A Solution to the Problem of Apollonius Using Vector Dot Products

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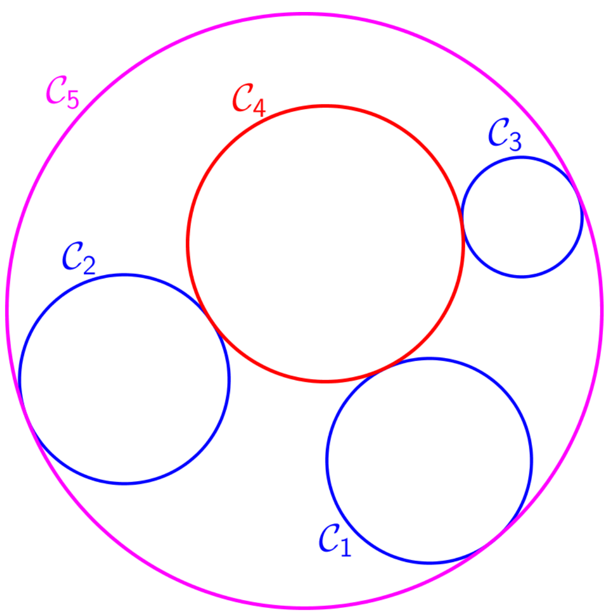
Abstract

To the collections of problems solved via Geometric Algebra (GA) in [1]-[11], this document adds a solution, using only dot products, to the Problem of Apollonius. The solution is provided for completeness and for contrast with the GA solutions presented in [4].



The Problem of Apollonius: *Given three coplanar circles, construct the circles that are tangent to all three of them, simultaneously.*

1 Introduction

The Problem of Apollonius reads,

> *"Given three coplanar circles, construct the circles that are tangent to all three of them, simultaneously."*

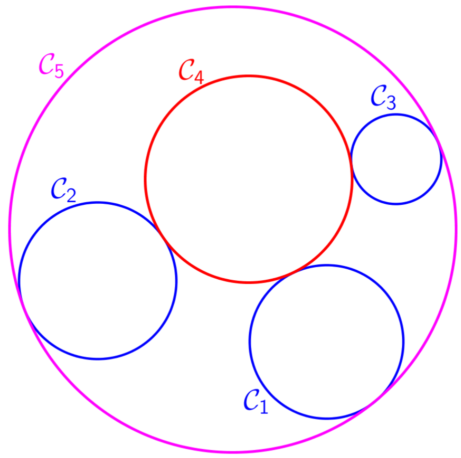


Figure 1: The Problem of Apollonius: *"Given three coplanar circles (e.g. $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$), construct the circles that are tangent to all three of them, simultaneously."* There are eight solution circles, two of which ($\mathcal{C}4$ and $\mathcal{C}_5$) are shown here.

Because this problem, along with several of its limiting cases, has been treated at length in the references, the solution presented here will leave the details to the reader. We will identify only the solution circle that encloses none of the givens. (I.e., $\mathcal{C}_4$ in Fig. 1.)

2 Solution

2.1 Defining Variables that are Amenable to Treatment via GA

The variables that we will use are shown in Fig. 2.

2.2 Solution Strategy

Our strategy will be to express the vector $\boldsymbol{c}_4$ in two ways, from which we will then derive expressions for $\hat{\boldsymbol{t}}$ and r_4.

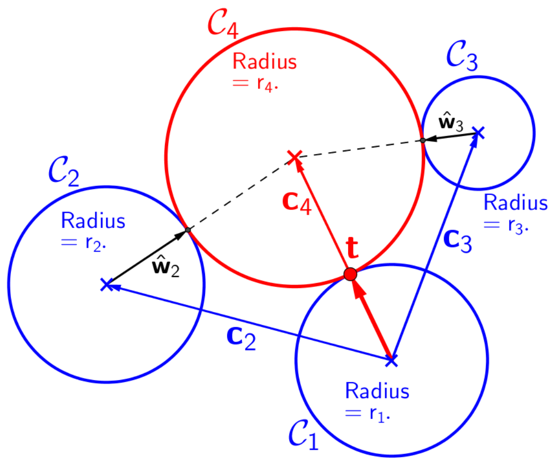


Figure 2: Definition of variables used for identifying the solution circle $\mathcal{C}_4$.

2.3 Formulating and Solving the Equations

We begin by equating two expressions for the vector $\boldsymbol{c}_4$

$$\boldsymbol{c}_4 = (r_1 + r_4)\,\hat{\boldsymbol{t}} = \boldsymbol{c}_2 + (r_2 + r_4)\,\hat{\boldsymbol{w}}_2$$
$$\therefore \quad (r_1 + r_4)\,\hat{\boldsymbol{t}} - \boldsymbol{c}_2 = (r_1 + r_4)\,\hat{\boldsymbol{w}}_2.$$

Now, we'll square both sides to eliminate the unknown vector $\hat{\boldsymbol{w}}_2$, after which we'll solve for r_4:

$$r_4 = \frac{{c_2}^2 + {r_1}^2 - {r_1}^2 - 2r_1\boldsymbol{c}_2 \cdot \hat{\boldsymbol{t}}}{2\left(r_2 - r_1 + \boldsymbol{c}_2 \cdot \hat{\boldsymbol{t}}\right)}. \tag{1}$$

Similarly,

$$\boldsymbol{c}_4 = (r_1 + r_4)\,\hat{\boldsymbol{t}} = \boldsymbol{c}_3 + (r_3 + r_4)\,\hat{\boldsymbol{w}}_3;$$
$$(r_1 + r_4)\,\hat{\boldsymbol{t}} - \boldsymbol{c}_3 = (r_3 + r_4)\,\hat{\boldsymbol{w}}_3; \text{ and}$$
$$r_4 = \frac{{c_3}^2 + {r_3}^2 - {r_1}^2 - 2r_1\boldsymbol{c}_3 \cdot \hat{\boldsymbol{t}}}{2\left(r_3 - r_1 + \boldsymbol{c}_3 \cdot \hat{\boldsymbol{t}}\right)}. \tag{2}$$

Equating the expressions for r_4 from Eqs. (1) and (2),

$$\frac{{c_2}^2 + {r_1}^2 - {r_1}^2 - 2r_1\boldsymbol{c}_2 \cdot \hat{\boldsymbol{t}}}{r_2 - r_1 + \boldsymbol{c}_2 \cdot \hat{\boldsymbol{t}}} = \frac{{c_3}^2 + {r_3}^2 - {r_1}^2 - 2r_1\boldsymbol{c}_3 \cdot \hat{\boldsymbol{t}}}{r_3 - r_1 + \boldsymbol{c}_3 \cdot \hat{\boldsymbol{t}}}.$$

After cross-multiplying, simplifying, and rearranging as explained in the references, we obtain

$$\underbrace{\left\{\left[{c_3}^2 - (r_3 - r_1)^2\right]\boldsymbol{c}_2 - \left[{c_2}^2 - (r_2 - r_1)^2\right]\boldsymbol{c}_3\right\}}_{\textit{We'll call this vector "}\boldsymbol{z}\textit{".}} \cdot \hat{\boldsymbol{t}} = (r_3 - r_1)\left({c_2}^2 - {r_2}^2\right)$$
$$- (r_2 - r_1)\left({c_3}^2 - {r_3}^2\right)$$
$$+ (r_3 - r_2)^2\,{r_1}^2. \tag{3}$$

The geometric interpretation of Eq. (3) is shown in Fig. 3: Our solution method has found the points of tangency of two tangent circles. One of them encloses all three of the givens, while the other encloses none of them.

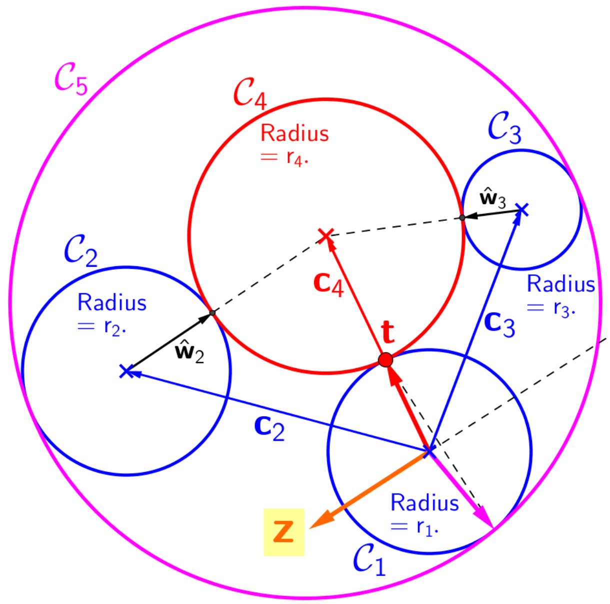


Figure 3: The geometric interpretation of Eq. (3): Our solution method has found the points of tangency of two tangent circles. One of them encloses all three of the givens, while the other encloses none of them.

References

[1] J. Smith, 2015, "From two dot products, determine an unknown vector using Geometric (Clifford) Algebra", https://www.youtube.com/watch?v=2cqDVtHcCoE .

[2] J. Smith, 2016, "Rotations of Vectors Via Geometric Algebra: Explanation, and Usage in Solving Classic Geometric 'Construction' Problems" (Version of 11 February 2016). Available at http://vixra.org/abs/1605.0232 .

[3] J. Smith, 2016,"Solution of the Special Case 'CLP' of the Problem of Apollonius via Vector Rotations using Geometric Algebra". Available at http://vixra.org/abs/1605.0314.

[4] J. Smith, 2016,"The Problem of Apollonius as an Opportunity for Teaching Students to Use Reflections and Rotations to Solve Geometry Problems via Geometric (Clifford) Algebra". Available at http://vixra.org/abs/1605.0233.

[5] J. Smith, 2016,“A Very Brief Introduction to Reflections in 2D Geometric Algebra, and their Use in Solving ‘Construction’ Problems”. Available at http://vixra.org/abs/1606.0253.

[6] J. Smith, 2016,“Three Solutions of the LLP Limiting Case of the Problem of Apollonius via Geometric Algebra, Using Reflections and Rotations”. Available at http://vixra.org/abs/1607.0166.

[7] J. Smith, 2016,“Simplified Solutions of the CLP and CCP Limiting Cases of the Problem of Apollonius via Vector Rotations using Geometric Algebra”. Available at http://vixra.org/abs/1608.0217.

[8] J. Smith, 2016, “Additional Solutions of the Limiting Case ‘CLP’ of the Problem of Apollonius via Vector Rotations using Geometric Algebra”. Available at http://vixra.org/abs/1608.0328.

[9] J. Smith, 2016, “Some Solution Strategies for Equations that Arise in Geometric (Clifford) Algebra”. Available at http://vixra.org/abs/1610.0054.

[10] J. Smith, “Geometric Algebra of Clifford, Grassman, and Hestenes”, https://www.youtube.com/playlist?list=PL4P20REbUHvwZtd1tpuHkziU9rfgY2xOu.

[11] J. Smith, “Geometric Algebra (Clifford Algebra)”, https://www.geogebra.org/m/qzDtMW2q#chapter/0.