+10\,\hat{y})}{\sqrt{10^2+10^2}}\right] + \left[10\cdot\sqrt{1^2+1^2}\cdot\frac{(1\,\hat{x}+1\,\hat{y})}{\sqrt{1^2+1^2}}\right] \tag{65}$$

By comparing the (63) with (65), it is easy to find out that the essential reason to lead to the inappropriate mapping (64.1) is that the terms $10\,\hat{x}+10\,\hat{y}$ and $1\,\hat{x}+1\,\hat{y}$ in (63) are not well normalized.

B. Modal normalization.

Based on the same reason explained above, to normalize the CMs is necessary for establishing the appropriate mapping from the $\{\xi\}_{\xi=1}^{\Xi}$ to the real number set whose elements quantitively depict the modal weights in whole modal expansion formulation. Based on (8), the modal total field $\bar{F}_\xi^{tot}$ on V is normalized as follows

$$\tilde{\bar{F}}_\xi^{tot}(r) \triangleq \frac{\bar{F}_\xi^{tot}(r)}{(1/\sqrt{2})\langle \bar{F}_\xi^{tot}, \bar{F}_\xi^{tot}\rangle_V^{1/2}}\ ,\quad (\bar{r}\in V) \tag{66}$$

here $F = E$ or H, and it depends on that the basic variable is selected as whom.

Based on the (5)-(7) and (66), the CMs are automatically normalized as follows

$$\left.\begin{aligned}\tilde{\bar{E}}_\xi^X(r) &= \bar{E}_\xi^X(r)\Big/(1/\sqrt{2})\langle \bar{F}_\xi^{tot},\bar{F}_\xi^{tot}\rangle_V^{1/2}\\ \tilde{\bar{H}}_\xi^X(r) &= \bar{H}_\xi^X(r)\Big/(1/\sqrt{2})\langle \bar{F}_\xi^{tot},\bar{F}_\xi^{tot}\rangle_V^{1/2}\end{aligned}\right\}\ ,\quad (\bar{r}\in V) \tag{67.1}$$

$$\left.\begin{aligned}\tilde{\bar{E}}_\xi^{sca}(r) &= \bar{E}_\xi^{sca}(r)\Big/(1/\sqrt{2})\langle \bar{F}_\xi^{tot},\bar{F}_\xi^{tot}\rangle_V^{1/2}\\ \tilde{\bar{H}}_\xi^{sca}(r) &= \bar{H}_\xi^{sca}(r)\Big/(1/\sqrt{2})\langle \bar{F}_\xi^{tot},\bar{F}_\xi^{tot}\rangle_V^{1/2}\end{aligned}\right\}\ ,\quad (\bar{r}\in \mathbb{R}^3) \tag{67.2}$$

$$\left.\begin{aligned}\tilde{\bar{J}}_\xi^Y(r) &= \bar{J}_\xi^Y(r)\Big/(1/\sqrt{2})\langle \bar{F}_\xi^{tot},\bar{F}_\xi^{tot}\rangle_V^{1/2}\\ \tilde{\bar{M}}_\xi^{vm}(r) &= \bar{M}_\xi^{vm}(r)\Big/(1/\sqrt{2})\langle \bar{F}_\xi^{tot},\bar{F}_\xi^{tot}\rangle_V^{1/2}\end{aligned}\right\}\ ,\quad (\bar{r}\in V) \tag{67.3}$$

and

$$\tilde{P}_\xi^{inp} = \frac{P_\xi^{inp}}{(1/2)\langle \bar{F}_\xi^{tot},\bar{F}_\xi^{tot}\rangle_V} \tag{68}$$

here $X = inc, tot$, and $Y = vo, vp, vop$.

Obviously, when the CMs are normalized as (67)-(68), the expansion coefficient (62) automatically becomes the following version.

$$\tilde{c}_\xi^{inp} = \begin{cases} \dfrac{1}{\tilde{P}_\xi^{inp}}\cdot\dfrac{1}{2}\langle \tilde{\bar{J}}_\xi^{vop},\bar{E}^{inc}\rangle_V & ,\ (\Delta\mu=0,\Delta\varepsilon_c\neq 0)\\ \left[\dfrac{1}{\tilde{P}_\xi^{inp}}\cdot\dfrac{1}{2}\langle \bar{H}^{inc},\tilde{\bar{M}}_\xi^{vm}\rangle_V\right]^* & ,\ (\Delta\mu\neq 0,\Delta\varepsilon_c=0)\\ \dfrac{1}{\tilde{P}_\xi^{inp}}\cdot\dfrac{1}{2}\langle \tilde{\bar{J}}_\xi^{vop},\bar{E}^{inc}\rangle_V = \left[\dfrac{1}{\tilde{P}_\xi^{inp}}\cdot\dfrac{1}{2}\langle \bar{H}^{inc},\tilde{\bar{M}}_\xi^{vm}\rangle_V\right]^* & ,\ (\Delta\mu,\Delta\varepsilon_c\neq 0)\\ 0 & ,\ (\Delta\mu,\Delta\varepsilon_c=0)\end{cases} \tag{69}$$

C. The generalized Modal Significance (MS) for system total power.

Based on the (68) and (69), the input power expansion formulation (57.1) can be equivalently rewritten as follows

$$\begin{aligned}P^{inp} &= \sum\nolimits_{\xi=1}^{\Xi}|\tilde{c}_\xi^{inp}|^2\,\tilde{P}_\xi^{inp}\\ &= \sum\nolimits_{\xi=1}^{\Xi}\frac{1}{|\tilde{P}_\xi^{inp}|}\cdot\left|\frac{1}{2}\langle \tilde{\bar{J}}_\xi^{vop},\bar{E}^{inc}\rangle_V\right|^2\cdot\frac{\tilde{P}_\xi^{inp}}{|\tilde{P}_\xi^{inp}|}\end{aligned} \tag{70}$$

Obviously, the magnitude of term $\tilde{P}_\xi^{inp}/|\tilde{P}_\xi^{inp}|$ is unit just like the terms $(10\,\hat{x}+10\,\hat{y})/\sqrt{10^2+10^2}$ and $(1\,\hat{x}+1\,\hat{y})/\sqrt{1^2+1^2}$ in (65), and then two mappings can be established as the following (71) just like the (64.2).

$$\{\xi\}_{\xi=1}^{\Xi} \leftrightarrow \{\mathrm{SMS}_\xi^{sys,tot}\}_{\xi=1}^{\Xi} \tag{71.1}$$

$$\{\xi\}_{\xi=1}^{\Xi} \leftrightarrow \{\mathrm{GMS}_\xi^{sys,tot}\}_{\xi=1}^{\Xi} \tag{71.2}$$

here

$$\text{SMS}_\xi^{sys,tot} \triangleq \frac{1}{\left|\tilde{P}_\xi^{inp}\right|}\cdot\left|\frac{1}{2}\left\langle\tilde{\vec{J}}_\xi^{vop},\vec{E}^{inc}\right\rangle_V\right|^2 \tag{72.1}$$

$$\text{GMS}_\xi^{sys,tot} \triangleq \frac{1}{\left|\tilde{P}_\xi^{inp}\right|} \tag{72.2}$$

Based on the (70)-(72), it is easily found out that:

(1) When a specific field $\vec{F}^{inc}$ incidents on scatterer, only the first several terms, whose $\text{SMS}_\xi^{sys,tot}$ are relatively large, are necessary to be included in the truncated modal expansion formulation.

(2) Generally speaking, when the sufficient terms, which have relatively large $\text{GMS}_\xi^{sys,tot}$, are included in the modal expansion formulation, the truncated expansion formulation can basically coincide with the full-wave solution for any external excitation. However, it must be clearly pointed out that this conclusion is not always right. For example, when the expansion formulation only includes N terms, and the external incident field satisfies that $\vec{F}^{inc}=\vec{F}_{N+M}^{inc}$ on V (here $M>0$), the truncated expansion formulation cannot coincide with the full-wave solution, because only the (N+M)-th modal component is excited, whereas this component is not included in the truncated expansion formulation.

Based on the above discussions, it is obvious that the $\text{SMS}_\xi^{sys,tot}$ and $\text{GMS}_\xi^{sys,tot}$ quantitively depict the modal weight in whole modal expansion formulation, so the $\text{SMS}_\xi^{sys,tot}$ is called as "the Modal Significance (MS) corresponding to the specific excitation" or simply called as "Special MS (SMS)", and the $\text{GMS}_\xi^{sys,tot}$ is called as "the MS corresponding to general excitation" or simply called as "General MS (GMS)". The SMS and GMS are collectively referred to as "the generalized MS for system total power", and this is the reason why the superscripts "sys,tot" are used in them.

D. The generalized Modal Significance (MS) for system active and reactive powers.

The real and imaginary parts of power P^{inp} is the active power $P^{inp,act}=\text{Re}\{P^{inp}\}$ and the reactive power $P^{inp,react}=\text{Im}\{P^{inp}\}$. Because the $\left|\tilde{c}_\xi^{inp}\right|^2$ in (70) is always real, the following expansions can be derived.

$$\begin{aligned} P^{inp,act} &= \sum_{\xi=1}^{\Xi}\left|\tilde{c}_\xi^{inp}\right|^2\text{Re}\{\tilde{P}_\xi^{inp}\} \\ &= \sum_{\xi=1}^{\Xi}\frac{1}{\left|\tilde{P}_\xi^{inp}\right|}\cdot\left|\frac{1}{2}\left\langle\tilde{\vec{J}}_\xi^{vop},\vec{E}^{inc}\right\rangle_V\right|^2\cdot\frac{\text{Re}\{\tilde{P}_\xi^{inp}\}}{\left|\tilde{P}_\xi^{inp}\right|} \end{aligned} \tag{73.1}$$

$$\begin{aligned} P^{inp,react} &= \sum_{\xi=1}^{\Xi}\left|\tilde{c}_\xi^{inp}\right|^2\text{Im}\{\tilde{P}_\xi^{inp}\} \\ &= \sum_{\xi=1}^{\Xi}\frac{1}{\left|\tilde{P}_\xi^{inp}\right|}\cdot\left|\frac{1}{2}\left\langle\tilde{\vec{J}}_\xi^{vop},\vec{E}^{inc}\right\rangle_V\right|^2\cdot\frac{\text{Im}\{\tilde{P}_\xi^{inp}\}}{\left|\tilde{P}_\xi^{inp}\right|} \end{aligned} \tag{73.2}$$

Similarly to the generalized MS for system total power, the following "generalized MS for system active and reactive powers" are introduced to quantitively depict the modal weights in whole system active and reactive powers.

$$\text{SMS}_\xi^{sys,act} \triangleq \text{SMS}_\xi^{sys,tot}\cdot\frac{\text{Re}\{\tilde{P}_\xi^{inp}\}}{\left|\tilde{P}_\xi^{inp}\right|}=\frac{1}{\left|\tilde{P}_\xi^{inp}\right|^2}\cdot\left|\frac{1}{2}\left\langle\tilde{\vec{J}}_\xi^{vop},\vec{E}^{inc}\right\rangle_V\right|^2\cdot\text{Re}\{\tilde{P}_\xi^{inp}\} \tag{74.1}$$

$$\text{GMS}_\xi^{sys,act} \triangleq \text{GMS}_\xi^{sys,tot}\cdot\frac{\text{Re}\{\tilde{P}_\xi^{inp}\}}{\left|\tilde{P}_\xi^{inp}\right|}=\frac{1}{\left|\tilde{P}_\xi^{inp}\right|^2}\cdot\text{Re}\{\tilde{P}_\xi^{inp}\} \tag{74.2}$$

for the active power, and

$$\text{SMS}_\xi^{sys,react} \triangleq \text{SMS}_\xi^{sys,tot}\cdot\frac{\text{Im}\{\tilde{P}_\xi^{inp}\}}{\left|\tilde{P}_\xi^{inp}\right|}=\frac{1}{\left|\tilde{P}_\xi^{inp}\right|^2}\cdot\left|\frac{1}{2}\left\langle\tilde{\vec{J}}_\xi^{vop},\vec{E}^{inc}\right\rangle_V\right|^2\cdot\text{Im}\{\tilde{P}_\xi^{inp}\} \tag{75.1}$$

$$\text{GMS}_\xi^{sys,react} \triangleq \text{GMS}_\xi^{sys,tot}\cdot\frac{\text{Im}\{\tilde{P}_\xi^{inp}\}}{\left|\tilde{P}_\xi^{inp}\right|}=\frac{1}{\left|\tilde{P}_\xi^{inp}\right|^2}\cdot\text{Im}\{\tilde{P}_\xi^{inp}\} \tag{75.2}$$

for the reactive power. The reason to use the superscript "sys" in the $\text{SMS}_\xi^{sys,act/react}$ and $\text{GMS}_\xi^{sys,act/react}$ is the same as the $\text{SMS}_\xi^{sys,tot}$ and $\text{GMS}_\xi^{sys,tot}$ in Sec. VII-C, and the reason to use the superscripts "act" and "$react$" is evident.

E. The modal characteristic to allocate active and reactive powers and the modal ability to couple energy from external excitation.

From the above discussions, it is obvious that the modal ability to transform the energy provided by external excitation to a part of the system active and reactive powers depends on the following three aspects:

(1) The modal ability to contribute system total power is quantitively depicted by the $\text{GMS}_\xi^{sys,tot}$ defined in (72.2).

(2) The modal characteristic to allocate the total modal output power to its active and reactive parts is quantitively depicted by "the Modal characteristic to Allocate modal Output Power (MAOP)" defined as the following (76).

$$\text{MAOP}_\xi^{mod,act} \triangleq \text{Re}\{\tilde{P}_\xi^{inp}\}\big/\left|\tilde{P}_\xi^{inp}\right| \tag{76.1}$$

$$\text{MAOP}_\xi^{mod,react} \triangleq \text{Im}\{\tilde{P}_\xi^{inp}\}\big/\left|\tilde{P}_\xi^{inp}\right| \tag{76.2}$$

Here, the relation $\tilde{P}_\xi^{inp}=\tilde{P}_\xi^{out}$ has been considered, and it will be further discussed in Sec. VIII-A.

(3) The modal ability to couple energy from external excitation is quantitively depicted by "the Modal Ability to Couple Excitation (MACE)" defined as the following (77).

$$\text{MACE}_\xi^{mod} \triangleq \left|(1/2)\left\langle\tilde{\vec{J}}_\xi^{vop},\vec{E}^{inc}\right\rangle_V\right|^2 \tag{77}$$

The superscript "mod" in (76) and (77) is to emphasize that to introduce the MAOP and MACE is to depict the characteristic of the mode itself, but not to depict the modal weight in whole system power.

F. The characteristic quantities and non-characteristic quantities.

Obviously, the $\text{GMS}_\xi^{sys,tot}$, $\text{GMS}_\xi^{sys,act/react}$, and $\text{MAOP}_\xi^{mod,act/react}$ are independent of the specific external excitation, so they are collectively referred to as the *modal characteristic quantities* (or simply called as *characteristic quantities*). However, the $\text{SMS}_\xi^{sys,tot}$, $\text{SMS}_\xi^{sys,act/react}$, and MACE_ξ^{mod} depend on the specific external excitation, so they are collectively referred to as the *modal non-characteristic quantities* (or simply called as *non-characteristic quantities*). The characteristic quantities and non-characteristic quantities are collectively referred to as *modal quantities*.

VIII. DISCUSSIONS

Some necessary discussions related to the theory developed in this paper are provided in this section.

A. The discussions for the powers related to material bodies.

Various electromagnetic powers can be divided into the following four categories: the lossy powers, the radiated powers carried by radiative fields, the reactive powers due to the energies stored in non-radiative fields (simply called as reactively stored powers in fields), and the reactive powers due to the energies stored in matter (simply called as reactively stored powers in matter) [15], [21]. The former two kinds are collectively referred to as active powers, and the latter two kinds are collectively referred to as reactive powers.

If the material scatterer is regarded as a whole object, there exist only two kinds of fields in $\mathbb{R}^3$, that are the $\bar{F}^{sca}$ generated by scatterer and the $\bar{F}^{inc}$ generated by external excitation and environment. The $\bar{F}^{sca}$ and $\bar{F}^{inc}$ respectively contribute all kinds of powers mentioned above, and they are detailedly listed as follows.

1) The active powers

(1.1) The radiated powers include the $P^{sca,rad}$ carried by $\bar{F}^{sca}$, the $P^{inc,rad}$ carried by $\bar{F}^{inc}$, and the $P^{coup,rad}$ corresponding to the coupling between $\bar{F}^{sca}$ and $\bar{F}^{inc}$ on surface S_∞. The mathematical expression for the power $P^{sca,rad}$ has been given in (14.2), and the mathematical expressions for the power $P^{inc,rad}$ and the power $P^{coup,rad}$ are expressed as the following (78) and (79) respectively.

$$P^{inc,rad} = \frac{1}{2}\oiint_{S_\infty}\left[\bar{E}^{inc}\times\left(\bar{H}^{inc}\right)^*\right]\cdot d\bar{S} \tag{78}$$

$$P^{coup,rad} = \frac{1}{2}\oiint_{S_\infty}\left[\bar{E}^{sca}\times\left(\bar{H}^{inc}\right)^*\right]\cdot d\bar{S}+\frac{1}{2}\oiint_{S_\infty}\left[\bar{E}^{inc}\times\left(\bar{H}^{sca}\right)^*\right]\cdot d\bar{S} \tag{79}$$

(1.2) The lossy powers include the $P^{sca,loss}$ dissipated by $\bar{F}^{sca}$, the $P^{inc,loss}$ dissipated by $\bar{F}^{inc}$, and the $P^{coup,loss}$ corresponding to the coupling between $\bar{F}^{sca}$ and $\bar{F}^{inc}$. Their mathematical expressions are given in (20.1) and (21) respectively. In fact, it is obvious that

$$P^{tot,loss} = P^{sca,loss}+P^{inc,loss}+P^{coup,loss} \tag{80}$$

2) The reactive powers

(2.1) The reactively stored powers in fields include the $P^{sca,react,vac}$ in (13), the $P^{inc,react,vac}$, and the $P^{coup,react,vac}$ corresponding to the coupling between $\bar{F}^{sca}$ and $\bar{F}^{inc}$ in $\mathbb{R}^3$. The mathematical expressions for $P^{inc,react,vac}$ and $P^{coup,react,vac}$ are as follows

$$P^{inc,react,vac} = 2\omega\left[\frac{1}{4}\left\langle\bar{H}^{inc},\mu_0\bar{H}^{inc}\right\rangle_{\mathbb{R}^3}-\frac{1}{4}\left\langle\varepsilon_0\bar{E}^{inc},\bar{E}^{inc}\right\rangle_{\mathbb{R}^3}\right] \tag{81}$$

$$\begin{aligned}P^{coup,react,vac} = 2\omega\Big[&\frac{1}{4}\left\langle\bar{H}^{sca},\mu_0\bar{H}^{inc}\right\rangle_{\mathbb{R}^3}-\frac{1}{4}\left\langle\varepsilon_0\bar{E}^{sca},\bar{E}^{inc}\right\rangle_{\mathbb{R}^3}\\&+\frac{1}{4}\left\langle\bar{H}^{inc},\mu_0\bar{H}^{sca}\right\rangle_{\mathbb{R}^3}-\frac{1}{4}\left\langle\varepsilon_0\bar{E}^{inc},\bar{E}^{sca}\right\rangle_{\mathbb{R}^3}\Big]\end{aligned} \tag{82}$$

(2.2) The reactively stored powers in matter include the $P^{sca,react,mat}$, $P^{inc,react,mat}$, and $P^{coup,react,mat}$, and their mathematical expressions are given in (18) and (21) respectively. In fact, it is obvious that

$$P^{tot,react,mat} = P^{sca,react,mat}+P^{inc,react,mat}+P^{coup,react,mat} \tag{83}$$

Obviously, the $P^{sca,rad}$, $P^{sca,loss}$, $P^{inc,loss}$, $P^{coup,loss}$, $P^{sca,react,vac}$, $P^{sca,react,mat}$, $P^{inc,react,mat}$, and $P^{coup,react,mat}$ are intrinsically related to the material scatterer; however, the $P^{inc,rad}$, $P^{coup,rad}$, $P^{inc,react,vac}$, and $P^{coup,react,vac}$ are not intrinsically related to the scatterer. The reasons are listed as below:

(a) Only the $\bar{F}^{tot}$ and $\bar{F}^{inc}$ in V have the one-to-one correspondences with the scattering sources as illustrated in (7) and (86), whereas the $\bar{F}^{inc}$ in $\mathbb{R}^3\backslash V$ and the $\bar{F}^{inc}$ on S_∞ have not this kind of one-to-one correspondence, here the $\mathbb{R}^3\backslash V$ is the space exterior to V. Specifically, there exist some different $\bar{F}^{inc}$, such that they equal to each other in whole V, but don't equal to each other exterior to V and on S_∞.

(b) The $P^{sca,rad}$ and $P^{sca,react,vac}$ are generated by the scattering sources, and the $P^{tot,loss}=P^{sca,loss}+P^{inc,loss}+P^{coup,loss}$ and $P^{tot,react,mat}=P^{sca,react,mat}+P^{inc,react,mat}+P^{coup,react,mat}$ are inherently related to the material parameters.

In addition, based on the above discussions, the system *output power* P^{out} can be expressed as the following (84).

$$\begin{aligned}P^{out} &= P^{sca,rad}+P^{tot,loss}+j\left(P^{sca,react,vac}+P^{tot,react,mat}\right)\\&= (1/2)\left\langle\bar{J}^{vop},\bar{E}^{tot}\right\rangle_V+(1/2)\left\langle\bar{H}^{tot},\bar{M}^{vm}\right\rangle_V\\&\quad-(1/2)\left\langle\bar{J}^{vop},\bar{E}^{sca}\right\rangle_V-(1/2)\left\langle\bar{H}^{sca},\bar{M}^{vm}\right\rangle_V\end{aligned} \tag{84}$$

In fact, the physical essence of the second equality in (12) is the following conservation law of energy [10].

$$P^{out} = P^{inp} \tag{85}$$

B. The discussions for various CM sets.

Besides the objective powers discussed in Secs. III and IV, some other kinds of powers can also be selected as the objects to be optimized by Mat-EMP-CMT. The CM sets derived from different objective powers depict the inherent characteristics of electromagnetic system from different aspects, and various CM sets have their own merits. In this section, the features of three typical power-based CM sets are discussed.

1) The RaCM set constructed by orthogonalizing $\tilde{P}^{sca,rad}$

The elements in RaCM set are the necessary conditions for optimizing the system radiation as explained in [14], and then they are valuable in the antenna engineering community.

2) The InpCM set constructed by orthogonalizing P^{inp}

Among all powers discussed in Secs. III, IV, and VIII-A, the input power P^{inp} is the only one which includes all energies and powers intrinsically related to material body, and at the same time doesn't include any energy and power which is independent of material body, so the InpCM set is the most integrated description for the inherent characteristics of material body to utilize various electromagnetic energies.

3) The CM set constructed by orthogonalizing $P^{inp,\,part,\,rad}$

When the scatterer is lossy, the active power related to scatterer includes two parts, the radiated power and the lossy power. At this time, the InpCM set doesn't satisfy the radiated power orthogonality, but the CM set derived from orthogonalizing $P^{inp,\,part,\,rad}$ satisfies the radiated power orthogonality in (52.2).

In antenna engineering society, the radiation characteristic of antenna is more concerned than lossy power and total active power, so the CM set derived from $P^{inp,\,part,\,rad}$ is much more valuable than the InpCM set for analyzing and designing the material antennas with loss, but it comes at the cost of abandoning to descript the lossy characteristic of scatterer.

C. The discussions about the Poynting's theorem-based CMT.

The PEC-SEFIE-CMT [2] is rebuilt based on Poynting's theorem in [5], and a Poynting's theorem-based interpretation for the power characteristic of Mat-SIE-CMT [4] is given in [6]. In fact, to carefully analyze the power characteristics of various CM sets are indeed very important for both theoretical research and engineering application, so the power characteristics of various CMTs for material bodies are discussed in this subsection and the following Sec. VIII-D.

1) What is the reason why the symbol "$\triangleq$" in (17) and (19) doesn't appear in (18)?

It is well known that the Poynting's theorem is derived from Maxwell's equations. At the same time, it is obvious that the CMT for material bodies cannot be established based on the Poynting's theorems derived from the Maxwell's equations for incident fields $\{\bar{E}^{inc},\bar{H}^{inc}\}$ and total fields $\{\bar{E}^{tot},\bar{H}^{tot}\}$, because they will include some powers which are not inherently related to scatterer as discussed in Sec. VIII-A. However, the Maxwell's equations related to scattering fields $\{\bar{E}^{sca},\bar{H}^{sca}\}$ have only two different forms. The one has been illustrated in (1), and the other is as follows

$$\begin{aligned}\nabla\times\bar{H}^{sca} &= \bar{J}^{inc} + j\omega\varepsilon_c\bar{E}^{sca}\\ \nabla\times\bar{E}^{sca} &= -\bar{M}^{inc} - j\omega\mu\bar{H}^{sca}\end{aligned} \tag{86.1}$$

here

$$\begin{aligned}\bar{J}^{inc} &= j\omega\Delta\varepsilon_c\bar{E}^{inc}\\ \bar{M}^{inc} &= j\omega\Delta\mu\bar{M}^{inc}\end{aligned}\ ,\quad \left(\bar{r}\in V\right) \tag{86.2}$$

The Poynting's theorem derived from (1) has been illustrated in (13), and the Poynting's theorem derived from the above (86) is as follows

$$P^{sca} = P^{sca,\,rad} + P^{sca,\,loss} + j\,2\omega\left(W_m^{sca} - W_e^{sca}\right) \tag{87}$$

here the $P^{sca,\,rad}$ and $P^{sca,\,loss}$ are given in (14.2) and (20.1), and

$$P^{sca} = -(1/2)\left\langle \bar{J}^{inc},\bar{E}^{sca}\right\rangle_V - (1/2)\left\langle \bar{H}^{sca},\bar{M}^{inc}\right\rangle_V \tag{88.1}$$

$$W_m^{sca} = W_m^{sca,\,vac} + W_m^{sca,\,mat} \tag{88.2}$$

$$W_e^{sca} = W_e^{sca,\,vac} + W_e^{sca,\,mat} \tag{88.3}$$

in which the $W_m^{sca,\,vac}$ and $W_e^{sca,\,vac}$ are given in (14.3) and (14.4), and the $W_m^{sca,\,mat}$ and $W_e^{sca,\,mat}$ are given in (20.2) and (20.3).

The reason why the symbol "$\triangleq$" is not used in (18) is that the power P^{sca} is directly derived from Maxwell's equations as illustrated above, instead of artificially defined as (17) and (19) for practical destinations.

2) In addition, what is the reason why the $(1/2)\left\langle \bar{H}^{inc},\bar{M}^{vm}\right\rangle_V$ is used in (12) instead of the $(1/2)\left\langle \bar{M}^{vm},\bar{H}^{inc}\right\rangle_V$?

It is obvious to find out that the (12) is consistent with the Poynting's theorem, which is derived from Maxwell's equations. When the $(1/2)\left\langle \bar{H}^{inc},\bar{M}^{vm}\right\rangle_V$ is replaced by $(1/2)\left\langle \bar{M}^{vm},\bar{H}^{inc}\right\rangle_V$, the consistence will disappear. The author of this paper thinks that the mathematical expression which is consistent with the fundamental physical law is more credible.

D. The discussions for Mat-VIE-CMT [3].

In this subsection, the core principle of Mat-VIE-CMT [3] is summarized at first, and then some necessary discussions for Mat-VIE-CMT are provided.

The core principle of Mat-VIE-CMT

1) The main destination of Mat-VIE-CMT is to construct a transformation from any mathematically complete basis function set $\{\bar{b}_\xi\}_{\xi=1}^{\Xi}$ to the CM set $\{\bar{a}_\xi\}_{\xi=1}^{\Xi}$ which satisfies the power orthogonality given by the formulations (14) and (15) in paper [3].

2) The Mat-VIE-CMT achieves above destination by decomposing the impedance matrix $\bar{\bar{Z}}$ in terms of its real part $\bar{\bar{R}}$ and imaginary part $\bar{\bar{X}}$, and then solving the generalized characteristic equation $\bar{\bar{X}}\cdot\bar{a} = \lambda\bar{\bar{R}}\cdot\bar{a}$.

3) In fact, realizing the destination in 1) by using the method in 2) relies on the following necessary conditions.

(3.1) $P^{inp}(\bar{a}) = \bar{a}^H\cdot\bar{\bar{Z}}\cdot\bar{a}$;

(3.2) The $\bar{\bar{R}}$ and $\bar{\bar{X}}$ must be symmetric matrices to guarantee that $\mathrm{Re}\{P^{inp}(\bar{a})\} = \bar{a}^H\cdot\bar{\bar{R}}\cdot\bar{a}$ and $\mathrm{Im}\{P^{inp}(\bar{a})\} = \bar{a}^H\cdot\bar{\bar{X}}\cdot\bar{a}$;

(3.3) The $\bar{\bar{R}}$ must be positive definite [16].

4) To guarantee the conditions listed in 3), the Mat-VIE-CMT is realized as follows.

(4.1) To guarantee the condition (3.1), the $\bar{\bar{Z}}$ must be constructed by using inner product, and the basis function set and the testing function set must be the same, and a coefficient "$1/2$" should be contained;

(4.2) To guarantee the condition (3.2), the $\bar{\bar{Z}}$ must be constructed by using symmetric product, and the basis function set and the testing function set must be the same;

(4.3) To guarantee the condition (3.3), the material scatterer is required to radiate some electromagnetic energy.

To simultaneously achieve the above (4.1) and (4.2), it is necessary for the Mat-VIE-CMT to expand currents in terms of the real basis function set, and then the Mat-VIE-CMT can only construct the real characteristic currents, because in the vector space $\mathbb{R}^{\Xi}$ only the real characteristic vectors can be derived from the equation $\bar{\bar{X}} \cdot \bar{a} = \lambda \bar{\bar{R}} \cdot \bar{a}$.

The discussions for Mat-VIE-CMT and Mat-EMP-CMT

1) The Mat-VIE-CMT has not ability to provide the complex characteristic currents, but it is more suitable for some electrically large structures, such as travelling wave material antennas, to depict their inherent characteristics by using the complex characteristic currents [7].

In Mat-EMP-CMT, the matrix $\bar{\bar{P}}^{inp}$ is decomposed in terms of two Hermitian matrices, the $\bar{\bar{P}}^{inp,act}$ and the $\bar{\bar{P}}^{inp,react}$, as illustrated in (34.1), and they respectively correspond to the active power and reactive power as illustrated in (31.1) and (31.2). The theoretical foundation of decomposition (34.1) is the decomposition (28), and the (28) can always be realized, and doesn't need to restrict the basis function set $\{\bar{b}_\xi\}_{\xi=1}^{\Xi}$ to be real. Based on this, the characteristic currents are not restricted to be real under the Mat-EMP-CMT framework, and then this paper provides a possible way for constructing the complex characteristic currents.

2) The Mat-VIE-CMT doesn't provide any efficient method to research the non-radiative CMs, but it is more suitable for some material components, such as the material body filters, to depict their inherent characteristics by using the non-radiative CMs.

In Mat-EMP-CMT, for lossless material bodies, when the $\bar{\bar{P}}^{inp,act}$ is positive definite, the CM set is obtained by solving the generalized characteristic equation $\bar{\bar{P}}^{inp,react} \cdot \bar{a} = \lambda\, \bar{\bar{P}}^{inp,act} \cdot \bar{a}$ in (41), and this equation is the necessary condition to orthogonalize the modal powers as (43). When the $\bar{\bar{P}}^{inp,act}$ is positive semi-definite at frequency f_0 , the frequency f_0 is efficiently recognized by employing the method given in [14], and then the CM set at f_0 is obtained by using a "limiting method" given in (42).

3) When the sub-domain basis functions, such as the SWG [22], are employed, the symmetry of $\bar{\bar{Z}}$ given in Mat-VIE-CMT cannot be guaranteed, because of the existence of the matrix elements which correspond to the couplings between the interior basis functions and the boundary basis functions (which include the surface charges on the boundary of scatterer).

In fact, the theoretical foundation to prove the symmetry of $\bar{\bar{Z}}$ is so-called reciprocity theorem [23], but the theorem requires that the source distribution has enough continuity. However, the scattering currents are not continuous on the boundary of material scatterers [11]-[13].

In Mat-EMP-CMT, it is obvious that the decomposition (28) and then the decomposition (34.1) is valid for any kind of basis function set.

4) The integral equation used in Mat-VIE-CMT contains two parts, a volume EFIE and a volume MFIE. The physical nature of the MFIE part is as follows [3]

$$(1/2)\left\langle \bar{M}^{vm}, \bar{H}^{inc} \right\rangle_V = (1/2)\left\langle \bar{M}^{vm}, \bar{H}^{tot} \right\rangle_V - (1/2)\left\langle \bar{M}^{vm}, \bar{H}^{sca} \right\rangle_V \quad (89.1)$$

instead of the following

$$(1/2)\left\langle \bar{H}^{inc}, \bar{M}^{vm} \right\rangle_V = (1/2)\left\langle \bar{H}^{tot}, \bar{M}^{vm} \right\rangle_V - (1/2)\left\langle \bar{H}^{sca}, \bar{M}^{vm} \right\rangle_V \quad (89.2)$$

However, only the $(1/2)\left\langle \bar{H}^{inc}, \bar{M}^{vm} \right\rangle_V$ is the power done by $\bar{H}^{inc}$ on $\bar{M}^{vm}$ as explained in Sec. VIII-C, and the $(1/2)\left\langle \bar{M}^{vm}, \bar{H}^{inc} \right\rangle_V$ is the complex conjugate of this power, and then

(4.1) when the scatterer is only magnetic, the reactive power given in Mat-VIE-CMT is the opposite of the correct one;

(4.2) when the scatterer is both dielectric and magnetic, the reactive power given in Mat-VIE-CMT is not the correct one.

In fact, it was clearly claimed in [3] that: 'the imaginary part of $\left\langle f^*, Tf \right\rangle$ is not simply related to reactive power.' Here, $\left\langle f^*, Tf \right\rangle$ is the inner product defined in [3].

5) For the lossy cases, the complex matrix $\bar{\bar{X}}_V - j\bar{\bar{Z}}_M$ in [3] is not Hermitian, so the CMs derived from the characteristic equation $\left(\bar{\bar{X}}_V - j\bar{\bar{Z}}_M\right) \cdot \bar{a} = \lambda \bar{\bar{R}}_V \cdot \bar{a}$ in [3] cannot guarantee the power orthogonality (60) given in [3].

In the Mat-EMP-CMT, the CM set which has ability to orthogonalize the radiation patterns of lossy material bodies is constructed by introducing the power $P^{inp,part,rad}$ in (19).

6) When at least two of polarization, magnetization, and conduction phenomena exist, the different scattering currents should be expanded in terms of related basis functions. For example, when the $\bar{J}^{vp}$ is expanded as

$$\bar{J}^{vp}(\bar{r}) = \sum\nolimits_{\xi=1}^{\Xi} a_\xi^{J^{vp}} \bar{b}_\xi^{J^{vp}}(\bar{r}) \quad , \quad (\bar{r} \in V) \quad (90.1)$$

this implies that the other kinds of scattering currents must have the following expansion formulations.

$$\bar{J}^{vo}(\bar{r}) = \sum\nolimits_{\xi=1}^{\Xi} a_\xi^{J^{vp}} \bar{b}_\xi^{J^{vo}}(\bar{r}) \quad , \quad (\bar{r} \in V) \quad (90.2)$$

$$\bar{M}^{vm}(\bar{r}) = \sum\nolimits_{\xi=1}^{\Xi} a_\xi^{J^{vp}} \bar{b}_\xi^{M^{vm}}(\bar{r}) \quad , \quad (\bar{r} \in V) \quad (90.3)$$

here

$$\bar{b}_\xi^{J^{vo}}(\bar{r}) = \bar{J}^{vo}\left[\bar{J}_{vp}^{-1}\left(\bar{b}_\xi^{J^{vp}}\right); \bar{r}\right] \quad , \quad (\bar{r} \in V) \quad (91.1)$$

$$\bar{b}_\xi^{M^{vm}}(\bar{r}) = \bar{M}^{vm}\left[\bar{J}_{vp}^{-1}\left(\bar{b}_\xi^{J^{vp}}\right); \bar{r}\right] \quad , \quad (\bar{r} \in V) \quad (91.2)$$

In (91), the $\bar{J}_{vp}^{-1}$ is the inverse of the operator $\bar{J}^{vp}\left(\bar{F}^{tot}; \bar{r}\right)$ in (7). In fact, doing like this is very boring. This is the one of causes why this paper expands the basic variable $\bar{F}^{tot}$ instead of any kind of scattering currents.

In addition, if different scattering currents are expanded in terms of some unrelated basis function sets like Mat-VIE-CMT did for the bodies both dielectric and magnetic [3], neither positive definite $\mathrm{Re}\{\bar{\bar{Z}}\}$ nor positive semi-definite $\mathrm{Re}\{\bar{\bar{Z}}\}$ can be guaranteed, because the unrelated expansion formulations for different scattering currents will lead to non-physical fields.

E. The discussions for modal quantities and normalizations.

Obviously, the various modal quantities discussed in the Sec. VII and the modal component $\left|\tilde{c}_\xi^{inp}\right|^2 \tilde{P}_\xi^{inp}$ in the expansion formulation (70) satisfy the relation (92).

In fact, the $\text{MAOP}_{\xi}^{mod,act}$ in (76.1) is equivalent to the traditional MS as illustrated in the following (93).

$$\begin{aligned} \text{Traditional MS}_{\xi} &\triangleq \frac{1}{\left|1+j\,\lambda_{\xi}\right|} \\ &= \frac{1}{\left|1+j\left(\text{Im}\left\{\tilde{P}_{\xi}^{inp}\right\}/\text{Re}\left\{\tilde{P}_{\xi}^{inp}\right\}\right)\right|} \\ &= \frac{\text{Re}\left\{\tilde{P}_{\xi}^{inp}\right\}}{\left|\text{Re}\left\{\tilde{P}_{\xi}^{inp}\right\}+j\,\text{Im}\left\{\tilde{P}_{\xi}^{inp}\right\}\right|} \\ &= \frac{\text{Re}\left\{\tilde{P}_{\xi}^{inp}\right\}}{\left|\tilde{P}_{\xi}^{inp}\right|} \\ &= \text{MAOP}_{\xi}^{mod,act} \end{aligned} \tag{93}$$

In (93), the second equality is due to the relation (49), and the third equality originates from that $\text{Re}\left\{P_{\xi}^{inp}\right\}>0$.

It is easily found out from the (92) and (93) that the physical essence of the traditional MS is to quantitively depict the modal ability to allocate the total modal output power to its active part, instead of a quantitive depiction for the modal weight in whole modal expansion formulation.

The essential reason leading to the above problem of the traditional MS is carefully analyzed as below.

When the CM M is normalized by using the normalization way given in [2], i.e., the modal active power is normalized to be unit, the normalized CM is denoted as the symbol $\breve{M}$ to be distinguished from the normalized version $\tilde{M}$ used in this paper. The $\breve{M}$ version for (70) is the following (70').

$$\begin{aligned} P^{inp} &= \sum\nolimits_{\xi=1}^{\Xi}\left|\breve{c}_{\xi}^{inp}\right|^{2}\breve{P}_{\xi}^{inp} \\ &= \sum\nolimits_{\xi=1}^{\Xi}\frac{1}{\left|\breve{P}_{\xi}^{inp}\right|}\cdot\left|\frac{1}{2}\left\langle\breve{\bar{J}}_{\xi}^{vop},\bar{E}^{inc}\right\rangle_{V}\right|^{2}\cdot\frac{\breve{P}_{\xi}^{inp}}{\left|\breve{P}_{\xi}^{inp}\right|} \end{aligned} \tag{70'}$$

here [2]

$$\text{Re}\left\{\breve{P}_{\xi}^{inp}\right\} = 1 \tag{94.1}$$

$$\text{Im}\left\{\breve{P}_{\xi}^{inp}\right\} = \lambda_{\xi} \tag{94.2}$$

and

$$\text{Traditional MS}_{\xi} = \frac{1}{\left|\breve{P}_{\xi}^{inp}\right|} \tag{95}$$

Obviously, the magnitude of the term $\breve{P}_{\xi}^{inp}/\left|\breve{P}_{\xi}^{inp}\right|$ in (70') is unit just like the term $\tilde{P}_{\xi}^{inp}/\left|\tilde{P}_{\xi}^{inp}\right|$ in (70).

However the $\breve{\bar{J}}_{\xi}^{vop}$ in term $\left|(1/2)\left\langle\breve{\bar{J}}_{\xi}^{vop},\bar{E}^{inc}\right\rangle_{V}\right|^{2}$ is not well normalized. For example, the following case may be existed. $(1/2)\left\langle\breve{\bar{J}}_{\xi}^{vop},\breve{\bar{J}}_{\xi}^{vop}\right\rangle_{V}\gg(1/2)\left\langle\breve{\bar{J}}_{\zeta}^{vop},\breve{\bar{J}}_{\zeta}^{vop}\right\rangle_{V}$, though $\text{Re}\left\{\breve{P}_{\xi}^{inp}\right\}=1=\text{Re}\left\{\breve{P}_{\xi}^{inp}\right\}$. In fact, this problem will not exist for the normalization way used in this paper, because the modal total field $\bar{F}_{\xi}^{tot}$ is normalized as (66), and the modal current $\bar{J}_{\xi}^{vop}$ is linearly related to the $\bar{F}_{\xi}^{tot}$ as follows

$$\bar{J}_{\xi}^{vop} = \begin{cases} j\omega\Delta\varepsilon_{c}\bar{E}_{\xi}^{tot} &, \quad \left(\text{if basic variable is } \bar{F}^{tot}=\bar{E}^{tot}\right) \\ \left(\Delta\varepsilon_{c}/\varepsilon_{c}\right)\nabla\times\bar{H}_{\xi}^{tot} &, \quad \left(\text{if basic variable is } \bar{F}^{tot}=\bar{H}^{tot}\right) \end{cases} \tag{96}$$

In particular, when the material scatterer is homogeneous and the basic variable $\bar{F}^{tot}$ is selected as the $\bar{E}^{tot}$, it is obvious that

$$\begin{aligned} (1/2)\left\langle\tilde{\bar{J}}_{\xi}^{vop},\tilde{\bar{J}}_{\xi}^{vop}\right\rangle_{V} &= \frac{(1/2)\left\langle\bar{J}_{\xi}^{vop},\bar{J}_{\xi}^{vop}\right\rangle_{V}}{(1/2)\left\langle\bar{E}_{\xi}^{tot},\bar{E}_{\xi}^{tot}\right\rangle_{V}} \\ &= \left|j\omega\Delta\varepsilon_{c}\right|^{2} \\ &= \frac{(1/2)\left\langle\bar{J}_{\zeta}^{vop},\bar{J}_{\zeta}^{vop}\right\rangle_{V}}{(1/2)\left\langle\bar{E}_{\zeta}^{tot},\bar{E}_{\zeta}^{tot}\right\rangle_{V}} = (1/2)\left\langle\tilde{\bar{J}}_{\zeta}^{vop},\tilde{\bar{J}}_{\zeta}^{vop}\right\rangle_{V} \end{aligned} \tag{97}$$

for any $\xi,\zeta=1,2,\cdots,\Xi$.

In fact, the most essential reason to lead to the above problem in the traditional MS is that the normalization way used in [2] only focuses on the modal active power, but ignores the modal currents and the modal fields. However, the normalization way used in (66) focuses on the modal basic variable $\bar{F}_{\xi}^{tot}$, and it is obvious that

$$(1/2)\left\langle\tilde{\bar{F}}_{\xi}^{tot},\tilde{\bar{F}}_{\xi}^{tot}\right\rangle_{V} = \frac{(1/2)\left\langle\bar{F}_{\xi}^{tot},\bar{F}_{\xi}^{tot}\right\rangle_{V}}{(1/2)\left\langle\bar{F}_{\xi}^{tot},\bar{F}_{\xi}^{tot}\right\rangle_{V}} = 1 \tag{98}$$

for any $\xi=1,2,\cdots,\Xi$. The (98) implies that the basic variable $\bar{F}_{\xi}^{tot}$ is well normalized, so the modal currents, the modal fields, and the modal powers are automatically well normalized as (67)-(68).

IX. Conclusions

An electromagnetic-power-based CMT for material bodies, Mat-EMP-CMT, is established in this paper, and then some different kinds of power-based CM sets are constructed. Various CM sets have their own merits to reveal material bodies' inherent power characteristics from different aspects.

Among various CM sets, the InpCM set has the same physical essence as the CM set derived from Mat-VIE-CMT, but the former is more advantageous than the latter in some aspects, for example, the former has a more physically reasonable power characteristic than the latter. The CM set constructed by orthogonalizing the power $P^{inp,part,rad}$ satisfies

$$\begin{aligned} \left|\tilde{c}_{\xi}^{inp}\right|^{2}\tilde{P}_{\xi}^{inp} &= \text{SMS}_{\xi}^{sys,act}+j\,\text{SMS}_{\xi}^{sys,react} = \overbrace{\text{MACE}_{\xi}^{mod}\cdot\text{GMS}_{\xi}^{sys,tot}}^{\text{SMS}_{\xi}^{sys,tot}}\cdot\text{MAOP}_{\xi}^{mod,act}+j\,\overbrace{\text{MACE}_{\xi}^{mod}\cdot\text{GMS}_{\xi}^{sys,tot}}^{\text{SMS}_{\xi}^{sys,tot}}\cdot\text{MAOP}_{\xi}^{mod,react} \\ &= \text{MACE}_{\xi}^{mod}\cdot\underbrace{\text{GMS}_{\xi}^{sys,tot}\cdot\text{MAOP}_{\xi}^{mod,act}}_{\text{GMS}_{\xi}^{sys,act}}+j\,\text{MACE}_{\xi}^{mod}\cdot\underbrace{\text{GMS}_{\xi}^{sys,tot}\cdot\text{MAOP}_{\xi}^{mod,react}}_{\text{GMS}_{\xi}^{sys,react}} \end{aligned} \tag{92}$$

the radiation pattern orthogonality for both lossless and lossy material scatterers, and then it is valuable for designing the material antennas with loss.

Under the Mat-EMP-CMT framework, the complex characteristic currents and the non-radiative CMs can be constructed, and they are valuable for engineering applications.

Based on the new normalization way introduced in this paper, the traditional characteristic quantity, MS, is generalized, and some new modal quantities are introduced to depict the modal characteristics from different aspects. In addition, a variational formulation for the scattering problem of material scatterer is established based on the conservation law of energy.

ACKNOWLEDGEMENT

This work is dedicated to Renzun Lian's mother, Ms. Hongxia Zou. This work cannot be finished without her constant understanding, support, and encouragement.

In addition, R. Z. Lian would like to thank J. Pan, Z. P. Nie, X. Huang, and S. D. Huang.

Pan taught Lian some valuable theories on the electromagnetics and antenna. Nie taught Lian some valuable theories and methods for solving the electromagnetic problem related to material bodies.

In the early stage of this work, there were some valuable discussions between X. Huang and Lian, and X. Huang wrote some computer codes to verify the ideas of Lian.

In the middle and late stages of this work, S. D. Huang wrote some computer codes to verify Lian's ideas.

REFERENCES

[1] R. J. Garbacz, "Modal expansions for resonance scattering phenomena," *Proc. IEEE*, vol. 53, no. 8, pp. 856–864, August 1965.

[2] R. F. Harrington and J. R. Mautz, "Theory of characteristic modes for conducting bodies," *IEEE Trans. Antennas Propag.*, vol. AP-19, no. 5, pp. 622-628, May 1971.

[3] R. F. Harrington, J. R. Mautz, and Y. Chang, "Characteristic modes for dielectric and magnetic bodies," *IEEE Trans. Antennas Propag.*, vol. AP-20, no. 2, pp. 194-198, March 1972.

[4] Y. Chang and R. Harrington, "A surface formulation for characteristic modes of material bodies," *IEEE Trans. Antennas Propag.*, vol. 25, no. 6, pp. 789–795, November 1977.

[5] Y. K. Chen and C.-F. Wang, *Characteristic Modes: Theory and Applications in Antenna Engineering*. Hoboken: Wiley, 2015.

[6] Y. K. Chen, "Alternative surface integral equation-based characteristic mode analysis of dielectric resonator antennas," *IET Microw. Antennas Propag.*, vol. 10, no. 2, pp. 193-201, January 2016.

[7] C. A. Balanis, *Antenna Theory: Analysis and Design, 3rd*. Hoboken: Wiley, 2005.

[8] K.-M. Luk and K.-W. Leung, *Dielectric Resonator Antennas*. Hertfordshire U.K.: Research Studies Press, 2003.

[9] D. Kajfez and P. Guillon, *Dielectric Resonators, 2nd*. Noble Publishing Corporation, 1998.

[10] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics*. Addison-Wesley Publishing Company, 1964.

[11] A. F. Peterson, S. L. Ray, and R. Mittra, *Computational Methods for Electromagnetics*. New York: IEEE Press, 1998.

[12] J. V. Volakis and K. Sertel, *Integral Equation Methods for Electromagnetics*. SciTech, 2012.

[13] M. I. Sancer, K. Sertel, J. L. Volakis, and P. V. Alstine, "On volume integral equations," *IEEE Trans. Antennas Propag.*, vol. 54, no. 5, pp. 1488-1495, May 2006.

[14] R. Z. Lian, "Electromagnetic-power-based characteristic mode theory for perfect electric conductors," arXiv: 1610.05099.

[15] G. A. E. Vandenbosch, "Reactive energies, impedance, Q factor of radiating structures," *IEEE Trans. Antennas Propag.*, vol. 58, no. 4, pp. 1112–1127, April 2010.

[16] R. A. Horn and C. R. Johnson, *Matrix Analysis, 2nd*. New York: Cambridge University Press, 2013.

[17] R. E. Collin, *Field Theory of Guided Waves, 2nd*. New York: IEEE Press, 1991.

[18] C. E. Baum, "On the singularity expansion method for the solution of electromagnetic interaction problems," *AFWL Interaction Notes 88*, December 1971.

[19] J. D. Jackson, *Classical Electrodynamics, 3rd*. New York: Wiley, 1999.

[20] J. M. Jin, *The Finite Element Method in Electromagnetics, 2nd*. New York: Wiley, 2002.

[21] G. Y. Wen, "Stored energies and radiation Q," *IEEE Trans. Antennas Propag.*, vol. 63, no. 2, pp. 636-645, February 2015.

[22] D. H. Schaubert, D. R. Wilton, and A. W. Glisson, "A tetrahedral modeling method for electromagnetic scattering by arbitrarily shaped inhomogeneous dielectric bodies," *IEEE Trans. Antennas Propag.*, vol. 32, no. 1, pp. 77-85, January 1984.

[23] R. F. Harrington, *Time-Harmonic Electromagnetic Fields*. New York: IEEE Press, 2001.