Solutions for Euler and Navier-Stokes Equations in Powers of Time

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Abstract – We present a solution for the Euler and Navier-Stokes equations for incompressible case given any smooth (C^∞) initial velocity, pressure and external force in $N = 3$ spatial dimensions, based on expansion in Taylor's series of time. Without major difficulties, it can be adapted to any spatial dimension, $N \geq 1$.

Keywords – Lagrange, Mécanique Analitique, exact differential, Euler's equations, Navier-Stokes equations, Taylor's series, Cauchy, Mémoire sur la Théorie des Ondes, Lagrange's theorem, Bernoulli's law, non-uniqueness solutions.

§ 1

Let p, q, r be the three components of velocity of an element of fluid in the 3-D orthogonal Euclidean system of spatial coordinates (x, y, z) and t the time in this system.

Lagrange in his *Mécanique Analitique*, firstly published in 1788, proved that if the quantity $(p\,dx + q\,dy + r\,dz)$ is an exact differential when $t = 0$ it will also be an exact differential when t has any other value. If the quantity $(p\,dx + q\,dy + r\,dz)$ is an exact differential at an arbitrary instant, it should be such for all other instants. Consequently, if there is one instant during the motion for which it is not an exact differential, it cannot be exact for the entire period of motion. If it were exact at another arbitrary instant, it should also be exact at the first instant.[1]

To prove it Lagrange used

$$(1.1) \qquad \begin{cases} p = p^I + p^{II}t + p^{III}t^2 + p^{IV}t^3 + \cdots \\ q = q^I + q^{II}t + q^{III}t^2 + q^{IV}t^3 + \cdots \\ r = r^I + r^{II}t + r^{III}t^2 + r^{IV}t^3 + \cdots \end{cases}$$

in which the quantities p^I, p^{II}, p^{III}, etc., q^I, q^{II}, q^{III}, etc., r^I, r^{II}, r^{III}, etc., are functions of x, y, z but without t.

Here we will finally solve the equations of Euler and Navier-Stokes using this representation of the velocity components in infinite series, as pointed by Lagrange. We assume satisfied the condition of incompressibility, for brevity. Without it the resulting equations are more complicated, as we know, but the method of solution is essentially the same in both cases. We focus our attention in the general case of the Navier-Stokes equations, and for the Euler equations simply set the viscosity coefficient as $\nu = 0$.

To facilitate and abbreviate our writing, we represent the fluid velocity by its three components in indicial notation, i.e., $u = (u_1, u_2, u_3)$, as well as the external force will be $f = (f_1, f_2, f_3)$ and the spatial coordinates $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$. The pressure, a scalar function, will be represented as p. As frequently used in mathematics approach, the density mass will be $\rho = 1$.

The representation (1.1) is as the expansion of the velocity in a Taylor´s series in relation to time around $t = 0$, considering x, y, z as constant, i.e., for $1 \le i \le 3$,

$$u_i = u_i|_{t=0} + \frac{\partial u_i}{\partial t}|_{t=0}\, t + \frac{\partial^2 u_i}{\partial t^2}|_{t=0}\, \frac{t^2}{2} + \frac{\partial^3 u_i}{\partial t^3}|_{t=0}\, \frac{t^3}{6} + \cdots + \frac{\partial^k u_i}{\partial t^k}|_{t=0}\, \frac{t^k}{k!} + \cdots \tag{1.2}$$

or

$$u_i = u_i^0 + \sum_{k=1}^{\infty} \frac{\partial^k u_i}{\partial t^k}|_{t=0}\, \frac{t^k}{k!}. \tag{1.3}$$

For the calculation of $\frac{\partial u_i}{\partial t}, \frac{\partial^2 u_i}{\partial t^2}, \frac{\partial^3 u_i}{\partial t^3}, \ldots$ we use the values that are obtained directly from the Navier-Stokes equations and its derivatives in relation to time, i.e.,

$$\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + \nu\, \nabla^2 u_i + f_i, \tag{1.4}$$

and therefore

$$\frac{\partial^2 u_i}{\partial t^2} = -\frac{\partial^2 p}{\partial t\, \partial x_i} - \sum_{j=1}^{3} \left(\frac{\partial u_j}{\partial t} \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} \right) + \nu\, \nabla^2 \frac{\partial u_i}{\partial t} + \frac{\partial f_i}{\partial t}, \tag{1.5}$$

$$\frac{\partial^3 u_i}{\partial t^3} = -\frac{\partial^3 p}{\partial t^2\, \partial x_i} - \sum_{j=1}^{3} \left(\frac{\partial^2 u_j}{\partial t^2} \frac{\partial u_i}{\partial x_j} + 2 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2} \right) + \nu\, \nabla^2 \frac{\partial^2 u_i}{\partial t^2} + \frac{\partial^2 f_i}{\partial t^2}, \tag{1.6}$$

$$\begin{aligned} \frac{\partial^4 u_i}{\partial t^4} &= -\frac{\partial^4 p}{\partial t^3\, \partial x_i} - \sum_{j=1}^{3} N_j^3 + \nu\, \nabla^2 \frac{\partial^3 u_i}{\partial t^3} + \frac{\partial^3 f_i}{\partial t^3}, \\ N_j^3 &= \frac{\partial}{\partial t} N_j^2,\ N_j^2 = \frac{\partial^2 u_j}{\partial t^2} \frac{\partial u_i}{\partial x_j} + 2 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2}, \\ N_j^3 &= \frac{\partial^3 u_j}{\partial t^3} \frac{\partial u_i}{\partial x_j} + 3 \frac{\partial^2 u_j}{\partial t^2} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + 3 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2} + u_j \frac{\partial}{\partial x_j} \frac{\partial^3 u_i}{\partial t^3}, \end{aligned} \tag{1.7}$$

$$\frac{\partial^5 u_i}{\partial t^5} = -\frac{\partial^5 p}{\partial t^4\, \partial x_i} - \sum_{j=1}^{3} N_j^4 + \nu\, \nabla^2 \frac{\partial^4 u_i}{\partial t^4} + \frac{\partial^4 f_i}{\partial t^4}, \tag{1.8}$$

$$N_j^4 = \frac{\partial}{\partial t} N_j^3 = \frac{\partial^4 u_j}{\partial t^4}\frac{\partial u_i}{\partial x_j} + 4\frac{\partial^3 u_j}{\partial t^3}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t} + 6\frac{\partial^2 u_j}{\partial t^2}\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2} +$$
$$+4\frac{\partial u_j}{\partial t}\frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3} + u_j\frac{\partial}{\partial x_j}\frac{\partial^4 u_i}{\partial t^4},$$

and using induction we come to

(1.9)
$$\frac{\partial^k u_i}{\partial t^k} = -\frac{\partial^k p}{\partial t^{k-1}\,\partial x_i} - \sum_{j=1}^3 N_j^{k-1} + \nu\,\nabla^2\frac{\partial^{k-1}u_i}{\partial t^{k-1}} + \frac{\partial^{k-1}f_i}{\partial t^{k-1}},$$
$$N_j^{k-1} = \frac{\partial}{\partial t}N_j^{k-2} = \sum_{l=0}^{k-1}\binom{k-1}{l}\partial_t^{k-1-l}u_j\,\frac{\partial}{\partial x_j}\partial_t^l u_i,$$
$$\partial_t^0 u_n = u_n,\ \partial_t^m u_n = \frac{\partial^m u_n}{\partial t^m},\ \binom{k-1}{l} = \frac{(k-1)!}{(k-1-l)!\,l!}.$$

In (1.2) and (1.3) it is necessary to know the values of the derivatives $\frac{\partial u_i}{\partial t}, \frac{\partial^2 u_i}{\partial t^2}, \dots, \frac{\partial^k u_i}{\partial t^k}$ in $t = 0$ then we must to calculate, from (1.4) to (1.9),

(1.10)
$$\frac{\partial u_i}{\partial t}\Big|_{t=0} = -\frac{\partial p^0}{\partial x_i} - \sum_{j=1}^3 u_j^0\frac{\partial u_i^0}{\partial x_j} + \nu\,\nabla^2 u_i^0 + f_i^0,$$

the superior index 0 meaning the value of the respective function at $t = 0$, and

(1.11)
$$\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} = -\frac{\partial^2 p}{\partial t\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^3 N_j^1\Big|_{t=0} +$$
$$+\nu\,\nabla^2\frac{\partial u_i}{\partial t}\Big|_{t=0} + \frac{\partial f_i}{\partial t}\Big|_{t=0},$$
$$N_j^1\Big|_{t=0} = \sum_{j=1}^3\left(\frac{\partial u_j}{\partial t}\Big|_{t=0}\frac{\partial u_i^0}{\partial x_j} + u_j^0\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\Big|_{t=0}\right),$$

(1.12)
$$\frac{\partial^3 u_i}{\partial t^3}\Big|_{t=0} = -\frac{\partial^3 p}{\partial t^2\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^3 N_j^2\Big|_{t=0} +$$
$$+\nu\,\nabla^2\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} + \frac{\partial^2 f_i}{\partial t^2}\Big|_{t=0},$$
$$N_j^2\Big|_{t=0} = \frac{\partial^2 u_j}{\partial t^2}\Big|_{t=0}\frac{\partial u_i^0}{\partial x_j} + 2\frac{\partial u_j}{\partial t}\Big|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\Big|_{t=0} +$$
$$+u_j^0\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0},$$

(1.13)
$$\frac{\partial^4 u_i}{\partial t^4}\Big|_{t=0} = -\frac{\partial^4 p}{\partial t^3\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^3 N_j^3\Big|_{t=0} +$$
$$+\nu\,\nabla^2\frac{\partial^3 u_i}{\partial t^3}\Big|_{t=0} + \frac{\partial^3 f_i}{\partial t^3}\Big|_{t=0},$$
$$N_j^3\Big|_{t=0} = \frac{\partial^3 u_j}{\partial t^3}\Big|_{t=0}\frac{\partial u_i^0}{\partial x_j} + 3\frac{\partial^2 u_j}{\partial t^2}\Big|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\Big|_{t=0} +$$
$$+3\frac{\partial u_j}{\partial t}\Big|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} + u_j^0\frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3}\Big|_{t=0},$$

(1.14) $$\frac{\partial^5 u_i}{\partial t^5}|_{t=0} = -\frac{\partial^5 p}{\partial t^4\,\partial x_i}|_{t=0} - \sum_{j=1}^{3} N_j^4\,|_{t=0} +$$

$$+\,\nu\,\nabla^2\frac{\partial^4 u_i}{\partial t^4}|_{t=0} + \frac{\partial^4 f_i}{\partial t^4}|_{t=0},$$

$$N_j^4|_{t=0} = \frac{\partial^4 u_j}{\partial t^4}|_{t=0}\frac{\partial u_i^0}{\partial x_j} + 4\frac{\partial^3 u_j}{\partial t^3}|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t} +$$

$$+\,6\frac{\partial^2 u_j}{\partial t^2}|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}|_{t=0} + 4\frac{\partial u_j}{\partial t}|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3}|_{t=0} +$$

$$+\,u_j^0\frac{\partial}{\partial x_j}\frac{\partial^4 u_i}{\partial t^4}|_{t=0},$$

and of generic form,

(1.15) $$\frac{\partial^k u_i}{\partial t^k}|_{t=0} = -\frac{\partial^k p}{\partial t^{k-1}\,\partial x_i}|_{t=0} - \sum_{j=1}^{3} N_j^{k-1}\,|_{t=0} +$$

$$+\,\nu\,\nabla^2\frac{\partial^{k-1} u_i}{\partial t^{k-1}}|_{t=0} + \frac{\partial^{k-1} f_i}{\partial t^{k-1}}|_{t=0},$$

$$N_j^{k-1}|_{t=0} = \sum_{l=0}^{k-1}\binom{k-1}{l}\,\partial_t^{k-1-l}u_j|_{t=0}\,\frac{\partial}{\partial x_j}\partial_t^l u_i|_{t=0},$$

$$\partial_t^0 u_n|_{t=0} = u_n^0,\ \partial_t^m u_n|_{t=0} = \frac{\partial^m u_n}{\partial t^m}|_{t=0}.$$

If the external force is conservative there is a scalar potential U such as $f = \nabla U$ and the pressure can be calculated from this potential U, i.e.,

(1.16) $$\frac{\partial p}{\partial x_i} = f_i = \frac{\partial U}{\partial x_i},$$

and then

(1.17) $$p = U + \theta(t),$$

$\theta(t)$ a generic function of time of class C^∞, so it is not necessary the use of the pressure p and external force f, and respective derivatives, in (1.4) to (1.15) if the external force is conservative. In this case, the velocity can be independent of the both pressure and external force, otherwise it will be necessary to use both the pressure and external force derivatives to calculate the velocity in powers of time.

The result that we obtain here in this development in Taylor's series seems to me a great advance in the search of the solutions of the Euler's and Navier-Stokes equations. It is possible now to know on the possibility of non-uniqueness solutions as well as breakdown solution respect to unbounded energy of another manner.

We now can choose previously an infinity of different pressures such that the calculation of $\frac{\partial u}{\partial t}$ and derivatives can be done, for a given initial velocity and external force, although such calculation can be very hard.

It is convenient say that Cauchy[2] in his memorable and admirable *Mémoire sur la Théorie des Ondes*, winner of the Mathematical Analysis award, year 1815, firstly does a study on the equations to be obeyed by three-dimensional molecules in a homogeneous fluid in the initial instant $t = 0$, coming to the conclusion which the initial velocity must be irrotational, i.e., a potential flow. Of this manner, after, he comes to conclusion that the velocity is always irrotational, potential flow, if the external force is conservative, which is essentially the Lagrange's theorem described in the begin of this article, but it is shown without the use of series expansion (a possible exception to the theorem occurs if one or two components of velocity are identically zero, when the reasonings on 3-D molecular volume are not valid). The solution obtained by Cauchy for Euler's equations is the Bernoulli's law, as almost always happens. Now a more generic solution is obtained, in special when it is possible a solution be expanded in polynomial series of time. Though not always a function can be expanded in Taylor's series, there is certainly an infinity of possible cases of solution where this is possible.

If the mentioned series is divergent in some point or region may be an indicative of that the correspondent velocity and its square diverge, again going to the case of breakdown solution due to unbounded energy. With the three functions initial velocity, pressure and external force belonging to Schwartz Space is expected that the solution for velocity also belongs to Schwartz Space, obtaining physically reasonable and well-behaved solution throughout the space.

The method presented here in this first section can also be applied in other equations, of course, for example in the heat equation, Schrödinger equation, wave equation and many others. Always will be necessary that the remainder in the Taylor's series goes to zero when the order k of the derivative tends to infinity (Courant[3], chap. VI). Applying this concept in (1.3) and (1.9), substituting t by τ, the remainder $R_{i,k}$ of order k for velocity component i is

(1.18) $$R_{i,k} = \frac{1}{k!}\int_0^t (t-\tau)^k \frac{\partial^{k+1} u_i}{\partial t^{k+1}}\, d\tau,$$

which can be estimated by Lagrange's remainder,

(1.19) $$R_{i,k} = \frac{t^{k+1}}{(k+1)!}\frac{\partial^{k+1} u_i}{\partial t^{k+1}}(\xi),$$

or by Cauchy's remainder,

(1.20) $$R_{i,k} = \frac{t^{k+1}}{k!}(1-\theta)^k \frac{\partial^{k+1} u_i}{\partial t^{k+1}}(\xi),$$

with $0 \le \xi \le t$ and $0 \le \theta \le 1$.

Note that if it is not possible to make a series around $t = 0$ (for example, to the functions $\log t$, $\sqrt[3]{t}$, e^{-1/t^2}, according Courant[3], chap. VI) an other instant t_0

of convergence and remainder $R_{i,k\to\infty}$ zero must be found, and then replacing t^k by $(t-t_0)^k$ and the calculations in $t=0$ by $t=t_0$ in previous equations.

§ 2

In this section we will build a series of powers of time solving the Navier-Stokes equations, differently than that used in the previous section. From theorem of uniqueness of series of powers (*A function $f(x)$ can be represented by a power series in x in only one way, if it all, i.e., the representation of a function by a power series is "unique"; Every power series which converges for points other than $x=0$ is the Taylor series of the function which it represents* (Courant[3], chap. VIII)), both solutions need be the same, for a same initial velocity, pressure, external force, compressibility condition and all boundary conditions.

Defining

(2.1) $$u_i = u_i^0 + X_{i,1}t + X_{i,2}t^2 + \cdots + X_{i,n}t^n + \cdots = \sum_{n=0}^{\infty} X_{i,n}t^n,$$
$$X_{i,0} = u_i^0 = u_i(x_1,x_2,x_3,0),$$

where each $X_{i,n}$ is a function of position (x_1,x_2,x_3), without t, and

(2.2) $$\frac{\partial p}{\partial x_i} = q_i^0 + q_{i,1}t + q_{i,2}t^2 + \cdots + q_{i,n}t^n + \cdots = \sum_{n=0}^{\infty} q_{i,n}t^n,$$
$$q_{i,0} = q_i^0 = \frac{\partial p^0}{\partial x_i},\ p^0 = p(x_1,x_2,x_3,0),$$

(2.3) $$f_i = f_i^0 + f_{i,1}t + f_{i,2}t^2 + \cdots + f_{i,n}t^n + \cdots = \sum_{n=0}^{\infty} f_{i,n}t^n,$$
$$f_{i,0} = f_i^0 = f_i(x_1,x_2,x_3,0),$$

we can put these series in the Navier-Stokes equation,

(2.4) $$\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} - \sum_{j=1}^{3} u_j\frac{\partial u_i}{\partial x_j} + \nu\,\nabla^2 u_i + f_i.$$

The velocity derivative in relation to time is

(2.5) $$\frac{\partial u_i}{\partial t} = X_{i,1} + 2X_{i,2}t + 3X_{i,3}t^2 + \cdots + nX_{i,n}t^{n-1} + \cdots =$$
$$= \sum_{n=0}^{\infty}(n+1)X_{i,n+1}t^n,$$

the nonlinear terms are, of order zero (constant in time)

(2.6) $$\sum_{j=1}^{3} u_j^0\frac{\partial u_i^0}{\partial x_j},$$

of order 1,

(2.7) $$\sum_{j=1}^{3}\left(u_j^0\frac{\partial X_{i,1}}{\partial x_j}+X_{j,1}\frac{\partial u_i^0}{\partial x_j}\right)t,$$

of order 2,

(2.8) $$\sum_{j=1}^{3}\left(u_j^0\frac{\partial X_{i,2}}{\partial x_j}+X_{j,1}\frac{\partial X_{i,1}}{\partial x_j}+X_{j,2}\frac{\partial u_i^0}{\partial x_j}\right)t^2,$$

of order 3,

(2.9) $$\sum_{j=1}^{3}\left(u_j^0\frac{\partial X_{i,3}}{\partial x_j}+X_{j,1}\frac{\partial X_{i,2}}{\partial x_j}+X_{j,2}\frac{\partial X_{i,1}}{\partial x_j}+X_{j,3}\frac{\partial u_i^0}{\partial x_j}\right)t^3,$$

and of order n, of generic form, equal to

(2.10) $$\sum_{j=1}^{3}\sum_{k=0}^{n}X_{j,k}\frac{\partial X_{i,n-k}}{\partial x_j}t^n,$$

with $X_{j,0}=u_j^0$, $\frac{\partial X_{i,0}}{\partial x_j}=\frac{\partial u_i^0}{\partial x_j}$.

Applying these sums in (2.4) we have

(2.11) $$\begin{aligned}&\sum_{n=0}^{\infty}(n+1)X_{i,n+1}t^n=-\sum_{n=0}^{\infty}q_{i,n}t^n-\\&-\sum_{n=0}^{\infty}\sum_{j=1}^{3}\sum_{k=0}^{n}X_{j,k}\frac{\partial X_{i,n-k}}{\partial x_j}t^n+\nu\sum_{n=0}^{\infty}\nabla^2X_{i,n}\,t^n+\\&+\sum_{n=0}^{\infty}f_{i,n}t^n,\end{aligned}$$

and then

(2.12) $$\begin{aligned}(n+1)X_{i,n+1}=-q_{i,n}-\sum_{j=1}^{3}\sum_{k=0}^{n}X_{j,k}\frac{\partial X_{i,n-k}}{\partial x_j}+\\+\nu\nabla^2X_{i,n}+f_{i,n},\end{aligned}$$

which allows us to obtain, by recurrence, $X_{i,1}$, $X_{i,2}$, $X_{i,3}$, etc., that is, for $1\le i\le 3$ and $n\ge 0$,

(2.13) $$\begin{aligned}&X_{i,n+1}=\frac{1}{n+1}S_n,\\&S_n=-q_{i,n}-\sum_{j=1}^{3}\sum_{k=0}^{n}X_{j,k}\frac{\partial X_{i,n-k}}{\partial x_j}+\nu\nabla^2X_{i,n}+f_{i,n}.\end{aligned}$$

You can see how much will become increasingly difficult calculate the terms $X_{i,n}$ with increasing the values of n, for example, will appear terms in $\nu^n,\nabla^2\nabla^2\ldots\nabla^2u_i^0$, etc. If $\nu>1$ certainly there is a specific problem to be studied with relation to convergence of the series, which of course also occurs in the representation given in section § 1. The same can be said for $t\to\infty$. In fact, I do not understand why a particle fluid initially in motion, without any collision with another particle and submitted to a permanent impulsive force need always be with finite velocity as $t \to \infty$. For example, a constant resulting force f, not equal to zero, applied all time on a body will produce an infinite velocity u to this body when $t \to \infty$, supposing possible such force and a way no obstacles, etc.

§ 3

The previous solutions show us that we need to have, for all integers $1 \leq i \leq 3$ and $n \geq 0$,

(3.1) $$\frac{1}{n!}\frac{\partial^n u_i}{\partial t^n}\Big|_{t=0} = X_{i,n},$$

and both members of this relation are very difficult to be calculated, either equation (1.15) as well as (2.13). Add to this difficulty the fact that besides the main Navier-Stokes equations (1.4)-(2.4) must be included the condition of incompressibility,

(3.2) $$\nabla \cdot u = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} u_i = 0.$$

Using (2.1) in (3.2) we have

(3.3) $$\nabla \cdot u = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \sum_{n=0}^{\infty} X_{i,n} t^n = \sum_{n=0}^{\infty} \left(\sum_{i=1}^{3} \frac{\partial}{\partial x_i} X_{i,n} \right) t^n = 0.$$

As this equation need be valid for all $t \geq 0$ we have

(3.4) $$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} X_{i,n} = \nabla \cdot X_n = 0,$$

defining $X_n = (X_{1,n},\ X_{2,n},\ X_{3,n})$, i.e., all coefficients X_n must obey the condition of incompressibility in the vector representation of velocity,

(3.5) $$u = \sum_{n=0}^{\infty} X_n\, t^n.$$

Following Lagrange[1], getting two differentiable and continuous functions α and β of class C^2 and defining

(3.6.1) $$u_1 = \frac{\partial \alpha}{\partial z}, \qquad u_2 = \frac{\partial \beta}{\partial z}, \qquad u_3 = -\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} \right),$$

(3.6.2) $$u_1^0 = \frac{\partial \alpha^0}{\partial z}, \qquad u_2^0 = \frac{\partial \beta^0}{\partial z}, \qquad u_3^0 = -\left(\frac{\partial \alpha^0}{\partial x} + \frac{\partial \beta^0}{\partial y} \right),$$

with $\alpha^0 = \alpha(t = 0)$ and $\beta^0 = \beta(t = 0)$, we have satisfied the condition (3.2), which it is easy to see. Other manner is when u is derived from a vector potential A, i.e., (3.7.1) $\quad u = \nabla \times A,$

(3.7.2) $\quad u^0 = \nabla \times A^0,$

with $A^0 = A(t = 0)$.

The relations (3.6) are very useful and easy to be implemented and we will use them to solve the Euler and Navier-Stokes equations when the incompressibility condition is required. Given any continuous, differentiable and integrable vector components u_1 and u_2 then

(3.8.1) $\quad \alpha = \int u_1 \, dz,$

(3.8.2) $\quad \beta = \int u_2 \, dz,$

and thus u_3 and u_3^0 need to be according

(3.9.1) $\quad u_3 = -\int \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right) dz = -\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y}\right),$

(3.9.2) $\quad u_3^0 = -\int \left(\frac{\partial u_1^0}{\partial x} + \frac{\partial u_2^0}{\partial y}\right) dz = -\left(\frac{\partial \alpha^0}{\partial x} + \frac{\partial \beta^0}{\partial y}\right),$

which reminds us that the components of the velocity vector maintains conditions to be complied to each other, i.e., it is not any initial velocity which can be used for solution of Euler and Navier-Stokes equations in incompressible flows case.

In the equations of the sections § 1 and § 2, instead u_1 we will use $\frac{\partial \alpha}{\partial z}$, instead u_2 will be $\frac{\partial \beta}{\partial z}$, and $-\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y}\right)$ instead u_3, as well as the correspondents initial values, replacing u_1^0 by $\frac{\partial \alpha^0}{\partial z}$, u_2^0 by $\frac{\partial \beta^0}{\partial z}$, and u_3^0 by $-\left(\frac{\partial \alpha^0}{\partial x} + \frac{\partial \beta^0}{\partial y}\right)$. Of this manner, we will be developing series for $\frac{\partial \alpha}{\partial z}$, $\frac{\partial \beta}{\partial z}$ and $-\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y}\right)$, so that $\nabla \cdot u = 0$. Then this is a preliminary problem to be solved, the calculation of α^0 and β^0 giving u_1^0, u_2^0 and u_3^0 when $\nabla \cdot u^0 = 0$ and it is necessary that $\nabla \cdot u = 0$, i.e.,

(3.10.1) $\quad \alpha^0 = \int u_1^0 \, dz,$

(3.10.2) $\quad \beta^0 = \int u_2^0 \, dz,$

with the validity of (3.9.2). Done this, the exact solution for the principal problem can be calculated from reasoning exposed here, if there is not an equivalent solution described in a most simplified formulation, for example, according Bernoulli's law and Laplace's equation.

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What good would living on a planet without destruction, greed and envy,
where the nations were dedicated to building a beautiful world
and to the salvation of those in need.
That there were no enemies and everyone could be happy where they live,
in their own way.

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