Using the stress function in the flow of generalized Newtonian fluids through conduits with non-circular or multiply connected cross sections

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Abstract

We investigate the possibility that the spatial dependency of stress in generalized Newtonian flow systems is a function of the applied pressure field and the conduit geometry but not of the fluid rheology. This possibility is well established for the case of a one-dimensional flow through simply connected regions, specifically tubes of circular uniform cross sections and plane thin slits. If it can also be established for the more general case of generalized Newtonian flow through non-circular or multiply connected geometries, such as the two-dimensional flow through conduits of rectangular or elliptical cross sections or the flow through annular circular pipes, then analytical or semi-analytical or highly accurate numerical solutions; regarding stress, rate of strain, velocity profile and volumetric flow rate; for these geometries can be obtained from the stress function, which can be easily obtained from the Newtonian case, in combination with the constitutive rheological relation for the particular non-Newtonian fluid, as done previously for the case of the one-dimensional flow through simply connected regions.

Keywords: fluid dynamics; rheology; generalized Newtonian fluid; non-Newtonian fluid; conduit with non-circular cross section; multiply connected flow region; universal stress function.

1 Introduction

The flow of generalized Newtonian fluids through conduits with circular and non-circular simply connected cross sections, such as those of elliptical or rectangular or triangular shape, and multiply connected cross sections like circular annulus, is commonplace in many biological systems and technological applications such as the transport of biological fluids in living organisms, the shipping of industrial liquids and the distribution of coolants in temperature regulating devices.

There are many investigations related to the flow of Newtonian fluids through conduits with non-circular cross sections or with multiple connectivity (e.g. [1–13]), and less on the flow of non-Newtonian fluids through such conduits (e.g. [14–19]). Several methods have been used in these investigations such as direct application of Laplace and Poisson equations, complex analysis, conformal mapping, variational methods, and numerical discretization techniques [3, 5, 7, 8, 14, 15, 20] as well as experimental examination [18].

In this paper we investigate the possibility that the stress function for generalized Newtonian fluids in multi-dimensional and multiple connectivity flow is universal, i.e. it is the same for Newtonian and non-Newtonian fluids, and hence the flow fields of generalized Newtonian fluids of non-Newtonian rheology through conduits with non-circular or multiply connected cross sections can be obtained by acquiring the stress, as a function of the spatial coordinates of the conduit cross section, from the Newtonian case. The stress function can then be utilized in combination with the rheological constitutive relation of the particular non-Newtonian fluid to obtain the flow field parameters which include the shear rate, as a function of the spatial coordinates of the cross section, and thereby the flow velocity profile and subsequently the volumetric flow rate. If this method is established, through the establishment of the universality of the stress function, it will be simple, general, reliable and easy to implement; moreover it can produce highly accurate solutions

for the flow of generalized Newtonian fluids of non-Newtonian rheology in those geometries.

The plan for this paper is that in section 2 we explain the method and the supporting argument for the universality of the stress function in general terms stating the relevant assumptions and restrictions. This will be followed in section 3 by a few examples of the stress function for conduits of non-circular or multiply connected cross sections which are obtained from the Newtonian flow case. The study will be concluded in section 4 with general briefing and discussion.

2 Method

Here, we assume a laminar, incompressible, steady state, rectilinear, isothermal, pressure-driven, fully-developed, creeping flow of a purely-viscous, time-independent generalized Newtonian fluid and hence history-dependent fluids, like viscoelastic and thixotropic, are excluded. We also exclude viscoplastic fluids, even if they are classified as generalized Newtonian fluids, due to the complications introduced by the presence of yield stress and the failure of the available viscoplastic models to account for these complications. The effect of any potential secondary flow under these conditions is negligible.

The effects of external body forces, such as gravity, as well as the edge effects at the entry and exit zones of the conduit are assumed insignificant. Dependencies on physical factors like temperature, which are not related to deformation, are also ignored assuming fixed conditions or negligible contribution from these factors. The flow is also assumed to be in shear mode with no significant extensional contributions. Moreover, the pressure is assumed to be a sole function of the axial dimension in the flow direction.

Concerning the type of conduit, we consider cylindrical ducts of uniform cross sections (i.e. having constant shape and size in the flow axial direction) with non-

circular simply or/and multiply connected cross section geometry. Rigid mechanical properties of the conduit wall are assumed and hence the conduit wall is not deformable under the considered range of pressure. As for the boundary conditions, no-slip at the conduit wall is assumed and hence a zero velocity condition at the fluid-solid interface is maintained.

We start from the momentum equation for the fluid flow which is given by [21, 22]

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p - \nabla . \boldsymbol{\tau} + \rho \mathbf{g} \tag{1}$$

where ρ is the fluid mass density, $\frac{D}{Dt}$ is the material derivative, $\mathbf{v}$ is the fluid velocity vector, t is the time, ∇ is the gradient operator, p is the pressure, $\boldsymbol{\tau}$ is the deviatoric or extra stress tensor, and $\mathbf{g}$ is the gravitational acceleration vector. Now, for a steady state creeping flow with negligible body forces we can neglect the time rate, convection and gravitational terms, and hence Equation 1 becomes

$$\nabla . \boldsymbol{\tau} = -\nabla p \tag{2}$$

This equation can be simplified for two dimensional flow in a Cartesian coordinates system, where the pressure gradient is in the stress-invariant z direction, into the following form of the z-component

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = -\frac{\partial p}{\partial z} \tag{3}$$

The stress in the last equation is dependent on the conduit spatial dimensions (geometry) and the applied pressure but is independent of the fluid rheology since the equation does not contain any rheological parameter. The independence of the stress function from the fluid rheology is clearly demonstrated by the absence of such dependency in the expressions of the stress function in the Newtonian flow

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case; examples of which are given in section 3.

For a given set of stress boundary conditions the solution of this equation should be unique. All we need to establish then is that the stress boundary conditions for generalized Newtonian fluids of non-Newtonian rheology are the same as those for the Newtonian rheology to conclude that the stress function (i.e. spatial dependency of stress over the conduit cross sectional region under steady flow conditions) for non-Newtonian rheology is the same as for Newtonian rheology. Now, the indifference of the stress boundary conditions between the non-Newtonian fluids and Newtonian fluids is well established for the case of one-dimensional simply connected flow which is represented by the case of circular cylindrical tubes and thin plane slits, as discussed in [23]. If the stress boundary conditions should depend on the rheology, the indifference cannot be established even in the case of one-dimensional simply connected flow. Hence, since we cannot see any particular consideration that can justify the restriction of such an indifference to the one-dimensional simply connected flow, we can assume that the indifference in the stress boundary conditions, and hence in the stress function as a whole, is applicable in general to multi-dimensional and multiply connected flows of the assumed type.

If the assumption that the stress function is the same for the Newtonian and non-Newtonian fluids is established, we can then obtain the stress function from the Newtonian flow case and use it for the non-Newtonian flow cases. In most circumstances, the stress function for the Newtonian flow is easily obtained analytically or semi-analytically (e.g. from infinite series solutions). As soon as the stress, τ, as a function of the spatial coordinates of the cross section is obtained, the rate of strain, γ, as a function of the spatial coordinates can be easily obtained from the fluid rheological constitutive relation which correlates the rate of shear strain to the shear stress as long as the relation can be put in the form $\gamma = \gamma(\tau)$ where the dependency of γ on τ can be explicit or implicit. If γ is an explicit function of τ, as

it is the case for example in the Ellis fluid (refer to Table 1), then γ can be obtained directly by a simple substitution in the rheological relation. If, on the other hand, γ is an implicit function of τ, as it is the case for example in the Cross fluid (refer to Table 1), then γ can be obtained numerically using a simple numerical solver based for instance on a bisection method. In both cases, the obtained rate of strain as a function of the spatial coordinates can be used to obtain the fluid velocity profile and subsequently the volumetric flow rate by consecutive integrations, as detailed in [23].

In Table 1 we present a few examples of the rheological constitutive relations for fluid models that can be used in conjunction with the stress function to obtain the flow field parameters. As indicated, for Carreau and Cross models, γ is given as an implicit function of τ and hence a simple numerical solver like bisection is required to obtain γ as a function of τ and hence as a function of the spatial coordinates of the conduit cross section. In the following section, we present some examples of analytical and semi-analytical stress functions obtained from the Newtonian flow case for some types of conduit geometry with non-circular or multiply connected cross sections.

3 Examples of Stress Functions for non-Circular and Multiply Connected Cross Sections

Here, we present some examples of non-circular and multiply connected cross sectional shapes of the flow conduits whose stress function can be obtained from their Newtonian flow case in the form of analytical or infinite series solutions (refer for instance to [5, 8, 20, 28–30]). Similar expressions of stress functions related to other cross sectional geometries, which can be derived from the Newtonian flow case, can also be obtained from references like the above.

Table 1: The rate of shear strain, γ, as a function of shear stress, τ, for a sample of five non-Newtonian fluids [21, 24–27] that can be employed in the investigated stress function approach for generalized Newtonian flow in conduits with non-circular or multiply connected cross sectional geometries. For Carreau and Cross models, γ is given as an implicit function of τ. The meanings of the symbols are given in Nomenclature § 5.

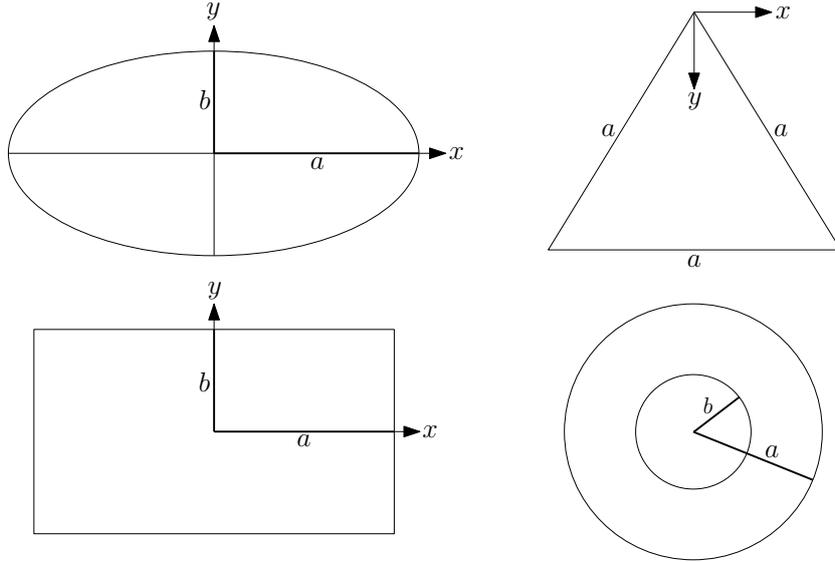| Model	Rate of Shear Strain
Power Law	$\gamma = \sqrt[n]{\frac{\tau}{k}}$
Ellis	$\gamma = \frac{\tau}{\mu_e}\left[1 + \left(\frac{\tau}{\tau_h}\right)^{\alpha-1}\right]$
Ree-Eyring	$\gamma = \frac{\tau_c}{\mu_r}\sinh\left(\frac{\tau}{\tau_c}\right)$
Carreau	$\gamma\left[\mu_i + (\mu_0 - \mu_i)\left(1 + \lambda^2\gamma^2\right)^{(n-1)/2}\right] = \tau$
Cross	$\gamma\left[\mu_i + \frac{\mu_0 - \mu_i}{1 + \lambda^m\gamma^m}\right] = \tau$

Figure 1: Schematics of the given examples of cross sectional shapes of non-circular and multiply connected conduits.

For a conduit centered on the origin of coordinates with an elliptical cross section of semi-major axis a along the x axis and semi-minor axis b along the y axis (refer to Figure 1) we have

$$\tau_{xz} = -\frac{\partial p}{\partial z}\frac{b^2 x}{a^2 + b^2} \tag{4}$$

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$$\tau_{yz} = -\frac{\partial p}{\partial z}\frac{a^2 y}{a^2 + b^2} \tag{5}$$

For a conduit with an equilateral triangular cross section of side a in the coordinates system given in Figure 1 we have

$$\tau_{xz} = -\frac{\partial p}{\partial z}\frac{\sqrt{3}}{a}\left(\frac{a\sqrt{3}}{2} - y\right)x \tag{6}$$

$$\tau_{yz} = -\frac{\partial p}{\partial z}\frac{1}{2\sqrt{3}a}\left(-3x^2 + 3y^2 - a\sqrt{3}y\right) \tag{7}$$

For a conduit centered on the origin of coordinates with a rectangular cross section of half length a along the x axis and half width b along the y axis (refer to Figure 1) we have

$$\tau_{xz} = -\frac{\partial p}{\partial z}\frac{8b}{\pi^2}\sum_{i=1,3,5,\ldots}^{\infty}\frac{(-1)^{(i-1)/2}}{i^2}\frac{\sinh\left(i\pi x/2b\right)}{\cosh\left(i\pi a/2b\right)}\cos\left(i\pi y/2b\right) \tag{8}$$

$$\tau_{yz} = -\frac{\partial p}{\partial z}\left[y - \frac{8b}{\pi^2}\sum_{i=1,3,5,\ldots}^{\infty}\frac{(-1)^{(i-1)/2}}{i^2}\frac{\cosh\left(i\pi x/2b\right)}{\cosh\left(i\pi a/2b\right)}\sin\left(i\pi y/2b\right)\right] \tag{9}$$

All these equations can be verified by substituting these expressions into Equation 3 which produces an identity in all cases.

Similarly, for a concentric circular annulus with an inner radius b and an outer radius a (refer to Figure 1), using a cylindrical coordinates system whose z-axis is oriented along the annulus axis of symmetry, we have

$$\tau_{rz} = -\frac{\partial p}{\partial z}\frac{1}{4}\left[2r + \frac{(a^2 - b^2)}{\ln\left(b/a\right)}\frac{1}{r}\right] \tag{10}$$

$$\tau_{\theta z} = 0 \tag{11}$$

The latter can be verified by substitution in the z-component of the cylindrical form of Equation 2, that is

$$\frac{1}{r}\frac{\partial\left(r\tau_{rz}\right)}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta} = -\frac{\partial p}{\partial z} \tag{12}$$

4 Conclusions

In this study we propose extending the stress function approach, which was established previously [23] for the case of one-dimensional simply connected flows, to obtain analytical or semi-analytical or highly accurate numerical solutions for the flow of generalized Newtonian fluids in conduits with non-circular or with multiply connected cross sections. The investigation is based on the assumption that the stress function for generalized Newtonian fluids is the same for Newtonian and non-Newtonian rheologies.

If this assumption, which is established for the one-dimensional simply connected flow geometries such as circular pipes and plane slits, can be established for the cases of non-circular and multiply connected flow geometries then the stress function, which normally can be easily obtained from the Newtonian flow case analytically or by series solutions or by other means, can be employed in combination with the non-Newtonian rheological constitutive relations that correlate, explicitly or implicitly, the rate of strain to the shear stress to obtain the rate of strain as a function of the spatial coordinates and hence the flow velocity profile and the volumetric flow rate.

In previous studies we investigated the optimization of total stress [31–34] and the minimization of transport energy [35–37] in the fluid flow phenomena. If these principles can be established for the flow through conduits with non-circular or with multiply connected cross sections then this will add more support to the proposal presented in the current study of the universality of the stress function since these principles are indifferent to the fluid rheology.

5   Nomenclature

a, b conduit geometric parameters

$\frac{D}{Dt}$ material derivative

g   gravitational acceleration vector

k viscosity coefficient in power law model

m indicial parameter in Cross model

n flow behavior index in power law and Carreau models

p pressure

r radius

t time

v   fluid velocity vector

x, y, z spatial coordinates

∇ gradient operator

α indicial parameter in Ellis model

γ rate of shear strain

θ azimuthal angle in cylindrical coordinates system

λ characteristic time constant in Carreau and Cross models

μ_0 zero-shear viscosity in Carreau and Cross models

μ_e low-shear viscosity in Ellis model

μ_i infinite-shear viscosity in Carreau and Cross models

μ_r characteristic viscosity in Ree-Eyring model

ρ fluid mass density

τ shear stress

$\boldsymbol{\tau}$ extra stress tensor

τ_c characteristic shear stress in Ree-Eyring model

τ_h shear stress when viscosity equals $\frac{\mu_e}{2}$ in Ellis model

τ_{xz}, τ_{yz} shear stress components

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