Bernoulli's law for an adiabatic ideal gas flow

By

Mugur Răuţ

Abstract. The aim of this paper is to establish a synthetic form for Bernoulli's law concerning an adiabatic ideal gas flow and to apply it to the expanding universe. Unfortunately, the obtained law is not applicable to the expansion of the universe, but it is applicable only to ordinary cases of fluid mechanics.

Introduction

A simple law that can describe, in terms of specific physical quantities, for instance temperature or pressure, an adiabatic flow of an ideal gas, with a, particularly, variable velocity, does not exist.

There is, in exchange, the law which links the cooling or the warming of a fluid in motion through a small section by the Joule-Thomson effect, (Reif, 1965).

In the following, we will try to obtain such a law. We will use a particular form of the Bernoulli's law, (Resnick & Halliday, 1960), and an expression that describes an adiabatic transform of an ideal gas, belonging as it is well-known to Poisson, (Bailyn, 1994).

Then, logically, we will discuss about the field of application of the result.

Theoretical model

For the beginning, let consider the Bernoulli's law, in which the gravitational pressure is null, of an ideal fluid flow:

(1) $$\frac{\rho v^2}{2}+p=\text{const.},$$

where ρ is, formally, the ideal fluid mass density, particularly an ideal gas mass density, v is the flow velocity and p is the static pressure.

Accordingly, if this ideal gas flow is adiabatic, then the complete description of it must include the law:

(2) $$T\,p^{(1-\gamma)/\gamma}=const.,$$

where T is the gas temperature, p is its static pressure and γ is the adiabatic exponent, $\gamma = c_p/c_v$.

Let's apply now to equation (1) the first derivative with respect to time. We obtain:

(3) $$\frac{v^2}{2}\frac{d\rho}{dt} + v\,\rho\,\frac{dv}{dt} + \frac{dp}{dt} = 0.$$

If the gas is ideal, then it is incompressible too, i.e. $d\rho/dt = 0$, variations of density with respect to time are zero. Considering this observation, equation (3) becomes:

(4) $$v\,\rho\frac{dv}{dt} + \frac{dp}{dt} = 0.$$

Then we must apply the first derivative with respect to time to equation (2) also:

$$p^{(1-\gamma)/\gamma}\,\frac{dT}{dt} + \frac{1-\gamma}{\gamma}\,T\,p^{(1-\gamma)/\gamma-1}\frac{dp}{dt} = 0.$$

After reduction with quantity $p^{(1-\gamma)/\gamma}$and introduction of remaining result in equation (4) we will find:

(5) $$\frac{dT}{dt} + \frac{1-\gamma}{\gamma}\,\frac{T}{p}\left(-\rho\,v\,\frac{dv}{dt}\right) = 0.$$

The ideal gas state equation is, (Perrot, 1998):

$$p = \rho\,(\gamma-1)\,e,$$

where

$$e = c_v\,T$$

Is the internal energy of the ideal gas. The report of the last two equations is:

$$\frac{\rho}{p} = \frac{1}{\gamma-1}(c_v\,T)^{-1}.$$

Taking into account this last result within equation (6), and the fact that $\gamma = c_p/c_v$, after the obvious simplifications, we have:

(6) $$\frac{dT}{dt} + \frac{v}{c_p}\,\frac{dv}{dt} = 0.$$

Equation (7) is a synthesis of equation (1) and (2), with a simpler form, is the essence of the two equations from which it is resulted.

In order to obtain the simplest form of this equation, we seek, hence, to eliminate the time-dependence. Accordingly, we multiply and divide with 2 the second term of equation (7). Next, if we consider $dv/dt = const.$ and $dc_p/dt = 0$, it results:

$$\frac{d}{dt}\left(T + \frac{v^2}{2c_p}\right) = 0,$$

equation that can be written more simple as:

(7) $$T + \frac{v^2}{2c_p} = const.,$$

which is the synthetic final form, the most simple resulting equation from (1) and (2), under conditions it was established.

Discussions

Equation (8) is a simplified synthetic writing of both equations (1) and (2), under conditions specified during its deduction. At first sight equation (8) sets a relation between gas temperature and the velocity of the gas flow. The higher is the velocity the lower is the temperature. In fluid mechanics, in the case of an ordinary adiabatic expansion, this would be translated, in a first phase, by a decrease of temperature. If the expansion would take place, in a first phase, with $a = const. > 0$, then, in a second phase, with $a = const. < 0$, the overall effect would be $T = const.$

The same conclusion would be valid in the case of an adiabatic compression. So that, it's hard to imagine an adiabatic flow that it produce a decrease or an increase of temperature only, in order to relate equation (8) with some practical applications. Maybe the solution is, in the second phase, when the velocity must fall more slowly than it was increased in the first phase, to put in contact the gas with a thermostat. Thus we could design a thermal machine whose operation have the result the cooling or the warming of a gas. But, under normal conditions of an adiabatic transform, in both ways the overall effect is null.

Let's see now, however, if equation (8) is suitable to the expansion of the universe. The universe is conceived as an adiabatic system as a whole, which is expanding with a constant, but time-dependent with a slow variability of epoch speed, the Hubble's constant. The actual speed, on each mega parsec, through which the expansion it is manifesting itself is 69,32$\pm$0,8 Km/s, (Bennet et al., 2013). If we approximate the space as a superfluid, (Liberati & Maccione, 2014) and we want to calculate the speed of its expansion, in the event of cooling from 2,7 K as the universe has now, to 1,7 K, in the future, we have an unpleasant surprise. Equation (8) does not apply in this case, because the value of specific heat at constant pressure is unknown.

In the case of a superfluid, c_p varies very much with temperature. Liquid helium, for instance, has $c_p \to 0$ for T=1 K and $c_p \to 15 J/Kg\,K$ at T=2,2 K, (Keesom & Keesom, 1935). Except this fact, even if we empirical approximate the specific heat at constant pressure for space, we have no clue about how this value varies with temperature, as any other superfluid. Our ideal gas, from which space is made of, should, logically, have a constant c_p with respect to temperature. Otherwise, if this physical quantity would vary decreasingly with respect to temperature, we get, to the situation when a decrease of one degree of the universe's temperature to cause an expansion with a smaller speed. What is contradictory with the actual data concerning the accelerated expansion of the universe.

So, until now, the overall problem in this case is the same as in the previous case, the unknown specific heat at constant pressure.

Nor in relativistic case the things seem to look better. Here, unlike the previous cases, we are on a ground of pure speculation. Because equation (8) was deduced only for non-relativistic speeds and a rigorous deduction of equation (8) for relativistic speeds assumes to consider the relativistic version of equation (1), which I acknowledge that does not exist, then the solution is to approximate empirically the equation (8) for relativistic speeds. We propose the formula:

$$T + \frac{c^2}{c_p}\left(1 - \frac{1}{\sqrt{1-v^2/c^2}}\right) = const. \tag{8}$$

where the second term tends to v^2/c_p when $v \ll c$. As the temperature concerns, the result of the microscopic process of a thermal movement, we adopted the same position as P. T. Landsberg, (Landsberg, 1967), in this matter: it does not vary in relativistic way.

With equation (9) we can treat inflation at $v \cong c$, or other cases corresponding to $v \ll c$ too. However, there are only conflicting results, due to the fact that the same specific heat at constant pressure is unknown. Now, we can't approximate the space as a superfluid, we better think of it as a very high energy "superplasma" with an unknown c_p. Either the specific heat at constant pressure varies with temperature or not, we are reaching to contradictory results, one way or another. Relativistic speeds after inflation or smaller temperatures as in standard theory, (deGrasse & Goldsmith, 2004). This situation illustrates that we are on a purely speculative ground and nothing more.

Conclusions

In this paper we obtained a synthetic form of Bernoulli's law concerning an adiabatic ideal gas flow. Unfortunately, the obtained law is not applicable to the expansion of the universe, but it is applicable only to ordinary cases of fluid mechanics.

There is the possibility to create a thermal machine capable to cool or warm a gas and more, and to be well described by the law we obtain. He hope there will be other applications of it too.

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