On the self-consistency of the Lorentz transformation for simultaneous events

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Abstract
We confirm the self-consistency of the Lorentz transformation by solving the mathematical problem encountered when one deal with some thought experiments involving simultaneous events.

Keywords: Lorentz transformation,Thought experiment, Mathematical problem, simultaneity.

At the beginning of the 20th century a new theory known as the special theory of relativity has revolutionized the history of human thinking. Emerging mainly from the works of Hendrik Lorentz [1-5], Henri Poincaré [5-9] and the founding article by Albert Einstein in 1905 [10]; this theory encompasses a new vision of space and time linked together for playing the same role in the description of the physical world which become the spacetime instead of the classical space where time is considered as an absolute quantity for all observers. Nonetheless the new concepts of length contraction and time dilation introduced by the Lorentz transformation were at the origin of many thought experiments imagined in order to put these new concepts into practice, to particularly see their effects and their feasibility at the macroscopic scale. It turned out that the analysis of such experiences leads in most cases to different kind of paradoxes [11-53] constituting a subject of controversy for some people, whereas for the majority of physicists these paradoxes are just the results of a wrong assumptions, a misinterpretation of the theory, or an application of the Lorentz transformation out of its domaine of validity like with the famous twin paradox [11][49-53] where in reality we cannot apply the theory to claim that there is a paradox since the spacecraft carrying the twin needs to make a U-turn for being able to return to earth inducing automatically an acceleration, while Lorentz transformation is applicable only for inertial observers. Thus a reasonable explanation for many paradoxes can be found and solving SR paradoxes is of significant importance not only for the pedagogical reasons but it constitutes also a strong response to the detractors of the theory. The aim of the present work is to solve the mathematical problem encountered when one deal with a certain kind of thought experiments like that proposed in the reference [54] for which a mathematical analysis with the Lorentz transformation leads to some contradictions. By following a mathematical approach based on the logical reasoning, we show that these contradictions are not due to the Lorentz transformation but the consequence of its application for a situation that actually cannot occur from the point of view of special relativity, confirming thereby the self consistency of the Lorentz transformation. To well understand the issue we propose the following thought experiment: Let $R(O,x,y,z)$ and $R'(O',x',y',z')$ be two inertial frame of reference in standard configuration, where O and O' represent physically two observers equipped with clocks so that at the origin of time, O and O'coincide, then O' moves in the positive direction of O with a relativistic speed V in a linear motion relative to O. A giant screen is hanging on the plane (Oyz) which can be considered as a wall at rest relative to O, we assume that the giant screen is large enough and fixed in a location so as to be visible from O and B which is a point on the axis (Ox) at rest relative to O and located at a distance $OB = L$ from O (see Fig.1). The point B can physically be the place of any object at rest relative to O clearly visible for O' like a tree, a statue or a traffic sign. Knowing that the giant screen broadcasts TV programs continuously, the two observers decide to watch simultaneously a scene broadcasted on the giant screen at the moment where O' reaches B.

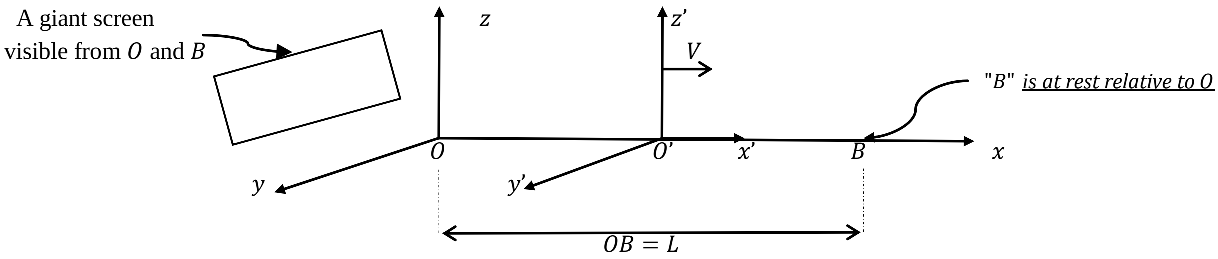

Figure 1:A giant- screen watched by two inertial observers O and O'

For that, O watches this scene when his clock shows the time $T = \frac{L}{V}$ corresponding exactly to the moment where O' arrives to B from where O' will also look to the screen, and his clocks shows at that instant the time $t' = T'$. The Lorentz transformation

$$(R'): \begin{cases} x' = \gamma(x - Vt) & (1) \\ y' = y & (2) \\ z' = z & (3) \\ t' = \gamma\left(t - \frac{V}{c^2}x\right) & (4) \end{cases} \Leftrightarrow (R): \begin{cases} x = \gamma(x' + Vt') & (5) \\ y = y' & (6) \\ z = z' & (7) \\ t = \gamma\left(t' + \frac{V}{c^2}x'\right) & (8) \end{cases}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}, \qquad c \text{ is the speed of light},$$

tells us that $T \neq T'$ and gives us the opportunity to get the relation betwen T and T'; Indeed the coordinates of the event: « The arrival of O' to B » relative to the frame of O and O' are respectively $(cT, L, 0,0), (cT', 0,0,0)$, so we deduce from (8) that

$$T = \gamma T'. \qquad (9)$$

But we too could get a relation between T and T' by considering the event: « The film scene watched on the screen by O and O' when O' encounters B» for which the coordinates in the frame of O and O' are respectively $(cT, 0, Y, Z), (cT', X', Y', Z')$, thus we deduce from (4)

$$T' = \gamma T. \qquad (10)$$

By combining equation (9) and (10) we obtain

$$\gamma = \pm 1, \qquad (11)$$

which contredict the hypothesis $\gamma > 1$, and we thus find the mathematical contradiction pointed out in the article [54] for which this thought experiment is an equivalent way to better understand the problem to solve. Although there is no doubt that (11) is a contradiction we should not rush to conclude that the Lorentz transformation is mathematically wrong. We must be aware that in mathematics when a general mathematical statement is used in a particular situation, leads to a contradiction it doesn't necessarily mean that this particular situation is a counterexample; it's possible that such particular case isn't allowed by the mathematical statement for having the right to assume it, so it's quite normal that one get a contradiction when one deal with such case, like when one assume that some object moves faster that the speed of light; even if the Lorentz transformation is written properly one should expect to get contradictions because the assumption is incompatible with the theory. We are exactly in this situation, the only difference is that the forbidden assumption is not easy to guess and everything appears to be perfect with this thought experiment the same as the one proposed in the reference [54]. In order to find the source of the contradiction (11), let's consider the more general case where the giant screen isn't necessarily in the plane (Oyz) but in an arbitrary position (x_s, y_s, z_s) relative to O, and (x'_s, y'_s, y'_s) relative to O' so that the screen remains at rest relative to O and clearly visible for both observers. If we write the Lorentz transformation (1) and (4) for the event : « The arivial of O' to B », we will have

$$\begin{cases} L = VT & (12) \\ T' = \gamma\left(T - \frac{V}{c^2}L\right), & (13) \end{cases}$$

and for the scene whatched on the screen

$$\begin{cases} x'_s = \gamma(x_s - VT) & (14) \\ T' = \gamma\left(T - \frac{V}{c^2}x_s\right). & (15) \end{cases}$$

We clearly see from these equations that all the contradictions disappear if

$$x_s = L, x'_s = 0, \qquad (16)$$

since equations (14) and (15) become identical to equations (12) and (13), as y_s, z_s are not necessarily equal to zero this doesn't mean that the two events should coincide, i.e the screen should necessarily be placed on B; it means that the screen couldn't be placed in any position and must be in the plane containing B which is perpendicular to the direction of motion and parallel to the plane (Oyz) (see Fig. 2).

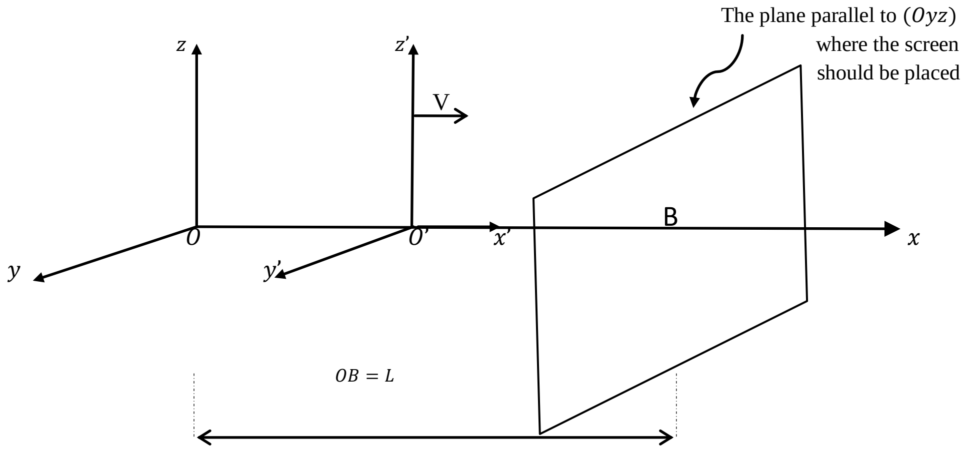

Figure 2:The right location for placing the screen in accordance with SR

Let's prove that the points of this plane are the only locations allowed by the Lorentz transformation for fixing the screen otherwise the special relativity will not be applicable. To achieve this we must consider two arbitrary events $A(ct_A, x_A, y_A, z_A)$ and $B(ct_B, x_B, y_B, z_B)$ in R for which the coordinates relative to R' are respectively$\left(ct'_A, x'_A, y'_A, z'_A\right)$, $B\left(ct'_B, x'_B, y'_B, z'_B\right)$ then by defining the separations $\Delta t = t_B - t_A, \Delta x = x_B - x_A, \Delta y = y_B - y_A, \Delta z = z_B - z_A, \Delta t' = t'_B - t'_A$, $\Delta x' = x'_B - x'_A, \Delta y' = y'_B - y'_A, \Delta z' = z'_B - z'_A$, and taking into account (1) (2) (3)(4) we get

$$\begin{cases} \Delta x' = \gamma(\Delta x - V\Delta t) & (17) \\ \Delta y' = \Delta y & (18) \\ \Delta z' = \Delta z & (19) \\ \Delta t' = \gamma\left(\Delta t - \frac{V}{c^2}\Delta x\right). & (20) \end{cases}$$

It's obvious that in special relativity two simultaneous events relative to an observer are necessarily not simultaneous for another one, nevertheless the arrival of O' to B and viewing the video on the screen are simultaneous for the two observers this is what motivate us to ask the question: When will two events be simultaneous for two relativistic inertial observers? In other words what conditions must be satisfied for two events for being considered as simultaneous in two relativistic inertial frames ? To answer this question let's suppose that the arbitrary events A and B have the same coordinates in R: $x_A = x_B$ and also in R' where $x'_A = x'_B$, so $\Delta x = \Delta x' = 0$, in this case (17) and (20) become

$$\begin{cases} \Delta t = 0 & (21) \\ \Delta t' = \gamma \Delta t, & (22) \end{cases}$$

as $\gamma \neq 0$ this suggested that

$$\Delta x = 0 \wedge \Delta x' = 0 \Rightarrow \Delta t = 0 \wedge \Delta t' = 0, \quad (23)$$

which means that: « For two relativistic inertial frame of reference in motion and in standard configuration the events occurring in a location having in each of these frames the same coordinates on the axis parallel to the direction of motion are necessarily simultanious in both frames ».

Let's show that the reciprocal is also true, indeed if the events A and B are simultaneous in R and R', we can write from equations (17) and (20)

$$\begin{cases} \Delta x' = \gamma \Delta x & (24) \\ \Delta x = 0, & (25) \end{cases}$$

that's to say

$$\Delta t = 0 \wedge \Delta t' = 0 \Rightarrow \Delta x = 0 \wedge \Delta x' = 0, \quad (26)$$

which translates into: « For two relativistic inertial frame of reference in standard configuration the coordinates on the axis parallel to the direction of motion of two simultaneous events in both frames coincide in each of the frames ».

Therefore we deduce from equations (23) and (26) that

$$\Delta t = 0 \wedge \Delta t' = 0 \Leftrightarrow \Delta x = 0 \wedge \Delta x' = 0, \quad (27)$$

which is the necessary and the sufficient condition for having two simultaneous events in two frames of reference in motion, stating: « Two events are simultaneous for two relativistic inertial frame of reference which are in standard configuration and in motion if and only if their coordinates on the axis parallel to the direction of motion coincide in each of the frames ».

And the contrapositive of (27)

$$\Delta x \neq 0 \vee \Delta x' \neq 0 \Leftrightarrow \Delta t \neq 0 \vee \Delta t' \neq 0, \quad (28)$$

constitutes the solution of the mathematical problem (11) and we can thus state:

*« **There can be no simultaneous events in two relativistic inertial frame of reference in motion for which the coordinates on the axis parallel to the direction of motion don't coincide** ».*

This is to be considered as a mathematical principle arising from the Lorentz transformation that one should respect when dealing with thought experiments involving simultaneous events. It's the most evident proof that the mathematical contradiction (11) is mainly due to the violation of this principle rather than an inconsistency in the theory. Although this principle confirms the self consistency of the Lorentz transformation, many questions about its physical consequences may be asked, for instance: What would be the physical interpretation of equation (16) which limit the location of the screen to a specific region of the space ? How would we integrate this principle with the obviousness and the logic that allows to conceive simultaneous events occurring anywhere for any moving observers as we did when we have placed the screen arbitrarily, especially that the giant screen remains visible for both observers by enabling them to watch the same scene simultaneously even when it's not placed in the required plane? A reasonable response would call to mind that this isn't a real experiment to be considered as a test for the Lorentz transformation. Only real experiments are able to validate a theory and until now all the experiments confirm successfully the correctness of special relativity [55-58] on the other hand even if this thought experiment will technically be feasible at the macroscopic scale there is nothing that says that this principle of simultaneity will necessarily be inconsistent with the experiment which may highlights new physical phenomenons like for instance a detectable relativistic effect that hide the screen to the observer if it's placed outside the allowed location, and the day when this experiment will be realisable several factors may come into play for providing a rationale explanation.

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