On the Eötvös effect

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The aim of this paper is to propose a new theory about the Eötvös effect. We develop a mathematical model which aloud us a better understanding of this effect. From the equation of motion the Eötvös term could arise naturally without supplementary assumptions. The Eötvös force and the Coriolis force are the vertical and horizontal projections of a force generated by the circular motion. Under these circumstances we can conceive the Eötvös effect like a vertical Coriolis effect. In addition we have deduced the Eötvös term from centrifugal force, classic hypothesis. The cosine function appears only due to spherical coordinates and express the variation of centrifugal force with altitude.

Introduction

During the early 1900s a scientific team from Posdam Institute of Geodesy performed gravity measurements on moving ships in the Pacific, Indian and Atlantic Oceans. Their results were astonishing. The measured values of the internal weigh of the gravimeter were lower when the boat moved eastward and higher when it moved westward. These results were then explained by the Hungarian physicist Lorand Eötvös after another set of measurements carried out in the Black Sea on two ships, in 1908. The results were in very good agreement with those calculated with formula:

$$a_r = 2\Omega u \cos\Phi + \frac{u^2 + v^2}{R} \tag{1}$$

Here a_r is the relative acceleration, Ω is the angular velocity of the Earth, u is the velocity (relative to the Earth) in latitudinal direction (west-east), v is the velocity in longitudinal direction (north-south), Φ is the latitude where the measurements are taken and R is the radius of the Earth.

The first right term in the formula (1) corresponds to the Eötvös effect. The second term represents the required centrifugal acceleration for the ship to follow the curvature of the Earth. It is independent of both the Earth's rotation and the direction of motion. Under normal conditions this term is negligible.

The formula (1) is the result of a simple reasoning. The attraction force of the Earth is the resultant of two forces: the gravitational force, according to Newton's law and the centrifugal force caused by the Earth rotation. Since the masses on the Earth's surface are uniform distributed and the angular velocity at which the Earth rotates are constant, the weight of the objects at rest on the Earth's surface is a constant.

In the case of moving objects the situation is different. Since the Earth's rotation is from west to east, the centrifugal force acting on a moving object is greater if its motion on Earth's surface is towards the east than towards the west. Therefore the specific weight of a moving eastwards object is decreasing, while the specific weight of a moving westwards object is increasing, [1].

However, this intuitive and simple theory is not the only one to explain the Eötvös effect. In the following section such a theory can be given on the basis of non-inertial motion.

The motion in non-inertial reference systems

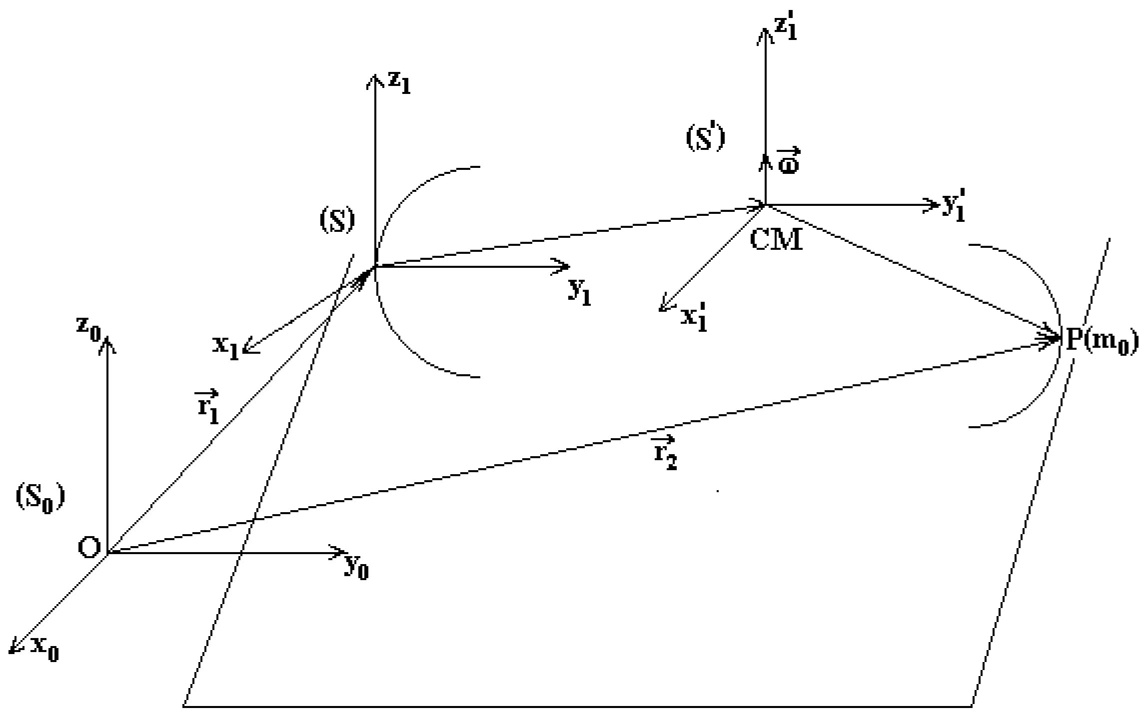

Figure 1

Consider the reference systems (S_0), (S) and (S') which are in the same plane (xy). The system (S') is the center of mass reference system of two co-moving masses (the Earth and a smaller body placed in P). The system (S) (Earth system) is not inertial because it is rotating with respect to the fixed system (S_0). Assume also that the system (S') it is rotating with respect to (S) and this rotation is described by the equation:

$$\boldsymbol{\omega}(t) = \omega(t)\mathbf{z}_1$$

In this case on point mass P (with mass m_0) are acting two forces:

$$m_0\ddot{\mathbf{r}} = \mathbf{F}_{IS} + \mathbf{F}_{rot}$$

The first right term is an attraction force due to the inertial motion:

$$\mathbf{F}_{IS} = -m_0 \frac{\mu}{r^3}\mathbf{u}$$

where $\mu = G(m_0 + m_1)$, $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ and $\mathbf{u} = \frac{\mathbf{r}}{r}$.

The second right term is due to rotation of (S') with respect to (S_0):

$$\mathbf{F}_{rot} = -m_0\left[\dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \dot{\mathbf{r}}\right]$$

Consequently the motion of point mass P is described by the equation:

$$m_0\ddot{\mathbf{r}} = -m_0\frac{\mu}{r^3}\mathbf{u} - m_0\left[\dot{\boldsymbol{\omega}}\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + 2\boldsymbol{\omega}\times\dot{\mathbf{r}}\right] \tag{2}$$

In order to be consequent with the problem we should now write equation (2) in more detail by using spherical coordinates. The motion we study is on spherical surface of the Earth, so it is normal to describe this motion in spherical coordinates. Nevertheless such a description goes nowhere. The fact we must write the pseudo vector ω in spherical coordinates eliminates it completely from intermediate calculi and from final results. And this is something we don't want to occur. It is essential for our study to express our results as a function of ω. The mathematical description of a non-inertial motion would be inconceivable without it. At ω=0 we have a inertial motion and specific equation for it. This is the reason why to simplify the problem we must evaluate the above equation in cylindrical coordinates, [2].

Accordingly we have:

$$\mathbf{r} = r\boldsymbol{\rho}$$
$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{r}}\boldsymbol{\rho} + r\dot{\theta}\boldsymbol{\eta}$$
$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\boldsymbol{\rho} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\boldsymbol{\eta}$$
$$\boldsymbol{\omega} = \omega\cdot\mathbf{z}_1$$

and:

$$\dot{\boldsymbol{\omega}}\times\mathbf{r} = \begin{vmatrix} \boldsymbol{\rho} & \boldsymbol{\eta} & \mathbf{z}_1 \\ 0 & 0 & \dot{\omega} \\ r & 0 & 0 \end{vmatrix} = r\dot{\omega}\boldsymbol{\eta}$$

$$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) = \omega\mathbf{z}_1\times(r\omega\boldsymbol{\eta}) = -r\omega^2\boldsymbol{\rho}$$

$$2\boldsymbol{\omega}\times\dot{\mathbf{r}} = 2\omega\mathbf{z}_1\times\left(\dot{r}\boldsymbol{\rho} + r\dot{\theta}\boldsymbol{\eta}\right) = +2\dot{r}\omega\boldsymbol{\eta} - 2r\omega\dot{\theta}\boldsymbol{\rho}$$

By replacing these partial results into equation (2) and making the projections onto polar axes we obtain:

$$a_\rho = -\frac{\mu}{r^2} - \ddot{r} + r\omega^2 + 2r\omega\dot{\theta} + r\dot{\theta}^2 \tag{3}$$

$$a_\eta = -r\ddot{\theta} - 2\dot{r}\dot{\theta} - r\dot{\omega} - 2\dot{r}\omega \tag{4}$$

The motion in spherical coordinates

First of all notice that if ω=0 equations (3) and (4) depict an inertial motion. The projection onto angular direction, equation (4), comprise the term corresponding to Coriolis force. The projection onto radial direction, equation (3), contains the terms corresponding to centrifugal forces. It is noticeable that the last right term of equation (3) resembles to Eötvös term (see formula (1)). If we denote ω=Ω (the angular velocity of the Earth) and (dθ/dt) r=u (the velocity relative to the Earth, in west-east direction) and neglect the contributions of centrifugal forces, then we have the Eötvös term. The only problem is the fact that in reality the Eötvös effect emerge in the motion on a sphere and our results are into cylindrical coordinates. Equations (3) and (4) are correct only if the Earth has a cylindrical form. In this case, no doubt, the rotation of it and of a body, in circles, upon its surface, generates both Eötvös and Coriolis effects. The z component of the motion does not count, so these effects are the same, under condition that vertical motion to be null.

In the case of Earth's motion around the Sun, in the context of galaxy rotation, this description is true. Perhaps the galaxy cross section near a cylinder is due to the Eötvös effect. But in the case of a motion upon a spherical surface, equations (3) and (4) do not hold. At this stage of our study is worthless to convert these two equations into spherical coordinates. We don't obtain the same results as we initially wrote these equations in spherical coordinates. And this is conceptually wrong.

Let now look for another way to infer the equation (3) and (4). These equations are describing a circular motion. It is simpler then to write acceleration in polar coordinates and follow the logical steps in order to find the equations of the same circular motion. It follows:

$$a_\rho = \ddot{r} - r(\dot{\theta} + \omega)^2 \tag{5}$$

and:

$$a_\eta = 2\dot{r}(\dot{\theta} + \omega) + r(\ddot{\theta} + \dot{\omega}) \tag{6}$$

The projections of forces acting on point mass P will look, after simplification and simple calculi:

$$a_\rho = -\frac{\mu}{r^2} - \ddot{r} + r\omega^2 + 2r\omega\dot{\theta} + r\dot{\theta}^2$$

$$a_\eta = -r\ddot{\theta} - 2\dot{r}\dot{\theta} - r\dot{\omega} - 2\dot{r}\omega$$

which are, without question, the same equations (3) and (4).

This observation is very important because it helps us to imagine a way to find out the correct form of the equations of motion in spherical coordinates in a simpler manner. The existence of previous section is justified not only by the deduction from other hypothesis than centrifugal force variation of Eötvös effect but to prove that equations (5) and (6) are correct. The manner in which ω was included in these equations can be extrapolated for the correspondent equations in spherical coordinates. Consequently we have for the radial component of acceleration:

$$a_r = \ddot{r} - r\dot{\theta}^2 - r(\dot{\phi} + \omega)^2 \sin^2\theta$$

The projected correspondent force acting on point mass P, after elementary calculi will be:

$$a_r = -\frac{\mu}{r^2} - \ddot{r} + r\dot{\theta}^2 + r\omega^2 \sin^2\theta + r\dot{\phi}^2 \sin^2\theta + 2r\omega\dot{\phi}\sin^2\theta \tag{7}$$

Assume that radial velocity of the motion is constant, therefore its derivative will be null. If ω is the angular velocity of the Earth then we can neglect also the correspondent centrifugal force. It remain the non-null terms due to non-inertial motion and the gravitational acceleration If we keep only the terms of interest then we can write:

$$a_E = r\dot{\theta}^2 + r\dot{\phi}^2 \sin^2\theta + 2r\omega\dot{\phi}\sin^2\theta$$

If we are taking into account that, for horizontal velocity of the body on the Earth's surface, its vertical velocity and the relation between altitude and elevation angle, the expressions are:

$$u = r\dot{\phi}\sin\theta$$

$$v = r\dot{\theta}$$

$$\Phi = 90 - \theta$$

then we find an expression similar to (1), in which the symbols have the same significations:

$$a_E = 2\omega u \cos\Phi + \frac{u^2 + v^2}{r} \tag{8}$$

The centrifugal effects are most of the time negligible, so this could not cause serious problems to the final evaluation. According to (7) and (8) the resulting force acting on a moving object to the Earth's surface it is smaller if its motion is eastwards than westwards. Therefore the specific weight of a moving eastwards object is decreasing, while the specific weight of a moving westwards object is increasing, with the same values as those calculated with (1).

Conclusions

We obtain the Eötvös term by making different assumptions. The study of non-inertial motion in cylindrical coordinates give rise to this possibility. So-called Eötvös force and Coriolis force are the radial and angular projections of a force generated by the circular motion. Thus the Eötvös effect it is no more the result of the centrifugal force variation. This model not fits to the motion on spherical surfaces, it is more appropriate to describe the revolution of cosmic bodies.

We have shown then that the similar results can be obtained if the centrifugal force variation hypothesis has been considered. The model is much simpler and drive to conclusion that the centrifugal force variation hypothesis fits the study of non-inertial motion in spherical coordinates too. The results are very conclusive in this matter, I think. The cosine function seems to be the small difference between the two models' results. It appears naturally due to variation of centrifugal force with altitude.

References

[1] R. Eötvös, Ann. d. Phys. **59** (1919), 743-752;
[2] C. I. Borş, Tensor **53** (1993), 271.