Towards a physical correlation between slow and fast brain timescales

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Brain electric activity exhibits two important features: oscillations with different timescales, characterized by diverse functional and psychological outcomes, and a temporal power law distribution, which appears as a straight line when plotted on logarithmic scales in a log power versus log frequency plot. In order to investigate the relationships between low- and high- frequency spikes in the brain, we used a variant of the Borsuk-Ulam theorem which states that, when we assess the nervous activity as embedded in a sphere equipped with a fractal dimension, we achieve two antipodal points with similar features (the slow and fast, scale-free oscillations). We demonstrate that slow and fast nervous oscillations are correlated and provide, through the Bloch theorem from solid-state physics, the possible equation which links the two timescale activities.

Brain electric activity displays a mean action potential rate of 4 Hz (Sengupta 2013). However, this is just a mean value which accounts for the diverse timescales of nervous activity. Indeed, the coexistence of very low-frequencies (<2 Hz) and high frequencies (>10) can be found in the brain (Buszaki). It is thought that these two timescales are associated with unrelated functional activities (Raichle 2010). It has been proposed that the anatomical, functional and psychological correlates of the slow cortical potentials are respectively the spontaneous brain activity (Raichle 2010) the default mode networks and the unconstrained, conscious cognition (i.e. mind-wandering or day dreaming propensities) (Andrews-Hanna; Kucyi) or the dreaming state (Fox, 2015). Many observations confirm the hypothesis that spontaneous fluctuations are the basic, "intrinsic," standard architecture of functional brain organization (Krueger; Yuste). Spontaneous low-frequency fluctuations of cerebral activity cannot be simply reduced to a background noise uncorrelated to the system response (de Arcangelis; de Pasquale), rather they occur during unconstrained "resting' states" (subjects left to themselves in a scanner, with no explicit task instruction, lying quietly with eyes closed or fixating on a cross) and thus represent neuronal activity that is intrinsically generated by the brain (Fox 2007). Spontaneous fluctuations have been observed not only in electric activity, but also in various haemodynamic and metabolic parameters, as well as spontaneous fluctuations in the membrane potential, spontaneous spikes (O'Donnell) and neurotransmitter release (Kavalali). Among the networks exhibiting coherent fluctuations in spontaneous activity during rest, of particular interest is the "default-mode network" (DMN), *i.e*., functionally and structurally connected regions that show high metabolic activity and blood flow at rest, but deactivate when specific goal-directed behavior or cognition is needed (Damoiseaux). On the other side, during stimuli-evoked activity or cognitive demands, the brain transitorily exhibits functional conformations - other than the spontaneous ones -, which are linked with specific psychological correlates (Cole). The high frequency activity has been correlated, *i.e*., with perceptual binding and feedback/feedforward waves which improve the perception of external inputs (Buszaki). The (environmental and internal) inputs cause changes in spike frequency in diverse cortical areas: the brain thus exhibits a high number of possible source configurations, such as gamma oscillations in somatomotor cortex during states of enhanced vigilance, or alpha waves in posterior zones with eyes open, and so on (Bastos).
In this paper we ask whether there is a correlation between the functionally heterogeneous low and high frequencies, and evaluate whether it is true that, when the former changes, the latter also changes. To assess this possibility, we bring to the table three powerful concepts from far-flung branches: the ubiquitous power laws, the Borsuk-Ulam theorem (BUT) from algebraic topology and the Bloch theorem from solid-state physics.

METHODS

Brain activity observed at both low and high temporal scales exhibits a $1/f^{\alpha}$ -like power spectrum (Newman), including not just macroscopic electric oscillations, electroencephalography, magnetoencephalography and functional magnetic resonance imaging signals (Heemail), but also microscopic membrane potentials and fluctuations in neurotransmitter release (Fox 2007, Milstein; Linkenkaer-Hansen). In particular, the temporal frequency spectrum of cerebral electric activity displays a scale-invariant behaviour $S(f)= 1/f^{\alpha}$, where $S(f)$ is the power spectrum, f is the frequency and α is an exponent that equals the negative slope of the line in a log power versus log frequency plot (Van de Ville; Pritchard). Pink noise can be regarded as an intrinsic property of the brain characterizing a large class of neuronal processes (Fraiman; He), suggesting the possibility that power law distributions contain information about how large-scale physiological and pathological outcomes arise from the interactions of many small-scale processes (de Arcangelis). The emergence of power law distributions in the brain has been also correlated with the spontaneous appearance of high frequency neuronal avalanches (Papo; Tinker; Beggs).

Therefore, both slow and fast oscillations are equipped with a power law structure. Now BUT and its variants come into play. The Borsuk-Ulam Theorem (Borsuk 1933; Dodson) points out that, if a sphere S^n is mapped continuously into a n-dimensional Euclidean space R^n, there is at least one pair of antipodal points on S^n which map onto the same point of R^n (Beyer). See Tozzi 2016a and Tozzi 2016b for further details about BUT and its variants. The notation S^n stands for an n-sphere, which is a generalization of the circle (Weeks). An n-sphere is a n-dimensional structure of constant curvature, embedded in a convex n+1 space (Marsaglia; Henderson). For example, a 2-sphere (S^2) is the 2-dimensional surface of a 3-dimensional ball (a beach ball is a good illustration). Examples of antipodal points are the poles of a sphere (Matousek). Tozzi (2016a) provides a mathematical treatment for technical readers.

The concept of antipodal points can be can be used not just for the description of simple topological points, but also of more complicated features, such as shapes of space (spatial patterns, i.e., area and diameter), of shapes of time (temporal patterns), vectors or tensors, functions, signals (Borsuk 1958-59; Borsuk 1969; Peters 2016). If we simply evaluate systems activity instead of "signals", BUT leads naturally to the possibility of a region-based, not simply point-based, geometry. We are thus allowed to describe systems features as antipodal points on a n-sphere. If we map the two points on a n-1 sphere, we obtain a single point. This means that signal shapes can be compared (Weeks; Peters 2016): the two antipodal points standing for systems features are assessed at one level of observation, while the single point at a lower level (Tozzi 2016a). The BUT can be used not just for the evaluation of antipodal, but also of non-antipodal points on an n-sphere. We can consider regions on an n-sphere that are either adjacent or far apart (Tozzi 2016a). And this BUT variants applies, provided there are a pair of regions on n-sphere with the same feature value. Therefore, the two points (or regions) do not need necessarily to be antipodal, in order to be described together (Peters 2016). This makes it possible to evaluate matching signals, even if they are not "opposite", but "near" each other: the antipodal points restriction from the "standard" BUT is no longer needed. Although BUT was originally described just in case of n being a natural number which expresses a structure embedded in a spatial dimension, nevertheless the value of n can stand for other types of numbers. The n value of S^n can be also cast as an integer, a rational or an irrational number (Tozzi 2016a). For example, we might regard functions or shapes as embedded in a sphere in which n does not stand for a spatial dimension, but for a fractal one. This makes it possible to use the n parameter as a tool for the description of nervous power laws.

In sum, the widespread brain scale-free can be evaluated in terms of algebraic topology. **Figure 1** illustrates an example of nervous temporal power laws embedded in a n-sphere, in which n stands for the brain power slope. The n-sphere, in this case, is equipped with a n-value corresponding to the fractal dimension α (which could be detected by several neurofunctional techniques: see (Pritchard; He). The figure clearly shows that the low- and high- frequency brain oscillations exhibit a matching description: indeed, the scale-free distribution of the brain spikes allows us to consider the slow and fast frequencies as homotopies or affine connections. Embracing brain fractals in the framework of algebraic topology (Willard; Dodson) means that power laws at different timescales (the antipodal points) can be described as functions on "abstract" structures: the BUT perspective permits the nervous scale-free property located in the real space (the brain geometric space) to be translated to an abstract space and *vice-versa,* enabling us to achieve maps from one level to another.

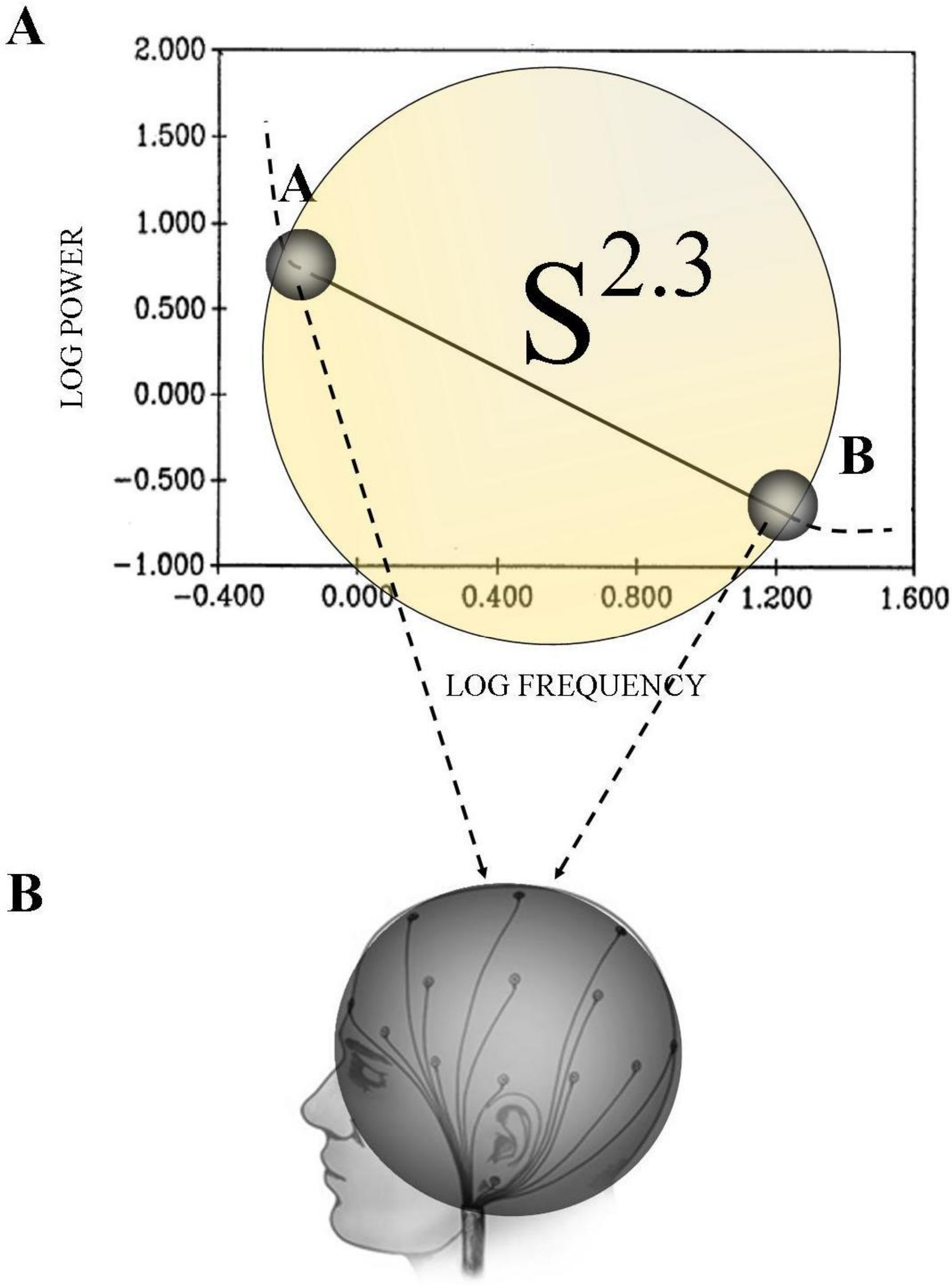


Figure 1A. Log amplitude versus log frequency scatter plot of brain spikes detected by EEG techniques (modified from Pritchard). The Figure displays on the x axis the frequency (in Hz) and on the y axis the power (in μV^2) of the electric spikes. Note that the scale is logarithmic: it means that on the x axis, for example, -0.400 = 0,39 Hz, 0 =1 Hz, 0.4 = 2.52 Hz, 1.2 = 16 Hz and so on. In turn, on the y axis, -1.000 = 0,1 μV^2, -0.5 = 0,32 μV^2, 0.000 = 1 μV^2, 1.000 = 10 μV^2, and so on. The figure also depicts a fractal dimension, in this case equipped with the slope $\alpha = 2.3$.
The BUT now comes into play: the black circles A and B, which respectively depict power laws at low- and high- brain frequency, stand for two antipodal points with matching description, embedded in a n-sphere with fractal exponent α (in this case, 2.3). Note that the Borsuk-Ulam theorem is not valid at the slope's tails, where the α exponent is lost (dotted lines on the right and left of the main slope).
Figure 1B. The total fractal activity detected by different functional neurotechniques stands for the projection of the two antipodal points on a n-1 -manifold. Ah head mounted EEG is showed for sake of simplicity.

RESULTS

In sum, our topological approach showed that low- and high-frequency fractal spikes are correlated in the brain, because they display the same features. By knowing the values of an antipodal points, the other one can be detected and measured on the opposite side of the diameter (corresponding to the fractal slope). Once established that a matching description occurs between slow and fast brain oscillations, we looked for a feasible equation able to link their reciprocal activities. A theory from an apparent far-flung branch, solid-state physics, might indeed help us to improve our knowledge of different nervous oscillations' relationships.

In a periodically repeating environment (such as a lattice), the Bloch wave is a linear oscillation obtained by the product of (Floquet, Taversa, Ahn):

a) a periodic function with the same periodicity as the lattice,
b) a plane wave.

In such a solid-state physics' system, the wavefunction Ψ for a particle has the form:

$$\Psi(x) = e^{ikr}u(x).$$

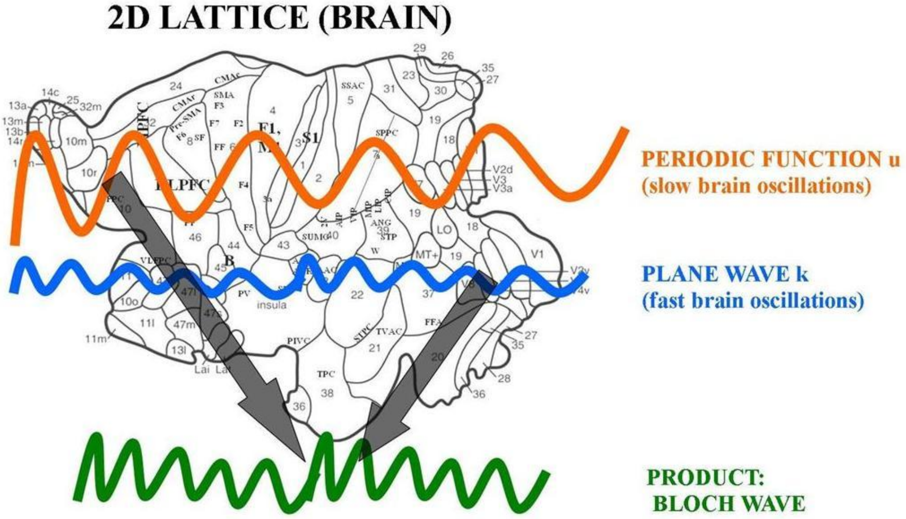
In which Ψ is called a Bloch wave, x is its position, e^{ikr} is a plane wave (in which e is the Euler's number, i is the imaginary unit, k is a vector of real numbers called the *crystal wave vector*), and $u(x)$ is a periodic function with the same periodicity as the lattice. A mathematical treatment for technical readers is provided in the **Appendix**.

In a brain framework, we are allowed to look for the nervous correlates of the Bloch wave's equation (**Figure 2A**):

1) We may consider just the real part and put aside the imaginary part of the Bloch theorem.
2) The brain stands for a 2-D lattice. We are indeed allowed to unfold and flatten cerebral hemispheres into a two-dimensional reconstruction by computerized procedures (Van Essen, 2005).
3) The periodic functions u stand for the spontaneous, slow brain oscillations which take place on the 2D brain lattice. We may conventionally state that u has the same periodicity of the brain lattice, because it corresponds to the nervous spontaneous oscillations.
4) The plane waves (the vectors of real numbers k) stand for the fast, task-evoked brain oscillations.
5) The Bloch waves stand for an unknown brain parameter able to link together the parameters u and k.

Figure 2B provides a simulation of the Bloch theorem applied to brain function. The picture shows that a change in slow oscillations leads to a change in fast oscillations, and vice versa.

A



B

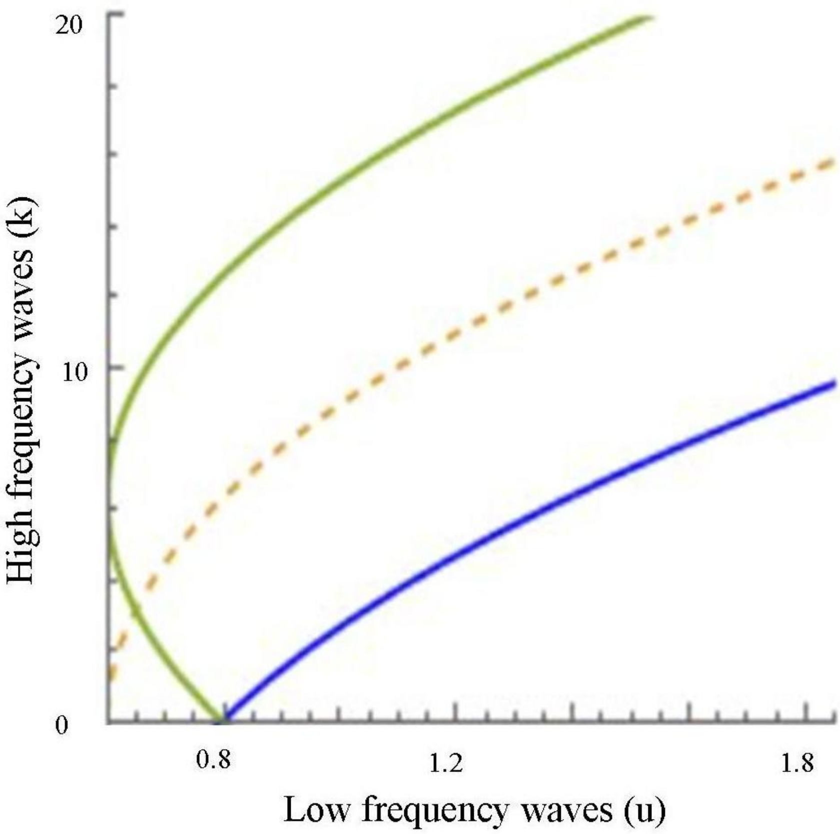


Figure 2A. A rough sketch depicting the brain correlates of the Bloch theorem. See the text for further details.
Figure 2B. Relationships between slow and fast brain oscillatory activity, according to the Bloch Theorem. A plot of Bloch wave forms, for *k* (high frequency waves) on the *y*-axis, and *u* (low frequency waves) on the *x*-axis, is displayed. The brain spike frequencies are expressed in Hz (note that this time the scale is non-logaritmic). The different lines correspond to diverse values of the Bloch wave. Just the slow frequencies included in the fractal range are displayed, while the non-fractal slope tail on the left of **Figure 1A** is excluded.

CONCLUSIONS

The aim of this paper was to investigate the possible functional relationships between slow and fast brain oscillations. We elucidated the correlations between nervous fluctuations with different timescales and functional activity (the low- and the high- frequency spikes) and proposed a possible equation, based on the Bloch theorem, in order to quantify their connections. Indeed, if we replace the periodic function with the scale-free ultra-slow waves of the default mode networks, and the plane wave with the fast brain spikes, we obtain a Bloch wave function which takes into account their correlations. In such a way, the increase in frequency of the slow electric oscillations leads to a predictable increase of the fast electric ones. Our simulation showed that spontaneous fluctuations in cortical excitability might have a remarkable effect on the field potentials activity frequency spectrum as well as the spiking activity of neurons. This coupling, which serves an important coordinating role, provides a logical structure for the integration of brain functional activity (Buszaki; Raichle). Recent results demonstrate indeed that gamma-band activity in the alert monkey is largely an emergent property of cortex from the resting state waves (Bastos). This hypothesis is made more salient by the observation that many types of behavioral variability follow a 1/*f* frequency distribution similar to that of spontaneous BOLD, meaning that there is increasing power at lower frequencies (Fox).

The Bloch theorem might shed new light on the interpretation of the data from functional neuroimaging. If we were able to calculate the Bloch wave from the experimental records, we could obtain a simple factor which summarizes the behaviour of the brain's dynamical system - and which could also be compared with the real data from pairwise entropy studies (Watanabe) -. We thus achieve a sort of single "order parameter" which could allow a better comprehension of cortical dynamics. This method could be used in the study of EEGs as well as fMRI neuroimaging. We also suggest other feasible applications of the Bloch theorem in different fields of neuroscience: for example, if we replace the brain with a lattice, we might evaluate the first Brillouin zone and quantify the Bloch waves in different functional states (for further details, see **Appendix**). The fact that the same Bloch wave may be obtained from different types of oscillations could explain the apparently chaotic behaviour of cortical fluctuations. If the Bloch waves change in different functional states (i.e., sensations, perceptions, emotions, mind wandering and so on), we might be able detect the underlying activity, starting just from the knowledge of the Bloch wave. Furthermore, the well-known relationships between Bloch waves, Floquet multipliers, Lyapunov exponents and limit-cycle attractors (see **Appendix**) allow us to evaluate the brain oscillations' spatial fractals/temporal power laws in the context of dynamical system theories. To make an example, starting from the Lyapunov exponents endowed in the metastable brain at the edge of chaos (Beggs) -, we might achieve a more manageable linear system, which depends just on the various brain timescales.

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APPENDIX: BLOCH WAVES AND FLOQUET THEOREM

The Bloch theorem states that, if you multiply a plane wave by a periodic function, you obtain a Bloch wave, which expresses the energy eigenstates for a particle in a lattice, written as ψn_k, where *n* is a discrete index.
There are different Bloch waves with the same *k*, each one with a different periodic component *u*. Further, the same Bloch wave can be built in different ways, involving different vectors *k* and different periodic functions *u*. However, if we take into account just the first Brillouin zone of our lattice, we obtain that every Bloch state has a unique *k*.
The concept of Bloch theorem from solid-state physics - a second order differential equation - is about crystals in any number of spatial dimensions and deals in particular with the Schrödinger equation. However, it can be also applied in theory of ordinary differential equations, through the Floquet theorem. Indeed, the two theorems are almost equivalent (Floquet). The Floquet theorem, through a coordinate change in the lattice, transforms the original periodic system into a more manageable traditional linear system with constant, real coefficients. In other words, we map a fundamental matrix solution into a matrix function depending on the time, giving rise to a time-dependent change of coordinates. The Floquet theorem holds for any homogeneous, linear system of first order differential equations with a periodic coefficient matrix. Through a coordinate change in the lattice, the Floquet's theorem transforms the periodic system into a traditional linear system with constant, real coefficients.
We start from a linear first order differential equation:

$$x = A(t)\,x,$$

where *x*(*t*) is a column vector of length n, and *A*(*t*) is an $n \times n$ periodic matrix with period *T*.
Then, for all $t \in R$:

$$\Phi(t+T) = \Phi(t)\,\Phi^{-1}(0)\,\Phi(T),$$

In which $\Phi(t)$ is a fundamental matrix solution of the above differential equation $x = A(t)\,x$, and $\Phi^{-1}(0)\,\Phi(T)$ is the monodromy matrix.
Now let's consider the $n \times n$ matrices: *B, P, Q, R*:
For each matrix B such that:

$$E^{TB} = \Phi^{-1}(0)\,\Phi(T),$$

there is a periodic (period *T*) matrix function $t \to P(t)$ such that:
$\Phi(t) = P(t)\,E^{tB}$ for all $t \in R$. This representation is the Floquet normal form for the fundamental matrix $\Phi(t)$.
There is also a real matrix R and a real periodic function (period -2T) matrix function $t \to Q(t)$ - which is continuous and periodic - such that:

$$\Phi(t) = Q(t)\,E^{tR} \text{ for all } t \in R.$$

The latter mapping gives rise to a time-dependent change of coordinates:

$$y = Q^{-1}(t)\,x,$$

under which the original system becomes a linear system with real constant coefficients $y = Ry$. The mapping of a fundamental matrix solution for such a differential equation into a (time-dependent) matrix function gives rise to a time-dependent change of coordinates, under which the original periodic system becomes a linear system with real constant coefficients $y = Ry$.

The eigenvalues of e^{TB} are called the characteristic multipliers of the system, while the characteristic exponent, called the Floquet exponent, is a complex μ such that $e^{\mu T}$ is a characteristic multiplier of the system. Floquet exponents are not unique and their real parts correspond to the Lyapunov exponents.
The linear differential equations with periodic coefficients have been widely used in many scientific fields. In particular, they provide a versatile tool for the stability analysis of physical systems equipped with a periodic steady-state and infinite memory, such as Brownian particles and circuit resonators (Traversa). The Floquet multipliers have been also used to assess the stability of periodic motion in natural rhythmic - movements in humans and machines -, not just in linear systems, but also in stochastic noise and in limit-cycle, nonlinear oscillators (Ahn).