The Number of Primes Between Consecutive Squares: A Proof of Brocard's Conjecture

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Abstract:

In this paper, we are going to find the number of primes between consecutive squares. We are going to prove a special case: Brocard's conjecture which states between the square of two consecutive primes greater than 2 at least four primes will present; means $\pi(p_{m+1}{}^2)-\pi(p_m{}^2) \geq 4$ where $p_m \geq 3$. Here, $\pi(x)$ is the prime counting function. Subsequently, we will prove that the number of primes between consecutive square is greater than big O of $\frac{n}{\log(n)}$.

Definition

According to Moivre-Stirling Approximation of factorial [1]:

$$\int_1^n \log(x)dx \; < \; \log(n!) \; < \; \int_1^{n+1} \log(x)dx \; , \; \int \log(x)dx \; = \; xlog(x)-x+C \quad , \ldots\ldots (1)$$

If we assume $\Delta_2=\frac{1}{2}\log(\frac{2}{1})$, $\Delta_3=\frac{1}{2}\log(\frac{3}{2})$,…, $\Delta_{n-1}=\frac{1}{2}\log(\frac{n-1}{n-2})$, $\Delta_n=\frac{1}{2}\log(\frac{n}{n-1})$,

we can get better Moivre-Stirling Approximation using simple geometric arguments from figure below [1]:

$$\int_{n-1}^{n} \log(n)dn-\log(n-1) \; = \; nlog(\frac{n}{n-1})-1 \; = \; 2n\Delta_n-1 \; \leq \; \Delta_{n-1} \quad \text{for } n > 2$$

$$\text{because} \quad \frac{d}{dn}(2n\Delta_n-\Delta_{n-1}) \; > 0 \text{ and } \lim_{n\to\infty}(2n\Delta_n-\Delta_{n-1}) \; = 1$$

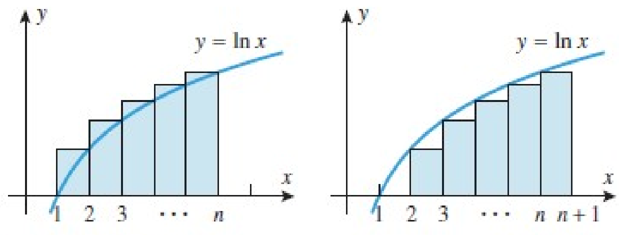
so,
$$\int_2^{n+1} \log(x)dx \; - \; (\Delta_2+\Delta_3+\ldots+\Delta_{n-1}+\Delta_n) \; < \; \log(n!)$$

$$< \; \int_1^n \log(x)dx \; + \; (\Delta_2+\Delta_3+\ldots+\Delta_{n-1}+\Delta_n)$$

hence,
$$\int_2^{n+1} \log(x)dx \; - \; \frac{1}{2}\log(n) \; < \; \log(n!) \; < \; \int_1^n \log(x)dx \; + \; \frac{1}{2}\log(n)$$

or,
$$(n+1)\log(n+1)-n-\frac{1}{2}\log(n)-b+1 \; < \; \log(n!) \; < \; nlog(n)-n+\frac{1}{2}\log(n)+1 \quad , \ldots\ldots (2)$$

where $b = log(4)$



▲ Figure Ex-30

We define function $\upsilon(x)$ and $\psi(x)$ conventionally as [2]:

$$\upsilon(x) = \sum_{p \le x} \log(p) = \log \prod_{p \le x} p \quad , \quad \psi(x) = \sum_{p^m \le x} \log(p)$$

Since $p^2 \le x, p^3 \le x, \ldots$ are equivalent to $p \le x^{1/2}, p \le x^{1/3}, \ldots$, we have [2], [3] :

$$\psi(x) = \upsilon(x) + \upsilon(x^{1/2}) + \upsilon(x^{1/3}) + \ldots = \sum_{m \ge 1} \upsilon(x^{1/m}) \quad , \ldots.. (3)$$

and so, $\psi(2n) = \upsilon(2n) + \upsilon((2n)^{1/2}) + \upsilon((2n)^{1/3}) + \ldots = \sum_{m \ge 1} \upsilon((2n)^{1/m})$

Proof

We know [3]: $\log((2n)!) = \psi(2n) + \psi(n) + \psi(\frac{2n}{3}) + \ldots$, ….. (4)

Let, $N_{2n} = \frac{(2n)!}{n!n!}$, then from (4) [3]: $\log(N_{2n}) = \psi(2n) - \psi(n) + \psi(\frac{2n}{3}) - \ldots$, ….. (5)

As $\psi(x)$ is a steadily increasing function,

$$\log(\frac{N_{(n+1)^2}}{N_{n^2}}) = (\psi((n+1)^2) - \psi(n^2)) - (\psi(\frac{(n+1)^2}{2}) - \psi(\frac{n^2}{2})) + (\psi(\frac{(n+1)^2}{3}) - \psi(\frac{n^2}{3})) - \ldots$$

$$< (\psi((n+1)^2) - \psi(n^2)) \quad , \ldots.. (6)$$

From (2) and (5), we get:

$$\log(\frac{N_{(n+1)^2}}{N_{n^2}}) >$$

$$((n+1)^2+1)\log((n+1)^2+1) - (n+1)^2 - \log(n+1) - b + 1 - n^2\log(n^2) + n^2 - \log(n) - 1$$

$$-(n+1)^2\log(\frac{(n+1)^2}{2})+(n+1)^2-\log(\frac{(n+1)^2}{2})-2+(n^2+2)\log(\frac{n^2}{2}+1)-n^2-\log(\frac{n^2}{2})-2b+2 \;=$$

$$(n+1)^2\log(\frac{2(n+1)^2+2}{(n+1)^2})-n^2\log(\frac{2n^2}{n^2+2})+2\log(n^2+2)+\log((n+1)^2+1)-3\log(n+1)-3\log(n)-3b$$

, ….. (7)

Again, form relation of $\upsilon(x)$ and $\psi(x)$, we get from (3):

$$\psi((n+1)^2)-\psi(n^2) \;=\; (\upsilon((n+1)^2)-\upsilon(n^2))+(\upsilon(n+1)-\upsilon(n))+(\upsilon((n+1)^{\frac{2}{3}})-\upsilon(n^{\frac{2}{3}}))+\ldots$$

We assume $2^m=(n+1)$, then $(\upsilon(n+1)-\upsilon(n))+(\upsilon((n+1)^{\frac{2}{3}})-\upsilon(n^{\frac{2}{3}}))+\ldots \;<$

$$mlog(2)+(\frac{2m}{3})\log(2)+\ldots+\log(2) \;=\; 2mlog(2)(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2m}) \;<\; 2mlog(2)\int_1^{(2m)}\frac{1}{x}dx \;=$$

$$2mlog(2)\log(2m) \;=\; 2\log(n+1)\log(2\,lg(n+1))$$ where $lg(x)=\frac{\log(x)}{\log(2)}$

because 1 = $(n+1)-n \;>\; (n+1)^{\frac{2}{3}}-n^{\frac{2}{3}} \;>\; (n+1)^{\frac{1}{2}}-n^{\frac{1}{2}} \;> \ldots$

hence, $(\psi((n+1)^2)-\psi(n^2)) \;<\; (\upsilon((n+1)^2)-\upsilon(n^2))+2\log(n+1)\log(2lg(n+1))$, ….. (8)

Finally, we get from (6), (7) and (8):

$$(\upsilon((n+1)^2)-\upsilon(n^2)) \;>$$

$$(n+1)^2\log(\frac{2(n+1)^2+2}{(n+1)^2})-n^2\log(\frac{2n^2}{n^2+2})+2\log(n^2+2)+\log((n+1)^2+1)-3\log(n+1)-3\log(n)-3b$$
$$-2\log(n+1)\log(2lg(n+1))$$,….. (9)

We assume there are *a* number of primes between n^2 and $(n+1)^2$

Hence, $\upsilon((n+1)^2)-\upsilon(n^2) \;<\; alog((n+1)^2)$, ….. (10)

Let, $F(n) \;=\; \upsilon((n+1)^2)-\upsilon(n^2)-2alog(n+1)$

Now, $F'(n) \;=\; 2nlog(\frac{(n^2+2)((n+1)^2+1)}{n^2(n+1)^2})+2\log(\frac{2(n+1)^2+2}{(n+1)^2})-\frac{3}{n+1}-\frac{3}{n}$

$$-\frac{2}{n+1}(\log(2lg(n+1)))-\frac{2}{n+1}-\frac{2a}{n+1}$$

For a particular value of *a*, $F'(n)$ is an increasing function of *n.*

If $a=4$, $F'(9)$ *> 0* implies $F'(n)$ *> 0* for *n > 8.*

Again, $F(33)$ *> 0*, hence, $F(n)$ for *n > 32.*

As $p_{m+1}-p_m \geq 1$:

$\pi(p_{m+1}{}^2)-\pi(p_m{}^2) \;\geq\; \pi((n+1)^2)-\pi(n^2)$ > 4 for $p_m \geq n \geq 33$.

It has actually proved Brocard's conjecture because the conjecture can be verified for $3 \leq p_m < 33$ easily.

Now, we assume the least value of *a* for which: $F(n)$ *< 0*

so, $a\log((n+1)^2)$ >

$$(n+1)^2\log(\frac{2(n+1)^2+2}{(n+1)^2})-n^2\log(\frac{2n^2}{n^2+2})+2\log(n^2+2)+\log((n+1)^2+1)-3\log(n+1)-3\log(n)-3b$$

$$-2\log(n+1)\log(2lg(n+1))$$

hence, *a* >

$$\frac{(n+1)^2}{2\log(n+1)}\log(\frac{2(n+1)^2+2}{(n+1)^2})-\frac{n^2}{2\log(n+1)}\log(\frac{2n^2}{n^2+2})+\frac{\log(n^2+2)}{\log(n+1)}+\frac{\log((n+1)^2+1)}{2\log(n+1)}-\frac{3}{2}$$

$$-\frac{3\log(n)}{2\log(n+1)}-\frac{3b}{2\log(n+1)}-\log(2lg(n+1))$$

$$= \frac{n^2}{2\log(n+1)}\log(\frac{(n^2+2)((n+1)^2+1)}{n^2(n+1)^2})+\frac{2n+1}{2\log(n+1)}\log(\frac{2(n+1)^2+2}{(n+1)^2})$$

$$+O(1)+O(1)-O(1)-O(1)-O(\frac{1}{\log(n)})-O(\log(2lg(n+1)))$$

$$= \frac{n^2}{2\log(n+1)}\int_{n^2(n+1)^2}^{(n^2+2)((n+1)^2+1)}\frac{1}{x}dx+\frac{2n+1}{2\log(n+1)}\int_{(n+1)^2}^{2(n+1)^2+2}\frac{1}{x}dx-O(\log(2lg(n+1))) >$$

$$\frac{n^2}{2\log(n+1)}\frac{(n^2+2)((n+1)^2+1)-n^2(n+1)^2}{(n^2+2)((n+1)^2+1)}+\frac{2n+1}{2\log(n+1)}\frac{2(n+1)^2+2-(n+1)^2}{2(n+1)^2+2}-O(\log(2lg(n+1)))$$

$$= O(\frac{1}{\log(n)})+O(\frac{n}{\log(n)})-O(\log(2lg(n+1))) = O(\frac{n}{\log(n)}) \quad , \ldots.. (10)$$

Similar result is provided in [4], true for infinitely positive integers. We integrate the result for any integer, *n > 1*.

Reference

[1] H. Anton, I. Bivens and S. Davis, "Calculus," John Wiley & Sons, Inc, NY, p. 528, p. 664, 2002.

[2] G. H. Hardy and E. M. Wright, "An Introduction to the Theory of Numbers," Oxford University Press, NY, pp. 340-341, 1979.

[3] Edited: G. H. Hardy, P. V. S. Aiyar and B. M. Wilson, "Collected Papers of Srinivasa Ramanujan," Cambridge University Press, pp. 208-209, 1927.

[4] M. Hassani, "Counting Primes In The Interval $(n^2,(n+1)^2)$ ", Transactions of the American Mathematical Society, 1997.