One-Dimensional Navier-Stokes Finite Element Flow Model

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Technical Report

April 9, 2013

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Abstract

This technical report documents the theoretical, computational, and practical aspects of the one-dimensional Navier-Stokes finite element flow model. The document is particularly useful to those who are interested in implementing, validating and utilizing this relatively-simple and widely-used model.

Keywords: one-dimensional flow; Navier-Stokes; Newtonian fluid; finite element; elastic vessel; interconnected network; blood flow in large vessels; branching flow; time-independent flow; time-dependent flow.

1 Introduction

The one-dimensional (1D) Navier-Stokes flow model in its analytic formulation and numeric implementation is widely used for calculating and simulating the flow of Newtonian fluids in large vessels and in interconnected networks of such vessels [1–5]. In particular, the model is commonly used by bioengineers to analyze blood flow in the arteries and veins. This model can be easily implemented using a numeric meshing technique, such as finite element method, to provide a computational framework for flow simulation in large tubes. The model can also be coupled with a pressure-area constitutive relation and hence be extended to elastic vessels and networks of elastic vessels. Despite its simplicity, the model is reliable within its validity domain and hence it can provide an attractive alternative to the more complex and costly multi-dimensional flow models in some cases of flow in regular geometries with obvious symmetry.

The roots of the 1D flow model may be traced back to the days of Euler who apparently laid down its mathematical foundations. In the recent years, the 1D model became increasingly popular, especially in the hemodynamics modeling. This is manifested by the fact that several researchers [2–4, 6–18] have used this model recently in their modeling and simulation work.

The '1D' label attached to this model stems from the fact that the θ and r dependencies of a cylindrically-coordinated vessel are neglected due to the axisymmetric flow assumption and the simplified consideration of the flow profile within a lumped parameter called the momentum correction factor. Therefore, the only dependency that is explicitly accounted for is the dependency in the flow direction, x.

The biggest advantages of the 1D model are the relative ease of implementation, and comparative low computational cost in execution. Therefore, the use of full multi-dimensional flow modeling, assuming its viability within the available

computational resources, is justified only when the 1D model fails to capture the essential physical picture of the flow system. However, there are several limitations and disadvantages of the 1D model that restrict its use. These limitations include, among other things, the Newtonian assumption, simplified flow geometry and the one-dimensional nature.

2 Theoretical Background

The widely-used one-dimensional form of the Navier-Stokes equations to describe the flow in a vessel; assuming laminar, incompressible, axi-symmetric, Newtonian, fully-developed flow with negligible gravitational body forces; is given by the following continuity and momentum balance relations with suitable boundary conditions

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad t \geq 0, \ \ x \in [0, L] \tag{1}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\alpha Q^2}{A}\right) + \frac{A}{\rho}\frac{\partial p}{\partial x} + \kappa\frac{Q}{A} = 0 \qquad t \geq 0, \ \ x \in [0, L] \tag{2}$$

In these equations, A is the vessel cross sectional area, t is the time, Q is the volumetric flow rate, x is the axial coordinate along the vessel, L is the length of the vessel, α ($= \frac{\int u^2 dA}{A\overline{u}^2}$ with u and $\overline{u}$ being the fluid local and mean axial speed respectively) is the momentum flux correction factor, ρ is the fluid mass density, p is the local pressure, and κ is a viscosity friction coefficient which is usually given by $\kappa = 2\pi\alpha\nu/(\alpha - 1)$ with ν being the fluid kinematic viscosity defined as the ratio of the dynamic viscosity μ to the mass density. These equations supported by appropriate compatibility and matching conditions are used to describe the 1D flow in a branched network of vessels. The equations, being two in three variables, Q A and p, are normally coupled with the following pressure-area relation in a distensible vessel to close the system and obtain a solution

$$p = p_o + f(A) \tag{3}$$

In this relation, p and p_o are the local and reference pressure respectively, and $f(A)$ is a function of area which may be modeled by the following relation

$$f(A) = \frac{\beta}{A_o} \left(\sqrt{A} - \sqrt{A_o} \right) \tag{4}$$

where

$$\beta = \frac{\sqrt{\pi} h_o E}{1 - \varsigma^2} \tag{5}$$

In these equations, A_o and h_o are respectively the vessel cross sectional area and wall thickness at reference pressure p_o, while E and ς are the Young's elastic modulus and Poisson's ratio of the vessel wall. Similar variants of this 1D flow model formulation can also be found in the literature (see for example [7, 10, 18, 19]).

The continuity and momentum equations are usually casted in matrix form [2, 11, 12, 16] which is more appropriate for numerical manipulation and discretization. In matrix form these equations are given by

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \mathbf{B} = \mathbf{0} \tag{6}$$

where

$$\mathbf{U} = \begin{bmatrix} A \\ Q \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} Q \\ \frac{\alpha Q^2}{A} + \int_A \frac{a}{\rho} \frac{df}{da} da \end{bmatrix} = \begin{bmatrix} Q \\ \frac{\alpha Q^2}{A} + \frac{\beta}{3\rho A_o} A^{3/2} \end{bmatrix} \tag{7}$$

and

$$\mathbf{B} = \begin{bmatrix} 0 \\ \kappa \frac{Q}{A} \end{bmatrix} \tag{8}$$

It should be remarked that the second term of the second row of the $\mathbf{F}$ matrix can be obtained from the third term of the original momentum equation as follow

$$\frac{A}{\rho}\frac{\partial p}{\partial x} = \frac{A}{\rho}\frac{\partial f}{\partial x} = \frac{A}{\rho}\frac{\partial f}{\partial A}\frac{\partial A}{\partial x} = \frac{\partial}{\partial x}\int_{x'}\frac{A}{\rho}\frac{\partial f}{\partial A}\frac{\partial A}{\partial x}\partial x$$

$$= \frac{\partial}{\partial x}\int_{A'}\frac{A}{\rho}\frac{\partial f}{\partial A}\partial A = \frac{\partial}{\partial x}\int_{A'}\frac{A}{\rho}\frac{df}{dA}dA = \frac{\partial}{\partial x}\left(\frac{\beta}{3\rho A_o}A^{3/2}\right) \tag{9}$$

3   Weak Form of 1D Flow Equations

On multiplying Equation 6 by weight functions and integrating over the solution domain, x, the following is obtained

$$\int_\Omega \frac{\partial \mathbf{U}}{\partial t}\cdot\boldsymbol{\omega}dx + \int_\Omega \frac{\partial \mathbf{F}}{\partial x}\cdot\boldsymbol{\omega}dx + \int_\Omega \mathbf{B}\cdot\boldsymbol{\omega}dx = \mathbf{0} \tag{10}$$

where Ω is the solution domain, and $\boldsymbol{\omega}$ is a vector of arbitrary test functions. On integrating the second term of Equation 10 by parts, the following weak form of the preceding 1D flow system is obtained

$$\int_\Omega \frac{\partial \mathbf{U}}{\partial t}\cdot\boldsymbol{\omega}dx - \int_\Omega \mathbf{F}\cdot\frac{d\boldsymbol{\omega}}{dx}dx + \int_\Omega \mathbf{B}\cdot\boldsymbol{\omega}dx + [\mathbf{F}\cdot\boldsymbol{\omega}]_{\partial\Omega} = \mathbf{0} \tag{11}$$

where $\partial\Omega$ is the boundary of the solution domain. This weak formulation, coupled with suitable boundary conditions, can be used as a basis for finite element implementation in conjunction with an iterative scheme such as Newton-Raphson method.

4   Finite Element Solution

There are two major cases to be considered in the finite element solution of the stated flow problem: single vessel and branched network where each one of these cases can be time-independent or time-dependent. These four cases are outlined in the following three subsections.

4.1   Single Vessel Time-Independent Flow

The single vessel time-independent model is based on dropping the time term in the continuity and momentum governing equations to obtain a steady-state solution. This should be coupled with pertinent boundary and compatibility conditions at the vessel inlet and outlet. The details are given in the following.

In discretized form, Equation 11 without the time term can be written for each node $N_i(A_i, Q_i)$ as

$$\mathbf{R}_i = \begin{bmatrix} f_i \\ g_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12}$$

where $\mathbf{R}$ is a vector of the weak form of the residuals and

$$f_i = \sum_{q=1}^{N_q} \left[-w_q Q(\zeta_q) \frac{\partial x}{\partial \zeta}(\zeta_q) \frac{d\omega_{A_i}}{d\zeta}(\zeta_q) \frac{d\zeta}{dx}(\zeta_q) \right] + Q(\partial\Omega)\omega_{A_i}(\partial\Omega) \tag{13}$$

and

$$
\begin{aligned}
g_i &= \sum_{q=1}^{N_q} w_q \frac{\partial x}{\partial \zeta}(\zeta_q) \left[-\left(\frac{\alpha Q^2(\zeta_q)}{A(\zeta_q)} + \frac{\beta}{3\rho A_o} A^{3/2}(\zeta_q) \right) \frac{d\omega_{Q_i}}{d\zeta}(\zeta_q)\frac{d\zeta}{dx}(\zeta_q) + \kappa\frac{Q(\zeta_q)}{A(\zeta_q)}\omega_{Q_i}(\zeta_q) \right] \\
&\quad + \left(\frac{\alpha Q^2(\partial\Omega)}{A(\partial\Omega)} + \frac{\beta}{3\rho A_o} A^{3/2}(\partial\Omega) \right) \omega_{Q_i}(\partial\Omega)
\end{aligned}
\tag{14}
$$

where q is an index for the N_q quadrature points, ζ is the quadrature point coor-

dinate and

$$A(\zeta_q) = \sum_i^n A_{c_i} \psi_{A_i}(\zeta_q) \qquad \& \qquad Q(\zeta_q) = \sum_i^n Q_{c_i} \psi_{Q_i}(\zeta_q) \qquad (15)$$

with n being the number of nodes in a standard element. Because of the non-linear nature of the problem, an iteration scheme, such as Newton-Raphson, can be utilized to construct and solve this system of equations based on the residual. The essence of this process is to solve the following equation iteratively and update the solution until a convergence criterion based on reaching a predefined error tolerance is satisfied

$$\mathbf{J} \, \Delta \mathbf{U} = -\mathbf{R} \qquad (16)$$

In this equation, $\mathbf{J}$ is the jacobian matrix, $\Delta\mathbf{U}$ is the perturbation vector, and $\mathbf{R}$ is the weak form of the residual vector. For a vessel with n nodes, the Jacobian matrix, which is of size $2n \times 2n$, is given by

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial A_1} & \frac{\partial f_1}{\partial Q_1} & \cdots & \frac{\partial f_1}{\partial A_n} & \frac{\partial f_1}{\partial Q_n} \\[6pt] \frac{\partial g_1}{\partial A_1} & \frac{\partial g_1}{\partial Q_1} & \cdots & \frac{\partial g_1}{\partial A_n} & \frac{\partial g_1}{\partial Q_n} \\[6pt] \vdots & \vdots & \ddots & \vdots & \vdots \\[6pt] \frac{\partial f_n}{\partial A_1} & \frac{\partial f_n}{\partial Q_1} & \cdots & \frac{\partial f_n}{\partial A_n} & \frac{\partial f_n}{\partial Q_n} \\[6pt] \frac{\partial g_n}{\partial A_1} & \frac{\partial g_n}{\partial Q_1} & \cdots & \frac{\partial g_n}{\partial A_n} & \frac{\partial g_n}{\partial Q_n} \end{bmatrix} \qquad (17)$$

where the subscripts stand for the node indices, while the vector of unknowns, which is of size $2n$, is given by

$$\mathbf{U} = \begin{bmatrix} A_1 \\ Q_1 \\ \vdots \\ A_n \\ Q_n \end{bmatrix} \tag{18}$$

In the finite element implementation, the Jacobian matrix is usually evaluated numerically by finite differencing, i.e.

$$\mathbf{J} \simeq \begin{bmatrix} \frac{\Delta f_1}{\Delta A_1} & \frac{\Delta f_1}{\Delta Q_1} & \cdots & \frac{\Delta f_1}{\Delta A_n} & \frac{\Delta f_1}{\Delta Q_n} \\ \frac{\Delta g_1}{\Delta A_1} & \frac{\Delta g_1}{\Delta Q_1} & \cdots & \frac{\Delta g_1}{\Delta A_n} & \frac{\Delta g_1}{\Delta Q_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\Delta f_n}{\Delta A_1} & \frac{\Delta f_n}{\Delta Q_1} & \cdots & \frac{\Delta f_n}{\Delta A_n} & \frac{\Delta f_n}{\Delta Q_n} \\ \frac{\Delta g_n}{\Delta A_1} & \frac{\partial g_n}{\Delta Q_1} & \cdots & \frac{\Delta g_n}{\Delta A_n} & \frac{\Delta g_n}{\Delta Q_n} \end{bmatrix} \tag{19}$$

The procedure to obtain a solution is summarized in the following scheme

1. Start with initial values for A_i and Q_i in the $\mathbf{U}$ vector.

2. The system given by Equation 16 is constructed where the weak form of the residual vector $\mathbf{R}$ may be calculated in each iteration $l \ (= 0, 1, \ldots, M)$ as

$$\mathbf{R}_l = \begin{bmatrix} f_1(\mathbf{U}_l) \\ g_1(\mathbf{U}_l) \\ \vdots \\ f_n(\mathbf{U}_l) \\ g_n(\mathbf{U}_l) \end{bmatrix} \tag{20}$$

3. The jacobian matrix is calculated from Equation 19.

4. System 16 is solved for $\Delta \mathbf{U}$, i.e.

$$\Delta \mathbf{U} = -\mathbf{J}^{-1}\mathbf{R} \qquad (21)$$

5. $\mathbf{U}$ is updated to obtain a new $\mathbf{U}$ for the next iteration, that is

$$\mathbf{U}_{l+1} = \mathbf{U}_l + \Delta \mathbf{U} \qquad (22)$$

6. The norm of the residual vector is calculated from

$$\mathfrak{N} = \frac{\sqrt{\epsilon_1^2 + \epsilon_2^2 + \cdots + \epsilon_N^2}}{N} \qquad (23)$$

where ϵ_i is the ith entry of the residual vector and N ($= 2n$) is the size of the residual vector.

7. This cycle is repeated until the norm is less than a predefined error tolerance (e.g. 10^{-8}) or a certain number of cycles is reached without convergence. In the last case, the operation will be aborted due to failure and may be resumed with improved finite element parameters.

With regard to the boundary conditions (BC), two types of Dirichlet conditions can be applied: pressure and volumetric flow rate, that is

$$A - A_{BC} = 0 \quad \text{(for area BC)} \quad \& \quad Q - Q_{BC} = 0 \quad \text{(for flow BC)} \qquad (24)$$

These conditions are imposed by replacing the residual function of one of the governing equations (the continuity equation in our model) for the boundary nodes with one of these constraints.

Imposing the boundary conditions as constraints in one of the two governing equations is associated with imposing compatibility conditions, arising from pro-

jecting the differential equations in the direction of the outgoing characteristic variables [20], at the inlet and outlet by replacing the residual function contributed by the other governing equation with these conditions. The compatibility conditions are given by

$$l_{1,2}^T \left(\mathbf{H}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{B} \right) = 0 \tag{25}$$

where $\mathbf{H}$ is the matrix of partial derivative of $\mathbf{F}$ with respect to $\mathbf{U}$, while the transposed left eigenvectors of $\mathbf{H}$ are given by

$$l_{1,2}^T = \left[\begin{array}{cc} -\alpha\frac{Q}{A} \pm \sqrt{\frac{Q^2}{A^2}\left(\alpha^2 - \alpha\right) + \frac{A}{\rho}\frac{df}{dA}} & 1 \end{array} \right] \tag{26}$$

that is

$$\mathbf{H}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{B} = \left[\begin{array}{cc} 0 & 1 \\ -\alpha\frac{Q^2}{A^2} + \frac{\beta}{2\rho A_o}A^{1/2} & 2\alpha\frac{Q}{A} \end{array} \right] \left[\begin{array}{c} \frac{\partial A}{\partial x} \\ \frac{\partial Q}{\partial x} \end{array} \right] + \left[\begin{array}{c} 0 \\ \kappa\frac{Q}{A} \end{array} \right] \tag{27}$$

i.e.

$$\mathbf{H}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{B} = \left[\begin{array}{c} \frac{\partial Q}{\partial x} \\ \left(-\alpha\frac{Q^2}{A^2} + \frac{\beta}{2\rho A_o}A^{1/2}\right)\frac{\partial A}{\partial x} + \left(2\alpha\frac{\partial Q}{\partial x} + \kappa\right)\frac{Q}{A} \end{array} \right] \tag{28}$$

Hence, Equation 25 reduces to

$$\left[\begin{array}{cc} -\alpha\frac{Q}{A} \pm \sqrt{\frac{Q^2}{A^2}\left(\alpha^2 - \alpha\right) + \frac{A}{\rho}\frac{df}{dA}} & 1 \end{array} \right] \left[\begin{array}{c} \frac{\partial Q}{\partial x} \\ \left(-\alpha\frac{Q^2}{A^2} + \frac{\beta}{2\rho A_o}A^{1/2}\right)\frac{\partial A}{\partial x} + \left(2\alpha\frac{\partial Q}{\partial x} + \kappa\right)\frac{Q}{A} \end{array} \right] = 0 \tag{29}$$

that is

$$\left(-\alpha\frac{Q}{A} \pm \sqrt{\frac{Q^2}{A^2}\left(\alpha^2 - \alpha\right) + \frac{A}{\rho}\frac{df}{dA}}\right)\frac{\partial Q}{\partial x} + \left(-\alpha\frac{Q^2}{A^2} + \frac{\beta}{2\rho A_o}A^{1/2}\right)\frac{\partial A}{\partial x} + \left(2\alpha\frac{\partial Q}{\partial x} + \kappa\right)\frac{Q}{A} = 0 \tag{30}$$

In the last relation, the minus sign is used for the inflow boundary while the plus sign for the outflow boundary. The compatibility conditions, given by Equation 30, replace the momentum residual at the boundary nodes.

4.2 Single Vessel Time-Dependent Flow

The aforementioned time-independent formulation can be extended to describe transient states by including the time terms in the residual equations in association with a numerical time-stepping method such as forward Euler, or backward Euler or central difference. The time-dependent residual will then be given (in one of the aforementioned schemes) by

$$\mathbf{R}_{TD}^{t+\Delta t} = \int_\Omega \frac{\mathbf{U}^{t+\Delta t} - \mathbf{U}^t}{\Delta t} \cdot \boldsymbol{\omega} dx + \mathbf{R}_{TI}^{t+\Delta t} = \mathbf{0} \tag{31}$$

where $\mathbf{R}$ is the weak form of the residual, TD stands for time-dependent and TI for time-independent. The time-dependent jacobian follows

$$\mathbf{J}_{TD}^{t+\Delta t} = \frac{\partial \mathbf{R}_{TD}^{t+\Delta t}}{\partial \mathbf{U}^{t+\Delta t}} \tag{32}$$

Again, we have

$$\Delta \mathbf{U} = -\mathbf{J}^{-1}\mathbf{R} \tag{33}$$

and

$$\mathbf{U}_{l+1} = \mathbf{U}_l + \Delta \mathbf{U} \tag{34}$$

where the symbols represent time-dependent quantities and l represents Newton iterations.

With regard to the boundary nodes, a steady-state or time-dependent boundary conditions could be applied depending on the physical situation while a time-dependent compatibility conditions should be employed by adding a time term to the time-independent compatibility condition, that is

$$C_{TD} = l_{1,2}^T \frac{\partial \mathbf{U}}{\partial t} + C_{TI} = 0 \qquad (35)$$

where C_{TI} is the time-independent compatibility condition as given by Equation 30, while C_{TD} is the time-dependent compatibility condition, that is

$$C_{TD} = \left[\ -\alpha \frac{Q}{A} \pm \sqrt{\frac{Q^2}{A^2}(\alpha^2 - \alpha) + \frac{A}{\rho}\frac{df}{dA}} \quad 1 \ \right] \begin{bmatrix} \frac{\partial A}{\partial t} \\ \frac{\partial Q}{\partial t} \end{bmatrix} + C_{TI} = 0 \qquad (36)$$

i.e.

$$C_{TD} = \left(-\alpha \frac{Q}{A} \pm \sqrt{\frac{Q^2}{A^2}(\alpha^2 - \alpha) + \frac{A}{\rho}\frac{df}{dA}} \right) \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial t} + C_{TI} = 0 \qquad (37)$$

where the time derivatives can be evaluated by finite difference, e.g.

$$\frac{\partial A}{\partial t} \simeq \frac{A^{t+\Delta t} - A^t}{\Delta t} \qquad \& \qquad \frac{\partial Q}{\partial t} \simeq \frac{Q^{t+\Delta t} - Q^t}{\Delta t} \qquad (38)$$

A sign convention similar to that outlined previously should be followed. An algorithmic code of the time-dependent module is presented in Algorithm 1.

4.3 Branched Network

To extend the time-independent and time-dependent single vessel model to time-independent and time-dependent branched network of interconnected vessels, matching constraints at the branching nodes are required. These nodes are treated as

Initialize time: $t = t_o$
Initialize $\mathbf{U}^{t_o}$
for $j \leftarrow 1$ to $numberOfTimeSteps$ **do**
   Increment time by Δt
   for $i \leftarrow 1$ to $MaximumNumberOfNewtonIterations$ **do**
     Find $\mathbf{R}_{TD}^{t+\Delta t} = \int_\Omega \frac{\mathbf{U}^{t+\Delta t}-\mathbf{U}^t}{\Delta t} \cdot \omega dx + \mathbf{R}_{TI}^{t+\Delta t} = \mathbf{0}$
     Find $\mathbf{J}_{TD}^{t+\Delta t} = \frac{\partial \mathbf{R}_{TD}^{t+\Delta t}}{\partial \mathbf{U}^{t+\Delta t}}$
     Find $\Delta \mathbf{U}^{t+\Delta t} = -(\mathbf{J}^{-1}\mathbf{R})_{TD}^{t+\Delta t}$
     Find $\mathbf{U}_{i+1}^{t+\Delta t} = \mathbf{U}_i^{t+\Delta t} + \Delta \mathbf{U}^{t+\Delta t}$
     Update: $\mathbf{U}_i^{t+\Delta t} = \mathbf{U}_{i+1}^{t+\Delta t}$
     if (convergence condition met) **then**
       Exit loop
     else
       if ($MaximumNumberOfNewtonIterations$ reached) **then**
         Declare failure
         Exit
       end if
     end if
   end for
    Update: $\mathbf{U}^t = \mathbf{U}^{t+\Delta t}$
  end for
Solution: $\mathbf{U}^{t+\Delta t}$
End

Algorithm 1: Algorithmic code for the time-dependent module.

discontinuous joints where each segment connected to that junction has its own index for that junction although they are spatially identical. The matching constraints are derived from the conservation of flow rate for incompressible fluid, and the Bernoulli energy conservation principle for inviscid flow. More specifically, at each n-segment branching node, n distinctive constraints are imposed: one represents the conservation of flow which involves all the segments at that junction, while the other $(n-1)$ constraints represent the Bernoulli principle with each Bernoulli constraint involving two distinctive segments. These constrains are summarized in

the following relations

$$\sum_{i=1}^{n} Q_i = 0 \tag{39}$$

and

$$p_k + \frac{1}{2}\rho u_k^2 - p_l - \frac{1}{2}\rho u_l^2 = 0 \tag{40}$$

where k and l are indices of two distinct segments, and $u \ (= \frac{Q}{A})$ is the fluid speed averaged over the vessel cross section. In Equation 39 a directional flow is assumed by attaching opposite signs to the inflow and outflow. The matching constraints, which replace the residuals of one of the governing equations (continuity), are coupled with compatibility conditions, similar to the ones used for the single vessel, where these conditions replace the residual of the other governing equation (momentum). The sign convention for these compatibility conditions should follow the same rules as for the boundary conditions, that is minus sign for inflow and plus sign for outflow. This branching model can be applied to any branching node with connectivity $n \geq 2$. The special case of $n = 2$ enables flexible modeling of discontinuous transition between two neighboring segments with different cross sectional areas. Suitable pressure or flux boundary conditions (which for the time-dependent case could be time-independent, or time-dependent over the whole or part of the time stepping process) should also be imposed on all boundary nodes of the network. With regard to the other aspects of the time-independence and time-dependence treatment, the network model should follow the same rules as for single vessel time-independent and time-dependent models which are outlined in the previous sections.

5 Non-Dimensionalized Form

To improve convergence, the aforementioned dimensional forms of the governing, boundary, compatibility, and matching equations can be non-dimensionalized by carefully-chosen scale factors. The following scale factors are commonly used to scale the model parameters:

$$Q \sim \pi R_o^2 U_o \quad A \sim \pi R_o^2 \quad p \sim \rho U_o^2 \quad x \sim \lambda \quad t \sim \frac{\lambda}{U_o} \tag{41}$$

where R_o, U_o, and λ are respectively typical values of the radius, velocity and length for the flow system. In the following we demonstrate non-dimensionalization of the flow equations by a few examples followed by stating the non-dimensionalized form of the others.

5.1 Non-Dimensionalized Navier-Stokes Equations

Continuity equation 1st form (Equation 1):

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{42}$$

$$\frac{\partial \left(\pi R_o^2 A' \right)}{\partial \left(\frac{\lambda}{U_o} t' \right)} + \frac{\partial \left(\pi R_o^2 U_o Q' \right)}{\partial \left(\lambda x' \right)} = 0 \tag{43}$$

that is

$$\frac{\partial A'}{\partial t'} + \frac{\partial Q'}{\partial x'} = 0 \tag{44}$$

where the prime indicates a non-dimensionalized value.

Continuity equation 2nd form (Equation 6):

Same as Equation 44.

Momentum equation 1st form (Equation 2):

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\alpha Q^2}{A}\right) + \frac{A}{\rho}\frac{\partial p}{\partial x} + \kappa\frac{Q}{A} = 0 \tag{45}$$

$$\frac{\partial\left(\pi R_o^2 U_o Q'\right)}{\partial\left(\frac{\lambda}{U_o}t'\right)} + \frac{\partial}{\partial\left(\lambda x'\right)}\left(\frac{\alpha\left(\pi R_o^2 U_o Q'\right)^2}{\left(\pi R_o^2 A'\right)}\right) + \frac{\left(\pi R_o^2 A'\right)}{\rho}\frac{\partial\left(\rho U_o^2 p'\right)}{\partial\left(\lambda x'\right)} + \kappa\frac{\left(\pi R_o^2 U_o Q'\right)}{\left(\pi R_o^2 A'\right)} = 0 \tag{46}$$

$$\frac{\pi R_o^2 U_o^2}{\lambda}\frac{\partial Q'}{\partial t'} + \frac{\alpha\pi^2 R_o^4 U_o^2}{\pi R_o^2\lambda}\frac{\partial}{\partial x'}\left(\frac{Q'^2}{A'}\right) + \frac{\pi R_o^2\rho U_o^2}{\lambda\rho}\frac{A'\partial p'}{\partial x'} + \frac{\kappa\pi R_o^2 U_o}{\pi R_o^2}\frac{Q'}{A'} = 0 \tag{47}$$

$$\frac{\pi R_o^2 U_o^2}{\lambda}\frac{\partial Q'}{\partial t'} + \frac{\alpha\pi R_o^2 U_o^2}{\lambda}\frac{\partial}{\partial x'}\left(\frac{Q'^2}{A'}\right) + \frac{\pi R_o^2 U_o^2}{\lambda}\frac{A'\partial p'}{\partial x'} + \frac{\kappa\lambda\pi R_o^2 U_o^2}{\lambda\pi R_o^2 U_o}\frac{Q'}{A'} = 0 \tag{48}$$

$$\frac{\partial Q'}{\partial t'} + \alpha\frac{\partial}{\partial x'}\left(\frac{Q'^2}{A'}\right) + \frac{A'\partial p'}{\partial x'} + \frac{\kappa\lambda}{\pi R_o^2 U_o}\frac{Q'}{A'} = 0 \tag{49}$$

$$\frac{\partial Q'}{\partial t'} + \alpha\frac{\partial}{\partial x'}\left(\frac{Q'^2}{A'}\right) + \frac{A'\partial p'}{\partial x'} + \frac{2\pi\alpha\nu\lambda}{(\alpha-1)\,\pi R_o^2 U_o}\frac{Q'}{A'} = 0 \tag{50}$$

that is

$$\frac{\partial Q'}{\partial t'} + \alpha\frac{\partial}{\partial x'}\left(\frac{Q'^2}{A'}\right) + \frac{A'\partial p'}{\partial x'} + \frac{2\alpha\nu\lambda}{(\alpha-1)\,R_o^2 U_o}\frac{Q'}{A'} = 0 \tag{51}$$

Momentum equation 2nd form (Equation 6):

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\alpha Q^2}{A}\right) + \frac{\beta}{3\rho A_o}\frac{\partial}{\partial x}A^{3/2} + \kappa\frac{Q}{A} = 0 \tag{52}$$

$$\frac{\pi R_o^2 U_o^2 \partial Q'}{\lambda \partial t'} + \frac{\left(\pi R_o^2 U_o\right)^2 \partial}{\lambda \pi R_o^2 \partial x'}\left(\frac{\alpha Q'^2}{A'}\right) + \frac{\left(\pi R_o^2\right)^{1/2}\beta}{\lambda 3\rho A_o'}\frac{\partial}{\partial x'}A'^{3/2} + \kappa\frac{\pi R_o^2 U_o Q'}{\pi R_o^2 A'} = 0 \tag{53}$$

$$\frac{\pi R_o^2 U_o^2 \partial Q'}{\lambda \partial t'} + \frac{\pi R_o^2 U_o^2 \partial}{\lambda \partial x'}\left(\frac{\alpha Q'^2}{A'}\right) + \frac{\left(\pi R_o^2\right)^{1/2}\beta}{\lambda 3\rho A_o'}\frac{\partial}{\partial x'}A'^{3/2} + \kappa\frac{U_o Q'}{A'} = 0 \tag{54}$$

$$\frac{\partial Q'}{\partial t'} + \frac{\partial}{\partial x'}\left(\frac{\alpha Q'^2}{A'}\right) + \frac{1}{\left(\pi R_o^2\right)^{1/2}U_o^2}\frac{\beta}{3\rho A_o'}\frac{\partial}{\partial x'}A'^{3/2} + \frac{\lambda}{\pi R_o^2 U_o}\kappa\frac{Q'}{A'} = 0 \tag{55}$$

5.2 Non-Dimensionalized Compatibility Condition

Time-independent term of compatibility condition:

$$\left(-\alpha\frac{U_o Q'}{A'} \pm \sqrt{\frac{U_o^2 Q'^2}{A'^2}\left(\alpha^2 - \alpha\right) + \frac{A'}{\rho\sqrt{\pi R_o^2}}\frac{\beta}{2A_o'\sqrt{A'}}}\right)\frac{U_o\partial Q'}{\lambda\partial x'}$$
$$+ \left(-\alpha\frac{U_o^2 Q'^2}{A'^2} + \frac{\beta}{2\rho\sqrt{\pi R_o^2}A_o'}A'^{1/2}\right)\frac{\partial A'}{\lambda\partial x'} + \left(2\alpha\frac{\pi R_o^2 U_o\partial Q'}{\lambda\partial x'} + \frac{2\pi\alpha\nu}{\alpha - 1}\right)\frac{U_o Q'}{\pi R_o^2 A'} = 0 \tag{56}$$

Time-dependent term of compatibility condition:

$$\left(-\alpha\frac{U_o Q'}{A'} \pm \sqrt{\frac{U_o^2 Q'^2}{A'^2}\left(\alpha^2 - \alpha\right) + \frac{A'}{\rho}\frac{\beta}{\left(\pi R_o^2\right)^{1/2}2A_o'\sqrt{A'}}}\right)\frac{\partial A'}{\partial t'} + \frac{U_o\partial Q'}{\partial t'} = 0 \tag{57}$$

5.3 Non-Dimensionalized Matching Conditions

Flow conservation:

$$Q'_1 - Q'_2 - Q'_3 = 0 \tag{58}$$

Bernoulli:

$$p'_k + \frac{1}{2}u'^2_k - p'_l - \frac{1}{2}u'^2_l = 0 \tag{59}$$

5.4 Non-Dimensionalized Boundary Conditions

$$A' - A'_{BC} = 0 \quad \text{(for area BC)} \quad \& \quad Q' - Q'_{BC} = 0 \quad \text{(for flow BC)} \tag{60}$$

6 Validation

The different modules of the the 1D finite element flow model are validated as follow

- Time-independent single vessel: the numeric solution should match the analytic solution as given by Equation 63 which is derived in Appendix A. Also, the boundary conditions should be strictly satisfied.

- Time-dependent single vessel: the solution should asymptotically converge to the analytic solution on imposing time-independent boundary conditions. Also, the boundary conditions should be strictly satisfied at all time steps.

- Time-independent network: four tests are used to validate the numeric solution. First, the boundary conditions should be strictly satisfied. Second, the conservation of mass (or conservation of volume for incompressible flow), as given by Equation 39, should be satisfied at all branching nodes (bridge,

bifurcation, trifurcation, etc.). A consequence of this condition is that the sum of the boundary inflow (sum of Q at inlet boundaries) should be equal to the sum of the boundary outflow (sum of Q at outlet boundaries). Third, the conservation of energy (Bernoulli's principle for inviscid flow), as given by Equation 40, should be satisfied at all branching nodes. Fourth, the analytic solution for time-independent flow in a single vessel, as given by Equation 63 in Appendix A, should be satisfied by all vessels in the network with possible exception of very few vessels with odd features (e.g. those with distorted shape such as extreme radius-to-length ratio, and hence are susceptible to large numerical errors). The fourth test is based on the fact that the single vessel solution is dependent on the boundary conditions and not on the mechanism by which these conditions are imposed.

- Time-dependent network: the solution is validated by asymptotic convergence to the time-independent solution, as validated by the four tests outlined in the previous item, on imposing time-independent boundary conditions.

The solutions may also be tested qualitatively by static and dynamic visualization for time-independent and time-dependent cases respectively to verify their physical sensibility. Other qualitative tests, such as comparing the solutions of different cases with common features, may also be used for validation.

It should be remarked that Equation 63 contains three variables: x A and Q, and hence it can be solved for one of these variables given the other two. Solving for A and Q requires employing a numeric solver, based for example on a bisection method; hence the best option is to solve for x and compare to the numeric solution. This in essence is an exchange of the role of independent and dependent variables which has no effect on validation. Alternatively, Equation 76 can be used to verify the solution directly by using the vessel inlet and outlet areas. In fact Equation 76 can be used to verify the solution at any point on the vessel axis by labeling the

area at that point as A_{ou}, as explained in Appendix A.

7 General Notes

7.1 Implementation

- The model described in this report was implemented and tested on both single vessels and networks of vessels for time-independent and time-dependent cases and it produced valid results. The implementation is based on a Galerkin method, where the test functions are obtained from the same space as the trial basis functions used to represent the state variables, with a Lagrange interpolation associated with a Gauss quadrature integration scheme (refer to Appendix B for Gauss quadrature tables). Many tests have been carried out to verify various aspects of the 1D model. These tests involved many synthetic and biological networks which vary in their size, connectivity, number and type of branching nodes, type of meshing, and so on. The tests also included networks with and without loops although the great majority of the networks contain loops. Some of the networks involved in these tests consist of very large number of vessels in the order of hundreds of thousands with much more degrees of freedom. Non-dimensional form, as well as dimensional form, was tested on single vessels and branched networks; the results, after re-scaling, were verified to be identical to those obtained from the dimensional form. The checks also included h and p convergence tests which demonstrated correct convergence behavior.

- To be on the safe side, the order of the quadrature should be based on the sum of orders of the interpolating functions, their derivatives and test functions. The adopted quadrature order scheme takes the highest order required by the terms.

- A constant delta may be used in the evaluation of the Jacobian matrix by finite difference. A suitable value for delta may be $\Delta A_i = \Delta Q_i = 10^{-7}$ or 10^{-8}.

7.2   Solution

- Negative flow in the solution means the flow direction is opposite to the vessel direction as indicated by the vessel topology, that is the flow rate of a segment indexed as N_1 N_2 will be positive if the flow is from N_1 to N_2 and negative if the flow is from N_2 to N_1.

- Interpolation polynomials of various degrees (p) in association with different meshing (h) should be used to validate the convergence behavior. The convergence to the correct solution should improve by increasing p and decreasing h. L^2 error norm may be used as a measure for convergence; it is given by

$$L^2 = \left(\int_X (S_a - S_n)^2 \, dx \right)^{1/2} \tag{61}$$

where S_a and S_n are the analytic and numeric solutions respectively, and X is the solution domain. The integration can be performed numerically using, for example, trapezium or Simpson's rules. The error norm should fall steadily as h decreases and p increases.

- With regard to the previously outlined implementation of the 1D model (see § 7.1), typical solution time on a typical platform (normal laptop or desktop) for a single time-independent simulation on a typical 1D network consisting of hundreds of thousands of degrees of freedom is a few minutes. The final convergence is normally reached within 5-7 Newton iterations. The solution time of a single time step for the time-dependent case is normally less than

the solution time of the equivalent time-independent case, and the number of the required Newton iterations of each time step in the time-dependent case is normally less than that for the corresponding time-independent case.

- Since there are many sources of error and wrong convergence, each acquired solution should be verified by the aforementioned validation tests (see § 6). The 1D finite element code should be treated as a device that suggests solutions which can be accepted only if they meet the validation criteria.

7.3 Non-Dimensionalization

- On implementing the non-dimensionalized form (as given in § 5) in the finite element code, all the user needs is to scale the primed input values either inside or outside the code; the results then should be scaled back to obtain the dimensionalized solution.

- Different length scales can be utilized as long as they are in different orientations (e.g. vessel length and vessel radius) and hence linearly independent; otherwise the physical space will be distorted in non-systematic way and hence may not be possible to restore by scaling back.

7.4 Convergence

A number of measures, outlined in the following points, can speedup convergence and help avoiding convergence failure.

- Non-dimensionalization which requires implementation in the finite element code (as given in § 5) where the input data is non-dimensionalized and the results are re-dimensionalized back to the physical space.

- Using different unit systems, such as m.kg.s or mm.g.s or m.g.s, for the input data and parameters.

- Scaling the model up or down to obtain a similarity solution which can then be scaled back to obtain the final results.

- Increasing the error tolerance of the solver for convergence criterion. However, the use of relatively large error tolerance can cause wrong convergence and hence should be avoided. It may be recommended that the maximum allowed error tolerance for obtaining a reliable solution must not exceed 10^{-5}. Anyway, the solution in all cases should be verified by the validation tests (see § 6) and hence it must be rejected if the errors exceed acceptable limits.

- For time-dependent cases, the required boundary condition value can be imposed gradually by increasing the inlet pressure, for instance, over a number of time steps to reach the final steady state value.

- The use of smaller time steps in the time-dependent cases may also help to avoid convergence failure.

It should be remarked that the first three strategies are based on the same principle, that is adjusting the size of the problem numbers to help the solver to converge more easily to the solution.

7.5 Boundary Conditions

- Dirichlet type boundary conditions are usually used for imposing flow rate and pressure boundary conditions. The previous formulation is based on this assumption.

- Pressure boundary conditions are imposed by adjusting the inlet or outlet area where p and A are correlated through Equation 3.

- While pressure boundary conditions can be imposed on both inlet and outlet boundaries simultaneously, as well as mixed boundary conditions (i.e. inlet

pressure with outlet flow or inlet flow with outlet pressure or mixed on one or both boundaries), it is not possible to impose flow boundary conditions on all inlet and outlet boundaries simultaneously because this is *either* a trivial condition repeating the condition of flow conservation (i.e. Equation 39) at the branching junctions if the inflow is equal to the outflow *or* it is a contradiction to the flow conservation condition if the inflow and outflow are different, and hence no solution can be found due to ambiguity and lack of constraints in the first case and to inconsistency in the second case.

- Zero Q boundary condition can be used to block certain inlet or outlet vessels in a network for the purpose of emulating a physical situation or improving convergence when the blockage does not affect the solution significantly.

- In some biological flow conditions there are no sufficient data to impose realistic pressure boundary conditions that ensure biologically sensible flow in the correct direction over the whole vascular network. In such situations a back flow may occur in some branches which is physically correct but biologically incorrect. To avoid this situation, an inlet pressure boundary condition with outlet flow boundary conditions where the total outflow is split according to a certain physical or biological model (such as being proportional to the area squared) can be used to ensure sensible flow in the right direction over the whole network. The total amount of the outflow can be estimated from the inlet flow which is usually easier to estimate as it normally comes from a single (or few) large vessel. This trick may also be applicable in some physical circumstances.

- Use may be made of an artificial single inlet boundary to avoid lack of knowledge about the pressure distribution in a multi-inlet network to ensure correct flow in the right direction. The inlets can be connected to a single artificial

node (e.g. located at their centroid) where the radii of the connecting artificial vessels is chosen according to a physical or biological model such as Murray's law. This node can then be connected through a single artificial vessel whose radius can be computed from a physical or biological model and whose length can be determined from a typical L/R ratio such as 10. The inlet of this vessel can then be used to impose a single p or Q biologically-sensible boundary condition. It should be remarked that Murray's biological law is given by

$$r_m^\gamma = \sum_i^n r_{d_i}^\gamma \tag{62}$$

where r_m is the radius of the mother vessel, r_{d_i} is the radius of the ith daughter vessel, n is the number of daughter vessels which in most cases is 2, and γ is a constant index which according to Murray is 3, but other values like 2.1 and 2.2 are also used in the literature.

- Time-dependent boundary conditions can be modeled by empirical signals (e.g. obtained from experimental data) or by closed analytical forms such as sinusoidal.

7.6 Initial Conditions

- The convergence usually depends on the initial values of area and flow rate. A good option for these values is to use unstressed area with zero flow for start.

7.7 Miscellaneous

- Apart from the interpolation nodes, there are two main types of nodes in the finite element network: segment nodes and finite element discretization

nodes. The connectivity of the second type is always 2 as these nodes connect two elements; whereas the connectivity of the first can be 1 for the boundary nodes, 2 for the bridge nodes connecting two segments, or ≥ 3 for the branching nodes (bifurcation, trifurcation, etc.). The mass and energy conservation conditions can be extended to include all the segment nodes with connectivity > 1 by including the bridge nodes.

- For networks, the vessel wall thickness at reference pressure, h_o, can be a constant or vary from vessel to vessel depending on the physical or biological situation. Using variable thickness is more sound in biological context where the thickness can be estimated as a fraction of the lumen or vessel radius. A fractional thickness of 10-15% of the radius is commonly used for blood vessels [6, 16, 20–26]. For more details, refer to Appendix D.

- The previous finite element formulation of the 1D model for single vessels and networks works for constant-radius vessels (i.e. with constant A_o) only and hence to extend the formulation to variable-radius vessels the previous matrix structure should be reshaped to include the effect of tapering or expanding of the vessels. However, the vessels can be straight or curved. The size of the vessels in a network can also vary significantly from one vessel to another as long as the 1D flow model assumptions (e.g. size, shape, etc.) do apply on each vessel.

- The networks used in the 1D flow model should be totally connected, that is any node in the network can be reached from any other node by moving entirely inside the network vessels.

- Different time stepping schemes, such as forward or backward Euler or central difference, can be used for implementing the time term of the time-dependent single vessel and network modules although the speed of convergence and

quality of solution vary between these schemes. The size of the time step should be chosen properly for each scheme to obtain equivalent results from these different schemes.

- Although the 1D model works on highly non-homogeneous networks in terms of vessels length without discretization, a scheme of homogeneous discretization may be employed by using a constant element length, h, over the whole network as an approximation to the length of the discretized elements. The length of the elements of each vessel is then obtained by dividing the vessel evenly to an integer number of elements with closest size to the given h. Although discretization is not a requirement, since the 1D model works even on non-discretized networks, it usually improves the solution. Moreover, discretization is required for obtaining a detailed picture of the pressure and flow fields at the interior points. Use of interpolation schemes higher than linear (with and without discretization) also helps in refining the variable fields. Also, for single vessel the solution can be obtained with and without discretization; in the first case the discretized elements could be of equal or varying length. The solution, however, should generally improve by discretization.

- Although the 1D model works on non-homogeneous networks in terms of vessels radius, an abrupt transition from one vessel to its neighbor may hinder convergence.

- In general, the time-dependent problem converges more easily than its equivalent time-independent problem. This may be exploited to obtain approximate time-independent solutions in some circumstances from the time-dependent module as the latter asymptotically approaches the time-independent solution.

- The correctness of the solutions mathematically may not guarantee physiological, and even physical, sensibility since the network features, boundary conditions, and model parameters which in general highly affect the flow pattern, may not be found normally in real biological and physical systems. The quality of any solution, assuming its correctness in mathematical terms, depends on the quality of the underlying model and how it reflects the physical reality.

- Because the 1D model depends on the length of the vessels but not their location or orientation, a 1D coordinate system, as well as 2D or 3D, can be used for coordinating the space. The vessels can be randomly oriented without effecting the solution. A multi-dimensional space may be required, however, for consistent and physically-correct description of the networks.

- The reference pressure, p_o, in Equation 3 is usually assumed zero to simplify the relation.

8 Conclusions

The one-dimensional Navier-Stokes formulation is widely used as a realistic model for the flow of Newtonian fluids in large vessels with certain simplifying assumptions, such as axi-symmetry. The model may also be coupled with a pressure-area constitutive relation and hence be extended to the flow in distensible vessels. Numerical implementation of this model based on a finite element method with suitable boundary conditions is also used to solve the time-independent and transient flow in single vessels and networks of interconnected vessels where in the second case compatibility and matching conditions, which include conservation of mass and energy, at branching nodes are imposed. Despite its comparative simplicity, the 1D flow model can provide reliable solutions, with relatively low computational cost, to many flow problems within its domain of validity. The current document outlined the analytical and numerical aspects of this model with theoretical and technical details related to implementation, performance, methods of improvement, validation, and so on.

Nomenclature

α correction factor for axial momentum flux

β parameter in the pressure-area relation

γ Murray's law index

ϵ residual error

ζ quadrature point coordinate

κ viscosity friction coefficient

μ fluid dynamic viscosity

ν fluid kinematic viscosity ($\nu = \frac{\mu}{\rho}$)

ρ fluid mass density

ς Poisson's ratio of vessel wall

ψ basis function for finite element discretization

$\boldsymbol{\omega}$ vector of test functions in the weak form of finite element

Ω solution domain

$\partial\Omega$ boundary of the solution domain

A vessel cross sectional area

A_{BC} boundary condition for vessel cross sectional area

A_{in} vessel cross sectional area at inlet

A_o vessel cross sectional area at reference pressure

$\mathbf{B}$ matrix of force terms in the 1D Navier-Stokes equations

E Young's modulus of vessel wall

$f(A)$ function in pressure-area relation

$\mathbf{F}$ matrix of flux quantities in the 1D Navier-Stokes equations

h	length of element
h_o	vessel wall thickness at reference pressure
$\mathbf{H}$	matrix of partial derivative of $\mathbf{F}$ with respect to $\mathbf{U}$
$\mathbf{J}$	Jacobian matrix
L	length of vessel
$\mathfrak{N}$	norm of residual vector
p	local pressure
p	order of interpolating polynomial
p_o	reference pressure
q	dummy index for quadrature point
Q	volumetric flow rate
Q_{BC}	boundary condition for volumetric flow rate
r	radius
$\mathbf{R}$	weak form of residual vector
S_a	analytic solution
S_n	numeric solution
t	time
Δt	time step
u	local axial speed of fluid
$\overline{u}$	mean axial speed of fluid
$\mathbf{U}$	vector of finite element variables
$\mathbf{\Delta U}$	vector of change in $\mathbf{U}$
x	vessel axial coordinate

References

[1] M.S. Olufsen; C.S. Peskin; W.Y. Kim; E.M. Pedersen; A. Nadim; J. Larsen. Numerical Simulation and Experimental Validation of Blood Flow in Arteries with Structured-Tree Outflow Conditions. *Annals of Biomedical Engineering*, 28(11):1281–1299, 2000. 5

[2] S.J. Sherwin; V. Franke; J. Peiró; K. Parker. One-dimensional modelling of a vascular network in space-time variables. *Journal of Engineering Mathematics*, 47(3-4):217–250, 2003. 5, 7, 46

[3] L. Formaggia; D. Lamponi; M. Tuveri; A. Veneziani. Numerical modeling of 1D arterial networks coupled with a lumped parameters description of the heart. *Computer Methods in Biomechanics and Biomedical Engineering*, 9(5):273–288, 2006. 5, 46

[4] J. Janela; A.B. de Moura; A. Sequeira. Comparing Absorbing Boundary Conditions for a 3D Non Newtonian Fluid-Structure Interaction Model for Blood Flow in Arteries. *Mecánica Computacional*, XXIX(59):5961–5971, 2010. 5, 46

[5] T. Sochi. Newtonian Flow in Converging-Diverging Capillaries. Submitted. 5

[6] L. Formaggia; J.F. Gerbeau; F. Nobile; A. Quarteroni. On the coupling of 3D and 1D Navier-Stokes equations for flow problems in compliant vessels. *Computer Methods in Applied Mechanics and Engineering*, 191(6-7):561–582, 2001. 5, 29, 46

[7] N.P. Smith; A.J. Pullan; P.J. Hunter. An Anatomically Based Model of Transient Coronary Blood Flow in the Heart. *SIAM Journal on Applied Mathematics*, 62(3):990–1018, 2002. 5, 7, 46

[8] W. Ruan; M.E. Clark; M. Zhao; A. Curcio. A Hyperbolic System of Equations

of Blood Flow in an Arterial Network. *SIAM Journal on Applied Mathematics*, 64(2):637–667, 2003. 5

[9] S. Urquiza; P. Blanco; G. Lombera; M. Venere; R. Feijoo. Coupling Multidimensional Compliant Models For Carotid Artery Blood Flow. *Mecánica Computacional*, XXII(3):232–243, 2003. 5, 46

[10] G. Pontrelli; E. Rossoni. Numerical modelling of the pressure wave propagation in the arterial flow. *International Journal for Numerical Methods in Fluids*, 43(6-7):651–671, 2003. 5, 7, 46

[11] V. Milišić; A. Quarteroni. Analysis of lumped parameter models for blood flow simulations and their relation with 1D models. *Mathematical Modelling and Numerical Analysis*, 38(4):613–632, 2004. 5, 7

[12] M.Á. Fernández; V. Milišić; A. Quarteroni. Analysis of a Geometrical Multiscale Blood Flow Model Based on the Coupling of ODEs and Hyperbolic PDEs. *Multiscale Modeling & Simulation*, 4(1):215–236, 2005. 5, 7

[13] L. Formaggia; A. Moura; F. Nobile. Coupling 3D and 1D fluid-structure interaction models for blood flow simulations. *Proceedings in Applied Mathematics and Mechanics, Special Issue: GAMM Annual Meeting 2006 - Berlin*, 6(1):27–30, 2006. 5, 46

[14] J. Alastruey; S.M. Moore; K.H. Parker; T. David; J. Peiró S.J. Sherwin. Reduced modelling of blood flow in the cerebral circulation: Coupling 1-D, 0-D and cerebral auto-regulation models. *International Journal for Numerical Methods in Fluids*, 56(8):1061–1067, 2008. 5

[15] J. Alastruey; K.H. Parker; J. Peiró; S.J. Sherwin. Lumped parameter outflow models for 1-D blood flow simulations: Effect on pulse waves and parameter

estimation. *Communications in Computational Physics*, 4(2):317–336, 2008. 5, 46

[16] J. Lee; N. Smith. Development and application of a one-dimensional blood flow model for microvascular networks. *Proceedings of the Institution of Mechanical Engineers, Part H: Journal of Engineering in Medicine*, 222(4):487–512, 2008. 5, 7, 29, 46

[17] T. Passerini; M. de Luca; L. Formaggia; A. Quarteroni; A. Veneziani. A 3D/1D geometrical multiscale model of cerebral vasculature. *Journal of Engineering Mathematics*, 64(4):319–330, 2009. 5, 46

[18] G. Papadakis. Coupling 3D and 1D fluid-structure-interaction models for wave propagation in flexible vessels using a finite volume pressure-correction scheme. *Communications in Numerical Methods in Engineering, Special Issue: Flow in collapsible tubes or over compliant surfaces for biomedical applications*, 25(5):533–551, 2009. 5, 7, 46

[19] A. di Carlo; P. Nardinocchi; G. Pontrelli; L. Teresi. A heterogeneous approach for modelling blood flow in an arterial segment. *Transactions of the Wessex Institute.* 7, 46

[20] L. Formaggia; D. Lamponi; A. Quarteroni. One-dimensional models for blood flow in arteries. *Journal of Engineering Mathematics*, 47(3/4):251–276, 2003. 13, 29, 46

[21] B.K. Podesser; F. Neumann; M. Neumann; W. Schreiner; G. Wollenek; R. Mallinger. Outer radius-wall thickness ratio, a postmortem quantitative histology in human coronary arteries. *Acta Anatomica*, 163(2):63–68, 1998. 29, 46

[22] X. Zhang; M. Fatemi; J.F. Greenleaf. Vibro-acoustography for modal analysis of arterial vessels. *IEEE International Symposium on Biomedical Imaging Proceedings 2002*, pages 513–516, 2002. 29, 46

[23] W.C.P.M. Blondel; B. Lehalle; M-N. Lercher; D. Dumas; D. Bensoussan; J-F. Stoltz. Rheological properties of healthy and atherosclerotic human arteries. *Biorheology*, 40(1-3):369–376, 2003. 29, 46

[24] L. Waite; J. Fine. *Applied Biofluid Mechanics*. McGraw-Hill, New York, 1st edition, 2007. 29, 46

[25] S. Badia; A. Quaini; A. Quarteroni. Coupling Biot and Navier-Stokes equations for modelling fluid-poroelastic media interaction. *Journal of Computational Physics*, 228(21):7986–8014, 2009. 29, 46

[26] C.N. van den Broek; A. van der Horst; M.C. Rutten; F.N. van de Vosse. A generic constitutive model for the passive porcine coronary artery. *Biomech Mod Mechanobiol*, 10(2):249–258, 2011. 29, 46

[27] A.P. Avolio. Multi-branched model of the human arterial system. *Medical and Biological Engineering and Computing*, 18(6):709–718, 1980. 46

[28] S. Čanić; E.H. Kim. Mathematical analysis of the quasilinear effects in a hyperbolic model blood flow through compliant axi-symmetric vessels. *Mathematical Methods in the Applied Sciences*, 26(14):1161–1186, 2003. 46

[29] N.P. Smith. A computational study of the interaction between coronary blood flow and myocardial mechanics. *Physiological Measurement*, 25(4):863–877, 2004. 46

[30] N. Koshiba; J. Ando; X. Chen; T. Hisada. Multiphysics Simulation of Blood Flow and LDL Transport in a Porohyperelastic Arterial Wall Model. *Journal of Biomechanical Engineering*, 129(3):374–385, 2007. 46

[31] H. Ashikaga; B.A. Coppola; K.G. Yamazaki; F.J. Villarreal; J.H. Omens; J.W. Covell. Changes in regional myocardial volume during the cardiac cycle: implications for transmural blood flow and cardiac structure. *American Journal of Physiology - Heart and Circulatory Physiology*, 295(2):H610–H618, 2008. 46

[32] L. Antiga. *Patient-Specific Modeling of Geometry and Blood Flow in Large Arteries.* PhD thesis, Milan Polytechnic, 2002. 46

[33] N. Westerhof; C. Boer; R.R. Lamberts; P. Sipkema. Cross-Talk Between Cardiac Muscle and Coronary Vasculature. *Physiological Reviews*, 86(4):1263–1308, 2006. 46

[34] A.B. de Moura. *The Geometrical Multiscale Modelling of the Cardiovascular System: Coupling 3D FSI and 1D Models.* PhD thesis, Dipartimento di Matematica Politecnico di Milano, 2007. 46

[35] X. Zhang; J.F. Greenleaf. Noninvasive estimation of local elastic modulus of arteries with the ring resonance measurement. *IEEE Ultrasonics Symposium, Vancouver, BC*, pages 1161–1164, 2006. 46

[36] I. Leventhal; P.C. Georges; P.A. Janmey. Soft biological materials and their impact on cell function. *Soft Matter*, 3:299–306, 2007. 46

[37] V.M. Calo; N.F. Brasher; Y. Bazilevs; T.J.R. Hughes. Multiphysics model for blood flow and drug transport with application to patient-specific coronary artery flow. *Computational Mechanics*, 43(1):161–177, 2008. 46

[38] K.S. Hunter; J.A. Albietz; P-F. Lee; C.J. Lanning; S.R. Lammers; S.H. Hofmeister; P.H. Kao; H.J. Qi; K.R. Stenmark; R. Shandas. In vivo measurement of proximal pulmonary artery elastic modulus in the neonatal calf model of pulmonary hypertension: development and ex vivo validation. *Journal of Applied Physiology*, 108(4):968–975, 2010. 46

[39] S.X. Deng; J. Tomioka; J.C. Debes; Y.C. Fung. New experiments on shear modulus of elasticity of arteries. *American Journal of Physiology - Heart and Circulatory Physiology*, 266(35):H1–H10, 1994. 46

[40] L. Formaggia; F. Nobile; A. Quarteroni; A. Veneziani. Multiscale modelling of the circulatory system: a preliminary analysis. *Computing and Visualization in Science*, 2(3):75–83, 1999. 46

[41] A. Quarteroni; S. Ragni; A. Veneziani. Coupling between lumped and distributed models for blood flow problems. *Computing and Visualization in Science*, 4(1):111–124, 2001. 46

[42] J. Sainte-Marie; D. Chapelle; R. Cimrman; M. Sorine. Modeling and estimation of the cardiac electromechanical activity. *Computers and Structures*, 84(28):1743–1759, 2006. 46

[43] P.J. Blanco; R.A. Feijóo; S.A. Urquiza. A unified variational approach for coupling 3D-1D models and its blood flow applications. *Computer Methods in Applied Mechanics and Engineering*, 196(41-44):4391–4410, 2007. 46

[44] M. Boulakia; S. Cazeau; M.A. Fernández; J.F. Gerbeau; N. Zemzemi. Mathematical Modeling of Electrocardiograms: A Numerical Study. *Annals of Biomedical Engineering*, 38(3):1071–1097, 2009. 46

[45] D. Chapelle; J.F. Gerbeau; J. Sainte-Marie; I.E. Vignon-Clementel. A poroelastic model valid in large strains with applications to perfusion in cardiac modeling. *Computational Mechanics*, 46(1):91–101, 2010. 46

9   Appendix A: Derivation of Time-Independent Analytical Solution for Single Vessel

The following analytical relation linking vessel axial coordinate x to cross sectional area A, cross sectional area at inlet A_{in}, and volumetric flow rate Q for time-independent flow can be derived and used to verify the finite element solution

$$x = \frac{\alpha Q^2 \ln\left(A/A_{in}\right) - \frac{\beta}{5\rho A_o}\left(A^{5/2} - A_{in}^{5/2}\right)}{\kappa Q} \tag{63}$$

The derivation is outlined in the following. For time-independent flow, the system given by Equation 6 in matrix form, will become

$$\frac{\partial Q}{\partial x} = 0 \qquad x \in [0, l], \ \ t \geq 0 \tag{64}$$

$$\frac{\partial}{\partial x}\left(\frac{\alpha Q^2}{A} + \frac{\beta}{3\rho A_o}A^{3/2}\right) + \kappa\frac{Q}{A} = 0 \qquad x \in [0, l], \ \ t \geq 0 \tag{65}$$

that is Q as a function of x is constant and

$$\frac{\partial}{\partial A}\left(\frac{\alpha Q^2}{A} + \frac{\beta}{3\rho A_o}A^{3/2}\right)\frac{\partial A}{\partial x} + \kappa\frac{Q}{A} = 0 \tag{66}$$

i.e.

$$\left(-\frac{\alpha Q^2}{A^2} + \frac{\beta}{2\rho A_o}A^{1/2}\right)\frac{\partial A}{\partial x} + \kappa\frac{Q}{A} = 0 \tag{67}$$

which by algebraic manipulation can be transformed to

$$\frac{\partial x}{\partial A} = \frac{-\frac{\alpha Q^2}{A} + \frac{\beta}{2\rho A_o}A^{3/2}}{-\kappa Q} \tag{68}$$

On integrating the last equation we obtain

$$x = \frac{\alpha Q^2 \ln A - \frac{\beta}{5\rho A_o} A^{5/2}}{\kappa Q} + C \tag{69}$$

where C is the constant of integration which can be determined from the boundary condition at $x = 0$ with $A = A_{in}$, that is

$$C = \frac{-\alpha Q^2 \ln A_{in} + \frac{\beta}{5\rho A_o} A_{in}^{5/2}}{\kappa Q} \tag{70}$$

i.e.

$$x = \frac{\alpha Q^2 \ln(A/A_{in}) - \frac{\beta}{5\rho A_o}\left(A^{5/2} - A_{in}^{5/2}\right)}{\kappa Q} \tag{71}$$

For practical reasons, this relation can be re-shaped and simplify to reduce the number of variables by the use of the second boundary condition at the outlet, as outlined in the following. When $x = L$, $A = A_{ou}$ where L is the vessel length and A_{ou} is the cross sectional area at the outlet, that is

$$L = \frac{\alpha Q^2 \ln(A_{ou}/A_{in}) - \frac{\beta}{5\rho A_o}\left(A_{ou}^{5/2} - A_{in}^{5/2}\right)}{\kappa Q} \tag{72}$$

which is a quadratic polynomial in Q i.e.

$$-\alpha \ln(A_{ou}/A_{in}) Q^2 + \kappa L Q + \frac{\beta}{5\rho A_o}\left(A_{ou}^{5/2} - A_{in}^{5/2}\right) = 0 \tag{73}$$

$$\alpha \ln(A_{in}/A_{ou}) Q^2 + \kappa L Q + \frac{\beta}{5\rho A_o}\left(A_{ou}^{5/2} - A_{in}^{5/2}\right) = 0 \tag{74}$$

with a solution given by

$$Q = \frac{-\kappa L \pm \sqrt{\kappa^2 L^2 - 4\alpha \ln(A_{in}/A_{ou}) \frac{\beta}{5\rho A_o}\left(A_{ou}^{5/2} - A_{in}^{5/2}\right)}}{2\alpha \ln(A_{in}/A_{ou})} \tag{75}$$

which is necessarily real for $A_{in} \geq A_{ou}$ which can always be satisfied for normal flow conditions by proper labeling. For a flow physically-consistent in direction with the pressure gradient, the root with the plus sign should be chosen, i.e.

$$Q = \frac{-\kappa L + \sqrt{\kappa^2 L^2 - 4\alpha \ln\left(A_{in}/A_{ou}\right) \frac{\beta}{5\rho A_o} \left(A_{ou}^{5/2} - A_{in}^{5/2}\right)}}{2\alpha \ln\left(A_{in}/A_{ou}\right)} \tag{76}$$

This, in essence, is a relation between flow rate and pressure drop (similar to the Hagen-Poiseuille law for rigid vessels) although for elastic vessels the flow rate, as given by Equation 76, does not depend on the pressure difference (as for rigid vessels) but on the actual inlet and outlet pressures.

Although Equation 76 may look a special case of Equation 63 as it involves only the vessel two end areas, A_{ou} may be assumed to be the area at any point along the vessel axis, with L being the distance form the vessel inlet to that point, and hence this relation can be used to verify the finite element solution at any point on the vessel.

10 Appendix B: Gauss Quadrature

In this appendix we list points and weights of Gauss quadrature for polynomials of order 1-10 which may not be easy to find.

Table 1: Gauss quadrature points and weights for polynomial order, p, 1-10 assuming a 0-1 master element.

p					Points					
1	0.500000000000000									
2	0.211324865405188	0.788675134594812								
3	0.112701665379259	0.500000000000000	0.887298334620741							
4	0.069431844202974	0.330009478207572	0.669990521792428	0.930568155797026						
5	0.046910077030669	0.230765344947159	0.500000000000000	0.769234655052841	0.953089922969331					
6	0.033765242898424	0.169395306766867	0.380690406958402	0.619309593041598	0.830604693233133	0.966234757101576				
7	0.025446043828621	0.129234407200303	0.297077424311301	0.500000000000000	0.702922575688699	0.870765592799697	0.974553956171380			
8	0.019855071751232	0.101666761293187	0.237233795041836	0.408282678752175	0.591717321247825	0.762766204958164	0.898333238706813	0.980144928248768		
9	0.015919880246187	0.081984446336682	0.193314283649705	0.337873288298095	0.500000000000000	0.662126711701905	0.806685716350295	0.918015553663318	0.984080119753813	
10	0.013046735741414	0.067468316655508	0.160295215850488	0.283302302935377	0.425562830509185	0.574437169490815	0.716697697064623	0.839704784149512	0.932531683344493	0.986953264258586
					Weights					
1	1.000000000000000									
2	0.500000000000000	0.500000000000000								
3	0.277777777777778	0.444444444444444	0.277777777777778							
4	0.173927422568727	0.326072577431273	0.326072577431273	0.173927422568727						
5	0.118463442528095	0.239314335249683	0.284444444444444	0.239314335249683	0.118463442528095					
6	0.085662246189585	0.180380786524069	0.233956967286345	0.233956967286345	0.180380786524069	0.085662246189585				
7	0.064742483084435	0.139852695744638	0.190915025252559	0.208979591836735	0.190915025252559	0.139852695744638	0.064742483084435			
8	0.050614268145188	0.111190517226687	0.156853322938943	0.181341891689181	0.181341891689181	0.156853322938943	0.111190517226687	0.050614268145188		
9	0.040637194180787	0.090324080347429	0.130305348201467	0.156173538520001	0.165119677500630	0.156173538520001	0.130305348201467	0.090324080347429	0.040637194180787	
10	0.033335672154344	0.074725674575291	0.109543181257991	0.134633359654998	0.147762112357376	0.147762112357376	0.134633359654998	0.109543181257991	0.074725674575291	0.033335672154344

11 Appendix C: Biological Parameters

In this appendix, we suggest some biologically-realistic values for the 1D flow model parameters in the context of simulating blood flow in large vessels.

1. Blood mass density (ρ): 1050 kg.m^{-3} [6, 7, 16, 25, 27–31].

2. Blood dynamic viscosity (μ): 0.0035 Pa.s [4, 6, 7, 15, 16, 20, 25, 27, 30, 32–34].

3. Young's elastic modulus (E): 100 kPa [4, 6, 13, 15, 16, 20, 22, 25, 27, 34–38]. Also see [25, 39] on shear modulus.

4. Vessel wall thickness (h_o): this, preferably, is vessel dependent, i.e. a fraction of the lumen or vessel radius according to some experimentally-established mathematical relation. The relation between wall thickness and vessel inner radius is somehow complex and vary depending on the type of vessel (e.g. artery or capillary). For arteries, the typical ratio of wall thickness to inner radius is about 0.1-0.15, and this ratio seems to go down in the capillaries and arterioles. Therefore a typical value of 0.1 seems reasonable [6, 16, 20–26].

5. Momentum correction factor (α): $4/3 = 1.33$ assuming Newtonian flow. A smaller value, e.g. 1.2, may be used to account for non-Newtonian shear-thinning effects [2, 3, 7, 16, 17, 20, 28, 40].

6. Time step (Δt): 1.0-0.1 ms [4, 7, 9, 10, 13, 15, 16, 18–20, 25, 29, 34, 41–45].

7. Pressure step (Δp): 1.0-5.0 kPa [4, 7, 16, 29].

8. Time of heart beat: 0.85 s assuming 70 beats per minute.

9. Poisson ratio (ς): 0.45 [2, 4, 6, 13, 22, 25, 27, 34, 37, 39, 41].