Some Solution Strategies for Equations that Arise in Geometric (Clifford) Algebra

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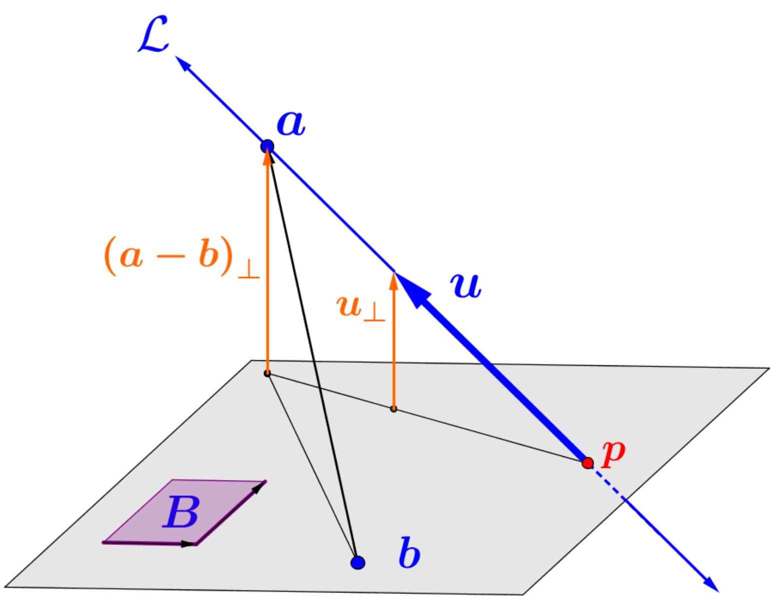
Abstract

Drawing mainly from Hestenes's *New Foundations for Classical Mechanics* ([2]), this document presents, explains, and discusses common solution strategies. Included are a list of formulas and a guide to nomenclature.



"Find the point of intersection of the line $(\boldsymbol{x}-\boldsymbol{a})\wedge\boldsymbol{u}=0$ *and the plane* $(\boldsymbol{y}-\boldsymbol{b})\wedge\boldsymbol{B}=0$*."*

1 Nomenclature and other comments

- Lower-case Greek letters (e.g, α, β, γ) represent scalars.
- Bold, lower-case Roman letters ($\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ etc.) represent vectors.
  EXCEPTION: The bolded, lower-case "$\boldsymbol{i}$" is the unit bivector.
- Bold, capitalized Roman letters ($\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$) represent bivectors.
  EXCEPTION: The bolded, upper-case "$\boldsymbol{I}$" is the pseudoscalar for the dimensionality of GA in use. When necessary, it is subscripted with that dimensionality. For example, $\boldsymbol{I}_4$ is the pseudoscalar for 4-dimensional GA. However, "$\boldsymbol{I}_2$" is represented by the bolded, lower-case "$\boldsymbol{i}$", and "$\boldsymbol{I}_3$" is represented by the non-bolded, lower-case "i".
- As noted above, the non-bolded, lower-case "i" represents the 3D GA pseudoscalar.
- Upper-case, non-bolded Roman letters (A, B, C) represent multivectors.

2 GA formulas and identities

The Geometric Product, and Relations Derived from It

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,
$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
$\boldsymbol{b} \wedge \boldsymbol{a} = -\boldsymbol{a} \wedge \boldsymbol{b}$
$\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b}$
$\boldsymbol{ba} = \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{a} \wedge \boldsymbol{b}$

$\boldsymbol{ab} + \boldsymbol{ba} = 2\boldsymbol{a} \cdot \boldsymbol{b}$
$\boldsymbol{ab} - \boldsymbol{ba} = 2\boldsymbol{a} \wedge \boldsymbol{b}$
$\boldsymbol{ab} = 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{ba}$
$\boldsymbol{ab} = 2\boldsymbol{a} \wedge \boldsymbol{b} - \boldsymbol{ba}$

For a vector $\boldsymbol{v}$ and a bivector $\boldsymbol{B}$,
$\boldsymbol{aB} = \boldsymbol{a} \cdot \boldsymbol{B} + \boldsymbol{a} \wedge \boldsymbol{B}$
$\boldsymbol{B} \cdot \boldsymbol{a} = -\boldsymbol{a} \cdot \boldsymbol{B}$
$\boldsymbol{B} \wedge \boldsymbol{a} = \boldsymbol{a} \wedge \boldsymbol{B}$

To clarify: When a vector $\boldsymbol{a}$ lies within the plane defined by $\boldsymbol{B}$, $\boldsymbol{a} \wedge \boldsymbol{B} = 0$; $\therefore \boldsymbol{aB} = \boldsymbol{a} \cdot \boldsymbol{B}$. In that case, $\boldsymbol{Ba} = -\boldsymbol{aB}$ because $\boldsymbol{B} \cdot \boldsymbol{a} = -\boldsymbol{a} \cdot \boldsymbol{B}$. This is why (as we'll see later) $\boldsymbol{ia} = -\boldsymbol{ai}$ in 2D GA.

For a vector $\boldsymbol{v}$ and the bivector $\boldsymbol{a} \wedge \boldsymbol{b}$,
$\boldsymbol{v} \cdot (\boldsymbol{a} \wedge \boldsymbol{b}) = (\boldsymbol{v} \cdot \boldsymbol{a})\, \boldsymbol{b} - (\boldsymbol{v} \cdot \boldsymbol{b})\, \boldsymbol{a}$.
$(\boldsymbol{a} \wedge \boldsymbol{b}) \cdot \boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{b})\, \boldsymbol{a} - (\boldsymbol{v} \cdot \boldsymbol{a})\, \boldsymbol{b}$.

For any bivector $\boldsymbol{a} \wedge \boldsymbol{b}$,
$[\boldsymbol{a} \wedge \boldsymbol{b}]^2 = (\boldsymbol{a} \cdot \boldsymbol{b})^2 - a^2 b^2 = -\|\boldsymbol{a} \wedge \boldsymbol{b}\|^2$.

Definitions of Inner and Outer Products ([1] , p. 101.)

The inner product
The inner product of a j-vector A and a k-vector B is

$A \cdot B = \langle AB \rangle_{k-j}$. Note that if $j>k$, then the inner product doesn't exist. However, in such a case $B \cdot A = \langle BA \rangle_{j-k}$ does exist.

The outer product

The outer product of a j-vector A and a k-vector B is
$A \wedge B = \langle AB \rangle_{k+j}$.

Relations Involving the Outer Product and the Unit Bivector, i.

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ lying within the plane of the bivector $\boldsymbol{i}$,
$\boldsymbol{ia} = -\boldsymbol{ai}$
$\boldsymbol{a} \wedge \boldsymbol{b} = [(\boldsymbol{ai}) \cdot \boldsymbol{b}]\, \boldsymbol{i} = -[\boldsymbol{a} \cdot (\boldsymbol{bi})]\, \boldsymbol{i} = -\boldsymbol{b} \wedge \boldsymbol{a}$.
See also the earlier comments regarding $\boldsymbol{a} \cdot \boldsymbol{B} = -\boldsymbol{B} \cdot \boldsymbol{a}$.

Some properties of the pseudoscalar $\boldsymbol{I}_n$ ([1], p. 105)
${\boldsymbol{I}_n}^2 = (-1)^{n(n-1)/2}$.
${\boldsymbol{I}_n}^{-1} = (-1)^{n(n-1)/2}\, \boldsymbol{I}_n$.
For vector $\boldsymbol{v}$ and pseudoscalar $\boldsymbol{I}_n$,
$\boldsymbol{v}\boldsymbol{I}_n = (-1)^{n-1}\, \boldsymbol{I}_n \boldsymbol{v}$.

Equality of Multivectors

For any two multivectors $\mathcal{M}$ and $\mathcal{N}$,
$\mathcal{M} = \mathcal{N}$ if and only if for all k, $\langle \mathcal{M} \rangle_k = \langle \mathcal{N} \rangle_k$.

Formulas Derived from Projections of Vectors and Equality of Multivectors

Any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written in the form of "Fourier expansions" with respect to a third vector, $\boldsymbol{v}$:

$\boldsymbol{a} = (\boldsymbol{a} \cdot \hat{\boldsymbol{v}})\, \hat{\boldsymbol{v}} + [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\, \hat{\boldsymbol{v}}\boldsymbol{i}$ and $\boldsymbol{b} = (\boldsymbol{b} \cdot \hat{\boldsymbol{v}})\, \hat{\boldsymbol{v}} + [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\, \hat{\boldsymbol{v}}\boldsymbol{i}$.

Using these expansions,

$$\boldsymbol{ab} = \{(\boldsymbol{a} \cdot \hat{\boldsymbol{v}})\, \hat{\boldsymbol{v}} + [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\, \hat{\boldsymbol{v}}\boldsymbol{i}\}\, \{(\boldsymbol{b} \cdot \hat{\boldsymbol{v}})\, \hat{\boldsymbol{v}} + [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\, \hat{\boldsymbol{v}}\boldsymbol{i}\}$$

Equating the scalar parts of both sides of that equation,
$\boldsymbol{a} \cdot \boldsymbol{b} = [\boldsymbol{a} \cdot \hat{\boldsymbol{v}}]\, [\boldsymbol{b} \cdot \hat{\boldsymbol{v}}] + [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\, [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]$, and
$\boldsymbol{a} \wedge \boldsymbol{b} = \{[\boldsymbol{a} \cdot \hat{\boldsymbol{v}}]\, [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})] - [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\, [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\}\, \boldsymbol{i}$.

Also, $a^2 = [\boldsymbol{a} \cdot \hat{\boldsymbol{v}}]^2 + [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]^2$, and $b^2 = [\boldsymbol{b} \cdot \hat{\boldsymbol{v}}]^2 + [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]^2$.

Reflections of Vectors, Geometric Products, and Rotation operators

For any vector $\boldsymbol{a}$, the product $\hat{\boldsymbol{v}}\boldsymbol{a}\hat{\boldsymbol{v}}$ is the reflection of $\boldsymbol{a}$ with respect to the direction $\hat{\boldsymbol{v}}$.

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\hat{\boldsymbol{v}}\boldsymbol{ab}\hat{\boldsymbol{v}} = \boldsymbol{ba}$, and $\boldsymbol{vabv} = v^2\boldsymbol{ba}$. Therefore, $\hat{\boldsymbol{v}}e^{\theta\boldsymbol{i}}\hat{\boldsymbol{v}} = e^{-\theta\boldsymbol{i}}$, and $\boldsymbol{v}e^{\theta\boldsymbol{i}}\boldsymbol{v} = v^2e^{-\theta\boldsymbol{i}}$.

A useful relationship that is valid only in plane geometry: $\boldsymbol{abc} = \boldsymbol{cba}$.

Here is a brief proof:

$$\begin{aligned} \boldsymbol{abc} &= \{\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b}\}\,\boldsymbol{c} \\ &= \{\boldsymbol{a}\cdot\boldsymbol{b} + [(\boldsymbol{ai})\cdot\boldsymbol{b}]\,\boldsymbol{i}\}\,\boldsymbol{c} \\ &= (\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{c} + [(\boldsymbol{ai})\cdot\boldsymbol{b}]\,\boldsymbol{ic} \\ &= \boldsymbol{c}\,(\boldsymbol{a}\cdot\boldsymbol{b}) - \boldsymbol{c}\,[(\boldsymbol{ai})\cdot\boldsymbol{b}]\,\boldsymbol{i} \\ &= \boldsymbol{c}\,(\boldsymbol{a}\cdot\boldsymbol{b}) + \boldsymbol{c}\,[\boldsymbol{a}\cdot(\boldsymbol{bi})]\,\boldsymbol{i} \\ &= \boldsymbol{c}\,(\boldsymbol{b}\cdot\boldsymbol{a}) + \boldsymbol{c}\,[(\boldsymbol{bi})\cdot\boldsymbol{a}]\,\boldsymbol{i} \\ &= \boldsymbol{c}\,\{\boldsymbol{b}\cdot\boldsymbol{a} + [(\boldsymbol{bi})\cdot\boldsymbol{a}]\,\boldsymbol{i}\} \\ &= \boldsymbol{c}\,\{\boldsymbol{b}\cdot\boldsymbol{a} + \boldsymbol{b}\wedge\boldsymbol{a}\} \\ &= \boldsymbol{cba}. \end{aligned}$$

3 The Equations, and Their Solutions

Unless stated otherwise, the unknown in each equation is $\boldsymbol{x}$.

3.1 Find the point of intersection of the lines $\boldsymbol{x} = \boldsymbol{a} + \alpha\boldsymbol{u}$ and $\boldsymbol{y} = \boldsymbol{b} + \beta\boldsymbol{v}$.

We'll solve this problem by identifying the values of scalars α and β for the vector $\boldsymbol{p}$ to the point of origin. First, we write

$$\boldsymbol{p} = \boldsymbol{a} + \alpha_p\boldsymbol{u} = \boldsymbol{b} + \beta_p\boldsymbol{v},$$

where α_p and β_p are the values of α and β for the point of intersection $\boldsymbol{p}$, specifically . Next, we use the exterior product with $\boldsymbol{v}$ to eliminate the β term, so that we may solve for α_p:

$$\begin{aligned} (\boldsymbol{a} + \alpha_p\boldsymbol{u})\wedge\boldsymbol{v} &= (\boldsymbol{b} + \beta_p\boldsymbol{v})\wedge\boldsymbol{v} \\ \therefore \quad \alpha_p &= [(\boldsymbol{b}-\boldsymbol{a})\wedge\boldsymbol{v}]\,(\boldsymbol{u}\wedge\boldsymbol{v})^{-1}. \end{aligned}$$

An alternative solution for α_p, starting from $(\boldsymbol{a} + \alpha_p\boldsymbol{u})\wedge\boldsymbol{v} = (\boldsymbol{b} + \beta_p\boldsymbol{v})\wedge\boldsymbol{v}$:

$$\begin{aligned} \alpha_p\boldsymbol{u}\wedge\boldsymbol{v} &= (\boldsymbol{b}-\boldsymbol{a})\wedge\boldsymbol{v}; \\ \alpha_p\,[(\boldsymbol{ui})\cdot\boldsymbol{v}]\,\boldsymbol{i} &= \{[(\boldsymbol{b}-\boldsymbol{a})\,\boldsymbol{i}]\cdot\boldsymbol{v}\}\,\boldsymbol{i}; \\ \therefore \alpha_p &= \frac{[(\boldsymbol{b}-\boldsymbol{a})\,\boldsymbol{i}]\cdot\boldsymbol{v}}{(\boldsymbol{ui})\cdot\boldsymbol{v}} \\ &= \frac{(\boldsymbol{b}-\boldsymbol{a})\cdot(\boldsymbol{vi})}{\boldsymbol{u}\cdot(\boldsymbol{vi})}. \end{aligned}$$

Similarly, we solve for β_p by using the exterior product with $\boldsymbol{u}$ to eliminate the α term:

$$\begin{aligned} (\boldsymbol{a} + \alpha_p\boldsymbol{u})\wedge\boldsymbol{u} &= (\boldsymbol{b} + \beta_p\boldsymbol{v})\wedge\boldsymbol{u} \\ \therefore \quad \beta_p &= [(\boldsymbol{a}-\boldsymbol{b})\wedge\boldsymbol{u}]\,(\boldsymbol{v}\wedge\boldsymbol{u})^{-1} \\ &= [(\boldsymbol{b}-\boldsymbol{a})\wedge\boldsymbol{u}]\,(\boldsymbol{u}\wedge\boldsymbol{v})^{-1}. \end{aligned}$$

3.2 Solve the simultaneous equations $\boldsymbol{a}\wedge\boldsymbol{x} = \boldsymbol{B}, \boldsymbol{c}\cdot\boldsymbol{x} = \alpha$, where $\boldsymbol{c}\cdot\boldsymbol{a} \neq 0$. ([2], p. 47, Prob. 1.5)

A straightforward solution method is to write $\boldsymbol{x}$ as the linear combination $\boldsymbol{x} = \lambda\boldsymbol{a} + \mu\boldsymbol{c}$. From there, we use the given information to derive a pair of equations for the scalars λ and μ:

$$\begin{aligned} \boldsymbol{x} &= \lambda \boldsymbol{a} + \mu \boldsymbol{c}; \\ \boldsymbol{a} \wedge \boldsymbol{x} = \boldsymbol{B} &= \mu \boldsymbol{a} \wedge \boldsymbol{c}; \\ \boldsymbol{c} \cdot \boldsymbol{x} = \alpha &= \lambda \boldsymbol{c} \cdot \boldsymbol{a} + \mu c^2, \end{aligned}$$

etc. A similar problem was treated at length in [3] as a vehicle for introducing and examining several GA themes.

3.3 Equation: $\alpha \boldsymbol{x} + \boldsymbol{a}\,(\boldsymbol{x} \cdot \boldsymbol{b}) = \boldsymbol{c}$ ([2], p. 47, Prob. 1.3)

Our first impulse might be to obtain an expression for $\boldsymbol{x} \wedge \boldsymbol{b}$, in order to combine it with $\boldsymbol{x} \cdot \boldsymbol{b}$ to form the geometric product $\boldsymbol{x}\boldsymbol{b}$. Instead, we'll derive an expression for $\boldsymbol{x} \cdot \boldsymbol{b}$, which we will then substitute into the given equation;

$$\begin{aligned} \alpha \boldsymbol{x} + \boldsymbol{a}\,(\boldsymbol{x} \cdot \boldsymbol{b}) &= \boldsymbol{c}; \\ \{\alpha \boldsymbol{x} + \boldsymbol{a}\,(\boldsymbol{x} \cdot \boldsymbol{b})\} \cdot \boldsymbol{b} &= \boldsymbol{c} \cdot \boldsymbol{b}; \\ \alpha \boldsymbol{x} \cdot \boldsymbol{b} + (\boldsymbol{a} \cdot \boldsymbol{b})\,(\boldsymbol{x} \cdot \boldsymbol{b}) &= \boldsymbol{b} \cdot \boldsymbol{c}; \\ (\alpha + \boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{x} \cdot \boldsymbol{b} &= \boldsymbol{b} \cdot \boldsymbol{c}; \\ \therefore\ \boldsymbol{x} \cdot \boldsymbol{b} &= \frac{\boldsymbol{b} \cdot \boldsymbol{c}}{\alpha + \boldsymbol{a} \cdot \boldsymbol{b}}. \end{aligned}$$

Now, we substitute that expression for $\boldsymbol{x} \cdot \boldsymbol{b}$ into the original equation:

$$\begin{aligned} \alpha \boldsymbol{x} + \boldsymbol{a} \left(\frac{\boldsymbol{b} \cdot \boldsymbol{c}}{\alpha + \boldsymbol{a} \cdot \boldsymbol{b}} \right) &= \boldsymbol{c}; \\ \boldsymbol{x} &= \frac{1}{\alpha} \left[\boldsymbol{c} - \boldsymbol{a} \left(\frac{\boldsymbol{b} \cdot \boldsymbol{c}}{\alpha + \boldsymbol{a} \cdot \boldsymbol{b}} \right) \right]. \end{aligned}$$

3.4 Equation: $\alpha \boldsymbol{x} + \boldsymbol{x} \cdot \boldsymbol{B} = \boldsymbol{a}$, where $\boldsymbol{B}$ is a bivector. ([2], p. 47, Prob. 1.4)

Following [2], p. 675, our strategy will be to obtain an expression for $\boldsymbol{x} \wedge \boldsymbol{B}$ in terms of known quantities, so that we may then add that expression to $\boldsymbol{x} \cdot \boldsymbol{B}$ $(= \boldsymbol{a} - \alpha \boldsymbol{x})$ to form an expression (again in terms of known quantities) that is equal to the geometric product $\boldsymbol{x}\boldsymbol{B}$. The key to the solution is recognizing that $(\boldsymbol{x} \cdot \boldsymbol{B}) \wedge \boldsymbol{B} = 0$. One way to recognize this is by factoring $\boldsymbol{B}$ as the exterior product of some two arbitrary vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ that lie within the plane that it defines:

For comparison, see the problem

$$\alpha \boldsymbol{x} + \boldsymbol{b} \times \boldsymbol{x} = \boldsymbol{a},$$

in Section 3.5

$$\begin{aligned} [\boldsymbol{x} \cdot \boldsymbol{B}] \wedge \boldsymbol{B} &= [\boldsymbol{x} \cdot (\boldsymbol{u} \wedge \boldsymbol{v})] \wedge (\boldsymbol{u} \wedge \boldsymbol{v}) \\ &= [(\boldsymbol{x} \cdot \boldsymbol{u})\,\boldsymbol{v} - (\boldsymbol{x} \cdot \boldsymbol{v})\,\boldsymbol{u}] \wedge \boldsymbol{u} \wedge \boldsymbol{v} \\ &= (\boldsymbol{x} \cdot \boldsymbol{u})\,\boldsymbol{v} \wedge \boldsymbol{u} \wedge \boldsymbol{v}\ - (\boldsymbol{x} \cdot \boldsymbol{v})\,\boldsymbol{u} \wedge \boldsymbol{u} \wedge \boldsymbol{v} \\ &= 0. \end{aligned}$$

Next, we make use of the relation we've just found (i.e., $(\boldsymbol{x}\cdot\boldsymbol{B})\wedge\boldsymbol{B}=0$) by "wedging" both sides of the given equation with $\boldsymbol{B}$:

$$\begin{aligned}\alpha\boldsymbol{x}+\boldsymbol{x}\cdot\boldsymbol{B}&=\boldsymbol{a}\\(\alpha\boldsymbol{x}+\boldsymbol{x}\cdot\boldsymbol{B})\wedge\boldsymbol{B}&=\boldsymbol{a}\wedge\boldsymbol{B};\\\alpha\boldsymbol{x}\wedge\boldsymbol{B}+0&=\boldsymbol{a}\wedge\boldsymbol{B};\\\therefore\ \boldsymbol{x}\wedge\boldsymbol{B}&=\frac{1}{\alpha}(\boldsymbol{a}\wedge\boldsymbol{B}).\end{aligned}$$

The equation to be solved: $\alpha\boldsymbol{x}+\boldsymbol{x}\cdot\boldsymbol{B}=\boldsymbol{a}$.

Now, we'll return to the equation that we are to solve. By adding $\boldsymbol{x}\wedge\boldsymbol{B}$ to the left-hand side, we'll form the term $\boldsymbol{x}\boldsymbol{B}$.

$$\begin{aligned}\alpha\boldsymbol{x}+\boldsymbol{x}\cdot\boldsymbol{B}+\boldsymbol{x}\wedge\boldsymbol{B}&=\boldsymbol{a}+\underbrace{\frac{1}{\alpha}(\boldsymbol{a}\wedge\boldsymbol{B})}_{=\boldsymbol{x}\wedge\boldsymbol{B}};\\\alpha\boldsymbol{x}+\boldsymbol{x}\boldsymbol{B}&=\boldsymbol{a}+\frac{1}{\alpha}(\boldsymbol{a}\wedge\boldsymbol{B});\\\boldsymbol{x}(\alpha+\boldsymbol{B})&=\boldsymbol{a}+\frac{1}{\alpha}(\boldsymbol{a}\wedge\boldsymbol{B});\\\boldsymbol{x}(\alpha+\boldsymbol{B})(\alpha+\boldsymbol{B})^{-1}&=\left[\boldsymbol{a}+\frac{1}{\alpha}(\boldsymbol{a}\wedge\boldsymbol{B})\right](\alpha+\boldsymbol{B})^{-1};\\\boldsymbol{x}&=\left[\frac{\alpha\boldsymbol{a}+\boldsymbol{a}\wedge\boldsymbol{B}}{\alpha}\right]\underbrace{\left[\frac{\alpha-\boldsymbol{B}}{\alpha^2+\|\boldsymbol{B}\|^2}\right]}_{=(\alpha+\boldsymbol{B})^{-1}};\\&=\frac{\alpha^2\boldsymbol{a}-\alpha\boldsymbol{a}\boldsymbol{B}+\alpha\boldsymbol{a}\wedge\boldsymbol{B}-(\boldsymbol{a}\wedge\boldsymbol{B})\boldsymbol{B}}{\alpha(\alpha^2+\|\boldsymbol{B}\|^2)}\\&=\frac{\alpha^2\boldsymbol{a}-\alpha\boldsymbol{a}\cdot\boldsymbol{B}-(\boldsymbol{a}\wedge\boldsymbol{B})\boldsymbol{B}}{\alpha(\alpha^2+\|\boldsymbol{B}\|^2)}.\end{aligned}$$

$-\boldsymbol{a}\boldsymbol{B}+\boldsymbol{a}\wedge\boldsymbol{B}=-\boldsymbol{a}\cdot\boldsymbol{B}$.

Checking that solution by substituting it into the given equation raises some instructive points. We'll begin by making the substitution, and some initial simplifications:

$$\begin{aligned}\alpha\boldsymbol{x}+\boldsymbol{x}\cdot\boldsymbol{B}&=\alpha\left\{\frac{\alpha^2\boldsymbol{a}-\alpha\boldsymbol{a}\cdot\boldsymbol{B}-(\boldsymbol{a}\wedge\boldsymbol{B})\boldsymbol{B}}{\alpha(\alpha^2+\|\boldsymbol{B}\|^2)}\right\}+\left\{\frac{\alpha^2\boldsymbol{a}-\alpha\boldsymbol{a}\cdot\boldsymbol{B}-(\boldsymbol{a}\wedge\boldsymbol{B})\boldsymbol{B}}{\alpha(\alpha^2+\|\boldsymbol{B}\|^2)}\right\}\cdot\boldsymbol{B}\\&=\frac{\alpha^3\boldsymbol{a}-\alpha^2\boldsymbol{a}\cdot\boldsymbol{B}-\alpha(\boldsymbol{a}\wedge\boldsymbol{B})\boldsymbol{B}+\alpha^2\boldsymbol{a}\cdot\boldsymbol{B}-\alpha(\boldsymbol{a}\cdot\boldsymbol{B})\cdot\boldsymbol{B}-[(\boldsymbol{a}\wedge\boldsymbol{B})\boldsymbol{B}]\cdot\boldsymbol{B}}{\alpha(\alpha^2+\|\boldsymbol{B}\|^2)}\\&=\frac{\alpha^3\boldsymbol{a}-\alpha(\boldsymbol{a}\wedge\boldsymbol{B})\boldsymbol{B}-\alpha(\boldsymbol{a}\cdot\boldsymbol{B})\cdot\boldsymbol{B}-[(\boldsymbol{a}\wedge\boldsymbol{B})\boldsymbol{B}]\cdot\boldsymbol{B}}{\alpha(\alpha^2+\|\boldsymbol{B}\|^2)}.\qquad(1)\end{aligned}$$

Here, we see a useful, common maneuver: $(\boldsymbol{a}\cdot\boldsymbol{B})\wedge\boldsymbol{B}=0$, so $(\boldsymbol{a}\cdot\boldsymbol{B})\cdot\boldsymbol{B}$ is in fact $(\boldsymbol{a}\cdot\boldsymbol{B})\boldsymbol{B}$. Similarly, when $(\boldsymbol{a}\cdot\boldsymbol{B})\cdot\boldsymbol{B}=0$, $(\boldsymbol{a}\cdot\boldsymbol{B})\wedge\boldsymbol{B}$ is $(\boldsymbol{a}\cdot\boldsymbol{B})\boldsymbol{B}$.

Let's look, now, at some of the terms in the numerator. First, the product $\boldsymbol{a}\cdot\boldsymbol{B}$ is a vector. The geometric product of a vector $\boldsymbol{v}$ and bivector $\boldsymbol{M}$ is the sum $\boldsymbol{v}\cdot\boldsymbol{M}+\boldsymbol{v}\wedge\boldsymbol{M}$. Therefore, the geometric product of the vector $\boldsymbol{a}\cdot\boldsymbol{B}$ and the bivector $\boldsymbol{B}$ is $(\boldsymbol{a}\boldsymbol{B})\cdot\boldsymbol{B}=(\boldsymbol{a}\cdot\boldsymbol{B})\cdot\boldsymbol{B}+(\boldsymbol{a}\cdot\boldsymbol{B})\wedge\boldsymbol{B}$. We've already established that $(\boldsymbol{a}\cdot\boldsymbol{B})\wedge\boldsymbol{B}=0$. Thus, $(\boldsymbol{a}\cdot\boldsymbol{B})\cdot\boldsymbol{B}=(\boldsymbol{a}\cdot\boldsymbol{B})\boldsymbol{B}$. Making this substitution in Eq. (1),

$$\alpha\boldsymbol{x} + \boldsymbol{x}\cdot\boldsymbol{B} = \frac{\alpha^3\boldsymbol{a} - \alpha\left(\boldsymbol{a}\wedge\boldsymbol{B}\right)\boldsymbol{B} - \overbrace{\alpha\left(\boldsymbol{a}\cdot\boldsymbol{B}\right)\boldsymbol{B}}^{=\alpha(\boldsymbol{a}\wedge\boldsymbol{B})\cdot\boldsymbol{B}} - \left[\left(\boldsymbol{a}\wedge\boldsymbol{B}\right)\boldsymbol{B}\right]\cdot\boldsymbol{B}}{\alpha\left(\alpha^2 + \|\boldsymbol{B}\|^2\right)}\ .$$

Next, we note that

$$\begin{aligned}\alpha\left(\boldsymbol{a}\wedge\boldsymbol{B}\right)\boldsymbol{B} + \alpha\left(\boldsymbol{a}\cdot\boldsymbol{B}\right)\boldsymbol{B} &= \alpha\left(\boldsymbol{a}\cdot\boldsymbol{B} + \boldsymbol{a}\wedge\boldsymbol{B}\right)\boldsymbol{B}\\ &= \alpha\left(\boldsymbol{a}\boldsymbol{B}\right)\boldsymbol{B}\\ &= -\alpha\boldsymbol{a}\|\boldsymbol{B}\|^2.\end{aligned}$$

Therefore,

$$\alpha\boldsymbol{x} + \boldsymbol{x}\cdot\boldsymbol{B} = \frac{\alpha^3\boldsymbol{a} + \alpha\boldsymbol{a}\|\boldsymbol{B}\|^2 - \left[\left(\boldsymbol{a}\wedge\boldsymbol{B}\right)\boldsymbol{B}\right]\cdot\boldsymbol{B}}{\alpha\left(\alpha^2 + \|\boldsymbol{B}\|^2\right)}.$$

Finally, let's look at the denominator's term $\left[\left(\boldsymbol{a}\wedge\boldsymbol{B}\right)\boldsymbol{B}\right]\cdot\boldsymbol{B}$. The product $\left(\boldsymbol{a}\wedge\boldsymbol{B}\right)\boldsymbol{B}$ evaluates to a vector; specifically, it's a scalar multiple of the component of $\boldsymbol{a}$ that is normal to $\boldsymbol{B}$. (In [2]'s words, of "$\boldsymbol{a}$'s rejection from $\boldsymbol{B}$", p. 65.) Thus, $\left[\left(\boldsymbol{a}\wedge\boldsymbol{B}\right)\boldsymbol{B}\right]\cdot\boldsymbol{B}$ is the inner product of the vector $\left(\boldsymbol{a}\wedge\boldsymbol{B}\right)\boldsymbol{B}$ and the bivector $\boldsymbol{B}$. For that reason,

$$\begin{aligned}\left[\left(\boldsymbol{a}\wedge\boldsymbol{B}\right)\boldsymbol{B}\right]\cdot\boldsymbol{B} &= \left\langle\left[\left(\boldsymbol{a}\wedge\boldsymbol{B}\right)\boldsymbol{B}\right]\boldsymbol{B}\right\rangle_1\\ &= \overbrace{\boldsymbol{B}^2}^{=-\|\boldsymbol{B}\|^2}\left\langle\boldsymbol{a}\wedge\boldsymbol{B}\right\rangle_1\\ &= 0.\end{aligned}$$

Using that result, we can write

$$\begin{aligned}\alpha\boldsymbol{x} + \boldsymbol{x}\cdot\boldsymbol{B} &= \frac{\alpha^3\boldsymbol{a} + \alpha\boldsymbol{a}\|\boldsymbol{B}\|^2 - \overbrace{\left[\left(\boldsymbol{a}\wedge\boldsymbol{B}\right)\boldsymbol{B}\right]\cdot\boldsymbol{B}}^{=0}}{\alpha\left(\alpha^2 + \|\boldsymbol{B}\|^2\right)}\\ &= \boldsymbol{a}.\end{aligned}$$

3.5 Equation: $\alpha\boldsymbol{x} + \boldsymbol{b}\times\boldsymbol{x} = \boldsymbol{a}$. ([2], p. 64, Prob. 3.10)

Even though the familiar cross product ("×") is not part of GA, [1] and [2] spend considerable time on that product for contact with the literature that uses conventional vector algebra. In GA terms, the product $\boldsymbol{u}\times\boldsymbol{v}$ (which evaluates to a vector) is said to be the "dual" of the bivector $\boldsymbol{a}\wedge\boldsymbol{b}$, and is related to it via

Note this problem's resemblance to that presented in Section (3.4).

$$\boldsymbol{u}\times\boldsymbol{v} = -i\boldsymbol{u}\wedge\boldsymbol{v}.$$

Therefore, the equation that we're asked to solve can be rewritten as

$$\begin{aligned}&\alpha\boldsymbol{x} - i\left(\boldsymbol{b}\wedge\boldsymbol{x}\right) = \boldsymbol{a}\ , or\\ &\alpha\boldsymbol{x} + i\left(\boldsymbol{x}\wedge\boldsymbol{b}\right) = \boldsymbol{a}.\end{aligned}$$

As [2] notes (p. 47), in such a problem we should try to eliminate one of the terms that involves the unknown. How might we do that here? We know from conventional vector algebra that $\boldsymbol{b}\times\boldsymbol{x}$ evaluates to a vector that's perpendicular to both $\boldsymbol{x}$ and $\boldsymbol{b}$. Therefore, $[i\,(\boldsymbol{x}\wedge\boldsymbol{b})]\cdot\boldsymbol{b}$ should be zero. Let's demonstrate that via GA, bearing in mind that $[i\,(\boldsymbol{x}\wedge\boldsymbol{b})]\cdot\boldsymbol{b}$ is a dot product of the two vectors $i\,(\boldsymbol{x}\wedge\boldsymbol{b})$ and $\boldsymbol{b}$:

$$\begin{aligned}
[i\,(\boldsymbol{x}\wedge\boldsymbol{b})]\cdot\boldsymbol{b} &= \langle [i\,(\boldsymbol{x}\wedge\boldsymbol{b})]\,\boldsymbol{b}\rangle_0 \\
&= \langle [i\,(\boldsymbol{x}\boldsymbol{b}-\boldsymbol{x}\cdot\boldsymbol{b})]\,\boldsymbol{b}\rangle_0 \\
&= \langle i\boldsymbol{x}b^2-(\boldsymbol{x}\cdot\boldsymbol{b})\,i\boldsymbol{b}\rangle_0 \\
&= 0,
\end{aligned}$$

because the product of i with any vector evaluates to a vector. Now, using the fact that $[i\,(\boldsymbol{x}\wedge\boldsymbol{b})]\cdot\boldsymbol{b}=0$, we can obtain an expression for $\boldsymbol{x}\cdot\boldsymbol{b}$ from the original equation:

$$\begin{aligned}
\alpha\boldsymbol{x}+\boldsymbol{b}\times\boldsymbol{x} &= \boldsymbol{a} \\
\alpha\boldsymbol{x}\cdot\boldsymbol{b}+\underbrace{[\boldsymbol{b}\times\boldsymbol{x}]\cdot\boldsymbol{b}}_{=0} &= \boldsymbol{a}\cdot\boldsymbol{b} \\
\therefore\quad \boldsymbol{x}\cdot\boldsymbol{b} &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\alpha}.
\end{aligned}$$

Whenever we can obtain an expression for $\boldsymbol{x}\cdot\boldsymbol{b}$ readily, we should consider trying to obtain one for $\boldsymbol{x}\wedge\boldsymbol{b}$ as well. In the problem on which we're working, we can obtain one from our "translation into GA" of the given equation:

$$\begin{aligned}
\alpha\boldsymbol{x}+i\,(\boldsymbol{x}\wedge\boldsymbol{b}) &= \boldsymbol{a} \\
i\,(\boldsymbol{x}\wedge\boldsymbol{b}) &= \boldsymbol{a}-\alpha\boldsymbol{x} \\
-ii\,(\boldsymbol{x}\wedge\boldsymbol{b}) &= -i\,(\boldsymbol{a}-\alpha\boldsymbol{x}) \\
\boldsymbol{x}\wedge\boldsymbol{b} &= \alpha i\boldsymbol{x}-i\boldsymbol{a}.
\end{aligned}$$

We may be concerned that the expression that we just obtained for $\boldsymbol{x}\wedge\boldsymbol{b}$ contains $\boldsymbol{x}$ itself. That might be a problem, but let's find out: let's add our expressions for $\boldsymbol{x}\cdot\boldsymbol{b}$ and $\boldsymbol{x}\wedge\boldsymbol{b}$, and see what happens.

$$\begin{aligned}
\boldsymbol{x}\cdot\boldsymbol{b}+\boldsymbol{x}\wedge\boldsymbol{b} &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\alpha}+\alpha i\boldsymbol{x}-i\boldsymbol{a} \\
\boldsymbol{x}\boldsymbol{b} &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\alpha}+\alpha i\boldsymbol{x}-i\boldsymbol{a} \\
\boldsymbol{x}\boldsymbol{b}-\alpha i\boldsymbol{x} &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\alpha}-i\boldsymbol{a} \\
\boldsymbol{x}\boldsymbol{b}-\boldsymbol{x}\alpha i &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\alpha}-i\boldsymbol{a}.
\end{aligned}$$

In that last line, we used the fact that $i\boldsymbol{x} = \boldsymbol{x}i$. Proceeding,

$$\begin{aligned}
\boldsymbol{x}(\boldsymbol{b} - \alpha i) &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\alpha} - i\boldsymbol{a} \\
\boldsymbol{x} &= \left[\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\alpha} - i\boldsymbol{a}\right]\underbrace{\left[\frac{\boldsymbol{b}+\alpha i}{\alpha^2 + b^2}\right]}_{=(\boldsymbol{b}-\alpha i)^{-1}} \\
&= \frac{(\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{b} + \alpha(\boldsymbol{a}\cdot\boldsymbol{b})\,i - \alpha i\boldsymbol{a}\boldsymbol{b} + \alpha^2\boldsymbol{a}}{\alpha(\alpha^2 + b^2)} \\
&= \frac{(\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{b} + \alpha i(\boldsymbol{a}\cdot\boldsymbol{b}) - \alpha i(\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b}) + \alpha^2\boldsymbol{a}}{\alpha(\alpha^2 + b^2)} \\
&= \frac{(\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{b} - \alpha i(\boldsymbol{a}\wedge\boldsymbol{b}) + \alpha^2\boldsymbol{a}}{\alpha(\alpha^2 + b^2)}.
\end{aligned}$$

A reminder:
"$\boldsymbol{i}$" is $\boldsymbol{I}_2$, but "i" is $\boldsymbol{I}_3$.
For any vector $\boldsymbol{v}$,
$\boldsymbol{i}\boldsymbol{v} = -\boldsymbol{v}\boldsymbol{i}$, but $i\boldsymbol{v} = \boldsymbol{v}i$.

3.6 Find the "directance" $\boldsymbol{d}$ from the line $(\boldsymbol{x} - \boldsymbol{a}) \wedge \boldsymbol{u} = 0$ to the line $(\boldsymbol{y} - \boldsymbol{b}) \wedge \boldsymbol{v} = 0$. ([2], p. 93, Prob. 6.7)

As [2], p. 93 tells us, "The *directance* from one point set to another can be defined as the chord of minimum length between points in the two sets, provided there is only one such chord." Reference [2] solves the present problem from a very "algebraic" point of view. In contrast, the solution presented here uses our visual capacities to help us. First, we'll look at two coplanar lines (Fig. 1) that have the directions $\boldsymbol{u}$ and $\boldsymbol{v}$, and intersect in some point $\boldsymbol{p}$.

In this solution, we'll see many places where the symbol for a vector (for example, $\boldsymbol{p}$) is used ambiguously, to refer either to the vector itself, or to its endpoint. In each such instance, we rely upon context to make the intended meaning clear. On a related theme, Hestenes's observations ([2], p. 80) on the correspondence between vectors and physical space are highly recommended reading.

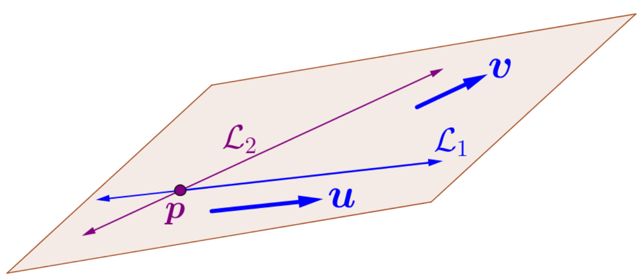


Figure 1: $\mathcal{L}_1$ and $\mathcal{L}_2$ are coplanar lines that have the directions $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively, and intersect in some point $\boldsymbol{p}$.

Now, let's examine a parallel plane (Fig. 2) through the point $\boldsymbol{p'}$ $(= \boldsymbol{p'} + \boldsymbol{d})$, where $\boldsymbol{d}$ is a vector perpendicular to both planes. Through $\boldsymbol{p}$, we've also drawn the line $\mathcal{L}_3$, which is parallel to $\mathcal{L}_2$. We'd almost certainly conclude, intuitively, that $\boldsymbol{d}$ is the directance from $\mathcal{L}_1$ to $\mathcal{L}_3$. However, let's demonstrate that our intuition is accurate, using GA.

Consider the distance between an arbitrary point $\boldsymbol{q}$ along $\mathcal{L}_1$, and an arbitrary point $\boldsymbol{s}$ along $\mathcal{L}_3$. Writing those points as $\boldsymbol{q} = \boldsymbol{p} + \lambda\boldsymbol{u}$, and $\boldsymbol{s} = \boldsymbol{p} + \boldsymbol{d} + \gamma\boldsymbol{v}$,

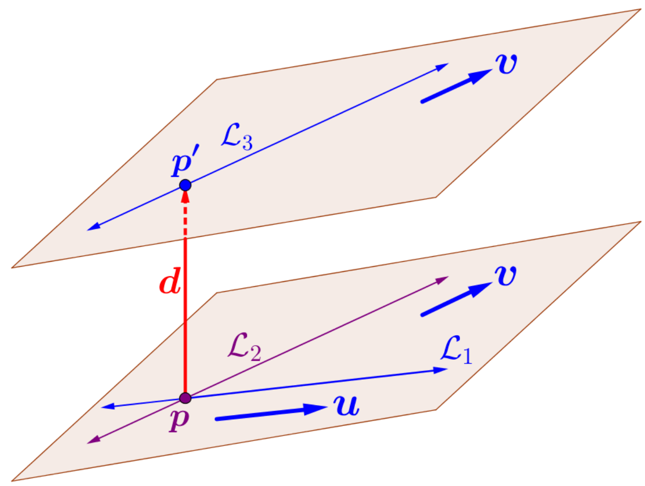


Figure 2: The plane shown in Fig. 1 plus a parallel plane that contains the point $\boldsymbol{p'}$ $(= \boldsymbol{p} + \boldsymbol{d})$, where $\boldsymbol{d}$ is some vector perpendicular to both planes. The line $\mathcal{L}_3$ is parallel to $\mathcal{L}_2$, and passes through the point $\boldsymbol{p'}$. In the text, we show that $\boldsymbol{d}$ is the directance from $\mathcal{L}_1$ to $\mathcal{L}_3$.

the square of the distance between them is

$$\begin{aligned}\|\,(\boldsymbol{p}+\boldsymbol{d}+\gamma\boldsymbol{v})-(\boldsymbol{p}+\lambda\boldsymbol{u})\,\|^2 &= \|\boldsymbol{d}+(\gamma\boldsymbol{v}-\lambda\boldsymbol{u})\,\|^2 \\ &= d^2 - 2\boldsymbol{d}\cdot(\gamma\boldsymbol{v}-\lambda\boldsymbol{u}) + (\gamma\boldsymbol{v}-\lambda\boldsymbol{u})^2 \,.\end{aligned}$$

Because $\boldsymbol{d}$ is perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$, it is perpendicular to any linear combination thereof. Therefore, the dot-product term is zero, making the square of the distance between $\boldsymbol{q}$ and $\boldsymbol{s}$ equal to $d^2 + (\gamma\boldsymbol{v}-\lambda\boldsymbol{u})^2$. That quantity is minimized when $\gamma = \lambda = 0$; that is, when the the points $\boldsymbol{q}$ and $\boldsymbol{s}$ are $\boldsymbol{p}$ and $\boldsymbol{p'}$.

Thus, the directance is, indeed, the vector $\boldsymbol{d}$ shown in Fig. 2. Now, we need to find out how we can determine $\boldsymbol{d}$ given that $\mathcal{L}_1 = (\boldsymbol{x}-\boldsymbol{a})\wedge\boldsymbol{u} = 0$ and $\mathcal{L}_3 = (\boldsymbol{y}-\boldsymbol{b})\wedge\boldsymbol{v} = 0$. Let's see what insights Fig. 3 might offer.

From that figure, we can see that $\boldsymbol{d}$ is the "rejection" (in the words of [2], p. 65) of the vector $\boldsymbol{b}-\boldsymbol{a}$ from the plane containing $\mathcal{L}_1$. That rejection is equal to $[(\boldsymbol{b}-\boldsymbol{a})\wedge\boldsymbol{B}]\,\boldsymbol{B}^{-1}$, where $\boldsymbol{B}$ can be any positive bivector that has the same orientation as the plane. One such bivector, clearly, is $\boldsymbol{u}\wedge\boldsymbol{v}$. Putting all of these ideas together,

$$\boldsymbol{d} = [(\boldsymbol{b}-\boldsymbol{a})\wedge(\boldsymbol{u}\wedge\boldsymbol{v})]\,(\boldsymbol{u}\wedge\boldsymbol{v})^{-1}\,.$$

3.7 Find the point of intersection of the line $(\boldsymbol{x}-\boldsymbol{a})\wedge\boldsymbol{u} = 0$ and the plane $(\boldsymbol{y}-\boldsymbol{b})\wedge\boldsymbol{B} = 0$. ([2], p. 93, Prob. 6.6)

Here, we use the fact that $(\boldsymbol{x}-\boldsymbol{a})\wedge\boldsymbol{u} = 0$ if and only if $\boldsymbol{x}-\boldsymbol{a}$ is a scalar multiple of $\boldsymbol{u}$.

One simple way to solve this problem is to write the line's equation as $\boldsymbol{x} = \boldsymbol{a}+\lambda\boldsymbol{u}$. Now, let $\boldsymbol{p} = \boldsymbol{a}+\lambda_p\boldsymbol{u}$ be the point of intersection. Then, from the

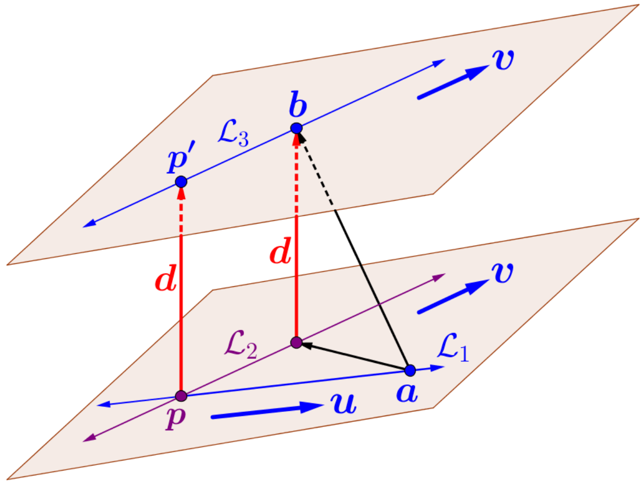


Figure 3: The same lines and planes as in Fig. 2, plus a point $\boldsymbol{a}$ along $\mathcal{L}_1$ and a point $\boldsymbol{b}$ along $\mathcal{L}_3$. Given these points, we can now write that $\mathcal{L}_1 = (\boldsymbol{x}-\boldsymbol{a})\wedge\boldsymbol{u} = 0$ and $\mathcal{L}_3 = (\boldsymbol{y}-\boldsymbol{b})\wedge\boldsymbol{v} = 0$. The text explains that $\boldsymbol{d}$, the directance from $\mathcal{L}_1$ to $\mathcal{L}_3$, is the "rejection" of the vector $\boldsymbol{b}-\boldsymbol{a}$ from the bivector $\boldsymbol{u}\wedge\boldsymbol{v}$.

equation for the plane,

$$(\boldsymbol{a}+\lambda_p\boldsymbol{u}-\boldsymbol{b})\wedge\boldsymbol{B} = 0,$$

from which

$$\begin{aligned} \lambda_p\boldsymbol{u}\wedge\boldsymbol{B} &= (\boldsymbol{b}-\boldsymbol{a})\wedge\boldsymbol{B}, \\ \lambda_p &= [(\boldsymbol{b}-\boldsymbol{a})\wedge\boldsymbol{B}]\,(\boldsymbol{u}\wedge\boldsymbol{B})^{-1}, \end{aligned}$$

and $\boldsymbol{p} = \boldsymbol{a} + \left\{[(\boldsymbol{b}-\boldsymbol{a})\wedge\boldsymbol{B}]\,(\boldsymbol{u}\wedge\boldsymbol{B})^{-1}\right\}\boldsymbol{u}$.

That result probably reminds us of the one presented in Section 3.6 , so let's seek a geometric interpretation of our solution. For example, Fig. 4 .

From that figure, we can deduce that the "λ_p" in $\lambda_p = [(\boldsymbol{b}-\boldsymbol{a})\wedge\boldsymbol{B}]\,(\boldsymbol{u}\wedge\boldsymbol{B})^{-1}$ must be the ratio (with appropriate algebraic sign) of the lengths of the vectors $(\boldsymbol{a}-\boldsymbol{b})_\perp$ and $\boldsymbol{u}_\perp$. Let's work with that idea a bit:

$$\begin{aligned} \lambda_p\boldsymbol{u}_\perp &= -(\boldsymbol{a}-\boldsymbol{b})_\perp; \\ \lambda_p(\boldsymbol{u}\wedge\boldsymbol{B})\boldsymbol{B}^{-1} &= -[(\boldsymbol{a}-\boldsymbol{b})\wedge\boldsymbol{B}]\boldsymbol{B}^{-1}; \\ \lambda_p(\boldsymbol{u}\wedge\boldsymbol{B}) &= [(\boldsymbol{b}-\boldsymbol{a})\wedge\boldsymbol{B}]; \\ \therefore\quad \lambda_p &= [(\boldsymbol{b}-\boldsymbol{a})\wedge\boldsymbol{B}]\,(\boldsymbol{u}\wedge\boldsymbol{B})^{-1}, \end{aligned}$$

Why the negative sign? Because the vertical displacement $\lambda\boldsymbol{u}_\perp$ from $\boldsymbol{a}$ to $\boldsymbol{p}$ is in the direction contrary to $(\boldsymbol{a}-\boldsymbol{b})_\perp$.

which agrees with the answer that we obtained algebraically.

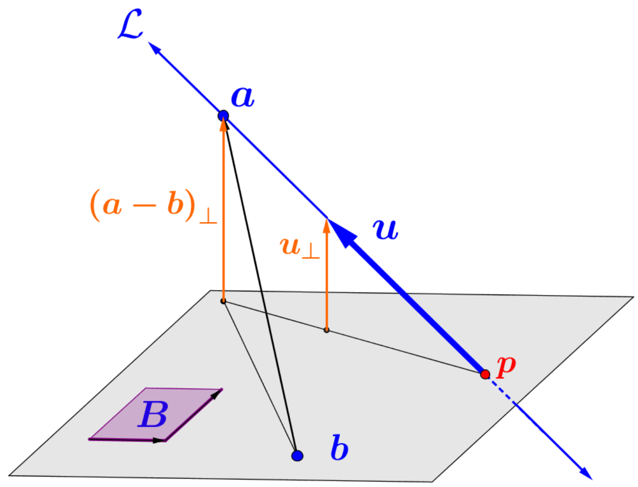


Figure 4: Point $\boldsymbol{p}$ is the intersection of the line $(\boldsymbol{x}-\boldsymbol{a})\wedge\boldsymbol{u}=0$ and the plane $(\boldsymbol{y}-\boldsymbol{b})\wedge\boldsymbol{B}=0$. The vector $\boldsymbol{u}_\perp$ is perpendicular to that plane, and is equal to $(\boldsymbol{u}\wedge\boldsymbol{B})\,\boldsymbol{B}^{-1}$. Similarly, $(\boldsymbol{a}-\boldsymbol{b})_\perp$ is equal to $[(\boldsymbol{a}-\boldsymbol{b})\wedge\boldsymbol{B}]\,\boldsymbol{B}^{-1}$.

3.8 Solve for x: $(\boldsymbol{a}-\boldsymbol{x})^{-1}(\boldsymbol{b}-\boldsymbol{x}) = \lambda e^{\theta\boldsymbol{i}} = 0$. ([2], p. 96, Prob. 6.22)

Here, the fact that $(\boldsymbol{a}-\boldsymbol{x})^{-1} = (\boldsymbol{a}-\boldsymbol{x})/\|\boldsymbol{a}-\boldsymbol{x}\|^2$ is unhelpful. Instead, we proceed as follows:

$$
\begin{aligned}
(\boldsymbol{a}-\boldsymbol{x})^{-1}(\boldsymbol{b}-\boldsymbol{x}) &= \lambda e^{\theta\boldsymbol{i}} \\
\boldsymbol{b}-\boldsymbol{x} &= \lambda(\boldsymbol{a}-\boldsymbol{x})\,e^{\theta\boldsymbol{i}} \\
\boldsymbol{x}\lambda e^{\theta\boldsymbol{i}}-\boldsymbol{x} &= \lambda\boldsymbol{a}-\boldsymbol{b} \\
\boldsymbol{x} &= (\lambda\boldsymbol{a}-\boldsymbol{b})\left(\lambda e^{\theta\boldsymbol{i}}-1\right)^{-1} \\
&= (\lambda\boldsymbol{a}-\boldsymbol{b})\left[\frac{\lambda e^{-\theta\boldsymbol{i}}-1}{2(1-\lambda\cos\theta)}\right].
\end{aligned}
$$

3.9 Assuming that we know $\|\boldsymbol{v}_0\|$, but not $\boldsymbol{v}_0$'s direction, determine t given that $\boldsymbol{g}\wedge\boldsymbol{r}=t(\boldsymbol{g}\wedge\boldsymbol{v}_0)$ and $\boldsymbol{r}=\frac{1}{2}\boldsymbol{g}t^2+\boldsymbol{v}_0t$. ([2], p. 133, Prob. 2.1)

See also the problem presented in 3.10.

This problem is from [2]'s treatment of constant-force problems; specifically, t is the time of flight of a projectile that will land at point $\boldsymbol{r}$ if fired at velocity $\boldsymbol{v}_0$.

Because we know $\|\boldsymbol{v}_0\|$, a reasonable first move is to make that quantity appear, somehow, by manipulating the given equations. Perhaps counterintuitively, we'll start from $\boldsymbol{g} \wedge \boldsymbol{r} = t\left(\boldsymbol{g} \wedge \boldsymbol{v}_0\right)$ rather than from $\boldsymbol{r} = \frac{1}{2}\boldsymbol{g}t^2 + \boldsymbol{v}_0 t$:

$$\begin{aligned}(\boldsymbol{g} \wedge \boldsymbol{r})^2 &= \left[t\left(\boldsymbol{g} \wedge \boldsymbol{v}_0\right)\right]^2 \\ (\boldsymbol{g} \cdot \boldsymbol{r})^2 - g^2 r^2 &= t^2\left[(\boldsymbol{g} \cdot \boldsymbol{v}_0)^2 - g^2 {v_0}^2\right].\end{aligned}$$

Are we making progress? We've obtained a ${v_0}^2$ term, but we've also introduced an unwanted quantity: $\boldsymbol{g} \cdot \boldsymbol{v}_0$. However, using the other equation that we were given, we can obtain an expression to substitute for $\boldsymbol{g} \cdot \boldsymbol{v}_0$:

$$\begin{aligned}\boldsymbol{g} \cdot \boldsymbol{r} &= \boldsymbol{g} \cdot \left[\tfrac{1}{2}\boldsymbol{g}t^2 + \boldsymbol{v}_0 t\right] \\ \therefore \quad \boldsymbol{g} \cdot \boldsymbol{v}_0 &= \frac{\boldsymbol{g} \cdot \boldsymbol{r} - \frac{1}{2}g^2 t^2}{t} \quad \text{and} \\ (\boldsymbol{g} \cdot \boldsymbol{v}_0)^2 &= \frac{(\boldsymbol{g} \cdot \boldsymbol{r})^2 - g^2 t^2 \boldsymbol{g} \cdot \boldsymbol{r} - \frac{1}{4}g^4 t^4}{t^2}.\end{aligned}$$

Making the substitution, then simplifying, we arrive at the quadratic equation (in t^2)

$$\tfrac{1}{4}g^2 t^4 - \left(\boldsymbol{g} \cdot \boldsymbol{r} + {v_0}^2\right) + r^2 = 0,$$

from which

$$t^2 = \frac{2}{g^2}\left\{{v_0}^2 + \boldsymbol{g} \cdot \boldsymbol{r} \pm \left[\left({v_0}^2 + \boldsymbol{g} \cdot \boldsymbol{r}\right) - g^2 r^2\right]\right\}, \text{ and}$$

$$t = \frac{\sqrt{2}}{g}\left\{{v_0}^2 + \boldsymbol{g} \cdot \boldsymbol{r} \pm \left[\left({v_0}^2 + \boldsymbol{g} \cdot \boldsymbol{r}\right) - g^2 r^2\right]\right\}^{1/2}.$$

3.10 Given $\boldsymbol{a}$, $\boldsymbol{c}$, $\|\boldsymbol{b}\|$, and β, solve the equation $\alpha\boldsymbol{a} + \beta\boldsymbol{b} = \boldsymbol{c}$ for α.

We begin by squaring both sides of $\alpha\boldsymbol{a} + \beta\boldsymbol{b} = \boldsymbol{c}$ to produce a $\|\boldsymbol{b}\|^2$ term:

$$\alpha^2 a^2 + 2\alpha\beta\boldsymbol{a} \cdot \boldsymbol{b} + \beta\|\boldsymbol{b}\|^2 = c^2,$$

Next, we "dot" both sides of $\alpha\boldsymbol{a} + \beta\boldsymbol{b} = \boldsymbol{c}$ with $\boldsymbol{a}$ to produce an expression that we can substitute for the unwanted quantity $\boldsymbol{a} \cdot \boldsymbol{b}$:

$$\begin{aligned}\boldsymbol{a} \cdot (\alpha\boldsymbol{a} + \beta\boldsymbol{b} = \boldsymbol{c}) &= \boldsymbol{a} \cdot \boldsymbol{c}; \\ \alpha a^2 + \beta\boldsymbol{a} \cdot \boldsymbol{b} &= \boldsymbol{a} \cdot \boldsymbol{c}; \\ \therefore \quad \boldsymbol{a} \cdot \boldsymbol{b} &= \frac{\boldsymbol{a} \cdot \boldsymbol{c} - \alpha a^2}{\beta}.\end{aligned}$$

After making the substitution, then simplifying, we obtain a quadratic in α:

$$a^2\alpha^2 - (2\boldsymbol{a} \cdot \boldsymbol{c})\,\alpha + c^2 - \beta^2 b^2 = 0;$$

$$\therefore \quad \alpha = \frac{\boldsymbol{a} \cdot \boldsymbol{c} \pm \sqrt{(\boldsymbol{a} \cdot \boldsymbol{c})^2 + a^2\left(\beta^2 b^2 - c^2\right)}}{a^2}.$$

See also the problem presented in 3.9. Extending these two problems a bit, we can see that if we had to solve for $\boldsymbol{x}$ in the equation $\boldsymbol{f} = \boldsymbol{g} + \boldsymbol{x}$, and knew $\boldsymbol{g}$ and $\|\boldsymbol{f}\|$, but not $\boldsymbol{f}$'s direction, we might consider forming the square of $\|\boldsymbol{f}\|$ by one of the following maneuvers:

- $\|\boldsymbol{f}\|^2 = \boldsymbol{f}\boldsymbol{f} = \boldsymbol{f}\,(\boldsymbol{g} + \boldsymbol{x})$;
- $\|\boldsymbol{f}\|^2 = (\boldsymbol{f})^2 = (\boldsymbol{g} + \boldsymbol{x})^2$;
- $\|\boldsymbol{f}\|^2 = \boldsymbol{f} \cdot \boldsymbol{f} = \boldsymbol{f} \cdot (\boldsymbol{g} + \boldsymbol{x})$.

4 Additional Problems

Several problems are solved in multiple ways in [3]-[12].

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