Some Representations of pi

Edgar Valdebenito

September 5 , 2016

abstract

This note presents a collection of mathematical formulas involving the mathematical constant pi :

$\pi = 3.14159265 \ldots$

Introducción

Recordamos algunas fórmulas integrales para pi :

$$\pi = \frac{3}{2} \int_1^3 \sqrt{\frac{4}{x} - 1}\, dx \tag{1}$$

$$\pi = 6 \int_0^1 \frac{x^2}{(2 - x^3)\sqrt{1 - x^3}}\, dx \tag{2}$$

$$\pi = 2 \int_0^\infty \frac{1}{(2 + x)\sqrt{1 + x}}\, dx \tag{3}$$

$$\pi = 6 \int_0^1 \frac{1 + 2x}{(2 + x + x^3)\sqrt{1 + x + x^2}}\, dx \tag{4}$$

$$\pi = \sqrt[4]{8}\,\sqrt{3} \int_0^1 \sqrt[4]{\sqrt{4x^{-1} - 3} - 1}\, dx \tag{5}$$

$$\pi = 2 + \int_2^\infty \left(1 - \sqrt{1 - 4x^{-2}}\right) dx \tag{6}$$

$$\pi = 2 \int_0^1 \ln\left(x^{-1} + \sqrt{x^{-2} - 1}\right) dx \tag{7}$$

$$\pi = 2 + 2 \int_0^1 \ln\left(1 + \sqrt{1 - x^2}\right) dx \tag{8}$$

$$\pi = \frac{3\sqrt{3}\,\ln 3}{2} + 6\int_u^1 \ln\left(x^{-1} + \sqrt{x^{-2} - 1}\right) dx\ ,\ u = \frac{\sqrt{3}}{2} \tag{9}$$

$$\pi = \int_0^\infty \left(-\frac{2}{3} + \sqrt[3]{f(x)} + \sqrt[3]{g(x)}\right) dx \tag{10}$$

donde

$$f(x) = \frac{2 + 27\,x^{-2}}{54} + \frac{x^{-1}\sqrt{12 + 81\,x^{-2}}}{18} \tag{11}$$

$$g(x) = \frac{2 + 27\,x^{-2}}{54} - \frac{x^{-1}\sqrt{12 + 81\,x^{-2}}}{18} \tag{12}$$

Notación : $z = x + i\,y \in \mathbb{C}$, $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$, $i = \sqrt{-1}$.

Fórmulas

fórmula 1.

$$\pi = 12\sum_{n=0}^{\infty}\left(\frac{c_{2n}}{2n+1} + \frac{c_{2n+1}}{30n+30}\right)\left(\frac{1}{5}\right)^{2n+1} \tag{13}$$

$$c_{n+2} = 4\,c_{n+1} - 5\,c_n\ ,\ c_0 = 1,\ c_1 = 4 \tag{14}$$

fórmula 2.

$$\pi = \frac{24\sqrt{2}}{5}\sum_{n=0}^{\infty}\left(\frac{c_{2n}}{2n+1} + \frac{c_{2n+1}}{6n+6}\left(\sqrt{2} - 1\right)\right)\left(\sqrt{2} - 1\right)^{2n+1} \tag{15}$$

$$c_{n+2} = 2\,c_{n+1} - 3\,c_n\ ,\ c_0 = 1,\ c_1 = 2 \tag{16}$$

fórmula 3.

$$\pi = \frac{9}{2}\sum_{n=0}^{\infty}\left(\frac{c_{2n}}{2n+1} + \frac{c_{2n+1}}{6n+6}\left(\sqrt{3} - 1\right)\right)\left(\sqrt{3} - 1\right)^{2n+1} \tag{17}$$

$$c_{n+2} = \left(\sqrt{3} - 1\right)c_{n+1} - \left(\frac{4 - \sqrt{3}}{2}\right)c_n\ ,\ c_0 = 1,\ c_1 = \sqrt{3} - 1 \tag{18}$$

fórmula 4.

$$\pi = 9\sum_{n=0}^{\infty}\left(\frac{c_{2n}}{2n+1} + \frac{c_{2n+1}}{6n+6}\right) \tag{19}$$

$$c_{n+2} = 2\,a\,c_{n+1} - d\,c_n\ ,\ c_0 = b,\ c_1 = 2\,a\,b \tag{20}$$

$$a = \frac{\left(2 - \sqrt{2}\right)\left(\sqrt{3} - 1\right)}{2}\ ,\ b = \frac{5\sqrt{2}}{2} + 2\sqrt{3} - \frac{3\sqrt{6}}{2} - 3 \tag{21}$$

$$d = 21\sqrt{6} - 30\sqrt{3} - 37\sqrt{2} + 53 \tag{22}$$

fórmula 5.

$$\pi = 6\tan^{-1}\left(\frac{1}{u}\right) - 3\sum_{n=1}^{\infty}\binom{2n}{n}\frac{10^{-n}}{n}\operatorname{Im}\left(\left(1+\frac{i}{\sqrt{3}}\right)^{n}\right) \tag{23}$$

$$u = 15\sqrt{\frac{\sqrt{93}}{90}+\frac{1}{10}}\left(1-\sqrt{\frac{\sqrt{93}}{30}+\frac{3}{10}}\right) \tag{24}$$

fórmula 6.

$$\pi = 6\tan^{-1}\left(\frac{1}{u}\right) - 3\sum_{n=1}^{\infty}\binom{2n}{n}\frac{(-10)^{-n}}{n}\operatorname{Im}\left(\left(1+\frac{i}{\sqrt{3}}\right)^{n}\right) \tag{25}$$

$$u = 15\sqrt{\frac{\sqrt{453}}{90}+\frac{7}{30}}\left(-1+\sqrt{\frac{\sqrt{453}}{30}+\frac{7}{10}}\right) \tag{26}$$

fórmula 7.

$$\pi = 4\tan^{-1}(u) + 2\sum_{n=1}^{\infty}\binom{2n}{n}\frac{10^{-n}}{n}\operatorname{Im}((1+i)^{n}) \tag{27}$$

$$u = 5\sqrt{\frac{\sqrt{13}}{10}+\frac{3}{10}} - \frac{\sqrt{13}+3}{2} \tag{28}$$

fórmula 8.

$$\pi = 4\tan^{-1}\left(\frac{2}{u}\right) - 2\sum_{n=1}^{\infty}\binom{2n}{n}\frac{(-10)^{-n}}{n}\operatorname{Im}((1+i)^{n}) \tag{29}$$

$$u = 7 + \sqrt{53} - 10\sqrt{\frac{\sqrt{53}}{10}+\frac{7}{10}} \tag{30}$$

fórmula 9.

$$\pi = 8\sum_{n=1}^{\infty}\binom{2n}{n}\frac{a^{n}}{n}\operatorname{Im}((1+i(1-a))^{n}) \tag{31}$$

$$a = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}+\sqrt{2-\sqrt{2+\sqrt{2}}}}, \quad i = \sqrt{-1} \tag{32}$$

fórmula 10.

$$\pi = 12\sum_{n=0}^{\infty}\left(\frac{c_{2n+1}}{2n+1}+\frac{c_{2n+2}}{6n+6}\right) \tag{33}$$

$$c_{n+2} = 2u\,c_{n+1} - \left(u^{2}+v^{2}\right)c_{n}, \quad c_{1} = v,\ c_{2} = 2uv \tag{34}$$

$$u = \frac{7}{6} - \frac{a}{6} - \frac{1}{a}, \quad v = \frac{a}{6} - \frac{1}{a} - \frac{1}{6}, \quad a = \sqrt{\frac{\sqrt{313} + 13}{2}} \tag{35}$$

fórmula 11.

$$\pi = 2 \sum_{n=1}^{\infty} \binom{2n}{n} \frac{1}{n} \left(12 \left(\frac{5814}{5^4\, 13^2} \right)^n a_n + 8 \left(\frac{46\,284}{5^6\, 13^2} \right)^n b_n - 5 \left(\frac{3\,412\,920}{13^8} \right)^n c_n \right) \tag{36}$$

$$a_n = \mathrm{Im}\left(\left(\frac{36}{323} + i \right)^n \right), \quad b_n = \mathrm{Im}\left(\left(\frac{57}{1624} + i \right)^n \right), \quad c_n = \mathrm{Im}\left(\left(\frac{239}{28\,560} + i \right)^n \right) \tag{37}$$

$$a_{n+2} = \frac{72}{323} a_{n+1} - \frac{5^4\, 13^2}{323^2} a_n, \quad a_1 = 1,\ a_2 = \frac{72}{323} \tag{38}$$

$$b_{n+2} = \frac{57}{812} b_{n+1} - \frac{5^6\, 13^2}{1624^2} b_n, \quad b_1 = 1,\ b_2 = \frac{57}{812} \tag{39}$$

$$c_{n+2} = \frac{239}{14\,280} c_{n+1} - \frac{13^8}{28\,560^2} c_n, \quad c_1 = 1,\ c_2 = \frac{239}{14\,280} \tag{40}$$

fórmula 12.

$$\pi = \frac{2}{b} \left(\ln 2 - x \ln y - a \ln\left(c^2 + d^2\right) + 2\, b \tan^{-1}\left(\frac{c - d}{c + d} \right) + \sum_{n=1}^{\infty} \frac{(-1)^n\, n!\, (2n)!}{n\, (3n)!} \right) \tag{41}$$

$$s = \frac{1}{3} \sqrt[3]{\frac{4}{31} + \frac{36}{31} \sqrt{\frac{1}{93}}}, \quad t = \frac{1}{3} \sqrt[3]{\frac{4}{31} - \frac{36}{31} \sqrt{\frac{1}{93}}} \tag{42}$$

$$p = \sqrt[3]{-4 + \frac{4}{3} \sqrt{\frac{31}{3}}}, \quad q = -\sqrt[3]{4 + \frac{4}{3} \sqrt{\frac{31}{3}}} \tag{43}$$

$$x = \frac{1}{3} + s + t, \quad y = 2 + p + q \tag{44}$$

$$a = \frac{1}{3} - \frac{s + t}{2}, \quad b = \frac{\sqrt{3}\,(s - t)}{2} \tag{45}$$

$$c = 2 - \frac{p + q}{2}, \quad d = \frac{\sqrt{3}\,(p - q)}{2} \tag{46}$$

fórmula 13.

$$\pi = 6 \sum_{n=1}^{\infty} \binom{2n}{n} \frac{1}{n} \left(\frac{9 - 5\sqrt{3}}{2} \right)^n \mathrm{Im}\left((1 + i)^n\right) \tag{47}$$

fórmula 14.

$$\pi = 6 \sum_{n=1}^{\infty} \binom{2n}{n} \left(\sum_{k=1}^{n} \frac{1}{k} \right) \mathrm{Im}\left(y\, x^n\right) \tag{48}$$

$$x = \frac{1057 - 172\sqrt{3}}{5329} + i\,\frac{434 + 247\sqrt{3}}{5329} \tag{49}$$

$$y = \frac{29 + 16\sqrt{3}}{73} - i\,\frac{20 + 6\sqrt{3}}{73} \tag{50}$$

$$c_n = \mathrm{Im}(y\,x^n) \tag{51}$$

$$c_{n+2} = 2\,a\,c_{n+1} - b\,c_n \tag{52}$$

$$a = \frac{1057 - 172\sqrt{3}}{5329}\,,\ \ b = \frac{296 - 28\sqrt{3}}{5329} \tag{53}$$

$$c_1 = \frac{582 + 233\sqrt{3}}{5329}\,,\ \ c_2 = \frac{594\,524 + 203\,586\sqrt{3}}{28\,398\,241} \tag{54}$$

fórmula 15.

$$\pi = 12\tan^{-1}\left(\frac{b}{1+a}\right) + 6\,A + 6\,B \tag{55}$$

$$A = a\sum_{n=1}^{\infty}\binom{4\,n-2}{2\,n-1}H_{2\,n-1}(-1)^{n-1}\left(\frac{1}{4\sqrt{3}}\right)^{2\,n-1} \tag{56}$$

$$B = b\sum_{n=1}^{\infty}\binom{4\,n}{2\,n}H_{2\,n}(-1)^{n-1}\left(\frac{1}{4\sqrt{3}}\right)^{2\,n} \tag{57}$$

$$a = \sqrt{\frac{1}{\sqrt{3}} + \frac{1}{2}}\,,\ \ b = \sqrt{\frac{1}{\sqrt{3}} - \frac{1}{2}}\,,\ \ H_n = \sum_{k=1}^{n}\frac{1}{k} \tag{58}$$

fórmula 16.

$$\pi = 16\tan^{-1}\left(\frac{b}{1+a}\right) + 8\,A + 8\,B \tag{59}$$

$$A = a\sum_{n=1}^{\infty}\binom{4\,n-2}{2\,n-1}H_{2\,n-1}(-1)^{n-1}\left(\frac{\sqrt{2}-1}{4}\right)^{2\,n-1} \tag{60}$$

$$B = b\sum_{n=1}^{\infty}\binom{4\,n}{2\,n}H_{2\,n}(-1)^{n-1}\left(\frac{\sqrt{2}-1}{4}\right)^{2\,n} \tag{61}$$

$$a = \sqrt{\frac{\sqrt{4-2\sqrt{2}}+1}{2}}\,,\ \ b = \sqrt{\frac{\sqrt{4-2\sqrt{2}}-1}{2}}\,,\ \ H_n = \sum_{k=1}^{n}\frac{1}{k} \tag{62}$$

fórmula 17.

$$\pi = \frac{5}{9}\sum_{n=0}^{\infty}\left(-\frac{13}{36}\right)^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{13}\right)^k\left(\frac{6}{2\,n+2\,k+1} + \frac{1}{2\,n+2\,k+3}\right) \tag{63}$$

fórmula 18.

$$\pi = \frac{13\sqrt{3}}{98}\sum_{n=0}^{\infty}\left(-\frac{85}{588}\right)^n \sum_{k=0}^{n}\binom{n}{k}\left(\frac{3}{85}\right)^k\left(\frac{14}{2n+2k+1}+\frac{1}{2n+2k+3}\right) \tag{64}$$

fórmula 19.

$$\pi = a\sum_{n=0}^{\infty}(-b)^n \sum_{k=0}^{n}\binom{n}{k}c^k\left(\frac{1\,136\,445}{2n+2k+1}-\frac{219}{2n+2k+3}\right) \tag{65}$$

$$a = \frac{4}{5^2\,239^2}\,,\quad b = \frac{57\,146}{5^2\,239^2}\,,\quad c = \frac{1}{57\,146} \tag{66}$$

fórmula 20.

$$\pi = a\sum_{n=0}^{\infty}(-b)^n \sum_{k=0}^{n}\binom{n}{k}c^k\left(\frac{\left(1+\sqrt{2}\right)\left(2+\sqrt{3}\right)}{2n+2k+1}+\frac{1}{2n+2k+3}\right) \tag{67}$$

$$a = \frac{24\left(3+\sqrt{2}+\sqrt{3}\right)}{5\left(1+\sqrt{2}\right)^2\left(2+\sqrt{3}\right)^2} \tag{68}$$

$$b = \left(\sqrt{2}-1\right)^2+\left(2-\sqrt{3}\right)^2 \tag{69}$$

$$c = \frac{1}{\left(\sqrt{2}+1\right)^2+\left(2+\sqrt{3}\right)^2} \tag{70}$$

fórmula 21.

$$\pi = 2\sqrt{3}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{\left(\sqrt{3}\right)^{-(n^2+n+2k)}}{n^2+n+2k+1}\,c(n,\,k-n) \tag{71}$$

$$c(0,\,k+3) = -c(0,\,k+2)-c(0,\,k+1)-c(0,\,k)\,,\quad k\in\mathbb{N}\cup\{0\} \tag{72}$$

$$c(0,\,0)=1\,,\quad c(0,\,1)=-1\,,\quad c(0,\,2)=0 \tag{73}$$

$$c(n,\,0)=1\,,\quad n\in\mathbb{N}\cup\{0\} \tag{74}$$

$$c(n+1,\,k)=c(n,\,k)\,,\quad 0\le k\le n+2\,,\quad n\in\mathbb{N}\cup\{0\} \tag{75}$$

$$c(n+1,\,k)=c(n,\,k)-c(n+1,\,k-n-3)\,,\quad k\ge n+3\,,\quad n\in\mathbb{N}\cup\{0\} \tag{76}$$

fórmula 22.

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n\sum_{k=0}^{n}\left(\frac{1}{3}\right)^{(m-2)k/2}\left(\frac{1}{2n+(m-2)k+1}+\frac{3^{-m/2}}{2n+(m-2)k+m+1}\right) \tag{77}$$

$$m\in\mathbb{N}$$

fórmula 23.

$$\pi = a\sum_{n=0}^{\infty}(-b)^n\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}c^k\,d^m\,f(n,\,k,\,m) \tag{78}$$

$$f(n, k, m) = \frac{248\,640}{2n + 2k + 2m + 1} + \frac{11\,748}{2n + 2k + 2m + 3} + \frac{132}{2n + 2k + 2m + 5} \tag{79}$$

$$a = \frac{1}{78\,400}, \; b = \frac{5961}{78\,400}, \; c = \frac{46}{1987}, \; d = \frac{1}{138} \tag{80}$$

fórmula 24.

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n \, 3^{-n-(2m-1)k}}{2n + (4m-2)k + 1} \, c(m, n-k) \tag{81}$$

$$m \in \mathbb{N}, \; c(m, n) = \begin{cases} 1 & , n = 1, 1, \ldots, 2m - 1 \\ 2 & , n \ge 2m \end{cases} \tag{82}$$

fórmula 25.

$$\pi = 6 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n \, 3^{-(m-1)k-(n+1)/2}}{n + (2m-2)k + 1} \tag{83}$$

$$m \in \mathbb{N}, \; c_{m,-1} = 0, \; c_{m,0} = 1 \tag{84}$$

$$c_{m,n} = \begin{cases} 1 - c_{m,2m-3} & , n = 2m - 1 \\ -c_{m,n-2} & , n \in \mathbb{N} - \{2m - 1\} \end{cases} \tag{85}$$

fórmula 26.

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} (-1)^n \binom{n+m}{n} \sum_{k=0}^{n} \binom{m}{k} \frac{3^{-n-k}}{2n + 2k + 1} \tag{86}$$

$$m \in \mathbb{N} \cup \{0\}$$

fórmula 27.

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} (-1)^n \left(\sum_{k=0}^{m-1} \binom{n+k+1}{n} \frac{3^{-n-k}}{2n + 2k + 1} + \binom{n+m}{n} \frac{3^{-n-m}}{2n + 2m + 1} \right) \tag{87}$$

$$m \in \mathbb{N}$$

fórmula 28.

$$\pi = 8 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{j=0}^{k+1} \binom{n}{k} \binom{k+1}{j} \frac{(-1)^n \left(\sqrt{2} - 1\right)^{mn+(2-m)k+mj+1}}{mn + (2-m)k + mj + 1} \tag{88}$$

$$m \in \mathbb{N}$$

fórmula 29.

$$\pi = 8 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{j=0}^{k+1} \binom{n}{k} \binom{k+1}{j} \frac{(-1)^{k+j} \left(\sqrt{2} - 1\right)^{mn+(2-m)k+mj+1}}{mn + (2-m)k + mj + 1} \tag{89}$$

$$m \in \mathbb{N}$$

fórmula 30.

$$\pi = 24 \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}\frac{(-1)^k\left(\left(1/\sqrt{3}\right)^{k+2j+2}-\left(\sqrt{2}-1\right)^{k+2j+2}\right)}{k+2j+2} \tag{90}$$

fórmula 31.

$$\pi = 4\sum_{n=0}^{\infty}\tan^{-1}\left(\frac{\operatorname{Im}\left((2+i)^{2^n}\right)}{3^{2^n}+\operatorname{Re}\left((2+i)^{2^n}\right)}\right) \tag{91}$$

fórmula 32.

$$\pi = 6\sum_{n=0}^{\infty}\tan^{-1}\left(\frac{\operatorname{Im}\left(\left(3+i\sqrt{3}\right)^{2^n}\right)}{6^{2^n}+\operatorname{Re}\left(\left(3+i\sqrt{3}\right)^{2^n}\right)}\right) \tag{92}$$

fórmula 33.

$$\pi = 4\sum_{n=0}^{\infty}\tan^{-1}\left(\frac{\operatorname{Im}\left(\left(4+i\left(4-\sqrt{7}\right)\right)^{2^n}\right)}{5^{2^n}+\operatorname{Re}\left(\left(4+i\left(4-\sqrt{7}\right)\right)^{2^n}\right)}\right) \tag{93}$$

fórmula 34.

$$\pi = 4\tan^{-1}\left(\frac{y}{x}\right) - 4\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\operatorname{Im}\left(\left(1+i\left(\sqrt{6}-1\right)\right)^{2n+2}\right)}{2^{2n+2}+\operatorname{Re}\left(\left(1+i\left(\sqrt{6}-1\right)\right)^{2n+2}\right)}\right) \tag{94}$$

$$y = \sum_{n=0}^{\infty}\operatorname{Im}(z_n) \tag{95}$$

$$x = \sum_{n=0}^{\infty}\operatorname{Re}(z_n) \tag{96}$$

$$z_{n+1} = z_n\frac{\left(1+i\left(\sqrt{6}-1\right)\right)^{2n+2}}{2^{2n+2}-\left(1+i\left(\sqrt{6}-1\right)\right)^{2n+2}},\ z_0 = 1 \tag{97}$$

fórmula 35.

$$\pi = 8\tan^{-1}\left(\frac{y}{x}\right) - 8\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\operatorname{Im}\left(\left(1+i\left(5\sqrt{2}-5\right)\right)^{2n+1}\right)}{2^{4n+2}+\operatorname{Re}\left(\left(1+i\left(5\sqrt{2}-5\right)\right)^{2n+1}\right)}\right) \tag{98}$$

$$y = \sum_{n=0}^{\infty}\operatorname{Im}(z_n) \tag{99}$$

$$x = \sum_{n=0}^{\infty}\operatorname{Re}(z_n) \tag{100}$$

$$z_{n+1} = z_n\frac{4\left(1+i\left(5\sqrt{2}-5\right)\right)^{2n+1}}{2^{4n+4}-\left(1+i\left(5\sqrt{2}-5\right)\right)^{2n+2}},\ z_0 = 1 \tag{101}$$

fórmula 36.

$$\pi = m\tan^{-1}\left(\frac{y}{x}\right) - m\tan^{-1}\left(x^9\right) - m\sum_{n=1}^{\infty}(-1)^n\left(\tan^{-1}\left(x^{10n+1}\right) + \tan^{-1}\left(x^{10n+9}\right)\right) \tag{102}$$

$$y = \sum_{n=0}^{\infty}\operatorname{Im}(z_n) \tag{103}$$

$$x = \sum_{n=0}^{\infty}\operatorname{Re}(z_n) \tag{104}$$

$$z_{n+1} = z_n\,\frac{(-1)^n x^{2n+1}\, i}{1 - x^{n+1}\, i^{n+1}}, \quad z_0 = 1 \tag{105}$$

$$\{(m, x)\} = \left\{\left(6, \frac{1}{\sqrt{3}}\right), \left(8, \sqrt{2} - 1\right), \left(12, 2 - \sqrt{3}\right)\right\} \tag{106}$$

fórmula 37.

$$\pi = m\tan^{-1}\left(\frac{y}{x}\right) - m\tan^{-1}\left(x^7\right) - m\sum_{n=1}^{\infty}(-1)^n\left(\tan^{-1}\left(x^{10n+3}\right) + \tan^{-1}\left(x^{10n+7}\right)\right) \tag{107}$$

$$y = \sum_{n=0}^{\infty}\operatorname{Im}(z_n) \tag{108}$$

$$x = \sum_{n=0}^{\infty}\operatorname{Re}(z_n) \tag{109}$$

$$z_{n+1} = z_n\,\frac{(-1)^{n+1} x^{2n+2}}{1 + (-1)^n x^{n+1}\, i^{n+1}}, \quad z_0 = 1 \tag{110}$$

$$\{(m, x)\} = \left\{\left(8, \sqrt[3]{\sqrt{2} - 1}\right), \left(12, \sqrt[3]{2 - \sqrt{3}}\right)\right\} \tag{111}$$

fórmula 38.

$$\pi = 4\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{(1 + \operatorname{Re}(z_{n+1}))\operatorname{Im}(z_n) - (1 + \operatorname{Re}(z_n))\operatorname{Im}(z_{n+1})}{(1 + \operatorname{Re}(z_n))(1 + \operatorname{Re}(z_{n+1})) + \operatorname{Im}(z_n)\operatorname{Im}(z_{n+1})}\right) \tag{112}$$

$$z_n = (-x + i(1 - x))^n, \quad n \in \mathbb{N} \tag{113}$$

$$z_{n+1} = z_1 z_n, \quad z_1 = -x + i(1 - x) \tag{114}$$

$$0 < x < 1$$

fórmula 39.

$$\pi = 6\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{(1 + \operatorname{Re}(z_{n+1}))\operatorname{Im}(z_n) - (1 + \operatorname{Re}(z_n))\operatorname{Im}(z_{n+1})}{(1 + \operatorname{Re}(z_n))(1 + \operatorname{Re}(z_{n+1})) + \operatorname{Im}(z_n)\operatorname{Im}(z_{n+1})}\right) \tag{115}$$

$$z_n = \left(x + i\,\frac{1 + x}{\sqrt{3}}\right)^n, \quad n \in \mathbb{N} \tag{116}$$

$$z_{n+1} = z_1 z_n, \quad z_1 = x + i\,\frac{1 + x}{\sqrt{3}} \tag{117}$$

$$-1 < x < 1/2$$

fórmula 40.

$$\frac{\ln 2}{2} + i\,\frac{\pi}{4} = \sum_{n=0}^{\infty} z^{3n+1}\left(\frac{1}{3n+1} + \frac{z}{3n+2} - \frac{2z^2}{3n+3}\right) \quad (118)$$

$$z = \frac{1}{2}\left(-1 + \sqrt{\frac{\sqrt{17}+1}{2}} + i\sqrt{\frac{\sqrt{17}-1}{2}}\right) \quad (119)$$

fórmula 41.

$$\frac{\ln(4/3)}{2} + i\,\frac{\pi}{6} = \sum_{n=0}^{\infty} z^{3n+1}\left(\frac{1}{3n+1} + \frac{z}{3n+2} - \frac{2z^2}{3n+3}\right) \quad (120)$$

$$z = \frac{1}{2}\left(-1 + \sqrt{\frac{\sqrt{19/3}+1}{2}} + i\sqrt{\frac{\sqrt{19/3}-1}{2}}\right) \quad (121)$$

fórmula 42.

$$i\,\frac{\pi}{4} = \sum_{n=0}^{\infty} z^{3n+1}\left(\frac{1}{3n+1} + \frac{z}{3n+2} - \frac{2z^2}{3n+3}\right) \quad (122)$$

$$z = \frac{1}{2}\left(-1 + \sqrt{\frac{a+2\sqrt{2}-3}{2}} + i\sqrt{\frac{a-2\sqrt{2}+3}{2}}\right) \quad (123)$$

$$a = \sqrt{25 - 12\sqrt{2}} \quad (124)$$

fórmula 43.

$$\frac{\ln\left(4 - 2\sqrt{2}\right)}{2} + i\,\frac{\pi}{8} = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}\, c_n \quad (125)$$

$$z = \frac{\sqrt{3}\,(a-b)}{2} + i\,\frac{(a+b)}{2} \quad (126)$$

$$a = \sqrt[3]{\frac{\sqrt{2}-1}{2} + \frac{1}{6}\sqrt{\frac{77}{3} - 18\sqrt{2}}} \quad (127)$$

$$b = \sqrt[3]{\frac{\sqrt{2}-1}{2} - \frac{1}{6}\sqrt{\frac{77}{3} - 18\sqrt{2}}} \quad (128)$$

$$c_{n+3} = -c_{n+2} - c_n\,,\ \ n \in \mathbb{N} \bigcup \{0\} \quad (129)$$

$$c_0 = 1\,,\ \ c_1 = -1\,,\ \ c_2 = 4 \quad (130)$$

fórmula 44.

$$\pi = 8\tan^{-1}\left(\frac{\mathrm{Im}(z_m)}{\mathrm{Re}(z_m)}\right) + 4\tan^{-1}\left(\frac{\mathrm{Im}(w_m)}{\mathrm{Re}(w_m)}\right) - 4\sum_{n=1}^{m-1}\left(2\tan^{-1}\left(3^{-n-1}\right) + \tan^{-1}\left(7^{-n-1}\right)\right) \tag{131}$$

$$z_{n+1} = z_n\left(1 + i\,3^{-n-1}\right),\ z_1 = 1 + \frac{i}{3},\ 1 \le n \le m-1 \tag{132}$$

$$w_{n+1} = w_n\left(1 + i\,7^{-n-1}\right),\ w_1 = 1 + \frac{i}{3},\ 1 \le n \le m-1 \tag{133}$$

$$m \in \mathbb{N} - \{1\}$$

fórmula 45.

$$\frac{\ln(4/3)}{2} + i\,\frac{\pi}{6} = \sum_{n=0}^{\infty}\frac{a^{n+1}\,i^{n+1}}{n+1}\,u_n \tag{134}$$

$$\frac{\ln\left(4 - 2\sqrt{2}\right)}{2} + i\,\frac{\pi}{8} = \sum_{n=0}^{\infty}\frac{b^{n+1}\,i^{n+1}}{n+1}\,u_n \tag{135}$$

$$\frac{\ln\left(8 - 4\sqrt{3}\right)}{2} + i\,\frac{\pi}{12} = \sum_{n=0}^{\infty}\frac{c^{n+1}\,i^{n+1}}{n+1}\,u_n \tag{136}$$

$$a^5 + a = \frac{1}{\sqrt{3}},\ a = 0.5339\ \ldots \tag{137}$$

$$\frac{1}{a} = \sqrt[5]{\sqrt{3} + \sqrt{3}\,\sqrt[5/4]{\sqrt{3} + \sqrt{3}\,\sqrt[5/4]{\sqrt{3} + \ldots}}} \tag{138}$$

$$b^5 + b = \sqrt{2} - 1,\ b = 0.4035\ \ldots \tag{139}$$

$$\frac{1}{b} = \sqrt[5]{x + x\sqrt[5/4]{x + x\sqrt[5/4]{x + \ldots}}}\ ,\ x = \sqrt{2} - 1 \tag{140}$$

$$c^5 + c = 2 - \sqrt{3},\ c = 0.2666\ \ldots \tag{141}$$

$$\frac{1}{c} = \sqrt[5]{x + x\sqrt[5/4]{x + x\sqrt[5/4]{x + \ldots}}}\ ,\ x = 2 - \sqrt{3} \tag{142}$$

$$u_{n+5} = -u_{n+4} - u_n,\ n \in \mathbb{N} \cup \{0\} \tag{143}$$

$$u_0 = 1,\ u_1 = -1,\ u_2 = 1,\ u_3 = -1,\ u_4 = 6 \tag{144}$$

fórmula 46.

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n\left(a_n + b_n\sqrt{3}\right)}{(2n+1)\,3^{2n+1}} \tag{145}$$

$$a_{n+2} = 126\,a_{n+1} - 81\,a_n,\ a_0 = 6,\ a_1 = 702 \tag{146}$$

$$b_{n+3} = 129\,b_{n+2} - 459\,b_{n+1} + 243\,b_n \tag{147}$$

$$b_0 = -2,\ b_1 = -402,\ b_2 = -50\,778 \tag{148}$$

fórmula 47.

$$\pi = 12 \sum_{n=0}^{\infty} \frac{(-1)^n \left(a_n + b_n \sqrt{3}\right)}{(2n+1)\, 3^{2n+1}} \tag{149}$$

$$a_{n+2} = 126\, a_{n+1} - 81\, a_n\,,\ \ a_0 = -6\,,\ \ a_1 = -702 \tag{150}$$

$$b_{n+3} = 129\, b_{n+2} - 459\, b_{n+1} + 243\, b_n \tag{151}$$

$$b_0 = 4\,,\ \ b_1 = 408\,,\ \ b_2 = 50\,796 \tag{152}$$

fórmula 48.

$$\pi = \frac{24}{5} \sum_{n=0}^{\infty} \frac{(-1)^n \left(a_n + b_n \sqrt{2} + c_n \sqrt{3}\right)}{2n+1} \tag{153}$$

$$a_{n+4} = 20\, a_{n+3} - 86\, a_{n+2} + 20\, a_{n+1} - a_n \tag{154}$$

$$a_0 = 1\,,\ \ a_1 = 19\,,\ \ a_2 = 321\,,\ \ a_3 = 4803 \tag{155}$$

$$b_{n+2} = 6\, b_{n+1} - b_n\,,\ \ b_0 = 1\,,\ b_1 = 5 \tag{156}$$

$$c_{n+2} = 14\, c_{n+1} - c_n\,,\ \ c_0 = -1\,,\ c_1 = -15 \tag{157}$$

fórmula 49.

$$\pi = 24 \sum_{n=0}^{\infty} \frac{(-1)^n \left(a_n + b_n \sqrt{2} + c_n \sqrt{3}\right)}{2n+1} \tag{158}$$

$$a_{n+4} = 20\, a_{n+3} - 86\, a_{n+2} + 20\, a_{n+1} - a_n \tag{159}$$

$$a_0 = -3\,,\ \ a_1 = -33\,,\ \ a_2 = -403\,,\ \ a_3 = -5281 \tag{160}$$

$$b_{n+2} = 6\, b_{n+1} - b_n\,,\ \ b_0 = 1\,,\ b_1 = 5 \tag{161}$$

$$c_{n+2} = 14\, c_{n+1} - c_n\,,\ \ c_0 = 1\,,\ c_1 = 15 \tag{162}$$

fórmula 50.

$$\pi = 4 \tan^{-1}\left(\frac{u}{v}\right) - 4 \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{\operatorname{Im}((1-i)^{n+1})}{2^{n+1} - \operatorname{Re}((1-i)^{n+1})}\right) \tag{163}$$

$$v = \sum_{n \in \mathbb{Z}} (-1)^n \left(\frac{1}{2}\right)^{n(3n+1)/2} \operatorname{Im}\left((1-i)^{n(3n+1)/2}\right) \tag{164}$$

$$u = \sum_{n \in \mathbb{Z}} (-1)^n \left(\frac{1}{2}\right)^{n(3n+1)/2} \operatorname{Re}\left((1-i)^{n(3n+1)/2}\right) \tag{165}$$

fórmula 51.

$$\pi = 8 \sum_{n=0}^{\infty} \frac{(-1)^n \left(a_n + b_n \sqrt{2}\right)}{2n+1} \tag{166}$$

$$a_{n+4} = \alpha\, a_{n+3} + \beta\, a_{n+2} + \gamma\, a_{n+1} + \delta\, a_n\,,\ n \in \mathbb{N} \bigcup \{0\} \tag{167}$$

$$b_{n+4} = \alpha\, b_{n+3} + \beta\, b_{n+2} + \gamma\, b_{n+1} + \delta\, b_n\,,\ n \in \mathbb{N} \bigcup \{0\} \tag{168}$$

$$\alpha = \frac{36\,303}{18\,818},\ \beta = -\frac{106\,337}{150\,544},\ \gamma = \frac{2031}{75\,272},\ \delta = -\frac{1}{150\,544} \tag{169}$$

$$a_0 = -\frac{66}{97},\ a_1 = -\frac{5\,715\,082}{3\,550\,733},\ a_2 = -\frac{3\,529\,127}{1\,410\,377},\ a_3 = -\frac{6\,312\,433}{1\,702\,136} \tag{170}$$

$$b_0 = \frac{74}{97},\ b_1 = \frac{4\,200\,793}{3\,650\,692},\ b_2 = \frac{886\,645}{500\,957},\ b_3 = \frac{7\,695\,878}{2\,934\,721} \tag{171}$$

fórmula 52.

$$\pi = \frac{24}{3\,a + 4\,b + 3\,a\sqrt{2} + 2\,b\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2\,n+1}\, c_n \tag{172}$$

$$c_{n+2} = \left(10 - 2\sqrt{2} - 4\sqrt{3}\right) c_{n+1} + \left(-21 + 14\sqrt{2} + 12\sqrt{3} - 8\sqrt{6}\right) c_n \tag{173}$$

$$c_0 = a + b,\ \ c_1 = 3\,a + 7\,b - 2\,a\sqrt{2} - 4\,b\sqrt{3} \tag{174}$$

$$a,\ b \in \mathbb{C}$$

fórmula 53.

$$\pi = 12 \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k+n}{k-n} \frac{(-1)^k}{(2\,n+1)\left(\sqrt{3}+1\right)^{k+n+1}} \tag{175}$$

$$\pi = 4 \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k+n}{k-n} \frac{(-1)^k}{(2\,n+1)\,3^{2\,n}} \left(\frac{\sqrt{3}-1}{3}\right)^{k-n} \tag{176}$$

$$\pi = 4 \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k+n}{k-n} \frac{(-1)^k}{(2\,n+1)\,2^{2\,n}} \left(\frac{\sqrt{2}-1}{2}\right)^{k-n} \tag{177}$$

$$\pi = 3 \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k+n}{k-n} \frac{(-1)^n \left(2-\sqrt{3}\right)^{k-n}}{(2\,n+1)\,2^{k+n}} \tag{178}$$

fórmula 54.

$$\pi = 16 \sum_{k=0}^{\infty} \frac{1}{c_k\, c_{k+1}} \sum_{n=0}^{k} \frac{(-1)^n}{2\,n+1}\, c_n \tag{179}$$

$$c_{n+2} = 6\,c_{n+1} - c_n,\ \ c_0 = 1,\ c_1 = 5 \tag{180}$$

fórmula 55.

$$\pi = 48 \sum_{k=0}^{\infty} \frac{1}{c_k\, c_{k+1}} \sum_{n=0}^{k} \frac{(-1)^n}{2\,n+1}\, c_n \tag{181}$$

$$c_{n+2} = 14\,c_{n+1} - c_n,\ \ c_0 = 1,\ c_1 = 15 \tag{182}$$

fórmula 56.

$$\pi = 5\sqrt{5 - 2\sqrt{5}} \left(1 + 2 \sum_{k=1}^{\infty} \frac{5^k}{c_k\, c_{k+1}} \sum_{n=1}^{k} \frac{(-1)^n}{2\,n+1}\, c_n\right) \tag{183}$$

$$c_{n+2} = 10\,c_{n+1} - 5\,c_n,\ \ c_0 = 2,\ c_1 = 20 \tag{184}$$

fórmula 57.

$$\pi = 2\sqrt{3}\sum_{k=0}^{\infty}\frac{(-1)^k\,(n+k)!}{n!\,k!}\sum_{m=0}^{n}\binom{n}{m}\frac{3^{-m-k}}{2m+2k+1}\,,\ n\in\mathbb{N} \tag{185}$$

fórmula 58.

$$\pi = 4\sum_{k=0}^{\infty}\frac{(-1)^k\,(n+k)!}{n!\,k!}\sum_{m=0}^{n}\binom{n}{m}\frac{\left(x^{2m+2k+1}+y^{2m+2k+1}\right)}{2m+2k+1} \tag{186}$$

$$n\in\mathbb{N}\,,\ 0<x<1\,,\ y=\frac{1-x}{1+x}$$

fórmula 59.

$$x_1 = 3\,,\ x_{n+1} = \frac{1}{3+x_n}\left(3\,x_n + 6\sum_{k=1}^{2^n}k^{-2}\right),\ n\in\mathbb{N} \tag{187}$$

$$\lim_{n\to\infty} x_n = \pi \tag{188}$$

$$x_{n+2} = \frac{1}{3+x_{n+1}}\left(6\,x_{n+1} - 3\,x_n + x_n\,x_{n+1} + 6\sum_{k=2^n+1}^{2^{n+1}}k^{-2}\right),\ n\in\mathbb{N} \tag{189}$$

$$x_1 = 3\,,\ x_2 = 11/4 \tag{190}$$

fórmula 60.

$$x_1 = 3\,,\ x_{n+1} = \frac{1}{3+x_n}\left(3\,x_n + 12\sum_{k=1}^{2^n}(-1)^{k-1}\,k^{-2}\right),\ n\in\mathbb{N} \tag{191}$$

$$\lim_{n\to\infty} x_n = \pi \tag{192}$$

$$x_{n+2} = \frac{1}{3+x_{n+1}}\left(6\,x_{n+1} - 3\,x_n + x_n\,x_{n+1} + 12\sum_{k=2^n+1}^{2^{n+1}}(-1)^{k-1}\,k^{-2}\right),\ n\in\mathbb{N} \tag{193}$$

$$x_1 = 3\,,\ x_2 = 3 \tag{194}$$

fórmula 61.

$$\pi = 4\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\operatorname{Im}(z^{2n+2}-z^{4n+2}-z^{6n+4})}{1+\operatorname{Re}(z^{2n+2}-z^{4n+2}-z^{6n+4})}\right) \tag{195}$$

donde

$$z = x + i\,y\,,\ i=\sqrt{-1}\,,\ x,\ y\in\mathbb{R} \tag{196}$$

$$x^4 + y^4 - 6\,x^2\,y^2 + 4\,x^3\,y - 4\,x\,y^3 - 1 = 0\,,\ x^2+y^2<1 \tag{197}$$

$$\text{Ejemplo}:\ z = \left(\frac{5+3\sqrt{3}}{16}\right)^{1/4} + \frac{i}{2^{3/4}\left(19+11\sqrt{3}\right)^{1/4}} \tag{198}$$

fórmula 62.

$$\pi = \lim_{n\to\infty}\left(\frac{n!\,((n+1)!)^3\,a_n}{81\cdot 2^{4n-8}\,b_n}\right) \tag{199}$$

$$a_n = \sum_{k=0}^{n}\binom{2k}{k}^3 2^{6n-6k}\ ,\ \ n\in\mathbb{N} \tag{200}$$

$$b_n = \prod_{k=1}^{n}(4k+3)\ ,\ n\in\mathbb{N} \tag{201}$$

fórmula 63.

$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = 0.5773502\ ... \tag{202}$$

$$k_n = \left[\frac{10^n}{\sqrt{3}}\right]\ ,\ \ n\in\mathbb{N} \tag{203}$$

$$\{k_n : n\in\mathbb{N}\} = \{5,\ 57,\ 577,\ 5773,\ 57\,735,\ ...\} \tag{204}$$

$$\pi = 6\tan^{-1}\left(\frac{k_1}{10}\right) + 6\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{10^n\,(k_{n+1}-10\,k_n)}{10^{2n+1}+k_n\,k_{n+1}}\right) \tag{205}$$

$$\pi = 6\tan^{-1}\left(\frac{1}{2}\right) + 6\tan^{-1}\left(\frac{7\cdot 10}{10^3+5\cdot 57}\right) + 6\tan^{-1}\left(\frac{7\cdot 10^2}{10^5+57\cdot 577}\right) + ... \tag{206}$$

fórmula 64.

$$\pi = 6\tan^{-1}\left(\frac{p\,a_0}{q}\right) + 6\sum_{n=0}^{\infty}\tan^{-1}\left(\frac{p\,q^{2n+1}(a_{n+1}-q^2\,a_n)}{q^{4n+4}+p^2\,a_n\,a_{n+1}}\right) \tag{207}$$

donde

$$p,\ q\ \in\mathbb{N} \tag{208}$$

$$\left|3\,p^2-q^2\right| < q^2 \tag{209}$$

$$a_n = \sum_{k=0}^{n}(-1)^k\binom{2k}{k}\left(3\,p^2-q^2\right)^k q^{2n-2k}\ ,\ \ n\in\mathbb{N}\cup\{0\} \tag{210}$$

fórmula 65.

$$\pi = 8\sum_{n=1}^{\infty}(-1)^{n-1}\tan^{-1}\left(\frac{1}{p_{2n}}\right) \tag{211}$$

$$p_{n+2} = 2\,p_{n+1} + p_n\ ,\ \ p_1 = 1,\ p_2 = 2 \tag{212}$$

fórmula 66.

$$\pi = 2 + 2\int_0^1\sqrt{\frac{1-x}{1+x}}\ dx \tag{213}$$

$$\pi = \frac{3\sqrt{3}}{2} + 3\int_{1/2}^{1}\sqrt{\frac{1-x}{1+x}}\,dx \tag{214}$$

$$\pi = 2\sqrt{2} + 4\int_{u}^{1}\sqrt{\frac{1-x}{1+x}}\,dx,\ u = \frac{1}{\sqrt{2}} \tag{215}$$

$$\pi = 3 + 6\int_{u}^{1}\sqrt{\frac{1-x}{1+x}}\,dx,\ u = \frac{\sqrt{3}}{2} \tag{216}$$

$$\pi = 6\left(\sqrt{2} - 1\right) + 12\int_{u}^{v}\sqrt{\frac{1-x}{1+x}}\,dx,\ u = \frac{1}{\sqrt{2}},\ v = \frac{\sqrt{3}}{2} \tag{217}$$

$$\pi = 3\left(\sqrt{3} - 1\right) + 6\int_{1/2}^{v}\sqrt{\frac{1-x}{1+x}}\,dx,\ v = \frac{\sqrt{3}}{2} \tag{218}$$

$$\pi = 6\sqrt{5 - 2\sqrt{6}} + 12\int_{1/2}^{v}\sqrt{\frac{1-x}{1+x}}\,dx,\ v = \frac{1}{\sqrt{2}} \tag{219}$$

$$\pi = 6 - 3\sqrt{3} + 6\int_{0}^{1/2}\sqrt{\frac{1-x}{1+x}}\,dx \tag{220}$$

$$\pi = 4 - 2\sqrt{2} + 4\int_{0}^{v}\sqrt{\frac{1-x}{1+x}}\,dx,\ v = \frac{1}{\sqrt{2}} \tag{221}$$

$$\pi = \frac{3}{2} + 3\int_{0}^{v}\sqrt{\frac{1-x}{1+x}}\,dx,\ v = \frac{\sqrt{3}}{2} \tag{222}$$

fórmula 67.

$$\pi = \frac{9 - 6\ln 2}{2\sqrt{3}} + \frac{3\sqrt{3}}{2}\int_{0}^{1}\sqrt[3]{\frac{1-x}{1+x}}\,dx \tag{223}$$

$$\pi = \frac{8 - 3\ln 3}{\sqrt{3}} + 3\sqrt{3}\int_{7/9}^{1}\sqrt[3]{\frac{1-x}{1+x}}\,dx \tag{224}$$

$$\pi = \frac{1 - 6\ln 2 + 3\ln 3}{\sqrt{3}} + 3\sqrt{3}\int_{0}^{7/9}\sqrt[3]{\frac{1-x}{1+x}}\,dx \tag{225}$$

fórmula 68.

$$\pi = \frac{3\sqrt{3}}{2} - 12\sum_{n=2}^{\infty}\int_{\pi/3}^{\pi/2}(\cos x)^n\cos(n\,x)\,dx \tag{226}$$

fórmula 69.

$$\pi = 4 \sum_{n=1}^{\infty} \frac{1}{2n-1} \operatorname{sen}\left((4n-2)\tan^{-1}\left(e^{-\pi/2}\right)\right) \tag{227}$$

$$\pi = 4 \sum_{n=1}^{\infty} \frac{1}{2n-1} \cos\left((4n-2)\tan^{-1}\left(e^{-\pi/2}\right)\right) \tag{228}$$

fórmula 70.

Sean c_n, $n \in \mathbb{N} \bigcup \{0\}$, definidos por :

$$c_0 = 1\,,\;\; c_1 = 1/2\,,\;\; c_n = \frac{(-1)^{n-1}(2n-1)}{(2n)!} - \sum_{k=2}^{n} \frac{(-1)^k\, 2^{2k-1}}{(2k)!}\, c_{n-k} \tag{229}$$

se tiene :

$$\pi = 4 \sum_{n=0}^{\infty} \frac{c_n}{2n+1}\, \alpha^{2n+1} \tag{230}$$

$$\alpha = \cos\alpha\,,\;\; \alpha = 0.73908513\ ... \tag{231}$$

$$\alpha = \cos(\cos(\cos(\ ...\ \cos(3/4)))) \tag{232}$$

$$\pi = 6 \sum_{n=0}^{\infty} \frac{c_n}{2n+1}\, \beta^{2n+1} \tag{233}$$

$$\beta = \frac{1}{\sqrt{3}} \cos\beta\,,\;\; \beta = 0.50522058\ ... \tag{234}$$

$$\beta = \frac{1}{\sqrt{3}} \cos\left(\frac{1}{\sqrt{3}} \cos\left(\frac{1}{\sqrt{3}} \cos\left(\ ...\ \frac{1}{\sqrt{3}} \cos(1/2)\right)\right)\right) \tag{235}$$

$$\pi = 8 \sum_{n=0}^{\infty} \frac{c_n}{2n+1}\, \gamma^{2n+1} \tag{236}$$

$$\gamma = \left(\sqrt{2} - 1\right) \cos\gamma\,,\;\; \gamma = 0.38404142\ ... \tag{237}$$

$$\gamma = \left(\sqrt{2} - 1\right) \cos\left(\left(\sqrt{2} - 1\right) \cos\left(\left(\sqrt{2} - 1\right) \cos\left(\ ...\ \left(\sqrt{2} - 1\right) \cos(1/3)\right)\right)\right) \tag{238}$$

$$\pi = 12 \sum_{n=0}^{\infty} \frac{c_n}{2n+1}\, \delta^{2n+1} \tag{239}$$

$$\delta = \left(2 - \sqrt{3}\right) \cos\delta\,,\;\; \delta = 0.25901138\ ... \tag{240}$$

$$\delta = \left(2 - \sqrt{3}\right) \cos\left(\left(2 - \sqrt{3}\right) \cos\left(\left(2 - \sqrt{3}\right) \cos\left(\ ...\ \left(2 - \sqrt{3}\right) \cos(1/4)\right)\right)\right) \tag{241}$$

fórmula 71.

$$\pi^2 = 12 \int_0^{\infty} e^{-x-x\,e^{-x-x\,e^{-x-\cdots}}}\, dx \tag{242}$$

$$\pi^2 = 12\ln\left(1 + e^{-1}\right) - 12\,\mathrm{Li}_2\left(-e^{-1}\right) - \frac{12}{1+e} + 12 \int_0^{u} e^{-x-x\,e^{-x-x\,e^{-x-\cdots}}}\, dx \tag{243}$$

$$u = \frac{1}{1 + e^{-1}} \tag{244}$$

$$\pi^2 = 12 \ln(2) \ln(3/2) - 12 \operatorname{Li}_2(-1/2) - 4 \ln 2 + 12 \int_0^u e^{-x-x\, e^{-x-x\, e^{-x-\cdots}}}\, dx \tag{245}$$

$$u = \frac{2 \ln 2}{3} \tag{246}$$

$$\operatorname{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \quad |x| \le 1 \tag{247}$$

fórmula 72.

$$\pi = \sqrt{\frac{3}{2}} \int_0^1 \sqrt{\frac{1 - x + \sqrt{1 + 2x - 3x^2}}{x}}\, dx \tag{248}$$

$$\pi = 3\sqrt{2} \int_0^1 \sqrt{\sqrt{45 + 4x^{-1}} - 7}\, dx \tag{249}$$

$$\pi = \frac{\sqrt{2}}{3} \int_0^1 \frac{\sqrt{2 + 7x + 3\sqrt{x(4 + 5x)}} - \sqrt{2 + 7x - 3\sqrt{x(4 + 5x)}}}{\sqrt{1 - x}}\, dx \tag{250}$$

fórmula 73.

$$\pi = \frac{3}{2}\sqrt{2} \sum_{n=0}^{\infty} \binom{4n+1}{2n} 2^{-6n} \int_0^1 x^{2n} \sqrt{1 + \sqrt{1 - \frac{x^2}{4}}}\, dx \tag{251}$$

$$\pi = 2\sqrt{2} \sum_{n=0}^{\infty} \binom{4n+1}{2n} 2^{-6n} \int_0^u x^{2n} \sqrt{1 + \sqrt{1 - \frac{x^2}{4}}}\, dx, \quad u = \sqrt{2 - \sqrt{2}} \tag{252}$$

fórmula 74.

$$\pi = 24 \int_0^{\infty} \frac{x^2}{a x^4 + 12 x^2 + 12}\, dx \tag{253}$$

$$a = 1 + \sqrt[3]{24\sqrt{78} + 181} - \sqrt[3]{24\sqrt{78} - 181} \tag{254}$$

fórmula 75.

$$\pi = \int_0^{\infty} \left(\frac{4x^2}{4x^4 + a_n x^2 + 4} \right)^{n+1} dx \tag{255}$$

$$a_n = \binom{2n}{n}^{2/(2n+1)} - 8, \quad n \in \mathbb{N} \cup \{0\} \tag{256}$$

fórmula 76.

$$\pi = \exp\left(2 \int_0^1 \ln\left(\Gamma\left(x + \frac{1}{2\sigma} \right) \right) dx \right) \tag{257}$$

$$\pi = \exp\left(2\int_0^1 \ln\left(\Gamma\left(x + \frac{1}{2\lambda}\right)\right) dx\right) \quad (258)$$

donde $\sigma = 0.213239 \ldots$ *y* $\lambda = 4.664380 \ldots$, son los ceros reales de la función :

$$f(x) = 2^x - 2\,e\,x \quad (259)$$

se tiene :

$$f(\sigma) = 0 \Longrightarrow \sigma = \frac{2^\sigma}{2\,e} \quad (260)$$

$$f(\lambda) = 0 \Longrightarrow \lambda = \frac{\ln(2\,e\,\lambda)}{\ln 2} = \log_2(2\,e\,\lambda) \quad (261)$$

El método del punto fijo :

$$\sigma_{n+1} = \frac{2^{\sigma_n}}{2\,e}\,,\ \ \sigma_0 = 0\,,\ \ n \in \mathbb{N} \bigcup \{0\} \ \Longrightarrow \sigma_n \to \sigma \quad (262)$$

$$\lambda_{n+1} = \log_2(2\,e\,\lambda_n)\ ,\ \ \lambda_0 = 1\,,\ \ n \in \mathbb{N} \bigcup \{0\} \ \Longrightarrow \lambda_n \to \lambda \quad (263)$$

$$\sigma = \frac{2^{-1+\frac{2^{-1+\frac{2^{-1+\cdots}}{e}}}{e}}}{e} \quad (264)$$

$$\lambda = \log_2(2\,e\log_2(2\,e\log_2(2\,e\ldots))) \quad (265)$$

$\Gamma(x)$ es la función gamma usual.

Referencias

A. Abramowitz, M. and Stegun, I.A. Handbook of Mathematical Functions.Nueva York:Dover,1965.

B. Gradshteyn, I.S. and Ryzhik, I.M. Table of Integrals,Series and Products.5th ed. , ed.Alan Jeffrey.Academic Press,1994.