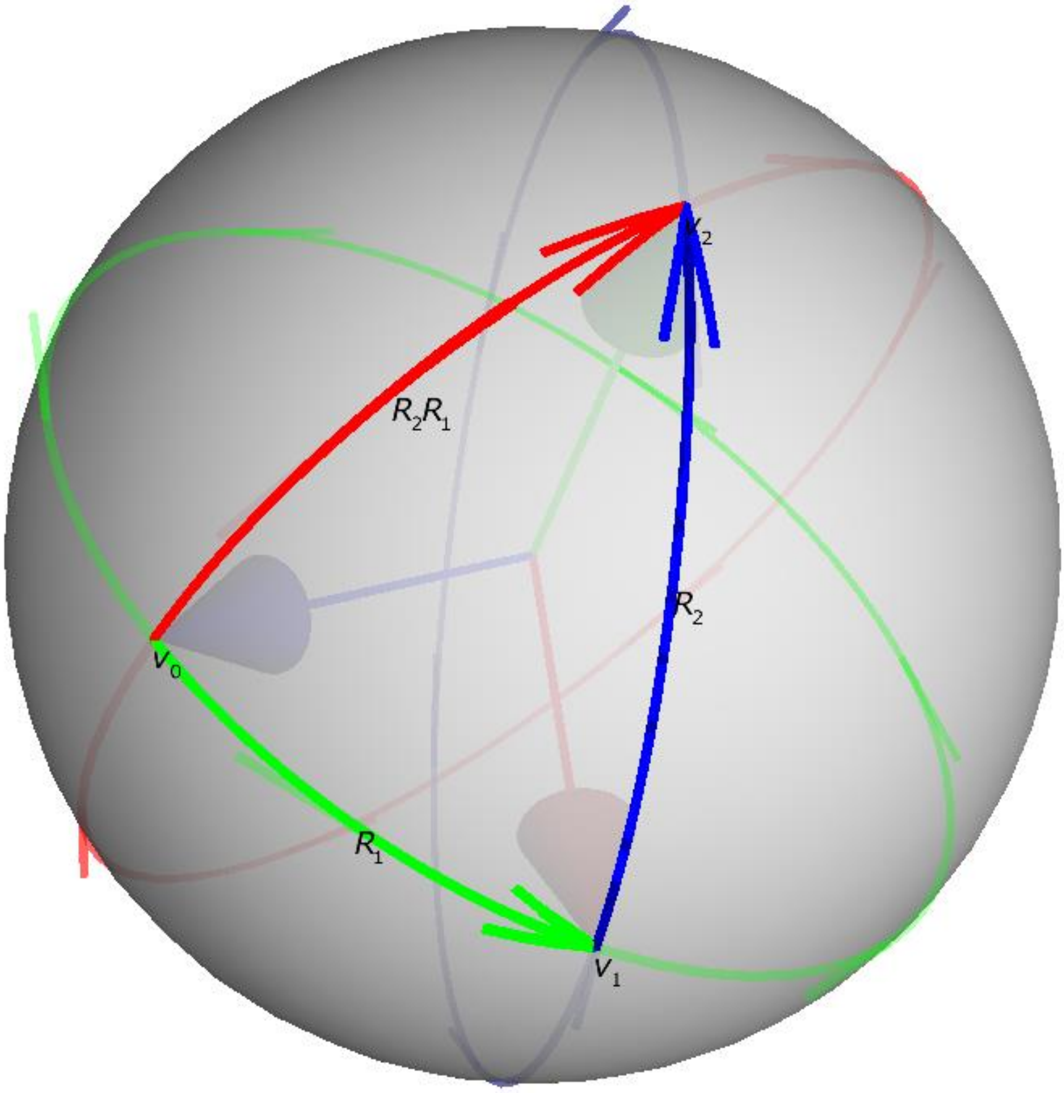
Miroslav Josipović

Multiplication of vectors and structure of 3D Euclidean space



„So God made the people speak many different languages …“

Virus

„The ***World Health Organization*** has announced a world-wide epidemic of the ***Coordinate Virus*** in mathematics and physics courses at all grade levels. Students infected with the virus exhibit compulsive ***vector avoidance*** behavior, unable to conceive of a vector except as a list of numbers, and seizing every opportunity to replace vectors by coordinates. At least two thirds of physics graduate students are severely infected by the virus, and half of those may be permanently damaged so they will never recover. The most promising treatment is a strong dose of ***Geometric Algebra***“. (*Hestenes*)

Cat

„When the spiritual teacher and his disciples began their evening meditation, the cat who lived in the monastery made such noise that it distracted them. So the teacher ordered that the cat be tied up during the evening practice. Years later, when the teacher died, the cat continued to be tied up during the meditation session. And when the cat eventually died, another cat was brought to the monastery and tied up. Centuries later, learned descendants of the spiritual teacher wrote scholarly treatises about the religious significance of tying up a cat for meditation practice.“ (*Zen story*)

Empty your cup

„A university professor went to visit a famous Zen master. While the master quietly served tea, the professor talked about Zen. The master poured the visitor's cup to the brim, and then kept pouring. The professor watched the overflowing cup until he could no longer restrain himself. - *It's overfull! No more will go in!* - the professor blurted. - *You are like this cup*,- the master replied, - *How can I show you Zen unless you first empty your cup*?“ (*Zen story*)

Division algebra

“Geometric algebra is, in fact, the largest possible associative division algebra that integrates all algebraic systems (algebra of complex numbers, vector algebra, matrix algebra, quaternion algebra, etc.) into a coherent mathematical language that augments the powerful geometric intuition of the human mind with the precision of an algebraic system.”

(Sabbata: *Geometric algebra and applications in physics* [28])

Preface

The aim of this paper is to introduce the interested reader to the world of geometric algebra. Why?

Alright, imagine the Neelix and Vulcan (from the starship Voyager) conversation. The goal is to sell a new product to the Vulcan (Tuvok). This can be achieved so that Neelix quickly intrigue the Vulcan, giving him as little information as possible, and the ultimate goal is that Vulcan, after using it, be surprised by the quality of the product and recommend it to the others. Let's start.

Neelix: "Mr Vulcan, would you like to rotate objects without matrices, in any dimension?"

Vulcan: "Mr Neelix, do you offering me quaternions?"

Neelix: "No, they only work in 3D, I have something much better. In addition you will be able to do spinors, too."

Vulcan: "Spinors? Come on, mr Neelix, you're not going to say that I will be able to work with complex numbers, too?"

Neelix: "Yes, mr Vulcan, the whole complex analysis, generalized to higher dimensions. And you will be able to get rid of tensors."

Vulcan: "Excuse me, what? I'm a physicist, it will not pass …"

Neelix: "It will, you do not need the coordinates. And you will be able to do the special theory of relativity and quantum mechanics using the same tool. And all integral theorems that you know, including the complex area, become a single theorem."

Vulcan: "Come on … nice idea … I work a lot with the Lie algebras and groups ..."

Neelix: "In the package …"

Vulcan: "Are you kidding me, mr Neelix? Ok, let's say that I believe you, how much would that product cost me?"

Neelix: "Pennyworth, mr Vulcan, You must multiply vectors differently."

Vulcan: "That's all? All of this you offer me for such a small price? What's trap?"

Neelix: "There is no one. But true, you will have to spend some time to learn to use the new tool".

Vulcan: "Time? Just do not have … And why would I ever forgo coordinates? You know, I am quite adept at juggling indices, I have my career …"

Neelix: "Do physical processes you are studying depend on the coordinate systems you choose?"

Vulcan: "I hope not."

Neelix: "There. Does a rotation by matrices provides you a clear geometric meaning when you do it?"

Vulcan: "No. I have to work hard to find it out."

Neelix: "Now you will not have to, it will be available to you at each step."

Vulcan: "Mr. Neelix, I'm curious, where did you get this new tool?"

Neelix: "Well, mr Vulcan, it is an old tool from Earth, 19th century, I think, invented by humans Grassmann and Clifford."

Vulcan: "What? How is that I'm not aware of it? Isn't it strange?"

Neelix: “Well, I think that human Gibbs and his followers had a hand in it. Allegedly, human Hestenes was trying to tell the other humans about it, but they did not listen to him. You will agree, mr Vulcan, that humans are really funny sometimes.”

Vulcan: “Mr Neelix, this is a rare occasion when I have to agree with you.”

Vulcan buys and lives long and prosper. And, of course, recommends the new tool to the captain …

This text is not intended as a textbook, it is more motivationally directed, to see „what's up“. It is intended mainly to young people. Also, intention here was to use simple examples and reader is referred to the independent problem solving. The active reading of the text is recommended, with paper and pencil in hand. There is a lot of literature, usually available at Internet, so, reader is referred to the independent research. The use of available computer programs is also recommended. There are reasons to think that geometric algebra is mathematics for future. Paradoxically, it has been established since the mid-19th century, but was ignored as a result of a series of (unfortunate) circumstances. It's hard to believe that those who have made careers will easily accept something new, hence belief that this text is mainly for young people. The background in physics and mathematics at the undergraduate level is necessary for some parts of the text, but it is somewhat possible to follow the exposure using Internet to find explanation for the less familiar terms. A useful source is the book [35], which can certainly help to those who are just starting with algebra and geometry. The book [20] is hard one and it is recommended to those who think seriously. But, read Hestenes' articles first.

It is important for the reader to adopt the idea that the vector multiplication here exposed is natural and justified. The rest are the consequences of such a multiplication. The reader can independently come up with arguments to justify the introduction of the geometric product. The goal is to understand that the geometric product is not just a "neat trick", but that naturally arises from the concept of vector. That changes a lot of mathematics. A simple setting that parallel vectors commute while orthogonal anti-commute produces an incredible amount of mathematics and unites many different mathematical disciplines into the language of geometric algebra.

You can send me comments or questions at:

miroslav.josipovic@gmail.com

Miroslav Josipović

Zagreb, 2017.

Geometric product (of vectors)

Vectors will generally be denoted in small letters in *italic* format, wherever there's no possibility of confusion. We will use the **bold** format also, if necessary. We will use the Greek alphabet for real numbers. *Multivectors* are denoted in the uppercase *italic* format. If we define the orthonormal basis in a vector space then the number of unit vectors which square to 1 is denoted with p, the number of those with square -1 with q and the number of those with square 0 with r. Then the common designation for such a vector space is $\Re(p,q,r)$ or $\Re^{p,q,r}$, while triplet (p,q,r) is referred as the *signature* (in literature this is also the sum $p+q$). For geometric algebra of 3D Euclidean vector space $\Re^3$ we use the abbreviation *Cl*3, which is motivated by the surname Clifford.

Here, when we say "vector", we do not refer to elements of an abstract vector space, we rather take that concept as an "oriented straight line". To add vectors we use the parallelogram rule. Vectors a and b that satisfy the relation $b=\alpha a,\quad \alpha\in\Re,\quad \alpha\neq 0$, are said to be *parallel*. For parallel vectors we say that they have the same *direction* (*attitude*), but could have the same or opposite *orientation*. We can resolve any vector b into the component in the direction of the vector a (*projection*) and the component without any part parallel to the vector a (*rejection*)

$$b=b_{\|}+b_{\perp},\quad b_{\|}=\alpha a,\quad \alpha\in\Re,\quad \alpha\neq 0\,.$$

Here we can immediately anticipate objections, like: „Yes, but if we talk about orthogonal vectors we need a scalar product …". Although we use the character „ $\perp$ ", here, for a moment, we are **not** talking about the orthogonality of vectors. Simply, by the fact that vectors can be added, we conclude that any vector can be written as a vector sum of two vectors, in an infinite number of ways. One of these possibilities is just given by the previous relation, so it can be seen as a question of **existence**, and not how to practically implement it. Namely, for $b_{\perp}=b-b_{\|}=b-\alpha a$, if we assume that the vector $b_{\perp}$ contains a component parallel to a we can write $b'_{\perp}+\beta a=b-\alpha\, a$, but then the vector $b'_{\perp}$ is our rejection. If there is no $b'_{\perp}$ then the vector b is parallel to the vector a. After, eventually, we succeed to define a new product of vectors, we can return to the question how to find $b_{\perp}$ practically, and that is what the new product of vectors should certainly enable to us.

Let's ask the question: **how to multiply vectors**? We will need to "forget" everything we have learned about the multiplication of vectors (i.e. scalar and cross products). Well, before we "forget" them, let's look at some of their properties. Can we uniquely solve the equation $\boldsymbol{a}\cdot\boldsymbol{x}=\alpha$ (here $\boldsymbol{a}\cdot\boldsymbol{x}$ is a scalar product)? The answer is, clearly, we cannot, because if $\boldsymbol{x}$ is a solution then each vector of the form $\boldsymbol{x}+\boldsymbol{b},\quad \boldsymbol{b}\cdot\boldsymbol{x}=0$ is a solution, too. And what about the equation $\boldsymbol{a}\times\boldsymbol{x}=\boldsymbol{b}$ (cross product)? It also cannot be uniquely solved, because if $\boldsymbol{x}$ is a solution then each vector of the form $\boldsymbol{x}+\beta\boldsymbol{a}$ is a solution, too. But, interesting, if we take into account both equations then we can find the unique solution. Notice that the scalar product is commutative, while the cross product is anti-commutative. For two unit vectors $\boldsymbol{m}$ and $\boldsymbol{n}$ in 3D we have

$$\boldsymbol{m}\cdot\boldsymbol{n}=\cos\alpha \quad\text{and}\quad \left|\boldsymbol{m}\times\boldsymbol{n}\right|=\sin\alpha\,,$$

which suggests that these two products are somehow related, because of

$$\sin^2\alpha+\cos^2\alpha=1\,.$$

An interconnection could be anticipated if we look at multiplication tables in 3D (e_i are orthonormal basis vectors):

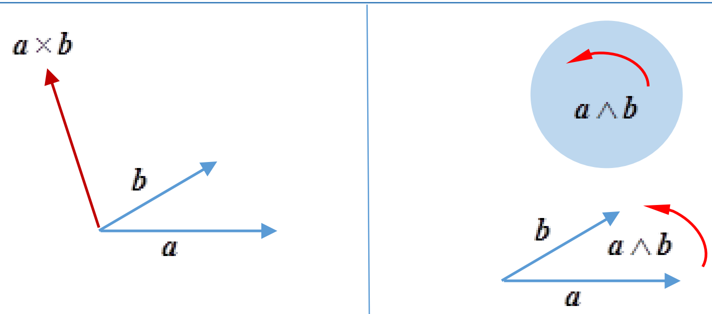
$\bullet$	e_1	e_2	e_3
e_1	1	0	0
e_2	0	1	0
e_3	0	0	1

$\times$	e_1	e_2	e_3
e_1	0	e_3	$-e_2$
e_2	$-e_3$	0	e_1
e_3	e_2	$-e_1$	0

We see that the scalar product has values different from zero only on the main diagonal, while the cross product has zeros on the main diagonal (due to anti-commutativity). Multiplication tables simply lure us to unite them. The form of both products suggests similarity with complex numbers that can be elegantly written in the trigonometric form, but for this we need a quantity which gives -1 squared, like the imaginary unit. But, it is not clear how to naturally relate the cross product to an imaginary unit like quantity. On the other hand, the cross product is anti-commutative, which suggests that it "should" have the feature to give -1 when squared. Namely, if we imagine any quantities that give a positive real value when squared and whose products are anti-commutative and associative we would have

$$(AB)^2 = ABAB = -ABBA = -A^2B^2 < 0 .$$

Let's look at an orthonormal basis in 3D, we can say that the vector e_1 is polar vector, while $e_2 \times e_3 = e_1$ is axial vector. So, what is e_1 like? Of course, we could play with more general definitions invoking tensors, but it is strange that in such an elementary example we immediately have a problem. Mathematicians would argue that the cross product can generally be defined in dimensions different from 3, but if you think about it a little and require a natural and simple definition, some questions arise immediately.



The **cross product** lives in 3D and have a lot of problems.

A **bivector** is an oriented part of a plane, lives in all dimensions greater than 1 and it's almost magical.

Let's look at a 2D world where plane physicists want to define the torque. If they do not wish to look for new dimensions outside "their world", they will not even try to define a cross product, there is no vector orthogonal to their world. But, we can see that the torque makes sense also in 2D world: it is proportional to the magnitude of both force and force arm, the two possible orientations of rotation are clearly defined, therefore, how to multiply a force arm vector and a force vector to provide the desired torque? The answer to that question is found already in 19th century by great mathematician Grassmann, underestimated and neglected in his time. He defined the anti-commutative exterior product of vectors and so got a **bivector**, an object contained in a plane, with orientation and module, so, it is ideal for our 2D problem. In addition, it can be easily generalized to higher dimensions. Grassmann himself and Clifford a little later managed to unite scalar and outer (exterior) product into one: *geometric product*, exactly what we are talking here about. The scalar product of vectors is not changed, but the cross product is replaced by the outer product and artificial difference between "axial" and "polar" vectors disappeared. All "axial" vectors are bivectors (magnetic field vector, for example, see in text).

Alright, now "forget" the scalar and the cross products and let's find how to define a **new** one. It is reasonable to require an associativity and distributivity of the new multiplication (like for real numbers), i.e.

$$a(bc)=(ab)c \text{ and } (\beta b+\gamma c)a=\beta ba+\gamma ca,\quad a(\beta b+\gamma c)=\beta ab+\gamma ac,\quad \beta,\gamma\in\Re .$$

Of course, we do not expect a commutativity of a vector multiplication, except for scalars (real numbers). After all, the definition of the cross product is motivated by the need for such a non-commutative constructs (like torque, or Lorentz force, …).

1) Let's consider the term a^2 first (a is a vector). We will assume that $a^2\in\Re$. Clarify immediately that we do **not** imply that $ab\equiv a\cdot b$, as usual, where we have the scalar product denoted by dot. This is important to note, as it would lead to confusion otherwise. We expect that the square of the vector does not depend on the vector direction, but depends on its length (we exclude the possibility of nonzero vectors with the length zero, for now).

2) We expect that the multiplication of the vector by a real scalar is commutative, which immediately results in that the multiplication of parallel vectors ($a \,||\, b$) is commutative:

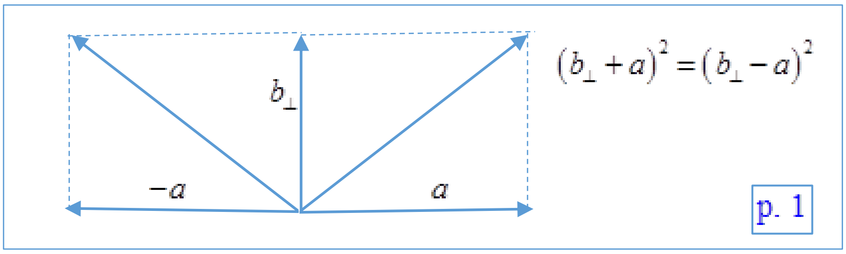
$$\lambda a=a\lambda \Rightarrow ab=a\lambda a=\lambda aa=ba,\quad \lambda\in\Re .$$

Actually, we could call principles of symmetry to help us, we immediately see that multiplication of parallel vectors must be commutative, because we have no criterion to distinguish which vector is the "first" and which is the "second". This is obvious if vectors have the same orientation, but if vectors have opposite orientations we can refer to the fact that all orientations in space are equal (*isotropy*). On the other hand, for perpendicular vectors we just expect anti-commutativity because we always know which vector in the product comes first and vectors define the same part of the plane (parallelogram) no matter in what order they are multiplied. Therefore, there remains the possibility that products with a different order of vectors differ in a sign. Our new product should also include multiplication of reals by reals.

3) Due to the independence of the square of the vector on direction we have (recall, $b_\perp$ has no component in the direction of a)

$$(b_\perp+a)^2-(b_\perp-a)^2=0=2(b_\perp a+ab_\perp),$$

meaning that vectors $b_\perp$ and a anti-commute. You can design other "arguments", but recall, we do not assume the scalar or cross product, we are looking for properties of the **new** product of vectors "from scratch". This example is **not** the proof, just an idea how we could think about it. In figure p. 1 we can easily see what we demanded: that the square of vectors does not depend on the direction.



We can, of course, after we assumed non-commutative multiplication, just use

$$\left(a+b_\perp\right)^2 = a^2 + b_\perp^2 + ab_\perp + b_\perp a$$

and immediately conclude that it must be $ab_\perp + b_\perp a = 0$ because we expect the Pythagorean theorem to be true. But figure p. 1 show us that we have symmetry here, namely, vectors a and $-a$ define a "straight line", here "right" and "left" is not important concept and we see that the direction of the vector $b_\perp$ suggests the symmetry in accordance with our intuitive concept of the orthogonality. Without this symmetry we enter a "skew land", but let pure mathematicians to go there.

4) Let us show now that, according to 3), a^2 commutes with *b* (without any assumption what a^2 is):

$$a^2 b = a^2\left(b_\parallel + b_\perp\right) = b_\parallel a^2 - ab_\perp a = b_\parallel a^2 + b_\perp a^2 = ba^2,$$

which justifies our (previous) assumption that $a^2 \in \Re$. Again, it is important to understand that we are **not** giving proofs, we are to **justify** the new product of vectors. It follows immediately that $ab_\parallel$ commutes with *b*, because of $b_\parallel = \alpha a,\quad \alpha \in \Re$. Now we have

$$ab + ba = ab_\perp + b_\perp a + 2ab_\parallel = 2ab_\parallel,$$

so $ab + ba$ commutes with b. It is clear that it commutes with a also, which means that commutes with any vector in the plane defined by vectors a and b. But it obviously commutes with any vector perpendicular to that plane, because a and $b_\parallel$ anti-commute.

We can always decompose any non-commutative product into the symmetric and anti-symmetric part:

$$ab = \frac{ab+ba}{2} + \frac{ab-ba}{2} = S + A,$$

where we have

$$S = ab_\parallel, \qquad A = ab_\perp.$$

Last two relations are very handy, for example we can see that

$$A^2 = ab_\perp ab_\perp = -a^2 b_\perp^2.$$

The symmetric part, we have seen, commutes with all vectors. This is also seen from

$$ab + ba = a^2 + b^2 - \left(a+b\right)^2,$$

because the square of a vector is commutative. Note that we have not defined precisely yet what a^2 is, but it is obvious that regardless of the explicit value of a^2 we have for vectors a and $b_\perp$

$$\left(a+b_{\perp}\right)^2 = a^2+b_{\perp}^2+ab_{\perp}+b_{\perp}a = a^2+b_{\perp}^2+ab_{\perp}-ab_{\perp} = a^2+b_{\perp}^2,$$

i.e. we have the Pythagorean theorem, here expressed through the new multiplication of vectors. If we define the term "orthogonal" as the relation between vectors in which the projection of one on another is zero ($b_{\perp} = a - b_{\parallel}$), we get the Pythagorean theorem, which now applies to orthogonal vectors regardless of the specific value of a^2, if we accept the arguments from the part 3). Let us recall that the Pythagorean theorem is, as a rule, expressed over the scalar product of vectors and that in this way we have a problem with negative signature (meaning that there are vectors whose square is negative), as is customary in the special theory of relativity. For any two vectors, the relation

$$\left(a+b\right)^2 = a^2+b^2+ab+ba$$

can be taken as the *cosine rule*, because the symmetric part of the new product commutes with all vectors, and thus is a "scalar" .

We assumed that a^2 is a real number equal to $\pm\left|a\right|^2$, where $\left|a\right|$ is the absolute length of the vector a (we say that we are introducing *metrics*). Now we can write for the symmetric part

$$a\cdot b \equiv \left(ab+ba\right)/2 = ab_{\parallel},$$

that we call the *inner product*. We see that it coincides with the usual scalar product of vectors, but here we need a little bit of caution: in geometric algebra we generally distinguish several types of "scalar" products, one of them is the *scalar product* (generally different than that of Gibbs), and there are more: *dot product, left contraction*, etc. For vectors, all types of "scalar" products coincide, but generally they are a little different (see literature). Here we are to work with the *inner product* and the *left contraction* (see in text).

For unit vectors of the orthonormal basis we have $e_i^2 = \pm 1$ (null-vectors are not included here), which means

$$e_ie_j + e_je_i = \pm 2\delta_{ij}.$$

Caution: do not confuse e_ie_j with $e_i\cdot e_j$! If you are wondering what e_ie_j is, the answer is: a **completely new type of object,** we will see about it in the text.

Let's look at 2D examples:

$$\Re^2: \quad e_1^2 = e_2^2 = 1 \Rightarrow \left(e_1+e_2\right)^2 = 1+1+e_1e_2+e_2e_1 = 2 = e_1^2+e_2^2,$$
$$\Re^{1,1}: \quad e_1^2 = -e_2^2 = 1 \Rightarrow \left(e_1+e_2\right)^2 = 1-1+e_1e_2+e_2e_1 = 0 = e_1^2+e_2^2,$$

so we see that in both cases the Pythagorean theorem is valid, but with the new multiplication of vectors.

For $\Re^3$ we have:

$$e_1^2 = e_2^2 = e_3^2 = 1, \quad e_ie_j + e_je_i = 2\delta_{ij},$$

but, here's a magic, there are known mathematical objects that meet precisely these relations: *Pauli matrices*, discovered in the glorious years of the development of quantum mechanics. We can say that Pauli matrices are 2D matrix representation of unit vectors in $\Re^3$, we only need vectors to be multiplied in a new manner, just described. That is to say, Pauli matrices have the same multiplication table as orthonormal basis vectors. Let's make sure of that. Pauli matrices are defined as

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so, for example

$$\hat{\sigma}_2\hat{\sigma}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\sigma}_1\hat{\sigma}_2 + \hat{\sigma}_2\hat{\sigma}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The designation $\hat{\sigma}_i$ is often used for Pauli matrices, so here we use σ_i for unit vectors in $\Re^3$. Pauli matrices are important to describe the spin in quantum mechanics, so we see that vectors could serve to this purpose as well, but with our new product of vectors. Indeed, quantum mechanics can be nicely formulated by such mathematics, **without matrices and the imaginary unit** (see below).

Note that by a transposition of Pauli matrices followed by a complex conjugation we get again the same matrix (*Hermitian adjoint*), for example

$$\hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

or simply $\hat{\sigma}_2^\dagger = \hat{\sigma}_2$. Also we have, for example, $(\hat{\sigma}_2\hat{\sigma}_3)^\dagger = \hat{\sigma}_3^\dagger\hat{\sigma}_2^\dagger = \hat{\sigma}_3\hat{\sigma}_2$ (*antiautomorphism*, show that). This exactly matches the operation *reverse* (see below) on products of vectors, for example $e_1e_2e_3 \to e_3e_2e_1$. Therefore, the character † is often used to denote the *reverse* operation (we will do so here).

Here we can immediately spot the important feature of the new multiplication of vectors. The vector is geometrically clear and intuitive concept, and the new product of vectors also has a clear geometric interpretation (see below). For example, we can clearly geometrically present the product e_1e_2 as the oriented area, it has the ability to rotate, unambiguously defines the plane spanned by vectors e_1 and e_2, etc. All this we can immediately conclude at a glance. For comparison, consider now the matrix representation of vectors σ_1 and σ_2 with their product:

$$\hat{\sigma}_1\hat{\sigma}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Can we derive similar conclusions about a geometric interpretation just by looking into the resultant matrix? Just looking certainly not, it would take a lot of effort, but we will often fail to get the clear geometrical interpretation. Which plane the resultant matrix defines (if any is to be defined)? **Pauli matrices cannot do all that vectors can**. In this text we will, hopefully, illuminate such a things in order to get an idea of the importance of the new multiplication of vectors.

It is time for the new multiplication of vectors to get the name "officially" (due to *Clifford*, *Hestenes*, …): *geometric product*. The symmetric and anti-symmetric parts of the geometric product of vectors have special insignia: $a \cdot b$ and $a \wedge b$ ($a \cdot b$ is the *inner* and $a \wedge b$ is the *outer product*), so we can write for vectors

$$ab = a \cdot b + a \wedge b.$$

An important concept, that we will often use, is the *grade*. Real numbers have the grade zero, vectors have the grade 1, all elements that are linear combinations of products $e_i \wedge e_j$, $i \neq j$, have the grade 2, and so on. Notice that geometric product of two vectors is a combination of grades 0 and 2, it is *even*, because its grades are even. What grades generally has the geometric product of three vectors?

A vector space over the real field with the geometric product (GP in text) becomes an algebra (*geometric algebra*, GA in text). Elements of geometric algebra obviously are not vectors only. Note that the inner product is zero for orthogonal vectors, for example, for orthonormal basis vectors we have

$$e_1 \cdot e_1 = \frac{e_1 e_1 + e_1 e_1}{2} = 1, \quad e_1 \cdot e_2 = \frac{e_1 e_2 + e_2 e_1}{2} = 0 \Rightarrow e_1 e_2 = e_1 \cdot e_2 + e_1 \wedge e_2 = e_1 \wedge e_2,$$

so for orthogonal vectors the geometric product is the same as the outer product. How about the anti-symmetric part? We have

$$e_1 \wedge e_2 = \frac{e_1 e_2 - e_2 e_1}{2} = \frac{e_1 e_2 + e_1 e_2}{2} = e_1 e_2, \quad e_1 \wedge e_1 = \frac{e_1 e_1 - e_1 e_1}{2} = 0.$$

Obviously, $e_1 e_2$ is not a scalar, it doesn't commute with all other vectors, for example

$$(e_1 \wedge e_2) \mathrm{e}_1 = (e_1 e_2) \mathrm{e}_1 = -e_1 e_1 e_2 = -e_1 (e_1 \wedge e_2),$$

but is neither a vector in $\Re^3$, it squares to -1:

$$(e_1 e_2)^2 = e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = -1,$$

so, we have a **new** type of mathematical object, it is like the **imaginary unit**, except that is non-commutative. The name for such an object is the *bivector*. Generally, we will define a bivector as element of algebra of form $a \wedge b$. Let's look at some more properties of the bivector $e_1 e_2$. We have

$$(e_1 e_2) \mathrm{e}_1 = -e_1 e_1 e_2 = -e_2, \quad (e_1 e_2) \mathrm{e}_2 = e_1,$$

so, acting from the left on vectors it rotates them by $-\pi / 2$. How it rotates vectors if acting from the right?

Recall the *reverse* operation on geometric product of vectors: $x = abc...d \rightarrow x^\dagger = d...cba$, so we have

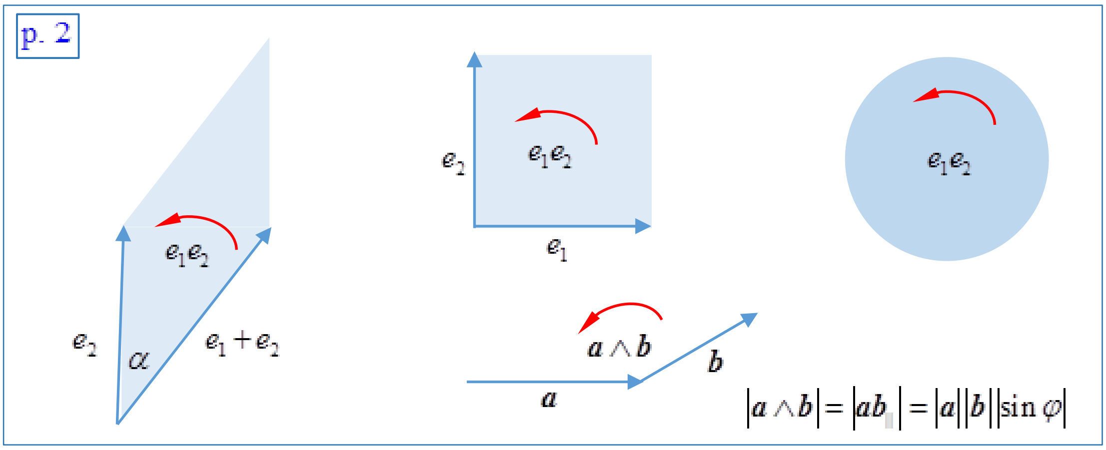
$$(e_1 e_2)(e_1 e_2)^\dagger = (e_1 e_2)(e_2 e_1) = -(e_1 e_2)^2 = 1,$$

therefore we call it a *unit bivector*. Generally, it is possible to find a module of bivectors, so, bivectors have the module and orientation. Furthermore, unit bivectors, like $e_1 e_2$, except for the module, orientation ($e_1 e_2 \neq e_2 e_1 = -e_1 e_2$) and the ability to rotate vectors, have another important feature, which the imaginary unit does not have, namely, it **defines** the plane spanned by vectors (here e_1 and e_2). Later we will see how this is implemented in practice by the outer product.

Now let's see how we can graphically present a (unit) bivector. The obvious option is to try with oriented parallelogram (square for $e_1 e_2$). But, the shape of an area which represents a bivector is not important, we should keep the magnitude of the area and orientation, therefore it is often a practical choice an oriented circle of radius $|e_1 e_2| / \sqrt{\pi} = 1 / \sqrt{\pi}$. To justify our claims, look at

$$e_1 e_2 = e_1 \wedge e_2 \Rightarrow (e_1 + e_2) \wedge e_2 = e_1 e_2,$$

it can illustrate the fact that the shape is not important.



Notice immediately that two vectors, except that define a plane, generally define a parallelogram, too. The outer product of such vectors (bivector) has the magnitude just equal to the parallelogram area (see below), while the direction we define as in figure p. 2. Find the area of the leftmost parallelogram in p. 2. Notice that bivector is just e_1e_2, but show that formula

$$1=|e_1e_2|=|e_1+e_2||e_2|\sin\alpha$$

gives the area of the parallelogram.

As previously for the symmetric part of the geometric product, we can write

$$ab-ba=ab_{\perp}-b_{\perp}a=2ab_{\perp}$$

and see immediately that it anti-commutes with a, $b_{\|}$ and $b_{\perp}$, so, it anti-commutes with b and, consequently, with all vectors from the plane defined by vectors a and b. Obviously, it commutes with vectors perpendicular to that plane. Also we have

$$(ab_{\perp})^2=ab_{\perp}ab_{\perp}=-aab_{\perp}b_{\perp}=-a^2b_{\perp}^2,$$

meaning that this quantity is negative in Euclidean space. So, the anti-symmetric part of the geometric product is not a vector, it can square to a negative real number and it is not a scalar, it anti-commutes with some vectors. Note that from

$$ab+ba=2ab_{\|},$$

$$ab-ba=2ab_{\perp},$$

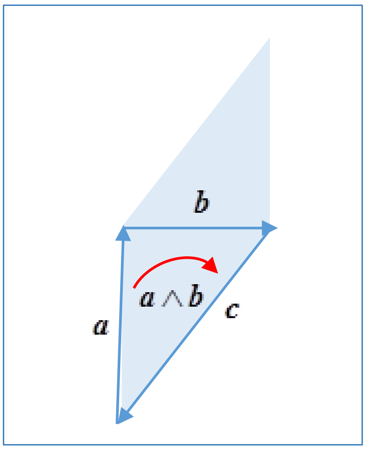
we can derive a lot of interesting properties of parts of geometric product, including their magnitudes, just using

$$|b_{\|}|=|b||\cos\varphi| \text{ and } |b_{\perp}|=|b||\sin\varphi|.$$

Let's look at three vectors in $\Re^3$ which sum is zero (a triangle), from $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=0$ it follows that

$$\boldsymbol{a}\wedge\boldsymbol{b}=\boldsymbol{b}\wedge\boldsymbol{c}=\boldsymbol{c}\wedge\boldsymbol{a}.$$

To see this it is enough to look at expressions $(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})\wedge\boldsymbol{a}$ and $(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})\wedge\boldsymbol{b}$ (check), but we can see it easy without calculation, it is enough to look at the figure on the left: each pair of vectors defines the parallelogram of the area equal to the double area of the triangle, and all pairs give the



same orientation of a bivector. This is important, often we can draw conclusions simply from the geometric observations, without calculation. In the formula for the area of a parallelogram appears the sine function, so we see that previous equalities are just the *sine theorem*. If we recall that bivector is not shape depended, we see that all three our bivectors have the same factor I (the unit bivector). Now we have

$$Iab\sin\gamma = Ibc\sin\alpha = Iac\sin\beta\,,$$

$$\frac{\sin\gamma}{c} = \frac{\sin\alpha}{a} = \frac{\sin\beta}{b}\,.$$

Bivectors define a plane. Consider the outer product in *Cl*3

$$\left(e_1 \wedge e_2\right) \wedge \left(a_1 e_1 + a_2 e_2 + a_3 e_3\right) = a_3 e_1 e_2 e_3\,,$$

so, we can see that the outer product of a bivector with a vector gives the possibility to eliminate components of the vector that do not belong to the plane defined by the bivector. Therefore, the plane of the bivector B (2D subspace) is defined by the relation

$$B \wedge x = 0\,.$$

In our example, solutions are all vectors of the form $x = a_1 e_1 + a_2 e_2$.

Imagine a unit bivector in *Cl*3. It defines a plane and have properties of (non-commutative) imaginary unit (in that plane). This is powerful: we can use the formalism of complex numbers in any plane, in any dimension. How? Let's take back our bivector $e_1 e_2$ and the vector $xe_1 + ye_2$. If we multiply our vector by e_1 from the left we get

$$e_1\left(xe_1 + ye_2\right) = x + ye_1 e_2 = x + yI,\ \ I = e_1 e_2\,,$$

so, we have a complex number. What we get if we multiply from the right? For more details see below.

The reader may show that any linear combination of unit bivectors in *Cl*3 can be expressed as an outer product of two vectors. This is not necessarily true in 4D, take for example $e_1 e_2 + e_3 e_4$. Prove that there are no two vectors in 4D with the property $a \wedge b = e_1 e_2 + e_3 e_4$. In 3D, for each plane we have exactly one orthogonal unit vector (up to the sign), while that is not true in higher dimensions. For example, in 4D, the plane defined by the bivector $e_1 e_2$ has orthogonal unit vectors e_3 and e_4 (their linear combinations too).

Take the bivector $e_1 e_2$ in $\Re^3$ and multiply it by $-e_1 e_2 e_3 \equiv -j$: $\ -e_1 e_2 j = e_3$, one can see that we get exactly the cross product of vectors e_1 and e_2, or, for arbitrary vectors

$$\boldsymbol{a} \times \boldsymbol{b} = -j\boldsymbol{a} \wedge \boldsymbol{b}\,.$$

This is valid in 3D, but the expression $-I\boldsymbol{a} \wedge \boldsymbol{b}$ is valid in any dimension, where I is a general pseudoscalar. In 2D $-I\boldsymbol{a} \wedge \boldsymbol{b}$ is just a real scalar, while in 4D or higher we can take the advantage of the concept of *duality*. The cross product of vectors (*Gibbs*) requires the right hand rule and the use of perpendiculars to surfaces. With bivectors it will not be necessary, so, for example, we can completely omit objects such as "rotation axis ", etc. Find the geometric product of two vectors $\boldsymbol{a}=\sum_{i=1}^{3} a_i e_i$ and $\boldsymbol{b}=\sum_{i=1}^{3} b_i e_i$ in $\Re^3$ and show that it can be expressed as

$$\boldsymbol{ab}=\boldsymbol{a}\cdot\boldsymbol{b}+(\boldsymbol{a}\times\boldsymbol{b})e_1e_2e_3=\boldsymbol{a}\cdot\boldsymbol{b}+(\boldsymbol{a}\times\boldsymbol{b})\,j.$$

Algebra

Let's look again at a 2D example. All possible outer products of vectors expressed in the orthonormal basis can provide a linear combination of "numbers" 1, e_1, e_2 and $e_1\wedge e_2=e_1e_2$ (any linear combination of these "numbers" we will refer to as a *multivector*). The outer product is anti-commutative, so, all terms that have some unit vector repeated disappear. “Numbers“ 1, e_1, e_2 and e_1e_2 form the basis of 2^2 – dimensional linear space. In fact, we have the basis of the algebra (*Clifford algebra*). When the geometric meaning is in the forefront we refer it as the *geometric algebra* (due to Clifford himself). The element 1 is a real scalar. We have two vectors and one bivector (in the terminology of geometric algebra it is referred as the *pseudoscalar* in the algebra, namely, a member of the algebra with the maximum grade). Note that scalars make a subspace (real numbers, *grade* zero, see below), vectors define 1D subspaces (*grade* 1) and pseudoscalar defines a 2D subspace (the space itself, of *grade* 2).

In $\Re^3$ we have the basis of the algebra (*Cl*3):

$$1,\ e_1,\ e_2,\ e_3,\ e_1e_2,\ e_1e_3,\ e_2e_3,\ e_1e_2e_3,$$

here $j\equiv e_1\wedge e_2\wedge e_3=e_1e_2e_3$ is the unit *pseudoscalar*. Show that *j* commutes with all elements of the Clifford basis in *Cl*3 and that $j^2=-1$. Pseudoscalars in any dimension are all proportional to the unit pseudoscalar. Prove it, at least for j. So, pseudoscalar j is a perfect (commutative) imaginary unit in *Cl*3. Such a pseudoscalar will appear also in *Cl*7, *Cl*11, … This has far-reaching consequences. But here one should be careful, the commutativity property of the pseudoscalar means the geometric product, while in terms with other products one should be cautious. Real scalars do not have this "problem", they can "walk" through all products. For the pseudoscalar we have, for example

$$je_1e_3=e_1je_3=e_1e_3j=(e_1j)e_3=e_1(je_3),$$

i.e. the geometric product allows "walking", but this is not generally valid with, say, the inner product

$$(e_1\cdot e_3)\,j=0\neq e_1\cdot(je_3)=e_1\cdot(e_1e_2)=e_2,$$

here we have a *mixed product* (see below).

In 3D, for arbitrary four vectors we have $\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}\wedge\boldsymbol{d}=0$. The outer product has distributivity and associativity properties also (see literature or prove itself). If any two vectors here are parallel, relation is true due to anti-commutativity of the outer product. Otherwise we have, for example, $\boldsymbol{d}=\alpha\boldsymbol{a}+\beta\boldsymbol{b}+\gamma\boldsymbol{c},\ \alpha,\beta,\gamma\in\Re$, so, our statement is true due to distributivity and anti-commutativity.

The maximum grade of a multivector cannot be larger than the dimension of the vector space (show that). Show that number of elements in the Clifford basis with the grade k equals to the binomial coefficient

$$\begin{pmatrix} n \\ k \end{pmatrix},$$

where n is the dimension of the vector space. For real scalars we have $k=0$, so, there it is just one real scalar in the basis (i.e. 1). The same is for $k=n$, there is just one element with the grade n in the basis, which gave rise to the term "*pseudoscalar*". Show that the number of elements in the Clifford basis for n-dimensional vector space equals to 2^n.

An important concept is the *parity* of a multivector and refers to the parity of its grades. All elements with even grades define a subalgebra (the geometric product of any two of such elements is even, too, show that!), while this is not true for the odd part of the algebra.

Grades of a multivector M are usually written as $\langle M\rangle_r$, where *r* is the grade. For the grade 0 we use just $\langle M\rangle$, for example $a\cdot b=\langle ab\rangle$. The grade 0 is a real number and it does not depend on the order of multiplication, so we have $\langle AB\rangle=\langle BA\rangle$, which leads to the possibility of cyclical changes, like $\langle ABC\rangle=\langle CAB\rangle$. This is a beneficial relation, for example, consider the inner product $a\cdot b$ and ask ourselves what would happen if we apply the transformations $a\rightarrow nan$ and $b\rightarrow nbn$ (n is a unit vector). Note that the result of such a transformation is a vector (decompose the vector a into components parallel and orthogonal to n). The inner product of two vectors is just the zero grade of their geometric product, so we have, using cyclical changes

$$(nan)\cdot(nbn)=\langle nannbn\rangle=\langle nabn\rangle=\langle abnn\rangle=\langle ab\rangle=a\cdot b.$$

Such a transformation doesn't change the inner product, so we have an example of an orthogonal transformation (this one is a *reflection*). A transformation $X\rightarrow nXn$ (n is a unit vector) generally doesn't change the grade. For example, if we have $X=ab$ then

$$nabn=nannbn=(nan)(nbn),$$

i.e. we have a geometric product of two vectors again. This is a very important conclusion. To see that it is generally valid, recall that each multivector is a linear combination of elements of a Clifford basis. So we have, for example $e_1(e_1e_3)e_1=e_3e_1=-e_1e_3$, so, the grade is still 2. If a grade of element is changed by a transformation then we obtain a new type of element, but we don't want that generally. Rather, we usually want to transform vectors to vectors, bivectors to bivectors, etc.

Let's now discuss some important formulas in which mixed products appear. For example, let's look at the product

$$a(b\wedge c)=a(bc-cb)/2=(abc-acb)/2.$$

We can take an advantage of the obvious (and useful) relation $ab=2a\cdot b-ba$ and show that (left to the reader)

$$a(b\wedge c)-(b\wedge c)a=2(a\cdot b)c-2(a\cdot c)b.$$

Here we have a situation in which the grade of the bivector is downgraded, so it is customary to write such a relationship as the inner product, i.e. a kind of **contraction**

$$a\cdot B=(aB-Ba)/2,$$

(B is a bivector) or,

$$a\cdot(b\wedge c)=(a\cdot b)c-(a\cdot c)b=a\cdot bc-a\cdot cb,$$

where it is understood that the inner product is executed first. This is a useful and important formula. It is not difficult to show that

$$a \wedge B = (aB + Ba)/2,$$
$$aB = a \cdot B + a \wedge B.$$

Find $e_1 \cdot (e_1 e_2)$ and $e_1 \wedge (e_1 e_2)$.

Here is one more useful relation (without proof)

$$e_1 \cdot (\boldsymbol{a}_1 \wedge \ldots \wedge \boldsymbol{a}_n) = \sum_{k=1}^{n} (-1)^{k+1} e_1 \cdot \boldsymbol{a}_k (\boldsymbol{a}_1 \wedge \ldots \wedge \breve{\boldsymbol{a}}_k \wedge \ldots \wedge \boldsymbol{a}_n),$$

where $\breve{\boldsymbol{a}}_k$ means that the factor $\boldsymbol{a}_k$ is missing in the outer product. Find $e_1 \cdot (\boldsymbol{a} \wedge \boldsymbol{b})$.

It is straightforward to find the *projection* and the *rejection* (we announced this possibility earlier), for example, for a vector a, using the orientation of the unit vector n, we have

$$a = n^2 a = n(n \cdot a + n \wedge a) = nn \cdot a + nn \wedge a = a_{\parallel} + a_{\perp},$$

where the geometric product is to be executed last. For general formulas (for any elements of the algebra) see literature, or derive them using the concept of the inverse of a vector.

Important concepts

Before we dive into *Cl*3 let's look at some more general terms.

a) *versor* $\rightarrow$ geometric product of any number of vectors
b) *blade* $\rightarrow$ outer product of any number of vectors
c) *involution* $\rightarrow$ any function with the property f (f(x)) = f(x)
d) *inverse* $\rightarrow$ for an element *x* it is the element *y* such that $xy = 1$, $y = x^{-1}$
e) *nilpotent* $\rightarrow x^2 = 0$
f) *idempotent* $\rightarrow x^2 = x$
g) *zero divisors* $\rightarrow$ $xy = 0$, $x, y \neq 0$

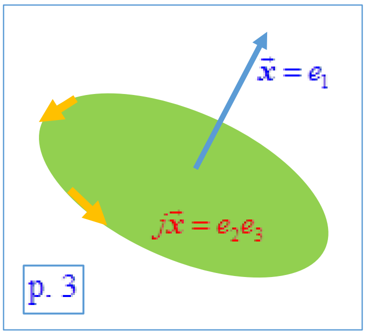
Let's explain those terms in some more details.

a) An example of a versor is abc, if factors are vectors. For geometric product of two vectors we generally have grades 0 and 2. For verification techniques that some multivector is a versor see Bouma and [19]. Show that the geometric product of versor and its reverse (for abc it is cba) is a real number.

b) An example of a blade is $a \wedge b \wedge c$, if factors are vectors. For verification techniques that some multivector is a blade see Bouma and [19]. A blade is *simple* if it can be reduced to the outer product of basis vectors (up to the real factor).

While the versor ab generally have grades 0 and 2, the blade $a \wedge b$ has the grade 2 and defines the 2D subspace. Show that any *homogeneous versor* (has single grade only) is a blade. Show that any blade can be transformed to a versor with orthogonal vectors as factors. Any blade in *Cl*3 which is an outer product of three linearly independent vectors is proportional to the unit pseudoscalar (show that, if you have not done it already).

Consider an arbitrary set of indices of unit vectors of an orthonormal basis, some of which can be repeated. Find an algorithm for sorting indices, so as to take into account skew-symmetry for different indices. The goal is to find the overall sign. After sorting, the unit vectors of the same index are multiplied and thus reduce to one unit vector (up to a sign) or a real number (± 1). Example: $e_2 e_3 e_1 e_2 = e_1 e_2 e_3 e_2 = -e_1 e_2 e_2 e_3 = (-)^s e_1 e_3$, $s = e_2^2$.



Elements of the Clifford basis are *simple* blades. We have seen that in *Cl*3 any linear combination of unit bivectors defines a plane (i.e. can be represented as an outer product of two vectors). Multiply every element of the Clifford basis by the pseudoscalar j. What you get? Figure p. 3 can help in thinking. You can use *GAViewer* and see how your products look like.

c) In geometric algebra the most commonly used are three *involutions*, and all of them come down to change the sign of elements in the Clifford basis.

Grade involution is obtained by changing the sign of each basis vector of the vector space (*inversion*). In this way all even elements remain unchanged, while odd ones change the sign. Consider a general multivector M in *Cl*3:

$$M = t + x_1 e_1 + x_2 e_2 + x_3 e_3 + B_1 e_{12} + B_2 e_{13} + B_3 e_{23} + bj\,,$$

where $e_{12} \equiv e_1 e_2$, etc. The grade involution gives

$$\hat{M} = t - x_1 e_1 - x_2 e_2 - x_3 e_3 + B_1 e_{12} + B_2 e_{13} + B_3 e_{23} - bj\,.$$

The grade involution is an *automorphism* (show that), which means

$$(MN)^{\wedge} = \hat{M}\hat{N}\,.$$

Elements $\left(M + \hat{M}\right)/2 \equiv \langle M \rangle_+$ and $\left(M - \hat{M}\right)/2 \equiv \langle M \rangle_-$ give the even and the odd parts of the multivector M (find them for general M in *Cl*3).

Reverse involution is an anti-*automorphism* ($(MN)^\dagger = N^\dagger M^\dagger$, show that):

$$M^\dagger = t + x_1 e_1 + x_2 e_2 + x_3 e_3 - X_1 e_{12} - X_2 e_{13} - X_3 e_{23} - bj\,.$$

Elements $\left(M + \hat{M}\right)/2 \equiv \langle M \rangle_R$ and $\left(M - \hat{M}\right)/2 \equiv \langle M \rangle_I$ give real and imaginary parts of the multivector M (see below, find them for general M in *Cl*3).

Clifford conjugation (*involution*) is an anti-*automorphism* ($\overline{MN} = \bar{N}\bar{M}$, show that):

$$\bar{M} = t - x_1 e_1 - x_2 e_2 - x_3 e_3 - X_1 e_{12} - X_2 e_{13} - X_3 e_{23} + bj\,.$$

Elements $\left(M + \hat{M}\right)/2 \equiv \langle M \rangle_S$ and $\left(M - \hat{M}\right)/2 \equiv \langle M \rangle_V$ give (complex) scalar and (complex) vector parts of the multivector M (see below, find them for general M in *Cl*3).

What we get applying all three involutions on a multivector, and what we get applying any two of them? Each involution changes a sign of some grades. If overall sign of the grade is given in the form $(-1)^{f(r)}$, r is a grade, find the function f for each involution. Often we need to check the properties of some product, sum, etc. What is multivector if $M = inv(M)$, where inv stands for any of three defined involutions? Show that for versors V relation $V = v_1 v_2 ... v_k \Rightarrow \hat{V} = (-v_1)(-v_2)...(-v_k)$ is valid. Show that the multivector $\hat{V} x V^\dagger$ is a vector if x is a vector.

d) An important consequence of the geometric multiplication of vectors is the existence of the inverse of a vector (and many other elements of algebra), i.e. we can divide by a vector. For vectors (null-vectors do not have an inverse) we have

$$a^{-1} = a / a^2,$$

which means that the unit vector is inverse to himself. The existence of the inverse has far-reaching consequences and significantly distinguishes the geometric product from the ordinary scalar and cross product. Now we can solve the equation:

$$ab = c \Rightarrow a = bc^{-1},$$

etc. We can define inverses of other multivectors, for example, it is easy to see what the inverse of the versor is:

$$(e_1 e_2)^{-1} = -e_1 e_2 / (e_1 e_2 e_2 e_1) = -e_1 e_2 = e_2 e_1 .$$

Here we are using the fact that geometric product of versor and his reverse is just a real number. There exist multivectors without the inverse, we will see it a little later. The existence and definition of an inverse isn't always simple and obvious, but in *Cl*3 that task is relatively easy. It is important to note that the existence of an inverse depends on a possibility to define module (norm, magnitude) of a multivector, and that is not always unique. For a general approach see references cited.

e) Geometric product allows the existence of multivectors different from zero, but whose square is zero. They are *nilpotents* in the algebra and have an important role here, for example, when formulated in *Cl*3, an electromagnetic wave in vacuum is just a nilpotent in the algebra. For example, we have

$$(e_1 + e_1 e_2)^2 = e_1 (1 + e_2) e_1 (1 + e_2) = e_1 e_1 (1 - e_2)(1 + e_2) = 0 .$$

Nilpotents don't have an inverse. If $N \neq 0$ is a nilpotent and M is its inverse, than from $NM = 1$ we have $N^2 M = N$, i.e. $0 = N$.

f) *Idempotents* have the simple property $p^2 = p$. Show that multivector $(1 + e_1)/2$ is the idempotent. In fact, every multivector of the form $(1 + \boldsymbol{f})/2,\ \boldsymbol{f}^2 = 1$, is an idempotent. Later in text we will find the general form of idempotents in *Cl*3. The trivial idempotent is 1. Show that the trivial idempotent is the only one with the inverse.

g) Multiply $(1 + e_1)(1 - e_1)$. There are multivectors different from zero that multiplied give zero (*zero divisors*). Although it differs from properties of real numbers, it turns out to be very useful in many applications.

We should mention that the addition of quantities like $\boldsymbol{x}$ and $j\boldsymbol{n}$ (or other expressions of different grades) is not a problem, as some people complain, we add objects of different grades, so, as with complex numbers, such a sum preserves separation of grades. Here sum is to be understood as a **relation** between different subspaces. Let us clarify this a little bit for *Cl*3. Real numbers have grade zero and define the subspace of "points". Vectors define oriented lines, bivectors define oriented plains and pseudoscalars define oriented volumes. For example, a bivector B defines an oriented plane by relation $B \wedge \boldsymbol{x} = 0$. In that plane we can find a unit bivector $\hat{B}$ which has a number of interesting properties: squares to -1, it is oriented, rotates vectors in the plane, etc. As an example, $B = e_1e_2 + e_2e_3 = e_2 \wedge (e_3 - e_1)$, so vectors e_2 and $e_3 - e_1$ span the plane. Relation $B \wedge \boldsymbol{x} = 0$ gives vectors $\boldsymbol{x}$ as a linear combinations of vectors e_2 and $e_3 - e_1$. Find $BB^{\dagger}$. We see, the (unit) bivector $\hat{B} = B / \sqrt{2}$ has a clear geometric interpretation, but it is also the operator which rotates vectors in the plane it defines. It can also serve as an imaginary unit for complex numbers defined in the plane it defines. A multivector of the form $\alpha + B$ is the sum of different grades, but there is no way to "blend" real scalars and bivectors in sums: they are always separated. But together, as a sum, they are powerful, as *rotors* or *spinors*, for example (see below).

Finally, any multivector can be expressed as a list of coefficients in a Clifford basis. As an example we can use the multivector $3 - e_2 + e_1e_2$ in 2D, the list of coefficients is $(3, 0, -1, 1)$. It is clear that we can add and subtract such lists, find a rule to multiply them, etc. Addition of elements of different grades is equivalent to making such a lists, as we are get used to do it for complex numbers. A complex number $\alpha + i\beta$ we express as an ordered pair of numbers (α, β).

Examples of solving equations

Let's find real numbers α and β such that $\boldsymbol{x} = \alpha\boldsymbol{a} + \beta\boldsymbol{b}$ in $\Re^3$. We have

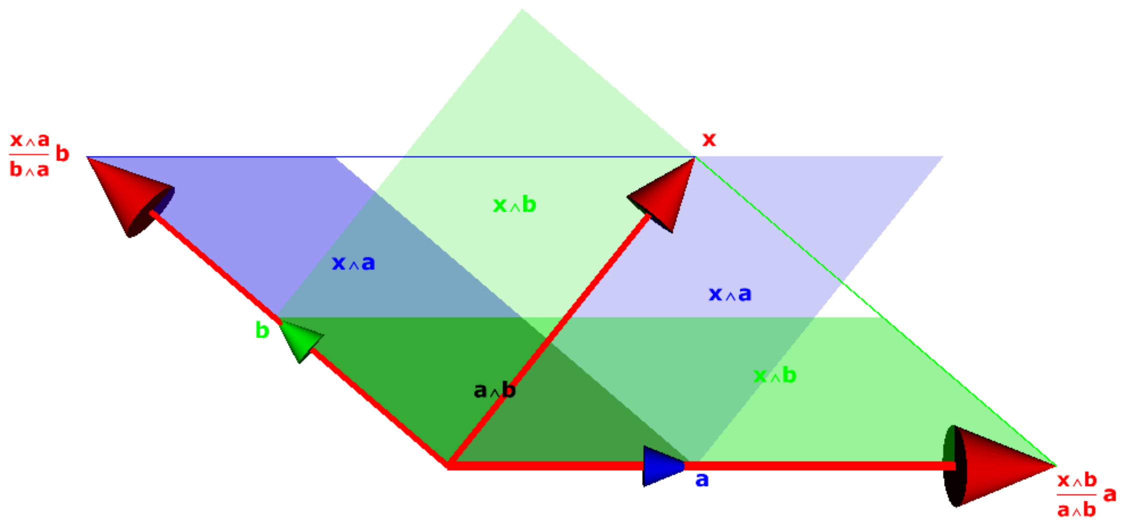
$$\boldsymbol{x} \wedge \boldsymbol{a} = \alpha\boldsymbol{a} \wedge \boldsymbol{a} + \beta\boldsymbol{b} \wedge \boldsymbol{a} = \beta\boldsymbol{b} \wedge \boldsymbol{a},$$

$$\boldsymbol{x} \wedge \boldsymbol{b} = \alpha\boldsymbol{a} \wedge \boldsymbol{b} + \beta\boldsymbol{b} \wedge \boldsymbol{b} = \alpha\boldsymbol{a} \wedge \boldsymbol{b}.$$

Note that bivectors $\boldsymbol{x} \wedge \boldsymbol{a}$ and $\boldsymbol{b} \wedge \boldsymbol{a}$ define the same plane and both are proportional to the unit bivector in that plane, i.e. their ratio is a real number (unit bivector divided by itself gives 1). Therefore we have

$$\boldsymbol{x} = \frac{\boldsymbol{x} \wedge \boldsymbol{b}}{\boldsymbol{a} \wedge \boldsymbol{b}}\boldsymbol{a} + \frac{\boldsymbol{x} \wedge \boldsymbol{a}}{\boldsymbol{b} \wedge \boldsymbol{a}}\boldsymbol{b}.$$

Let's use the *GAViewer* to show it graphically:



Now let's look the quadratic equation:

$$x^2 + x + 1 = 0.$$

Show that $x = -e^{\pm i\pi/3}$, $i = e_1 e_2$, is the solution. Can you find a solution for an arbitrary quadratic equation? Pay attention to the fact that we can interpret the expression $x^2 + x + 1$, with the above solution, as the operator which acting on some vector v gives zero. This means that we have the sum of a vector (v), a rotaded vector (xv) and a twice rotaded vector ($x^2 v$), three vectors that we can arange in the triangle. About rotations and an exponential form see below, here you can feel free to treat expressions like complex numbers with the imaginary unit $i = e_1 e_2$ (i.e. you can use the trigonometric form of the complex number). In the next chapter you will find an explanation for this approach.

Geometric product of vectors in the trigonometric form

Let's look at the square of a bivector in $\Re^n$ (for other signatures see literature, main ideas are the same),

$$\begin{aligned}&(a \wedge b)(a \wedge b) = (ab - a \cdot b)(a \cdot b - ba) = \\ &-ab^2 a - (a \cdot b)^2 + a \cdot b(ab + ba) = \\ &(a \cdot b)^2 - a^2 b^2 = -a^2 b^2 \sin^2 \theta,\end{aligned}$$

where we used $(a \cdot b)^2 = a^2 b^2 \cos^2 \theta$. The another way to see this is to start from the form $ab_\perp$

$$(ab_\perp)^2 = ab_\perp ab_\perp = -a^2 b_\perp^2.$$

We see that in $\Re^n$ the square of a bivector is a negative real number. Now we can define the magnitude of a bivector as

$$|a \wedge b| = |a||b||\sin \theta|.$$

We got a general expression for the square of a bivector, so we see that the geometric product of two vectors can be written as

$$ab = |a||b|\hat{a}\hat{b} = |a||b|\left(\hat{a} \cdot \hat{b} + \hat{a} \wedge \hat{b}\right) = |a||b|\left(\cos \theta + \hat{B} \sin \theta\right), \quad \hat{B} = \frac{\hat{a} \wedge \hat{b}}{\left|\hat{a} \wedge \hat{b}\right|}, \quad \hat{B}^2 = -1 ,$$

or

$$ab = |a||b| e^{\hat{B}\theta}.$$

Notice that we have a similar formula for complex numbers, but the situation is quite different here: the unit bivector $\hat{B}$ is not just an „imaginary unit“, it defines the plane spanned by vectors a and b. This is a great advantage compared to ordinary complex numbers, it brings the clear geometric meaning to expressions. For example, the formulation of quantum mechanics in geometric algebra uses real numbers, there is no need for $\sqrt{-1}$, and in every expression we can see the geometric meaning directly. This makes the new formulation more powerful, it provides new insights, which otherwise would be hidden or difficult to reach.

Here we have the opportunity to answer the question about multiplication tables. We have seen how multiplication tables for the scalar and cross product are almost complement. We know, the geometric product of two vectors can be decomposed into symmetric and anti-symmetric parts, then we can find their modules, they have functions sine and cosine as factors and that gives us “united” multiplication table. Here it is (note that, for example, $e_1e_2 = je_3$)

$$\begin{array}{cccc} \bullet & e_1 & e_2 & e_3 \\ e_1 & 1 & 0 & 0 \\ e_2 & 0 & 1 & 0 \\ e_3 & 0 & 0 & 0 \end{array} \quad \oplus \quad \begin{array}{cccc} \times & e_1 & e_2 & e_3 \\ e_1 & 0 & e_3 & -e_2 \\ e_2 & -e_3 & 0 & e_1 \\ e_3 & e_2 & -e_1 & 0 \end{array} \quad \rightarrow \quad \begin{array}{cccc} \mathrm{GP} & e_1 & e_2 & e_3 \\ e_1 & 1 & e_1e_2 & e_1e_3 \\ e_2 & -e_1e_2 & 1 & e_2e_3 \\ e_3 & -e_1e_3 & -e_2e_3 & 1 \end{array}$$

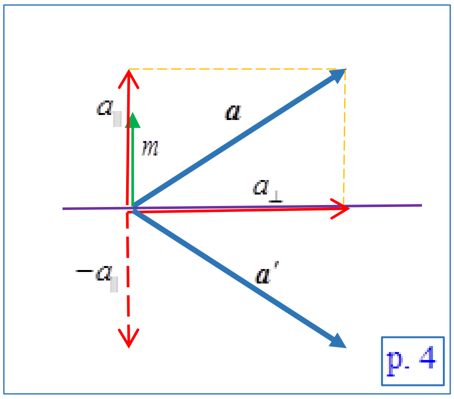
and we can see that the new multiplication table has bivectors as non-diagonal elements ($\oplus$ is just for fun). In fact, looking at those tables one can get nice insights about our 3D space and geometric algebra in general.

Reflections, rotations, spinors, quaternions …

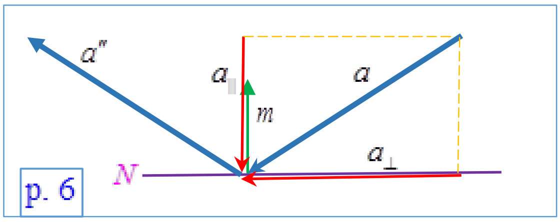
The reader is now, perhaps, convinced that the geometric product is really natural and, actually, inevitable way to multiply vectors. One way or another, the magic is still to come.

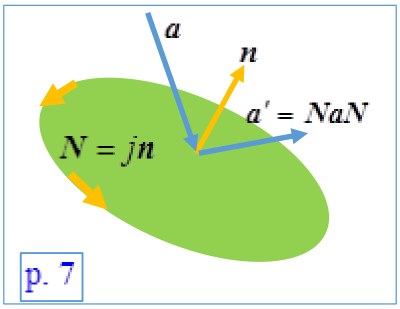
Consider now the powerful formalism of geometric algebra applied to reflections and rotations (we are still in $\Re^n$, details for other signatures can be found in the literature). For the vector a and the unit vector n in $\Re^3$ (just to imagine things easier, generalization is straightforward) we can find the *projection* (parallel to n) and the *rejection* (orthogonal to n) of the vector a, so, $a = a_\parallel + a_\perp$. Now we have

$$a' = -nan = -n\left(a_\parallel + a_\perp\right)n = -\left(a_\parallel - a_\perp\right)nn = a_\perp - a_\parallel,$$



which means that vector a is reflected on the plane orthogonal to n (generally a hyper plane**,** figure p. 4). We can omit the minus sign, then the reflection is on the vector n. Recall, reflections do not change the grade of a reflected object.

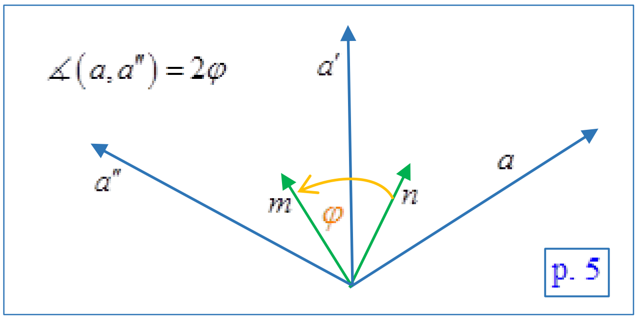




We should mention that in physics we are often interested in reflections on surfaces in 3D, so we can slightly adjust the pictures (p. 6, p. 7). We use the fact that $j^2 = -1$, so

$$\mathbf{a}' = -\mathbf{nan} = j^2\mathbf{nan} = \mathbf{jnajn} = \mathbf{NaN},$$

where the unit bivector $\mathbf{N}$ defines the reflection plane.



What if we apply two consecutive reflections, using two unit vectors m and n? There is a well-known theorem, which states that two consecutive reflections provide a rotation. In figure p. 5 we see that after the reflection on n we have $a \to a'$, then by reflection on m we have $a' \to a''$. If the angle between unit vectors m and n is φ then the rotation angle of the vector a is 2φ (prove it). Respectively, if we want to rotate the vector by the angle φ we need to use unit vectors which make the angle $\varphi/2$. We see how the half angle appears, so characteristic in the description of a spin in quantum mechanics. Here we see that there is nothing "quantum" in the half angle, it is simply a part of the geometry of our 3D space (with the geometric product). This will be discussed later. Now we can write an expression for a rotation as

$$a'' = m(nan)m = mnanm.$$

Another way to rotate a vector is to construct an operator which rotates and operates from the left. Thanks to the existence of an *inverse* of the vector this is easy to achieve:

$$a'' = \left(a''a^{-1}\right)a \equiv Oa, \quad O = a''a^{-1}.$$

But the method that uses reflections is very general and elegant (rotates any element of the algebra), has a "sandwich" form, which is actually common and preferable in geometric algebra, especially for generalizations to higher dimensions. Let's look more closely the term $mnanm$. Geometric products of two unit vectors consist generally of grades 0 and 2, so, it belongs to the even part of the algebra and makes a subalgebra, which means that the product of any two of these elements will result in an element of the even part of the algebra. We denote it as $R = mn$ (*rotor* in text). Now we have

$$a'' = RaR^{\dagger}, RR^{\dagger} = mnnm = 1 = R^{\dagger}R,$$

where $R^{\dagger} = R^{-1}$ means *reverse* ($mn \to nm$). For the rotation angle φ we need unit vectors with the angle $\varphi/2$ between them. We have $m\,n = m \cdot n + m \wedge n$, where $|m \wedge n| = |\sin(\varphi/2)|$. Using the unit bivector $\hat{B} \equiv n \wedge m / |n \wedge m|$ (note the order of vectors), we have

$$mn = m \cdot n + m \wedge n = \cos(\varphi/2) - \hat{B}\sin(\varphi/2) = \exp\left(-\hat{B}\varphi/2\right),$$

the minus sign here is due to the convention (positive rotation is counter clockwise). In *Cl*3 we can write a unit bivector $\hat{B}$ as $j\boldsymbol{w}$, where $\boldsymbol{w}$ is the unit vector defining the axis of rotation. The rotor inverse is

$$R^{\dagger} = nm = \exp\left(\hat{B}\varphi/2\right),$$

so the rotation is finally

$$a'' = RaR^{\dagger} = e^{-\frac{\varphi}{2}\hat{B}} a e^{\frac{\varphi}{2}\hat{B}}.$$

This is the general formula. If a commutes with $\hat{B}$ the rotation transformation has no effect on a. If a anti-commutes with $\hat{B}$ we have an operator form

$$a'' = e^{-\varphi\hat{B}}a.$$

For example, for $\hat{B} = e_1e_2$ the vector e_3 commutes with $\hat{B}$, while the vector e_1 anti-commutes.

The bivector $\hat{B}$ defines the rotation plane and it is clear that vectors orthogonal to that plane are not changed by the rotor. Notice, we do not need rotation matrices, Euler angles, or any other known mechanism. Once you define a unit bivector it will do all the necessary job. You can imagine it like a small spinning top that does exactly what we need. Notice that two different consecutive rotations make the rotation again (show that). This produces a *group structure*, but here we will not talk about it.

Example. Rotate the vector $e_1 + e_2 + e_3$ in the plane $e_1 e_2$ by an angle φ. We have

$$e^{-\frac{\varphi}{2}e_1e_2}\left(e_1 + e_2 + e_3\right)e^{\frac{\varphi}{2}e_1e_2},$$

so take the advantage of the fact that the vector e_3 commutes with the bivector $e_1 e_2$, while e_1 and e_2 anti-commute:

$$e^{-\frac{\varphi}{2}e_1e_2}\left(e_1 + e_2 + e_3\right)e^{\frac{\varphi}{2}e_1e_2} = e_3 e^{-\frac{\varphi}{2}e_1e_2}e^{\frac{\varphi}{2}e_1e_2} + e^{-\frac{\varphi}{2}e_1e_2}e^{-\frac{\varphi}{2}e_1e_2}\left(e_1 + e_2\right) =$$

$$e_3 + \left(\cos\varphi - e_1 e_2 \sin\varphi\right)\left(e_1 + e_2\right) = e_3 + \left(e_1 \cos\varphi + e_2 \sin\varphi\right) + \left(-e_1 \sin\varphi + e_2 \cos\varphi\right),$$

and for vectors in the plane $e_1 e_2$ we recognize the rotation matrix

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix},$$

where columns represent images of unit vectors. A rotation by an angle $-\varphi$ we get using the bivector $e_2 e_1 = -e_1 e_2$.

Consider the rotation

$$e^{-\frac{0.7\pi}{2}e_1e_2} a e^{\frac{0.7\pi}{2}e_1e_2}$$

and the corresponding rotation matrix

$$\begin{pmatrix} -0.588 & -0.809 \\ 0.809 & -0.588 \end{pmatrix}.$$

What can be said about the geometrical interpretation, that is, what you can conclude just looking at the matrix? Try now to make a rotation matrix for an arbitrary plane. Try to repeat all that in 4D. The easiness with which we perform rotations in geometric algebra is unseen before. There are no special cases, no vague matrices, just follow the simple application of rotors to any multivector. Many prefer quaternions, but they do not have the geometric clarity. And they are limited to 3D! If only elegance and power of rotations were results of using geometric algebra it would be worth of effort. But it gives us much, much more.

Notice how any rotor can be factored in small rotations

$$R = e^{I\varphi/2} = \underbrace{e^{I\varphi/2n} \ldots e^{I\varphi/2n}}_{n},$$

which can be used in practice, for example, when interpolating.

Let's look at the rotation of vector e_2 for a small angle in the plane $e_1 e_2$ (p. 8, p.9). Recall the definition

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n ,$$

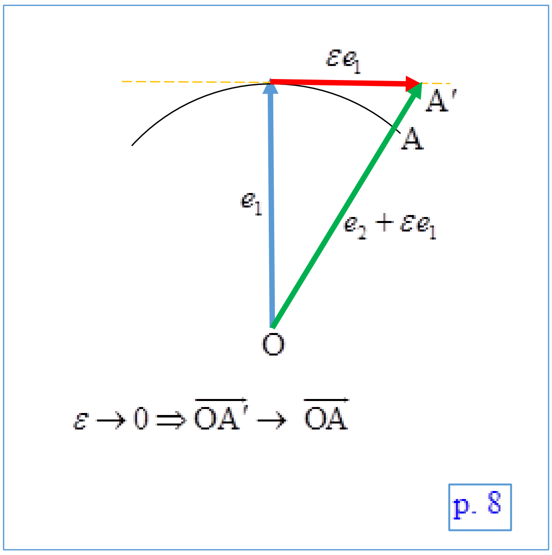
and let's construct the operator $1+\varepsilon e_1 e_2$, ε is a small real number. Acting from the left we have

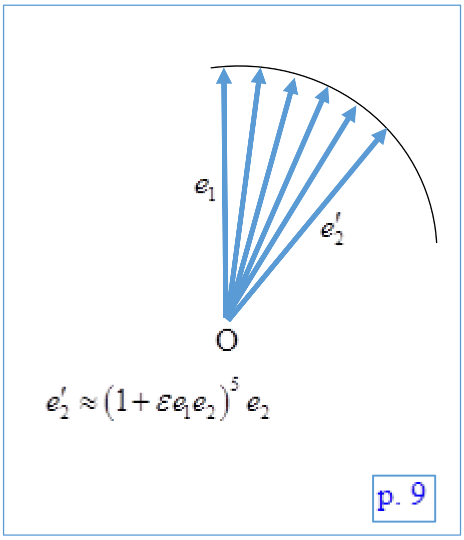
$$(1+\varepsilon e_1 e_2)e_2 = e_2 + \varepsilon e_1 ,$$

so we get an approximate small rotation of the vector e_2. Note the sign of the number ε, for $\varepsilon < 0$ we would have a counterclockwise rotation. Operator $1+\varepsilon e_1 e_2$ rotates all vectors in the plane for the same angle, so, by successive applications on e_2 we get rotated e_2 first, then rotated the newly established vector, etc (we can neglect the small change of the magnitude). This justifies the definition of the exponential form of the rotor: each rotation is the composition of a large number of small successive rotations. Of course, all this is well defined for infinitely small rotations, so for the bivector B we have

$$e^B = \lim_{n\to\infty}\left(1+\frac{B}{n}\right)^n .$$

Notice (or show it) that this rotor will not change the bivector $\hat{B}$, for example, so it is an invariant of the rotation. The fact that the blade can be invariant directly leads to the notion of a *proper blade* with real eigenvalues, which is a generalization of the common concept of eigenvectors and eigenvalues (see in *linear transformations*). Rotate the bivector $e_1 e_2$ in the plane spanned by vectors e_1 and e_2. What do you notice?



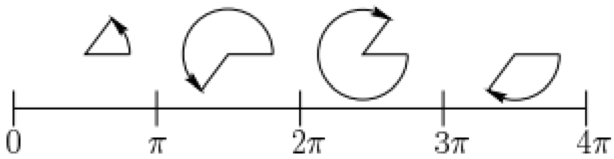


Rotations are linear, orthogonal transformations that are usually described by matrices in linear algebra. To find the invariants of these transformations we study the results of action of matrices on vectors **only**. For matrix A (that represents a linear transformation) we seek for vectors x such that $Ax = \lambda x$, which provides solutions for the eigenvalues $\lambda \in \mathbb{C}$. Here we see that in geometric algebra we can find invariants with respect to a bivector (or any blade). Instead of the concept of eigenvector we can introduce the concept of the eigenblade (which includes eigenvectors). This allows reduction of the set of eigenvalues of transformation to the set of real numbers and gives a geometric meaning to the concept of eigenblades. Linear transformations will be discussed later in the text.

Rotor $-R$ has the same effect as the rotor R, but the direction of the rotation is not the same, for example, the vector e_1 can be rotated to $-e_2$ clockwise by $\pi/2$ or counter clockwise by $3\pi/2$, so we see that rotor clearly shows the direction of rotation (try it with matrices!). For example

$$-e^{I\varphi/2}=e^{-I\pi}e^{I\varphi/2}=e^{-I(2\pi-\varphi)/2},$$

the minus disappears due to the "sandwich" form. For each rotation we have two possible rotors (find what *double cover* of a group is).



Note that, due to the half-angle, rotor

$$e^{-\frac{\varphi}{2}\hat{B}}=\cos(\varphi/2)-\hat{B}\sin(\varphi/2)$$

has periodicity of 4π instead of 2π. Often for such objects we are using the name *unit spinor*. Geometric algebra is an ideal framework to study all unusual properties of rotations, but it would take a lot of space.

Example: Let's rotate (see [18]) some object in 3D around e_1 by $\pi/2$, then around e_2 by $\pi/2$, what we get? Do that also using matrices.

$$e^{je_1\pi/4}e^{je_2\pi/4}=\frac{1}{\sqrt{2}}(1+je_1)\frac{1}{\sqrt{2}}(1+je_2)=\ldots=\frac{1}{2}+\frac{1}{2}\sqrt{3}j\frac{e_1+e_2-e_3}{\sqrt{3}}=e^{jv\pi/3},\quad \boldsymbol{v}=\frac{e_1+e_2-e_3}{\sqrt{3}},$$

so we have a rotation by $2\pi/3$ around the vector $\boldsymbol{v}$.

Question: What is the meaning of $e^{i\pi}=-1$? In 2D for $i=e_1e_2$ we have ($\boldsymbol{v}$ is a vector in the e_1e_2 plane, you can choose $\boldsymbol{v}=e_1$ if you like)

$$e^{i\pi/2}\boldsymbol{v}e^{-i\pi/2}=-\boldsymbol{v},$$

and using anti-commutativity

$$e^{i\pi/2}\boldsymbol{v}e^{-i\pi/2}=e^{i\pi}\boldsymbol{v}=-\boldsymbol{v},$$

then multiplying by $\boldsymbol{v}^{-1}$ on the right we get a clear meaning. The rotor $e^{i\pi/2}$ transforms the vector $\boldsymbol{v}$ to the vector $-\boldsymbol{v}$, i.e. rotates it by $-\pi$ (the sign is not important here). Of course, we also recognize the rotational properties of the imaginary unit in the complex plane (selected in advance), but bivector **defines** the rotation plane and we could write identical relations, without change, in any dimension, in any plane. In fact, a bivector in the exponent of the rotor could depend on time, formulas are still valid, the rotation plane changes with the bivector. Try to do that with the “square root of minus one”.

Let's say you want to find the rotor in 3D that will transform the orthonormal coordinate basis e_i to the orthonormal coordinate basis f_i (see [18]). We need a rotor with the property $f_i=Re_iR^\dagger$. Let's define $R=\alpha-\beta\hat{B}$, where $\hat{B}$ is a unit bivector, then $R^\dagger=\alpha+\beta\hat{B}$. Notice two simple and useful relations in 3D

$$\sum_i e_i^2=3 \quad \text{and} \quad \sum_i e_i\hat{B}e_i=-\hat{B}$$

(prove them). It follows

$$\sum_i e_i R^\dagger e_i = 3\alpha - \beta\hat{B} = 4\alpha - R^\dagger$$

and

$$\sum_i f_i e_i = \sum_i R e_i R^\dagger e_i = R\left(4\alpha - R^\dagger\right) = 4\alpha R - 1,$$

so

$$R = \frac{1+\sum_i f_i e_i}{\left|1+\sum_i f_i e_i\right|} = \frac{A}{\sqrt{A\bar{A}}}, \quad A = 1 + \sum_i f_i e_i .$$

The rotation by π can be treated as a special case. Show that the rotor can be expressed using Euler angles as

$$e^{-e_{12}\phi/2} e^{-e_{23}\theta/2} e^{-e_{12}\psi/2} .$$

Let's comment the historical role of Hamilton, who in the 19th century found a similar mechanism for rotations: *quaternions*. There is a connection between quaternions and formalism described here, namely, quaternions can be easily related to unit bivectors in *Cl*3. However, quaternions are like extended complex numbers, they do not have a clear geometrical interpretation. Moreover, they exist only in 3D. Hamilton wanted to give a geometric meaning to unit quaternions, trying to treat them as vectors, which did not gave expected results, but unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ inherited their names due to these attempts. The formalism of geometric algebra is valid for any dimension. Every calculation in which we use quaternions can be easily translated into the language of geometric algebra, while the reverse is not true. However, quaternions are still successfully used in the applications for calculating rotations, for example, in computers of military and space vehicles, as in robotics. If you implement the geometric algebra on computer, quaternions are not needed.

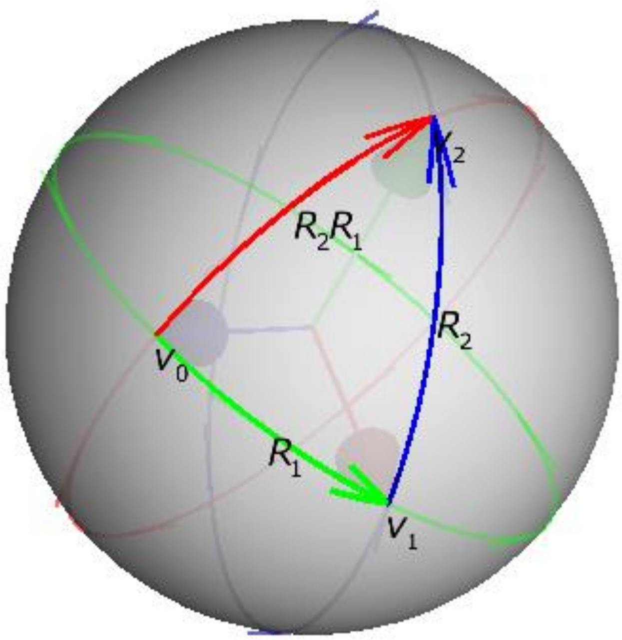
Unit quaternions have the property $\boldsymbol{ijk} = -1$ and the square of each of them is -1. It was enough to come up with objects that square to -1 and anti-commute to describe rotations in 3D successfully. The reader can check that replacements $\boldsymbol{i} \to -e_{23}$, $\boldsymbol{j} \to e_{13}$, $\boldsymbol{k} \to -e_{12}$ generate the quaternions multiplication table.

Certainly it is good to understand that bivector $-e_{12} = e_2 e_1$ has a very clear geometrical interpretation, while the unit quaternion $\boldsymbol{k}$ (like the imaginary unit or a matrix) has not. Unfortunately, the concept of geometric objects like bivectors is often strange to traditionally oriented people.

Once we know how to rotate vectors we can rotate any element of geometric algebra. Note especially nice feature of geometric algebra: objects that perform transformations ("operators") are also elements of the algebra. Let's look at the rotation of a versor

$$RabcR^\dagger = RaR^\dagger RbR^\dagger RcR^\dagger = \left(RaR^\dagger\right)\left(RbR^\dagger\right)\left(RcR^\dagger\right),$$

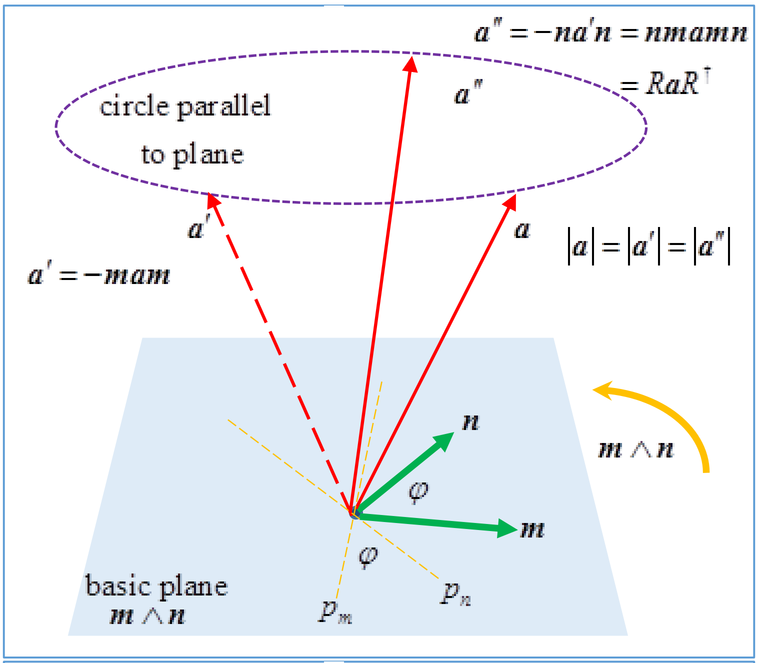
which clearly shows how the rotation of versor can be reduced to rotations of individual vectors and vice versa. Every multivector is a linear combination of elements of Clifford basis which elements are simple blades, so, they are versors. We see that our last statement is always true, due to the linearity. The reader is advised to do rotations of different objects in *Cl*3. Find on Internet the term *„gimbal lock*" (it is fun, really).

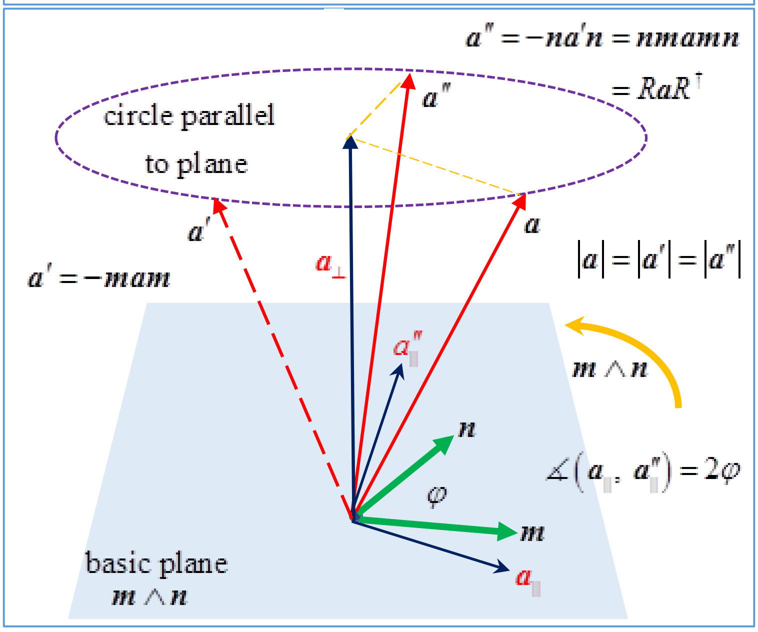


It is interesting to look at the unit sphere in 3D and unit vectors starting at the center of the sphere. Each rotation of the unit vector defines the arc on some main circle. Such arches, if we take into account their orientation, can become a kind of vectors on the sphere, and composition of two rotations can be reduced to an addition (non-commutative) of such vectors. See [4].

If we take an arbitrary element of the even part of an algebra (for example in 3D), not only the rotors, except for a rotation we get the additional effect: dilatation, which is exactly the property of spinors. Spinors are closely associated with the even part of the algebra. Geometric algebra hides within itself an unusual amount of mathematics which is branched out in different disciplines. It's amazing how the redefinition of the multiplication of vectors integrates many different branches of mathematics into a single formalism. Spinors, tensors, Lie groups and algebras, various theorems of integral and differential calculus are united, …, theory of relativity (special and general), quantum mechanics, theory of quantum information, … one almost cannot believe. Many complex results of physical theories here become simple and get a new meaning. Maxwell's equations are reduced to three letters, with the possibility of inverting the derivation operator over the Green functions, hard problems in electromagnetism become solvable (see [2]), the Kepler problem is elegantly reduced to the problem of the harmonic oscillator, Dirac theory in *Cl*3 or the minimal standard model in *Cl*7 are nicely formulated ([34]), not to list further.

Geometric algebra has a good chance to become mathematics of future. Unfortunately, it is difficult to break through the traditional university (and especially high school) programs.





One can study following pictures to better understand rotations.

- $\boldsymbol{m} \wedge \boldsymbol{n}$ defines the plane, direction of rotation and the rotation angle
- $\boldsymbol{a}_{\perp}$ is invariant to rotation, only $\boldsymbol{a}_{\|}$ is rotated by 2φ
- The same picture is valid in any dimension (in dimensions higher than 3 there is a subspace invariant to rotation).
- It is easy to obtain any composition of rotations in the same manner.
- Geometric product of vectors gives us the possibility to maintain rotations easily.

Contractions

We defined the inner product that for vectors coincides with the usual scalar multiplication of vectors. In general, in geometric algebra we can define various products that lower grades of elements (outer product raises them). It appears that the best choice is the *left contraction*. For vectors it is just as the inner product, but generally it allows avoiding various special cases, such as, for example, the inner product of a vector with a real number. Here we will mention just few properties of the left contraction, see [19] for more details. The idea is that for any two blades (including real numbers) we define a „scalar" multiplication that will generally reduce the grade of the blade that is on the right in the product:

$$grade\left(A \rfloor B\right) = grade\left(B\right) - grade\left(A\right),$$

whence immediately follows that the left contraction is zero if $grade(B) < grade(A)$. For vectors we have

$$a \rfloor b \equiv a \cdot b,$$

and generally for blades we have

$$(A \wedge B) \rfloor C = A \rfloor (B \rfloor C).$$

The useful relation for vectors is

$$x \rfloor (a \wedge b) = (x \cdot a) b - (x \cdot b) a,$$

while in general we can write for any multivector

$$A \rfloor B = \sum_{k,l} \left\langle \langle A \rangle_k \langle B \rangle_l \right\rangle_{l-k},$$

where we have a geometric product between homogeneous (of the same grade) parts of multivectors. The left contraction for blades A and B ($A \rfloor B$) is the subspace in B orthogonal to A. If the vector x is orthogonal to all vectors from the subspace defined by the blade A then $x \rfloor A = 0$. The left contraction can help us to define an angle between subspaces. Because of the generality, the clear geometric interpretation and benefits for use on computers (there are no exceptions, so *if* loops are not needed) the left contraction should be used instead of the "ordinary" inner product. We can also define the right contraction, however, due to the properties of duality, it is not really necessary.

Commutators and orthogonal transformations

Let's define the *commutator* as a new kind of product of multivectors (here we use the character $\otimes$ to avoid a possible confusion with the cross product)

$$A \otimes B \equiv (AB - BA)/2.$$

This product is not associative, i.e. $(A \otimes B) \otimes C \neq A \otimes (B \otimes C)$, but we have the Jacobi identity

$$(A \otimes B) \otimes C + (C \otimes A) \otimes B + (B \otimes C) \otimes A = 0.$$

We have (prove it) general formulas (A is a bivector, not necessarily a blade, X is a multivector, α is a real scalar, x is a vector)

$$\alpha X = \alpha \wedge X$$

$$xX = x \rfloor X + x \wedge X$$

$$AX = A \rfloor X + A \wedge X + A \otimes X.$$

Here we are particularly interested in commutators with a bivector as one of factors. Namely, commutators with the bivector keep the grade of multivector (if they do not commute with it):

$$grade(B) = 2 \Rightarrow grade(X \otimes B) = grade(X), \quad X \otimes B \neq 0.$$

Instead of proving it let us look at examples. For the bivector $B = e_1 e_2$ the vector e_3 commutes with B, but for the vector e_1 (grade 1) we have

$$B \otimes e_1 = (e_1 e_2 e_1 - e_1 e_1 e_2)/2 = -e_2,$$

the grade 1 again. Let us take the series expansion

$$e^{-B/2}Xe^{B/2}=X+X\otimes B+\left(X\otimes B\right)\otimes B/2+\left(\left(X\otimes B\right)\otimes B\right)\times B/3!\ +\ \dots\ ,$$

so if we take a small bivector of the form $\varepsilon\hat{B},\quad \hat{B}^2=-1$, we see that we can keep only two terms

$$e^{-\varepsilon\hat{B}/2}Xe^{\varepsilon\hat{B}/2}\approx X+\varepsilon X\otimes\hat{B}.$$

Preservation of grades is important here, because we want to, after the transformation, have a geometric object of the same type. The last transformation we see as an orthogonal transformation which will slightly change the initial multivector. Here we must mention that we look for the orthogonal transformation connected to the identity transformation, which means that they can be implemented in small steps. Reflections do not meet this requirement, we cannot perform "a little of reflection". Such small transformations are called *perturbations*. Therefore, we can conclude that small perturbations of elements of geometric algebra are to be performed by rotors.

Note that orthogonal transformations do not permit to just add a small vector δx to the vector x, orthogonal transformations must keep the vector length. So we must have $x\cdot\delta x=0$. Generally, such an element (δx) of geometric algebra has the form $\delta x=x\rfloor\delta B$, where δB is a small bivector. We can show it

$$x\cdot\left(x\rfloor\delta B\right)=x\rfloor\left(x\rfloor\delta B\right)=\left(x\wedge x\right)\rfloor\delta B=0.$$

It follows now that

$$\delta x=x\rfloor\delta B=\left(x\delta B-\delta Bx\right)/2=x\otimes\delta B$$

and we have the desired shape in the form of a commutator. It may seem that the restriction on the rotations is too strict, it looks as if we cannot do a simple translation of a vector. However, here it just means that we need to find a way to describe translations by rotations. It is possible in geometric algebra, but we will not show it here (see [19]).

Here we will stop, but noting that a small portion of formalism just shown leads to Lie groups and algebras. It can be shown that every finite Lie group or algebra can be directly described in the context of geometric algebra. The infinite case is not yet absolutely clear, but it would be unusual for a result to be different. Anyway, another nice part of mathematics fits perfectly into the geometric algebra. Anyone who seriously studies the geometric algebra was initially probably astonished by the fact that different branches of mathematics show a new light in the language of geometric vector multiplication, but with time one gets used to it and does not expect exceptions. One cannot help wonder what our science would look like if the power of this magical language of mathematics was understood and accepted a century ago. And it was all at our fingertips.

Complex numbers

Let's specify the vector in 2D $\boldsymbol{r}=xe_1+ye_2$. Using the existence of the inverse we have

$$\boldsymbol{r}=e_1\left(x+ye_1e_2\right)=e_1\left(x+yi\right),\quad i=e_1e_2,$$

and we see that we get a complex number $x+yi$, but with a non-commutative "imaginary unit". The first thing to complain about is: „Yes, but your imaginary unit is not commutative, and quantum mechanics cannot be formulated without an imaginary unit …". Immediately you see that the "critic" commented something he knows almost nothing about, because, first, quantum mechanics works nicely (and even better) with real numbers, without the imaginary unit, but one should learn geometric algebra, then learn the formulation of quantum mechanics in the language of geometric algebra ... Not only that we can without using the imaginary unit, but many relations obtain a clear geometric meaning and thus provide a new insights into the theory in the language of geometric algebra. And second, non-commutativity of our bivector $i = e_1 e_2$ actually becomes an advantage, it enriches the theory of complex numbers and, as we are repeating until you get bored, gives it a clear geometric meaning. For our complex number $z = e_1 \boldsymbol{r}$ we have (due to anti-commutativity) $z^* = \boldsymbol{r} e_1$, so

$$zz^* = e_1 \boldsymbol{r}\boldsymbol{r} e_1 = r^2 e_1 e_1 = r^2 = x^2 + y^2, \quad \text{or}$$

$$z + z^* = e_1 \boldsymbol{r} + \boldsymbol{r} e_1 = 2 e_1 \cdot \boldsymbol{r} = 2x,$$

$$z - z^* = e_1 \boldsymbol{r} - \boldsymbol{r} e_1 = 2 e_1 \wedge \boldsymbol{r} = 2yi,$$

etc. We see that operations on complex numbers are, without any problem, confined to the operations in geometric algebra. Define the derivative operator in 2D

$$\nabla f \equiv e_1 \frac{\partial f}{\partial x} + e_2 \frac{\partial f}{\partial y},$$

and introduce a complex field $\psi(x, y) = u(x, y) + iv(x, y)$, $i = e_1 e_2$. Simple calculation shows (do it) that the derivation of the field is

$$\nabla \psi = \nabla u + \nabla (iv) = \nabla u - i \nabla v,$$

$$\nabla \psi = e_1 \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + e_2 \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right).$$

So, if we want the derivative to be identically zero (*analyticity*), *Cauchy-Riemann* equations immediately follow. Note how anti-commutativity of unit vectors gives correct signs. So, analyticity condition in geometric algebra has a simple form $\nabla \psi = 0$, and we can immediately generalize it to higher dimensions. And yes, this is just a right moment to stop and think. Let advocates of the traditional approach do all that using just the commutative imaginary unit. Actually, it's amazing how this old, good imaginary unit has made a lot of work, given the modest possibilities! But, it is time to rest a little, let bivectors, pseudoscalars ... do the job. It should be noted, to make no confusion, the choice of the plane $e_1 e_2$ is unimportant here. We can take the bivector like $(e_1 + e_2)(e_1 - e_3)$, normalize it and we get a new „imaginary unit", but in the new plane. We can do that in 4D also, and take, for example $i = e_3 e_4$, all formulas will be valid. The plane $e_1 e_2$ is just one of infinity of them, but geometrical relationships in each of them are the same. We can solve the problem in the plane $e_1 e_2$, and then rotate all to the plane we want, we have powerful rotors in geometric algebra. And when it is said „ powerful" then it literally means that we do not have to be experts in matrix calculations, here something like that an advanced high school student can make. We can rotate any object, not only the vectors. Linear algebra is the mathematics of vectors and operators, geometric algebra is mathematics of subspaces and operations on them. Anyone who uses mathematics should understand how important this is.

We will show here that one can get solutions of the equation $\nabla \psi = 0$ by using series in z. Notice first an easy relation for vectors

$$abc + bac = (ab + ba)c = 2a \cdot bc,$$

where the inner product has priority. The operator ∇ is acting as a vector (expressions like $\boldsymbol{r}\nabla$ are possible, but then we usually write $\dot{\boldsymbol{r}}\dot{\nabla}$, which does not mean the time derivative, but indicates the element on which the derivation operator acts on and gives desired order in products of unit vectors), so take the advantage of the previous relation (a very useful calculation)

$$\nabla z = \nabla\left(e_1\boldsymbol{r}\right) = 2e_1 \cdot \nabla \boldsymbol{r} - e_1 \nabla \boldsymbol{r} = 2e_1 - 2e_1 = 0 .$$

Now we have

$$\nabla\left(z - z_0\right)^n = n\nabla\left(e_1\boldsymbol{r} - z_0\right)\left(z - z_0\right)^{n-1} = 0 ,$$

so, the Taylor expansion about z_0 automatically gives an analytical function. Again, in any plane, in any dimension. It is not only that geometric algebra contains all the theory of functions of complex variables (including integral theorems, as a special case of the fundamental theorem of integral calculus in geometric algebra), but also extends and generalizes it to any dimension. Is not this a miracle? And we were just wondering how to multiply vectors. If one still have a desire to pronounce the sentence „Yes, but …“, he could go back to the beginning of the text and see how all this began. Time of geometric algebra is yet to come, hopefully. Dark Ages of matrices and coordinates will disappear and it will be replaced by the time of synergy of algebra and intuitively clear geometry. Students will learn much faster and be superior to today's "experts". And when we learn computers to "think" in this magical language (imagine a computer that knows how to perform operations on subspaces) children will be able to play with geometric shapes as now play a car racing or other computer games. The properties of triangles, circles, spheres and other shapes we will learn through a play, on computers, interactive. Language of geometric algebra is so powerful that it can "automate" even the process of proving theorems (there's still a lot of work to do, but possibilities are there). We have reasons to think that geometric algebra is not just "another formalism", but it offers the possibility of deep questioning the very concept of a number.

Spinors

Let's look at elements of algebra which in the "sandwich" form do not change the grade of a vector (i.e. a vector transform to a vector). Among them are transformations which rotate and dilate vectors, we usually call them *spinors*. Let's look at multivectors ψ with the property ($\boldsymbol{v}$ is a vector)

$$\psi \boldsymbol{v} \psi^\dagger = \rho R \boldsymbol{v} R^\dagger, \quad \rho \in \Re, \quad RR^\dagger = 1 ,$$

which is precisely the rotation of the vector with dilatation. If we define $U \equiv R^\dagger \psi$, the previous relation becomes

$$U \boldsymbol{v} U^\dagger = \rho \boldsymbol{v} ,$$

and we will find the element U. Show that pseudoscalars of odd dimensions commute and of even dimensions anti-commute with vectors. Other grades do not possess such a general property (real scalars commute). We see that an element U induces a pure dilation of the vector $\boldsymbol{v}$ and can commute or anti-commute with $\boldsymbol{v}$, so it follows that the element U is, generally, a real scalar, or pseudoscalar, or combination of both: $U = \lambda_1 + \lambda_2 I$. Now, using the definition of U, we get

$$U = \lambda_1^2 \boldsymbol{v} + \lambda_1 \lambda_2 \left(I\boldsymbol{v} + \boldsymbol{v} I^\dagger\right) + \lambda_2^2 I \boldsymbol{v} I^\dagger = \rho \boldsymbol{v} .$$

In *Cl*3 ($p = 3, \quad q = 0$) the pseudoscalar $I = j$ commutes with all elements of the algebra and the reverse is $I^\dagger = -j$, the middle term disappears, so we have

$$\lambda_1^2+\lambda_2^2=\rho \Rightarrow \psi = R\left(\lambda_1+j\lambda_2\right),$$

and it is easy to check

$$\psi \boldsymbol{v}\psi^\dagger = R\left(\lambda_1+j\lambda_2\right)\boldsymbol{v}\left(\lambda_1-j\lambda_2\right)R^\dagger = \left(\lambda_1+j\lambda_2\right)\left(\lambda_1-j\lambda_2\right)R\boldsymbol{v}R^\dagger = \left(\lambda_1^2+\lambda_2^2\right)R\boldsymbol{v}R^\dagger = \rho R\boldsymbol{v}R^\dagger .$$

In general, note that

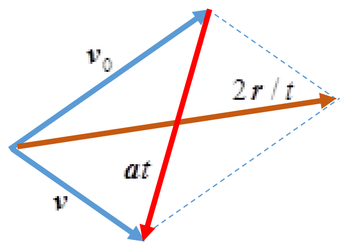
$$\boldsymbol{v}I^\dagger = \left(-1\right)^{n-1} I^\dagger \boldsymbol{v},\quad \boldsymbol{v}I^\dagger = \left(-1\right)^{(n-1)(n-2)/2} I\boldsymbol{v},\quad II^\dagger = \left(-1\right)^q ,$$

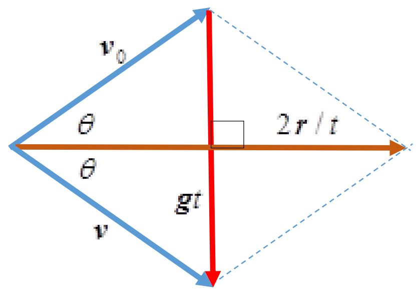
(prove it, at least for signatures (3, 0) and (1, 3)) and we ca find solutions (find them) dependent on the parity of the number $\left(n-1\right)\left(n-2\right)/2$.

Spinors in geometric algebra, as elsewhere, can be defined by (left) ideals of the algebra, but here we will not deal with it ([7]).

A little of "ordinary" physics

Let's see how we can solve a kinematic problem in its generality using simple calculations and intuitively clear. Consider the problem of an accelerated motion with a constant acceleration.





The problem is easily reduced to relations

$$\boldsymbol{v}=\boldsymbol{v}_0+\boldsymbol{a}t, \qquad \boldsymbol{v}+\boldsymbol{v}_0=2\boldsymbol{r}/t,$$

wherein the second relation defines the average speed vector $\overline{\boldsymbol{v}}=\boldsymbol{r}/t$, so we have

$$\left(\boldsymbol{v}+\boldsymbol{v}_0\right)\left(\boldsymbol{v}-\boldsymbol{v}_0\right)=2\boldsymbol{r}\boldsymbol{a}\Rightarrow$$

$$v^2-v_0^2+\boldsymbol{v}_0\boldsymbol{v}-\boldsymbol{v}\boldsymbol{v}_0=v^2-v_0^2+2\boldsymbol{v}_0\wedge\boldsymbol{v}=2\left(\boldsymbol{r}\cdot\boldsymbol{a}+\boldsymbol{r}\wedge\boldsymbol{a}\right),$$

where by comparison of the scalar and bivector parts we get

$$v^2-v_0^2=2\boldsymbol{r}\cdot\boldsymbol{a},$$

$$\boldsymbol{v}_0\wedge\boldsymbol{v}=\boldsymbol{r}\wedge\boldsymbol{a},$$

i.e. the law of conservation of energy and the surface of the parallelogram theorem. For the projectile motion problem ($\boldsymbol{a}=\boldsymbol{g}$) we have (figure on the right)

$$\boldsymbol{r}\cdot\boldsymbol{g}=0\Rightarrow v^2=v_0^2\Rightarrow\left|\boldsymbol{v}_0\wedge\boldsymbol{v}\right|=v_0^2\sin\left(2\theta\right)=\left|\boldsymbol{r}\wedge\boldsymbol{g}\right|=rg\Rightarrow$$
$$r=\frac{v_0^2}{g}\sin\left(2\theta\right)$$

and this is the known relation for the range. Notice how properties of geometric product lead to simple manipulations. Another example is the Kepler problem. Immediately after setting the problem, after a few lines, we obtain non-trivial conclusions that textbooks usually put as hard part at the end. Examples here are to show how to obtain solutions without coordinate systems and coordinates. Unfortunately, research shows ([21]) that many physics students see vectors mainly as a series of numbers (coordinates) and it is a sad reflection of the current education systems, regardless of the place on the planet. The connection of linear algebra and geometry is usually quite neglected. With the geometric product algebra and geometry go hand in hand. Instead of treating vectors as key elements of the algebra we have a whole range of objects that are not vectors and have a very clear geometric meaning. We are calculating with subspaces! And in any dimension. Something like that is impossible to achieve just manipulating by coordinates. Emphasize this, impossible! Russian physicist Landau, famous for his math skills, ended up in Stalin's prison. After his release from the prison, he said that his prison was welcome, because he had learned to run tensor calculus "in the head". Physicists of the future will be more skilled than Landau, they will use linear transformations in geometric algebra instead of tensor calculus. They will calculate faster, regardless of the dimension of space, without using coordinates and with a clear geometric interpretation at every step. Landau was also famous by the method of accepting students. He would said to a young candidate: "Here, solve the integral." Many have failed. In geometric algebra, there is a theorem (fundamental theorem) about integration that combines all known integral theorems used in physics, including complex area. Just imagine, Landau would be really surprised! He was a typical representative of the mathematics of the 20th century, although in his time already existed the new mathematics. It existed, but almost completely neglected and forgotten. Part of the price paid (and we still pay it) is a rediscovery of what is neglected and forgotten. Pauli discovered its matrices – we have continued to use matrices. It is often said that the geometric algebra is non-commutative and that this discourages people. What about matrices? Not only that they are non-commutative, they are unintuitive. Then Dirac discovered his matrices, ideal for geometric algebra. Again, we continued with matrices. And many authors, on various occasions, rediscovered spinors, even giving them different names. Then we decided to make fast spacecrafts equipped with computers and found that we have problems with matrices. Then we started to use quaternions and improved things in some extent. We can find a number of other indications, and, after all, it is obvious that many of problems simply disappear when geometric product is introduced instead of products of Gibbs. In spite of everything, one of the great authors in the field of geometric algebra, Garret Sobczyk, wrote in an e-mail:

„*I am surprised that after 45 years working in this area, it is still not generally recognized in the scientific community. But I think that you are right that it deserves general recognition ... Too bad Clifford died so young, or maybe things would be different now*."

Words and sentences

Let's look, just for illustration, how „words" in geometric algebra can have a geometric content. For example, the „word" ***abba***. From $\boldsymbol{ab}=S+A$ (symmetric and anti-symmetric parts)

$$\boldsymbol{abba}=a^2b^2=\left(S+A\right)\left(S-A\right)=S^2-A^2=S^2+\left|A\right|^2=$$
$$\left(\boldsymbol{a}\cdot\boldsymbol{b}\right)^2+\left|\boldsymbol{a}\wedge\boldsymbol{b}\right|^2=a^2b^2\left(\cos^2\theta+\sin^2\theta\right)$$

and we have the well-known trigonometric identity. This is, of course, just a game, but in geometric algebra it is important to develop intuition about the geometric content written in expressions. Due to properties of geometric product a structure of expressions is quickly manifested, as for relations between subspaces, to be an element of a subspace, orthogonality, to be parallel, etc.

Let's compare exposed to the matrix approach. We have seen that in 3D we can represent vectors by Pauli matrices. Try to imagine that we are not aware of it, but we know about Pauli matrices (from quantum mechanics). We could write the word ***abba*** in the language of matrices, we could resolve matrices in symmetric and anti-symmetric parts (it is custom), but try to derive the sine and cosine of the angle and the basic trigonometric identity. If you succeed (it is possible), how would you interpret that angle? And more important, how to even come up with the idea to look for an angle, just looking at matrices? It is hard, for sure, but with vectors it is natural and straightforward. That is the main idea: **the language of matrices hides an important geometric content**. True, physicists know that Pauli matrices have to do something with the orientation of the spin, but generally, the problem of geometric interpretation still remains. Here is one more example. We have unit vectors $m=\left(e_1+e_2\right)/\sqrt{2}$ and $n=\left(e_2+e_3\right)/\sqrt{2}$ in 3D. It is not difficult to imagine or draw them, there is the plane spanned and bivector $m\wedge n$ in it (bivector defines the plane). Image again that we are using Pauli matrices, but, as before, without awareness that they represent vectors in 3D (we cannot even know it if we do not accept the geometric product of vectors). Someone could really investigate a linear combinations of the Pauli matrices, even come to the idea to look at anti-symmetric part of products of matrices, something like $\left(\hat{\sigma}_m\hat{\sigma}_n-\hat{\sigma}_n\hat{\sigma}_m\right)/2$, where $\hat{\sigma}_m=\left(\hat{\sigma}_1+\hat{\sigma}_2\right)/\sqrt{2}$ and $\hat{\sigma}_n=\left(\hat{\sigma}_2+\hat{\sigma}_3\right)/\sqrt{2}$. We should now calculate this, so, we can compare the needed calculation with matrices and simple calculation of the outer product (in fact, there is no need to calculate the outer product, we have the geometric picture without effort). Whatever, the bivector is

$$m\wedge n=\left(e_1+e_2\right)\wedge\left(e_2+e_3\right)/2=\left(e_1e_2+e_1e_3+e_2e_3\right)/2.$$

Fortunately, computer can help here with matrices (you see the problem?), so, the anti-symmetric part of the matrix product is

$$\begin{pmatrix} i & -1+i \\ 1+i & -i \end{pmatrix}/2.$$

Now, how, without connecting with vectors in 3D, to interpret this matrix as a plane? Or find the angle between - what? It is easy to express formulas from *Cl*3 via Pauli matrices, but the matrix form to vectors – it could be tricky, especially for blades of higher grades, or general multivectors. **The language of matrices blurs the geometric content**! In quantum mechanics with Pauli matrices we need the imaginary unit, and people say that the imaginary unit is necessary to formulate theories of the subatomic world. This often leads to a philosophical debates and questions about the „real nature“ of the world we live in. In the language of geometric algebra the **imaginary unit is absolutely not necessary**, quantum mechanics can be beautifully and elegantly formulated using just real numbers, with the clear geometric interpretation. Besides the real numbers, complex numbers and quaternions could be of interest in quantum mechanics, but it is clear now, they all are natural part of *Cl*3, as we discussed earlier. In the article [1], author comments: "… *instead of being distinct alternatives, real, complex and quaternionic quantum mechanics are three aspects of a single unified structure*." There are useful remarks on the *Frobenius–Schur indicato*r in this article. True, there is no geometric algebra in the cited article, although there is the term "*division algebra*" in the title. Rather than comment, here is the sentence from [28], one that should be known to all mathematician and physicists. Unfortunately, it is not.

"*Geometric algebra is, in fact, the largest possible associative division algebra that integrates all algebraic systems (algebra of complex numbers, vector algebra, matrix algebra, quaternion algebra,* *etc.) into a coherent mathematical language that augments the powerful geometric intuition of the human mind with the precision of an algebraic system*." To be honest, division algebra or not – it is unimportant. **It unifies and it works!**

Linear transformations

Often we are interested in transformations of elements of an algebra (eg, vectors, bivectors, …) to other elements in the same space. Among them are certainly the most interesting linear transformations. Let's look a linear transformation F which translates vectors into vectors, with the property

$$\mathrm{F}(\alpha a+\beta b)=\alpha\mathrm{F}(a)+\beta\mathrm{F}(b),\quad \alpha,\ \beta\in\Re .$$

We can imagine that the result of such a transformation is, for example, the rotation of the vector with the dilatation. For such a simple picture we do not need vector components. Another example may be a rotation:

$$\mathrm{F}(a)=\mathrm{R}(a)\equiv RaR^{\dagger} .$$

We have seen that the effect of rotation of the blade is the same as the action of the rotation on each vector in the blade, so we require that all of our linear transformations have that property, which means

$$\mathrm{F}(a\wedge b)=\mathrm{F}(a)\wedge\mathrm{F}(b).$$

A linear transformation acting on vector gives back a vector and we see that the form of outer product is preserved. Such a transformation have the special name: *outermorphism*. The action of two successive transformations can be written as $\mathrm{F}(\mathrm{G}(a))\equiv\mathrm{FG}a$, which is handy for manipulating expressions.

If for linear transformation $\mathrm{F}:V\to\mathrm{W}$ there is an adequate linear transformation $\overline{\mathrm{F}}:W\to V$, we'll call it a *transposed transformation* (*adjoint*). Here we will restrict to transformations $\mathrm{F}:V\to V$. We say that they are transposed because we can always find a matrix representation in some basis and see that the matrix of $\overline{\mathrm{F}}$ is just the transposed matrix of F (see [22]). Here is an implicit definition of the adjoint

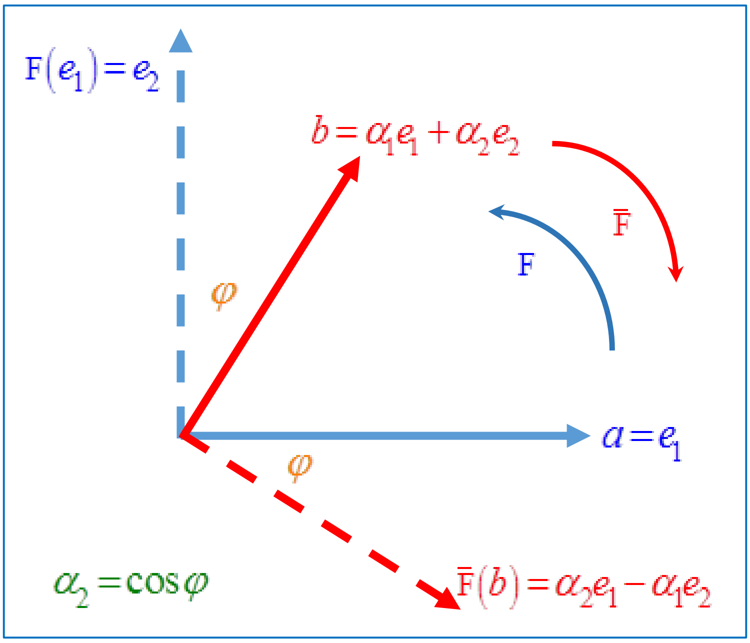
$$\mathrm{a}\cdot\overline{\mathrm{F}}(b)=\mathrm{F}(a)\cdot b ,$$

for any two vectors a and b. To see what this means we can imagine a simple example. Let $a=e_1$, $b=\alpha_1 e_1+\alpha_2 e_2$ and imagine that F rotates vectors by $\pi/2$ in the plane $e_1 e_2$. Then we have $\mathrm{F}(a)=ae_1e_2$, for example, $\mathrm{F}(e_1)=e_1e_1e_2=e_2$. This gives

$$\mathrm{F}(e_1)\cdot(\alpha_1 e_1+\alpha_2 e_2)=e_2\cdot(\alpha_1 e_1+\alpha_2 e_2)=\alpha_2 .$$

According to the figure bellow we see that our linear transformation transforms the vector $a=e_1$ to $\mathrm{F}(e_1)=e_2$ and the inner product gives $\alpha_2=\cos\varphi$. But it is clear that we can take the vector b, rotate it and the inner product with the vector $a=e_1$ will give us the same result. So, adjoint operation is just the rotation by $-\pi/2$ in the plane e_1e_2 (for a more general result for rotations see below). Then we have

$$\overline{\mathrm{F}}(b)=e_1e_2 b ,$$



$$\overline{\mathrm{F}}(b)=e_1 e_2(\alpha_1 e_1+\alpha_2 e_2)=\alpha_2 e_1-\alpha_1 e_2$$

Define now reciprocal basis vectors e^i with the property

$$e^i\cdot e_j=\delta_{ij}.$$

Here we are using an orthonormal basis of positive signature, so

$$e^i=e_j\Rightarrow e^i\cdot e_j=e^i e_j=\delta_{ij}$$

and the definition is motivated by two facts: first, we want to use the Einstein summation convention

$$e^i e_i\equiv\sum_{i=1}^{n}e^i e_i$$

and, second, we want the ability to generalize easily. The explicit form of the transposed transformation can be found using the orthonormal basis

$$e_i\cdot\overline{\mathrm{F}}(a)=\mathrm{F}(e_i)\cdot a,$$

so we have (recall, the inner product has priority)

$$\overline{\mathrm{F}}(a)=e^i a\cdot\mathrm{F}(e_i),$$

where summation is understood and the inner product has a priority. The designation $\overline{\mathrm{F}}$ is not common, F^{T} or $\mathrm{F}^{\dagger}$ is, but sometimes we use $\underline{\mathrm{F}}$ for linear transformations, so nice symmetry in expressions could occur if we use $\overline{\mathrm{F}}$. Furthermore, $\overline{\mathrm{F}}(a)$ is not a matrix or a tensor, so designation highlights the difference. There cannot be confusion with Clifford conjugation in the text, we are consistently using format *italic* for multivectors. For a transposed transformation of the "product" of transformations we have

$$\overline{\mathrm{FG}}(a)=\overline{\mathrm{G}}\overline{\mathrm{F}}(a),$$

(see literature). Transformations with the property $\overline{\mathrm{F}}=\mathrm{F}$ are *symmetric*. Important symmetric transformations are $\overline{\mathrm{F}}\mathrm{F}$ and $\mathrm{F}\overline{\mathrm{F}}$ (show that).

Let I to be a unit pseudoscalar. The determinant of a linear transformation is defined as

$$\mathrm{F}(I)\equiv I\det\mathrm{F},\quad\det\mathrm{F}\in\Re.$$

This definition is in full compliance with the usual definition. Notice that this relation looks like an eigenvalue relation. In fact, that is true, the pseudoscalar is invariant (eigenblade) and the determinant is an eigenvalue (real!). An example is the 3D rotation

$$\mathrm{R}(j)=RjR^{\dagger}=jRR^{\dagger}\Rightarrow\det\mathrm{R}=RR^{\dagger}=1,\quad j=e_{123},$$

what we expect for rotors (for rotation matrices, too). Again, notice the power of formalism: without components, without matrices, by a simple manipulation, we get an important result. The pseudoscalar represents an oriented volume, so the linear transformation of the pseudoscalar is simply reduced to its multiplication by a real number. The determinant of the transposed transformation is

$$\mathrm{F}(I)=I\det\mathrm{F}\Rightarrow$$
$$\det\mathrm{F}=\mathrm{F}(I)I^{-1}=\left\langle\mathrm{F}(I)I^{-1}\right\rangle=\left\langle I\overline{\mathrm{F}}\left(I^{-1}\right)\right\rangle=\det\overline{\mathrm{F}},$$

where we take an advantage of the fact that determinant is a real number, therefore has the grade zero. For the composition of transformations we have

$$(\mathrm{FG})(I)=\mathrm{FG}(I)=\mathrm{F}(I\det\mathrm{G})=(\det\mathrm{G})\mathrm{F}(I)=I\det\mathrm{F}\det\mathrm{G}$$

and it is the well known rule for determinants, but recall how much effort we need to prove that in the matrix theory. Here, proof is almost trivial. A beginner needs pretty much time to become skilled with matrices. Finally she(he) gets a tool that cannot effectively cope even with rotations. That time he could use to learn the basics of geometric algebra and get a powerful tool for many branches of mathematics. And geometric algebra today, thanks to Grassmann, Clifford, Artin, Hestenes, Sobczyk, Baylis and many other smart and hardworking people (see detailed list at the end of the text) has become a well-developed theory, with applications in many areas of mathematics, physics, engineering, including biology, studies of brain functions, computer graphic, robotics, etc.

We will state without a proof (the reader can prove it) some useful relations. For bivectors we have

$$B_1\cdot\overline{\mathrm{F}}(B_2)=\mathrm{F}(B_1)\cdot B_2 .$$

This can be extended to arbitrary multivectors as

$$\left\langle A\overline{\mathrm{F}}(B)\right\rangle=\left\langle\mathrm{F}(A)B\right\rangle .$$

Now we will define the inverse of a linear transformation. For a multivector M we have

$$IM\det\mathrm{F}=\mathrm{F}(I)M=\mathrm{F}\left(I\overline{\mathrm{F}}(M)\right),$$

(see in literature). Let's take the multivector $A=IM$ so we get

$$A\det\mathrm{F}=\mathrm{F}\left(I\overline{\mathrm{F}}\left(I^{-1}A\right)\right),$$

and a similar relation can be written for $\overline{\mathrm{F}}$. It follows

$$\mathrm{F}^{-1}(A)=I\overline{\mathrm{F}}\left(I^{-1}A\right)(\det\mathrm{F})^{-1},$$
$$\overline{\mathrm{F}}^{-1}(A)=I\mathrm{F}\left(I^{-1}A\right)(\det\mathrm{F})^{-1}.$$

For rotors in *Cl*3 we have $\mathrm{R}(a)=RaR^{\dagger}$, applied to any multivector gives $\mathrm{R}(M)=RMR^{\dagger}$ and $\overline{\mathrm{R}}(M)=R^{\dagger}MR$, so using $\det\mathrm{R}=1$ we have

$$\mathrm{R}^{-1}(M)=jR^{\dagger}j^{-1}MR=R^{\dagger}MR=\overline{\mathrm{R}}(M),$$

i.e. the inverse of rotation is equal to the transposed rotation. This is actually the definition of each orthogonal transformation (transformation with the determinant ±1). For nice examples see [18].

Eigenvectors and eigenblades

The concept of eigenvalues and eigenvectors should be known to the reader. Briefly, for an operator (matrix) m we can define eigenvalues λ_i and eigenvectors v_i as follows

$$mv_i = \lambda_i v_i, \quad \lambda_i \in \mathbb{C}.$$

In geometric algebra we say that a linear transformation (function) has *eigenvector* e and *eigenvalue* $\lambda \in \Re$ if

$$\mathrm{F}(e) = \lambda e,$$

which entails

$$\det(\mathrm{F} - \lambda I) = 0,$$

so, we have a polynomial equation (the *secular equation*). Generally, the secular equation has roots over the complex field, but, we have algebra over the field of real numbers and it is not desirable to spread to a complex area. For example, how to interpret the product $\sqrt{-1}e_1$, which is not an element of the algebra? Fortunately, this is not necessary in geometric algebra, because we can give a whole new meaning to complex solutions. For this purpose, we introduce the concept of an *eigenblade*. Namely, vectors are just elements of the algebra with the grade 1, but we have grades 2, 3, … in geometric algebra, which are not defined in the ordinary theory of vector spaces. It is therefore natural to extend the definition of eigenvalues to other elements of the algebra. For a blade B_r with grade r we define

$$\mathrm{F}(B_r) = \lambda B_r, \quad \lambda \in \Re.$$

In fact, we already have such a relationship, namely, for $B_r = I$ we have an eigenvalue $\det \mathrm{F}$, because of $\mathrm{F}(I) = I \det \mathrm{F}$. Accordingly, pseudoscalars are eigenblades of linear transformations. To explain the concept of an eigenblade let's look at the following example (see [18]). Let's specify the linear function with the property

$$\mathrm{F}(e_1) = e_2, \quad \mathrm{F}(e_2) = -e_1,$$

(recognize rotation?) so, it is not difficult to find a solution using matrices. The matrix of transformation is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with eigenvalues $\pm i$, $i = \sqrt{-1}$, and eigenvectors $e_1 \pm i e_2$ (use secular equation and prove that). In geometric algebra, for the blade $e_1 \wedge e_2$ we have (notice the elegance)

$$\mathrm{F}(e_1 \wedge e_2) = \mathrm{F}(e_1) \wedge \mathrm{F}(e_2) = e_2 \wedge (-e_1) = e_1 \wedge e_2,$$

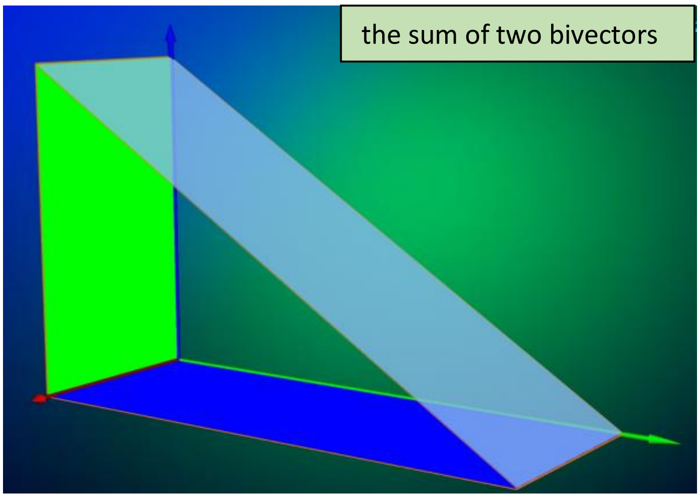
so, the blade $e_1 \wedge e_2$ is the eigenblade with the (real) eigenvalue 1. Our blade is an invariant, but we know that from rotors formalism! There is no need for a imaginary unit, we have our blade. Notice that vectors in the plane defined by $e_1 \wedge e_2$ are changed by transformation, but the unit bivector is not. You see a simple mathematics and the important result. In standard methods, using matrices, there is no "blades" at all. Why? Simple, there is no geometric product. So, try to find such a result using matrices. All those who like to comment on geometric algebra by sentences as „*Yes, but imaginary unit in* *quantum mechanics* …“ should think twice about this simple example, and when they come to the conclusion that „*it does not make sense* …“, well, what to say? Just think again. This is the question of how do we understand the very concept of a number. Probably, Grassmann and Clifford directed us well and their time is yet to come.

If orthonormal basis vectors e_i and e_j are eigenvectors of a linear transformation F, then

$$e_i \cdot \mathrm{F}(e_j) = e_i \cdot (\lambda_j e_j) = \lambda_j e_i \cdot e_j .$$

Apply the previous relations to symmetric linear transformations and show that their eigenvectors with different eigenvalues must be orthogonal.

Euclidean 3D geometric algebra (*Cl*3)



Generally, a multivector in *Cl*3 can be rewritten as

$$M = t + \boldsymbol{x} + j\boldsymbol{n} + bj, \quad t, b \in \Re, \quad j = e_1e_2e_3,$$

where for three-dimensional vectors we are using the **bold** format here. We have seen already that the unit pseudoscalar j commutes with all elements of the algebra and squares to -1, making it an ideal replacement for the imaginary unit (there are many "imaginary units" in GA). A pseudoscalar with such properties will appear again in *Cl*7, *Cl*11, … Here we use once more, a very useful, form of a multivector:

$$M = Z + \boldsymbol{F}, \quad Z = t + bj, \quad \boldsymbol{F} = \boldsymbol{x} + j\boldsymbol{n}.$$

The element Z obviously commutes with all elements of the algebra (belongs to the *center of the algebra*). This feature makes it a *complex scalar*. A complex scalar is really acting as a complex number, as we shall see below. This is the reason that we write $Z \in \mathbb{C}$, although, obviously, we have to change the meaning of the symbol $\mathbb{C}$, i.e. we replace the ordinary imaginary unit by the pseudoscalar. An element $\boldsymbol{F}$ is a *complex vector*, with *real vectors* as components. The choice of designation ($\boldsymbol{F}$), as well as for complex scalars, is not without significance, namely, due to a complex mixture of electric and magnetic field in electromagnetism. Here, when we say "real", we mean a real scalar, or a 3D vector, or their linear combination. When a real element is multiplied by the pseudoscalar j we get an imaginary element, so, the sum of real and imaginary elements gives a complex one. For example, $\boldsymbol{x}$ (*vector*) is real, $t + \boldsymbol{x}$ (*paravector*) is real, $t + j\boldsymbol{n}$ (*spinor*) is complex, $j\boldsymbol{n}$ (*bivector*) is imaginary, $\boldsymbol{F} = \boldsymbol{x} + j\boldsymbol{n}$ (*complex vector*) is complex, etc. Note that a multivector could be written as

$$M = t + \boldsymbol{x} + j\boldsymbol{n} + bj = t + \boldsymbol{x} + j(b + \boldsymbol{n}),$$

so, it is just a complex number, with real components (*paravectors*). Use an involution (which?) to extract the real (imaginary) part of a multivector. How about Z and $\boldsymbol{F}$? Or $t + j\boldsymbol{n}$? The reader is suggested to write all three described involutions in this new form. You can use a complex conjugation. As an example we look at Clifford involution (i.e. *Clifford conjugation*, *main involution*) $\bar{M} = Z - \boldsymbol{F}$

$$\left(M + \bar{M}\right)/2 = Z \equiv \langle M \rangle_S, \quad \text{(scalar part)}$$

$$\left(M - \bar{M}\right)/2 = \boldsymbol{F} \equiv \langle M \rangle_V \quad \text{(vector part).}$$

Due to commutativity of the complex scalar Z we have

$$M\bar{M} = (Z + \boldsymbol{F})(Z - \boldsymbol{F}) = Z^2 - \boldsymbol{F}^2 = (Z - \boldsymbol{F})(Z + \boldsymbol{F}) = \bar{M}M,$$

where

$$Z^2 = t^2 - b^2 + 2tbj\,,\quad \boldsymbol{F}^2 = x^2 - n^2 + j\left(\boldsymbol{xn} + \boldsymbol{nx}\right) = x^2 - n^2 + 2j\boldsymbol{x}\cdot\boldsymbol{n}\,.$$

Here is the result to remember: **the square of a complex vector is a complex scalar**. It means that the element $M\bar{M}$ is a complex scalar. It can be shown that $M\bar{M}$ is the **only** element of form $M\breve{M}$ (here $\breve{M}$ stands for **any** involution of M, $M\breve{M}$ is referred as the square of the *amplitude*) that satisfies $M\breve{M} = \breve{M}M \in \mathbb{C}$. We have

$$\left(Z + \boldsymbol{F}\right)\left(\breve{Z} + \breve{\boldsymbol{F}}\right) = Z\breve{Z} + Z\breve{\boldsymbol{F}} + \breve{Z}\boldsymbol{F} + \boldsymbol{F}\breve{\boldsymbol{F}}\,,$$

so we have two possibilities

$$\breve{Z} = Z,\quad \breve{\boldsymbol{F}} = -\boldsymbol{F} \text{ or } \breve{Z} = -Z,\quad \breve{\boldsymbol{F}} = \boldsymbol{F}\,,$$

which differ only in the overall sign. Any involution that changes the complex vector the other way, changes (up to overall sign) the bivector or vector part, so

$$\boldsymbol{F}\breve{\boldsymbol{F}} = \left(\boldsymbol{x} + j\boldsymbol{n}\right)\left(\boldsymbol{x} - j\boldsymbol{n}\right) = x^2 + n^2 + j\left(\boldsymbol{nx} - \boldsymbol{xn}\right) = x^2 + n^2 - 2j\boldsymbol{x}\wedge\boldsymbol{n}\,,$$

and we get the outer product of real vectors which cannot be canceled, it is absent in $Z\breve{Z} + Z\breve{\boldsymbol{F}} + \breve{Z}\boldsymbol{F}$ so it must be $\breve{M} = \bar{M}$. We already found that $M\bar{M} = \bar{M}M$, but we can show that from a demand that the amplitude (any) belongs to the center of the algebra follows commutativity

$$M\breve{M} \in \mathbb{C} \Rightarrow M\left(M\breve{M}\right) = \left(M\breve{M}\right)M = M\left(\breve{M}M\right) \Rightarrow M\left(M\breve{M} - \breve{M}M\right) = 0\,,$$

due to associativity and distributivity. In a special case the expression in parentheses need not to be zero because there are zero divisors in the algebra, but we need general commutativity, so it must be zero. The scalar $M\bar{M}$ is referred as the *amplitude of multivector* (MA in text, in fact this is the square of the amplitude, but that will not make a confusion).

Using MA we can define the inverse of a multivector, if $M\bar{M} \neq 0$:

$$M^{-1} \equiv \bar{M} \,/\, M\bar{M}\,.$$

To find $1/M\bar{M}$ we use complex numbers technique

$$\frac{1}{M\bar{M}} = \frac{\left(M\bar{M}\right)^*}{M\bar{M}\left(M\bar{M}\right)^*}\,,$$

where * stands for a complex conjugation, which means $j \to -j$. The technique is the same, but an interpretation is not, namely, the pseudoscalar j is the oriented unit volume, it has an intuitive geometric interpretation.

Example: $1/\left(1+j\right)$? We have $1/\left(1+i\right) = \left(1-i\right)/2 \;\Rightarrow\; 1/\left(1+j\right) = \left(1-j\right)/2$.

Of course, this „trick“ is justified

$$\frac{1}{1+j} = \frac{1-j}{(1+j)(1-j)} = \frac{1-j}{2}\,.$$

We'll see that this procedure sometimes is not enough to find all possible solutions in geometric algebra, e.g. solutions for roots of complex numbers can be extended to complex vectors. A simple example is $\sqrt{1} = e_1$.

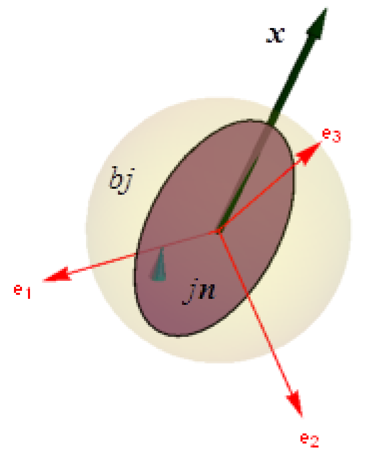
An important concept is the *dual* of a multivector M is defined as

$$M^* \equiv -jM$$

(do not confuse with a complex conjugation *). Note that with the dual operation a real scalar becomes a pseudoscalar, a vector becomes a bivector (and vice versa). As mentioned, an element $j\boldsymbol{n}$ is a bivector. We suggest the reader to express $j\boldsymbol{n}$ in an orthonormal basis and interpret it. Also, take any two vectors in orthonormal basis and make their outer product. Then find the dual of obtained bivector and check that this dual is just the cross product of your vectors. It follows that the cross product is

$$\boldsymbol{x} \times \boldsymbol{y} = -j\boldsymbol{x} \wedge \boldsymbol{y},$$

but, we can use it in 3D only, although the term on the right can be defined in any dimension.



multivector in 3D:

- **disk** represents an oriented bivector
- transparent **sphere** represents a pseudoscalar, orientation is given by color

p. 10

From the general form of a multivector in *Cl*3

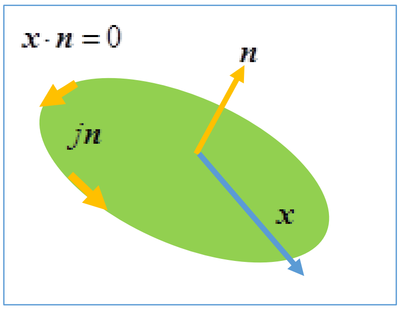
$$M = t + \boldsymbol{x} + j\boldsymbol{n} + bj = Z + \boldsymbol{F}$$

we see that it is essentially determined by two real numbers (t, b) and two vectors ($\boldsymbol{x}$, $\boldsymbol{n}$). Bivectors are usually represented by oriented disks, while the pseudoscalar can be represented by a sphere with two possible colors to give the orientation, so we can imagine a simple image that represents a multivector (p. 10). It helps a lot. Figure p. 10 is created in the program *Mathematica*. For the reader, except an imagination, we certainly suggest the *GAViewer*.

Let's look at properties of a complex scalar $\boldsymbol{F}^2 = x^2 - n^2 + 2j\boldsymbol{x}\cdot\boldsymbol{n}$. In particular, for orthogonal vectors ($\boldsymbol{x}\cdot\boldsymbol{n} = 0$) we have $\boldsymbol{F}^2 \in \Re$ and values -1, 0 and 1 are of particular interest.

Recall that $j\boldsymbol{n}$ is a bivector which defines the plane orthogonal to the vector $\boldsymbol{n}$, so, for $\boldsymbol{x}\cdot\boldsymbol{n} = 0$ the vector $\boldsymbol{x}$ belongs to that plane. This is an often used situation (e.g. a complex vector of the electromagnetic field in an empty space), so it is important to imagine a clear picture. Note that in this case the real value of $\boldsymbol{F}^2 = x^2 - n^2$ is determined by lengths of vectors $\boldsymbol{x}$ and $\boldsymbol{n}$. On the next picture you can see the situation described. There is no a special name for this kind of a complex vector in the literature (probably?), so we suggest the term *whirl* (short of whirligig).

Nilpotents and dual numbers



1) $\boldsymbol{F}^2 = 0$

This means that such a complex vector is a *nilpotent*. Let's find the general form of nilpotents in *Cl*3 (recall, $\boldsymbol{F}^2 \in \mathbb{C}$):

$$\left(Z+\boldsymbol{F}\right)^2 = Z^2 + 2Z\boldsymbol{F} + \boldsymbol{F}^2 = 0 \Rightarrow$$
$$Z = 0 \Rightarrow \boldsymbol{F}^2 = 0 \Rightarrow x = n, \quad \boldsymbol{x}\cdot\boldsymbol{n} = 0,$$

(we excluded the trivial case $\boldsymbol{F} = 0$). Notice how often we use the form $Z+\boldsymbol{F}$ to draw conclusions here, it is not a coincidence. It is a good practice to avoid habits of some authors to frequently express multivectors by components, so formulas look opaque. Here the focus is on the structure of a multivector, and that structure reflects geometrical properties.

One simple example of a nilpotent is $e_1 + je_2$ (check it). Functions with a nilpotent as an argument is easy to find using the series expansion, almost all terms just disappear. For example, from $\boldsymbol{N}^2 = 0$ follows $e^{\boldsymbol{N}} = 1 + \boldsymbol{N}$ (see below).

Nilpotents are welcome in physics, for example, an electromagnetic wave in vacuum is a nilpotent in *Cl*3 formulation, a field is a complex vector $\boldsymbol{F} = \boldsymbol{E} + j\boldsymbol{B}, \quad E = B, \quad c = 1$, here $\boldsymbol{E}$ and $\boldsymbol{B}$ are vectors of electric and magnetic field. We can define the *direction of the nilpotent* $\boldsymbol{N} = \boldsymbol{x} + j\boldsymbol{n}$ as $\hat{\boldsymbol{k}} = -j\hat{\boldsymbol{x}}\wedge\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}}\times\hat{\boldsymbol{n}}, \quad \hat{\boldsymbol{x}}^2 = \hat{\boldsymbol{n}}^2 = 1$, so we have

$$\hat{\boldsymbol{k}}\boldsymbol{N} = -\boldsymbol{N}\hat{\boldsymbol{k}} = \boldsymbol{N}, \; \left(1+\hat{\boldsymbol{k}}\right)\boldsymbol{N} = 2\boldsymbol{N}.$$

All this is not difficult to prove whether we recall that

$$\boldsymbol{x}\perp\boldsymbol{n} \Rightarrow \boldsymbol{x}\wedge\boldsymbol{n} = \boldsymbol{x}\boldsymbol{n}, \quad \hat{\boldsymbol{x}} = \boldsymbol{x}/x, \quad x = n.$$

There are many other interesting relations (see literature). These relations have a direct application in electromagnetism, for example.

Let us now comment the possibility of defining the *dual numbers*. For the nilpotent $\boldsymbol{N} = \boldsymbol{x} + j\boldsymbol{n}$ we have $x = n, \quad \boldsymbol{x}\cdot\boldsymbol{n} = 0$, so let's define a „unit nilpotent" (nilpotents have a zero MA)

$$\boldsymbol{\varepsilon} \equiv \boldsymbol{N}/x = \hat{\boldsymbol{x}} + j\hat{\boldsymbol{n}}, \quad \boldsymbol{\varepsilon}^2 = 0.$$

Now we can define dual numbers as $\alpha + \beta\varepsilon, \quad \alpha, \beta \in \Re$. The addition of these numbers is similar to complex numbers, while for multiplication we have

$$\left(\alpha_1 + \beta_1\varepsilon\right)\left(\alpha_2 + \beta_2\varepsilon\right) = \alpha_1\alpha_2 + \left(\alpha_1\beta_2 + \alpha_2\beta_1\right)\varepsilon,$$

so, for $\alpha_1\beta_2 + \alpha_2\beta_1 = 0$ it is a real number. If $\alpha_1 = 0, \; \alpha_2 = 0$ the product is zero, which distinguishes dual and complex numbers. For a dual number z specified as $z = \alpha + \beta\varepsilon$ we define the conjugation $\bar{z} = \alpha - \beta\varepsilon$ (notice, it is again just the Clifford involution), it follows

$$z\bar{z} = \left(\alpha + \beta\varepsilon\right)\left(\alpha - \beta\varepsilon\right) = \alpha^2,$$

and the module of a dual number is $|z| = \alpha$ (could be negative). Notice that there is no dependence on β. For $\alpha \neq 0$ we have the polar form

$$z = \alpha + \beta\varepsilon = \alpha\left(1 + \varphi\varepsilon\right), \quad \varphi = \beta/\alpha,$$

here φ is an *argument* of a dual number. Check that

$$\left(1+\varphi\varepsilon\right)\left(1-\varphi\varepsilon\right)=1 \quad \text{and} \quad \left(1+\varphi\varepsilon\right)^n=\left(1+n\varphi\varepsilon\right),\ \ n\in\mathbb{N}.$$

For polynomials we have (check that)

$$\begin{aligned}&P\left(\alpha+\beta\varepsilon\right)=p_0+p_1\left(\alpha+\beta\varepsilon\right)+\ldots+p_n\left(\alpha+\beta\varepsilon\right)^n=\\&P\left(\alpha\right)+\beta P'\left(\alpha\right)\varepsilon,\end{aligned}$$

where P' is the first derivation of the polynomial. This may be extended to analytic functions (see below), or to maintain the *automatic derivation*. The division by dual numbers is also defined as

$$\frac{\alpha+\beta\varepsilon}{\gamma+\delta\varepsilon}=\frac{\left(\alpha+\beta\varepsilon\right)\left(\gamma-\delta\varepsilon\right)}{\left(\gamma+\delta\varepsilon\right)\left(\gamma-\delta\varepsilon\right)}=\frac{\left(\alpha+\beta\varepsilon\right)\left(\gamma-\delta\varepsilon\right)}{\gamma^2},\quad \gamma\neq 0.$$

Especially,

$$\begin{aligned}&\frac{1}{1+\varphi\varepsilon}=\frac{1-\varphi\varepsilon}{\left(1+\varphi\varepsilon\right)\left(1-\varphi\varepsilon\right)}=1-\varphi\varepsilon\Rightarrow\\&\left(\frac{1}{1+\varphi\varepsilon}\right)^n=\left(1-\varphi\varepsilon\right)^n=1-n\varphi\varepsilon,\end{aligned}$$

and we see that the Moivre's formula is valid

$$z^n=\alpha^n\left(1+\varphi\varepsilon\right)^n=\alpha^n\left(1+n\varphi\varepsilon\right),\ \ n\in\mathbb{Z}.$$

Dual numbers are of some interest in physics, for example, let's define the special dual number („*event*“) $t+x\varepsilon$, where coordinates of time and position are introduced, and the *proper velocity* („*boost*“) as $u\equiv 1+v\varepsilon,\ u\bar{u}=1$. A speed v is the argument of the dual number, $\varphi=x/t$. It follows

$$\left(t+x\varepsilon\right)u=\left(t+x\varepsilon\right)\left(1+v\varepsilon\right)=t+\left(x+vt\right)\varepsilon=t'+x'\varepsilon,$$

which means $t'=t$, $x'=x+vt$, so we have Galilean transformations. The velocity addition rule follows immediately

$$\begin{aligned}&u=1+v\varepsilon=u_1u_2=\left(1+v_1\varepsilon\right)\left(1+v_2\varepsilon\right)=1+\left(v_1+v_2\right)\varepsilon\Rightarrow\\&v=v_1+v_2.\end{aligned}$$

Here we have a problem, namely, the velocity vector is not defined properly (there is no orientation), but if we recall a nilpotent direction, we can use it to specify the velocity vector $\mathbf{v}=-jv\varepsilon$. The proper velocity now becomes $u\equiv 1-jv\varepsilon,\ u\bar{u}=1$. For an „event“ we then use $t-jx\varepsilon$. It follows

$$\left(t-jx\varepsilon\right)\left(1-jv\varepsilon\right)=t-j\left(x+vt\right)\varepsilon=t'-jx'\varepsilon,$$

and we have Galilean transformations again.

So, as for Lorentz transformations (hyperbolic numbers, vectors and complex vectors, see below) and rotors (complex numbers, bivectors), Galilean transformations (dual numbers, nilpotents) also have a common place in the geometry of $\mathfrak{R}^3$. Because all this is a part of larger structure (*Cl*3), one can get an idea that Galilean transformations are not just approximation of Lorentz transformations for small velocities, but that have some deeper physical content, independent of a speed. But, such an idea is just due to our special choice of components of the dual number (*x*, *t*). Dual numbers like $t+x\varepsilon$ could be useful in non-relativistic physics, but certainly they are not in accordance with the special theory of relativity. In the chapter on special relativity it is shown that Galilean transformations (with rotations and Lorentz transformations) follow from simple symmetry assumptions about our world (homogeneity and isotropy). If there is a deeper physics behind this formalism then it certainly does not include an explicit space-time events. But what if we choose differently? For example, a typical nilpotent is the electromagnetic wave in vacuum. If we define $\phi = E = B$ and $\varepsilon = (\boldsymbol{E} + j\boldsymbol{B})/\phi$ one could investigate dual numbers like $\psi + \phi\varepsilon, \quad \psi \in \Re$, but then there is a question: how to interpret ψ ? According to the structure of expression it could be some sort of scalar field, but then we have another question: what is an argument of such a dual number, the ratio of a vector field (complex vector) and a scalar field should be a "velocity"? Ok, let's stop.

Idempotents and hyperbolic structure

2) $\boldsymbol{F}^2 = 1$

For $\boldsymbol{F}^2 = x^2 - n^2 = 1$ we can find a general form using the relation $\cosh^2\varphi - \sinh^2\varphi = 1$, so, generally, we have $\boldsymbol{F} \equiv \boldsymbol{f} = \boldsymbol{n}\cosh\varphi + j\boldsymbol{m}\sinh\varphi, \quad \boldsymbol{n}^2 = \boldsymbol{m}^2 = 1, \quad \boldsymbol{n} \perp \boldsymbol{m}$, where $\boldsymbol{f}$ is a unit complex vector. Example: $\boldsymbol{f} = e_1\cosh\varphi + je_2\sinh\varphi$. Such a complex vector can be obtained using $\sqrt{\boldsymbol{F}^2}$, check that the multivector $\boldsymbol{f} \equiv \boldsymbol{F}/\sqrt{\boldsymbol{F}^2}$ has requested properties. Check that $p = (1+\boldsymbol{f})/2 \Rightarrow p^2 = p$, so, we have an *idempotent*.

Theorem 1. All idempotents in *Cl*3 have the form $p = (1+\boldsymbol{f})/2$.

Proof:

$$\left(Z + \boldsymbol{F}\right)^2 = Z^2 + 2Z\boldsymbol{F} + \boldsymbol{F}^2 = Z + \boldsymbol{F} \Rightarrow Z = 1/2 \Rightarrow \boldsymbol{F}^2 = 1/4 \Rightarrow \boldsymbol{F} = \boldsymbol{f}/2.$$

Notice again the "$Z,\ \boldsymbol{F}$" form. The general form of idempotents is now

$$p = \left(1 + \boldsymbol{n}\cosh\varphi + j\boldsymbol{m}\sinh\varphi\right)/2, \quad \boldsymbol{n}^2 = \boldsymbol{m}^2 = 1, \quad \boldsymbol{n} \perp \boldsymbol{m}.$$

Idempotents like $(1+\boldsymbol{n})/2, \quad \boldsymbol{n}^2 = 1$ ($\boldsymbol{n}$ is a unit vector) are referred as *simple*.

Theorem 2: Each idempotent in *Cl*3 can be expressed as the sum of a simple idempotent and a nilpotent.

Proof:

For the simple idempotent $p = (1+\boldsymbol{n})/2$ and a nilpotent $\boldsymbol{N}$ we have

$$\left(p + \boldsymbol{N}\right)^2 = p + p\boldsymbol{N} + \boldsymbol{N}p = p + \boldsymbol{N} + \left(\boldsymbol{nN} + \boldsymbol{Nn}\right)/2,$$

so we can see that the statement is correct if $\boldsymbol{nN} + \boldsymbol{Nn} = 0$, which means that the vector $\boldsymbol{n}$ must to anti-commutate with vectors which are defining $\boldsymbol{N}$, i.e. must be orthogonal to them, or, parallel to the vector of the nilpotent direction: $\hat{\boldsymbol{n}} = \pm\hat{\boldsymbol{k}}$. The theorem is proved and we found conditions for the nilpotent.

Example: $p = (1+e_1)/2, \quad \boldsymbol{N} = (e_2 + je_3)/2$, $\hat{\boldsymbol{k}} = e_2 \times e_3 = e_1$.

Spectral decomposition and functions of multivectors

Let's define $u_{\pm}=(1\pm \boldsymbol{f})/2,\ \boldsymbol{f}^2=1$, with properties

$$u_{+}+u_{-}=1,\quad u_{+}-u_{-}=\boldsymbol{f},\ u_{+}u_{-}=u_{-}u_{+}=0,\ u_{\pm}^2=u_{\pm},\ \bar{u}_{+}=u_{-}.$$

Note that idempotents $u_{\pm}$ do not make a basis in *Cl*3 (for details about the *spectral basis* see [33]), and that we should write $\boldsymbol{f}=\boldsymbol{f}(M)$ and $u_{\pm}=u_{\pm}(M)$, but we omit that. We can express a general multivector with $\boldsymbol{F}^2\neq 0$ as

$$M=Z+\boldsymbol{F}=Z+\sqrt{\boldsymbol{F}^2}\,\boldsymbol{f}\equiv Z+Z_f\boldsymbol{f},\ \boldsymbol{f}^2=1,$$

so if we define complex scalars $M_{\pm}=Z\pm Z_f$ we get the form

$$M=M_{+}u_{+}+M_{-}u_{-}.$$

We say that we have a *spectral decomposition* of a multivector. The spectral decomposition gives us a magic opportunity

$$M^2=\left(M_{+}u_{+}+M_{-}u_{-}\right)^2=M_{+}^2u_{+}+M_{-}^2u_{-},$$

and we can immediately generalize this to any positive integer in the exponent, but to negative integers also if the inverse of the multivector exists. Prove that in the spectral basis the form $M\bar{M}=M_{+}M_{-}$ is valid.

For analytic functions we can utilize series expansion to find

$$f(M)=f(M_{+})u_{+}+f(M_{-})u_{-}.$$

Recall, to find $f(M_{\pm})$ we use the complex numbers theory, switch $j\rightarrow i=\sqrt{-1}$, find our function and switch again $i\rightarrow j$. For multivectors $M=\boldsymbol{F}=\sqrt{\boldsymbol{F}^2}\,\boldsymbol{f}$ we have

$$M_{\pm}=\pm\sqrt{\boldsymbol{F}^2}\Rightarrow f(M)=f\left(\sqrt{\boldsymbol{F}^2}\right)u_{+}+f\left(-\sqrt{\boldsymbol{F}^2}\right)u_{-}.$$

Now for even functions follows

$$f(\boldsymbol{F})=f\left(\sqrt{\boldsymbol{F}^2}\right)(u_{+}+u_{-})=f\left(\sqrt{\boldsymbol{F}^2}\right)$$

and for odd functions

$$f(\boldsymbol{F})=f\left(\sqrt{\boldsymbol{F}^2}\right)(u_{+}-u_{-})=f\left(\sqrt{\boldsymbol{F}^2}\right)\boldsymbol{f}.$$

Multivectors of form $M=z+\boldsymbol{F},\ \boldsymbol{F}^2=\boldsymbol{N}^2=0$ haven't the spectral decomposition, but using

$$M^n=(z+\boldsymbol{N})^n=z^n+nz^{n-1}\boldsymbol{N},$$

we have

$$f(z+\boldsymbol{N})=f(z)+\frac{df(z)}{dz}\boldsymbol{N}.$$

We can look at some special cases

$$f\left(u_{\pm}\right)=f\left(\pm 1\right)u_{\pm},$$

$$f\left(\boldsymbol{f}\right)=f\left(u_{+}-u_{-}\right)=f\left(1\right)u_{+}+f\left(-1\right)u_{-},$$

$$f\left(-j\boldsymbol{f}\right)=f\left(-ju_{+}+ju_{-}\right)=f\left(-j\right)u_{+}+f\left(j\right)u_{-}.$$

For the inverse function we have

$$f^{-1}\left(y\right)=x\Rightarrow f\left(x\right)=y\Rightarrow f\left(x_{\pm}\right)=y_{\pm}\Rightarrow x_{\pm}=f^{-1}\left(y_{\pm}\right).$$

If $M\bar{M}=0$ (a *light-like* multivector) we have

$$M=\mathrm{z}+\sqrt{\boldsymbol{F}}\,\boldsymbol{f}=\mathrm{z}+\mathrm{z}_{\boldsymbol{F}}\,\boldsymbol{f},\ \mathrm{z}^2-\boldsymbol{F}^2=0=\left(\mathrm{z}-\mathrm{z}_{\boldsymbol{F}}\right)\left(\mathrm{z}+\mathrm{z}_{\boldsymbol{F}}\right),$$

so we have two options:

1) $z=z_{\boldsymbol{F}}\Rightarrow M_{+}=2z_{\boldsymbol{F}},\ M_{-}=0\Rightarrow f\left(M\right)=f\left(2z_{\boldsymbol{F}}\right)u_{+},$
2) $z=-z_{\boldsymbol{F}}\Rightarrow M_{+}=0,\ M_{-}=-2z_{\boldsymbol{F}}\Rightarrow f\left(M\right)=f\left(-2z_{\boldsymbol{F}}\right)u_{-}.$

Let us now see some examples of elementary functions.

An inverse of a multivector ($M\bar{M}\neq 0$) is found easily

$$M^{-1}=\frac{1}{M_{+}u_{+}+M_{-}u_{-}}=\frac{M_{+}u_{-}+M_{-}u_{+}}{\left(M_{+}u_{+}+M_{-}u_{-}\right)\left(M_{+}u_{-}+M_{-}u_{+}\right)}=\frac{M_{+}u_{-}+M_{-}u_{+}}{M_{+}M_{-}}=\frac{u_{+}}{M_{+}}+\frac{u_{-}}{M_{-}},$$

with the power

$$M^{-n}=\frac{1}{\left(M_{+}u_{+}+M_{-}u_{-}\right)^{n}}=\frac{u_{+}}{\left(M_{+}\right)^{n}}+\frac{u_{-}}{\left(M_{-}\right)^{n}},\quad n\in\mathbb{N}.$$

The square root is simple, too (see [13] for a different form)

$$\sqrt{M}=S=S_{+}u_{+}+S_{-}u_{-}\Rightarrow M=M_{+}u_{+}+M_{-}u_{-}=\left(S_{+}\right)^{2}u_{+}+\left(S_{-}\right)^{2}u_{-}\Rightarrow S_{\pm}=\pm\sqrt{M_{\pm}},$$

or

$$M^{\pm 1/n}=S\Rightarrow S_{\pm}=\left(M_{\pm}\right)^{\pm 1/n},\ \ n\in\mathbb{N}.$$

Example:

$$\sqrt{\boldsymbol{e}_{i}}=\pm\left(j+\boldsymbol{e}_{i}\right)/\sqrt{2j}.$$

The exponential function is

$$e^{M}=e^{M_{+}}u_{+}+e^{M_{-}}u_{-},$$

so the logarithmic function is obtained as

$$\log M=X\Rightarrow e^{X}=M=M_{+}u_{+}+M_{-}u_{-}=\exp\left(X_{+}\right)u_{+}+\exp\left(X_{-}\right)u_{-}\Rightarrow X_{\pm}=\log M_{\pm}.$$

With the definition $\boldsymbol{I}\equiv\boldsymbol{F}/\left|\boldsymbol{F}\right|=-j\boldsymbol{f},\ \ \boldsymbol{I}^{2}=-1$, the logarithmic function has a form (*Chappell*)

$$\log M=\log\left|M\right|+\varphi\boldsymbol{I},\quad \varphi=\arctan\left(\left|\boldsymbol{F}\right|/Z\right),$$

but we can show that these two formulas are equivalent:

$$\log(M_+)u_+ + \log(M_-)u_- = \frac{\log(M_+)+\log(M_-)}{2} + \frac{\log(M_+)-\log(M_-)}{2}\boldsymbol{f} =$$

$$\log|M| - j\boldsymbol{I}\log\left(\sqrt{\left[1-j\left(|\boldsymbol{F}|/z\right)\right]/\left[1+j\left(|\boldsymbol{F}|/z\right)\right]}\right) = \log|M| + \boldsymbol{I}\arctan\left(|\boldsymbol{F}|/z\right) = \log|M| + \varphi\boldsymbol{I}.$$

Examples:

$$\boldsymbol{e_1}^{\boldsymbol{e_1}} = X \Rightarrow \boldsymbol{e_1}\log\boldsymbol{e_1} = \log X\,,\ \ (u_\pm = (1\pm\boldsymbol{e_1})/2),\ \ e_1 u_\pm = \pm u_\pm\,,$$

$$\boldsymbol{e_1} = u_+ - u_- \Rightarrow \log\boldsymbol{e_1} = u_+\log 1 + u_-\log(-1) = j\pi u_- \Rightarrow$$

$$\boldsymbol{e_1}\log\boldsymbol{e_1} = -j\pi u_- \Rightarrow X = \exp(-j\pi u_-) = \exp(-j\pi)u_- = -u_-\,.$$

We leave to the reader to explore possibilities, and to find expressions for trigonometric functions.

We can now take an example of the polynomial equation

$$M^2 + 1 = 0\,,$$

where solutions are all multivectors whose square is -1. We could try

$$(Z+\boldsymbol{F})^2 + 1 = 0 \Rightarrow Z^2 + 2Z\boldsymbol{F} + \boldsymbol{F}^2 + 1 = 0 \Rightarrow Z = 0 \Rightarrow \boldsymbol{F}^2 = -1\,,$$

and we know (see the next chapter) the general solution. Using the spectral decomposition we have

$$M^2 + 1 = (M_+u_+ + M_-u_-)^2 + u_+ + u_- = (M_+^2+1)u_+ + (M_-^2+1)u_- = 0 \Rightarrow$$

$$M_+^2 + 1 = 0,\ \ M_-^2 + 1 = 0\,,$$

so we get two equations with complex numbers. This was just a little demonstration of possibilities, but the reader should do complete calculations.

We have already pointed out that *Cl*3 has the complex and hyperbolic structures, the complex one due to j and other elements that square to -1, and hyperbolic due to elements that square to 1, unit vectors are hyperbolic, for example. There are also dual numbers here (using nilpotents). It is possible to efficiently formulate the special relativity theory using hyperbolic (*double*, *split-complex*) numbers, so, it should not be a surprise if it turns out that the theory is easy to formulate in *Cl*3 (see below). A unit complex vector $\boldsymbol{f}$ is the most general element of the algebra with features of hyperbolic unit. For two multivectors that have the same unit complex vector $\boldsymbol{f}$ (the same „direction“)

$$M_1 = z_1 + z_{1\boldsymbol{F}}\boldsymbol{f} \ \text{ and } \ M_2 = z_2 + z_{2\boldsymbol{F}}\boldsymbol{f}\,,$$

we can define the square of the *distance of multivectors* as

$$\bar{M}_1 M_2 \equiv (z_1 - z_{1\boldsymbol{F}}\boldsymbol{f})(z_2 + z_{2\boldsymbol{F}}\boldsymbol{f}) = z_1z_2 - z_{1\boldsymbol{F}}z_{2\boldsymbol{F}} + (z_1z_{2\boldsymbol{F}} - z_2z_{1\boldsymbol{F}})\boldsymbol{f} \equiv h_i + h_o\boldsymbol{f}\,,$$

where h_i and h_o are *hyperbolic inner* and *hyperbolic outer* products. If $M_1 = M_2 = M$ we have the *multivector amplitude*. For $h_o = 0$ we say that multivectors are *h*-parallel, while for $h_i = 0$ are *h*-orthogonal.

Lemma: *Let* $\bar{M}_1 M_2 = 0$ *for* $M_1 \neq 0$ *and* $M_2 \neq 0$. Then $M_1 M_2 \neq 0$ and vice versa.

$$M_1M_2 = (M_{1+}u_+ + M_{1-}u_-)(M_{2+}u_+ + M_{2-}u_-) = M_{1+}M_{2+}u_+ + M_{1-}M_{2-}u_-,$$

$$\bar{M}_1 M_2 = \left(M_{1+}u_- + M_{1-}u_+\right)\left(M_{2+}u_+ + M_{2-}u_-\right) = M_{1-}M_{2+}u_+ + M_{1+}M_{2-}u_-,$$

so $M_{1-}M_{2+} = 0$ or $M_{1+}M_{2-} = 0$, which means $M_{1-} = 0$ && $M_{2-} = 0$ or $M_{1+} = 0$ && $M_{2+} = 0$, but both cases imply $M_1 M_2 \neq 0$. The reverse statement is similar to prove.

What is $\sqrt{-1}$?

3) $\boldsymbol{F}^2 = -1$

Generally, this kind of a complex vector can be obtained by $\sqrt{-\boldsymbol{F}^2} = \sqrt{\boldsymbol{F}\bar{\boldsymbol{F}}} \equiv \left|\boldsymbol{F}\right|$, we have $\boldsymbol{I} \equiv \boldsymbol{F} / \left|\boldsymbol{F}\right| = -j\boldsymbol{f}, \quad \boldsymbol{I}^2 = -1$. The general form is

$$\boldsymbol{I} = \boldsymbol{n}\sinh\varphi + j\boldsymbol{m}\cosh\varphi, \quad \boldsymbol{n}^2 = \boldsymbol{m}^2 = 1, \quad \boldsymbol{n} \perp \boldsymbol{m}.$$

Note that we have a non-trivial solution for $\sqrt{-1}$. In order to further substantiate we can look for all possible solutions for $\sqrt{z} = \sqrt{c + jd}$, so we need to solve the equation $M^2 = z$. One solution is just the ordinary square root of a complex number (for $\boldsymbol{F} = 0$), but more generally

$$\left(Z + \boldsymbol{F}\right)^2 = z \Rightarrow Z^2 + 2Z\boldsymbol{F} + \boldsymbol{F}^2 = z \Rightarrow Z = 0 \Rightarrow \boldsymbol{F} = \sqrt{z} = \boldsymbol{v} + j\boldsymbol{w},$$

so

$$\sqrt{c + jd} = \boldsymbol{v} + j\boldsymbol{w} \Rightarrow c + jd = v^2 - w^2 + 2j\boldsymbol{v}\cdot\boldsymbol{w},$$

and $c = v^2 - w^2, \quad d = 2\,\boldsymbol{v}\cdot\boldsymbol{w}$. Amazing, the square root of a complex number is a complex vector (and this is expected because the square of a complex vector is a complex scalar)! The reader is proposed to explore different possibilities.

Trigonometric forms of multivectors

Recall that for $\boldsymbol{F}^2 = 0$ we defined dual numbers $z = \alpha + \beta\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^2 = 0$, $\alpha, \beta \in \Re$ and that for $\alpha \neq 0$ we found the polar form

$$z = \alpha + \beta\boldsymbol{\varepsilon} = \alpha\left(1 + \varphi\boldsymbol{\varepsilon}\right), \quad \varphi = \beta / \alpha,$$

where φ is an *argument* of the dual number.

Elements $\boldsymbol{f}$ and I can be utilized to define trigonometric forms of general multivectors. To take advantages of the theory of complex numbers we use I. So, we define the *argument of a multivector* as

$$\varphi = \arg M \equiv \operatorname{atan}\left(\frac{\left|\boldsymbol{F}\right|}{Z}\right).$$

Now we have (with conditions of existence), $|M| \equiv \sqrt{M\bar{M}}$,

$$\cos\varphi = \frac{Z}{|M|}\ ,\ \sin\varphi = \frac{|\boldsymbol{F}|}{|M|},$$

which gives

$$M = Z + \boldsymbol{F} = |M|\left(\cos\varphi + \boldsymbol{I}\sin\varphi\right).$$

Recalling that $\boldsymbol{I}^2 = -1$, the generalized Moivre's formula is valid

$$M^n = |M|^n\left[\cos(n\varphi) + \boldsymbol{I}\sin(n\varphi)\right].$$

Notice that we have a form as for complex numbers, but there is a substantial difference: the element I has a clear geometric meaning, it contains the properties that are determined by vectors which define the vector part of the multivector. Using $\boldsymbol{F} = \boldsymbol{I}|\boldsymbol{F}|$ and the series expansion we have

$$e^M = e^{Z+\boldsymbol{F}} = e^Z e^{\boldsymbol{F}} = e^Z\left(\cos|\boldsymbol{F}| + \boldsymbol{I}\sin|\boldsymbol{F}|\right),$$

which is possible due to the commutativity of a complex scalar Z . The case $\boldsymbol{F}^2 = 0$ we discussed earlier. There is an interesting article where multivector functions are defined starting right from the properties of the complex vector I ([13]).

To take advantages of the theory of hyperbolic numbers we use $\boldsymbol{f}$:

$$M = Z + \boldsymbol{F} = Z + Z_f\boldsymbol{f} = \rho\left(\frac{Z}{\rho} + \frac{Z_f}{\rho}\boldsymbol{f}\right) = \rho\left(\cosh\varphi + \boldsymbol{f}\sinh\varphi\right),\quad \rho = \sqrt{M\bar{M}} = \sqrt{Z^2 - Z_f^2}\ .$$

If $M\bar{M} = 0$ there is no polar form (*light-like* multivectors), but then we have $M = Z\left(1 \pm \boldsymbol{f}\right)$. Let's define a „velocity“ $\vartheta \equiv \tanh\varphi$, then follows

$$M = \rho\left(\cosh\varphi + \boldsymbol{f}\sinh\varphi\right) = \rho\gamma\left(1 + \vartheta\boldsymbol{f}\right),\quad \gamma^{-1} = \sqrt{1 - \vartheta^2}\ .$$

If we define the *proper velocity* $u = \gamma\left(1 + \vartheta\boldsymbol{f}\right),\ u\bar{u} = 1$, it follows the *„velocity addition rule“* as

$$\gamma_1\gamma_2\left(1 + \vartheta_1\boldsymbol{f}\right)\left(1 + \vartheta_2\boldsymbol{f}\right) = \gamma_1\gamma_2\left(1 + \vartheta_1\vartheta_2 + \left(\vartheta_1 + \vartheta_2\right)\boldsymbol{f}\right) =$$
$$\gamma_1\gamma_2\left(1 + \vartheta_1\vartheta_2\right)\left(1 + \boldsymbol{f}\left(\vartheta_1 + \vartheta_2\right)/\left(1 + \vartheta_1\vartheta_2\right)\right) \Rightarrow$$
$$\gamma = \gamma_1\gamma_2\left(1 + \vartheta_1\vartheta_2\right),\quad \vartheta = \left(\vartheta_1 + \vartheta_2\right)/\left(1 + \vartheta_1\vartheta_2\right),$$

which are formulas of the special theory of relativity. The proper velocity in a „rest reference system“ $\vartheta = 0$ is $u_0 = 1$, so we can transform to a new reference frame by $u_0 u = u$, or, as in the previous example $u_0 u_1 u_2 = u_1 u_2$. These formulas represent geometric relations and are more general than those of the special theory of relativity, namely, for SR we usually need just the real part of a multivector (*paravectors*, see next chapter), here we have bivectors too.

Using the spectral decomposition we have

$$M = \rho\gamma\left(1 + \vartheta\boldsymbol{f}\right) = k_+ u_+ + k_- u_- \Rightarrow k_\pm = \rho\gamma\left(1 \pm \vartheta\right) = \rho K^{\pm 1},$$

where (here we use $\ln x \equiv \log_e x$)

$$K=\sqrt{(1+\vartheta)/(1-\vartheta)},\quad \varphi=\ln \mathrm{K},$$

is the generalized *Bondi factor*. It follows

$$\left(K_1u_+ + u_- / K_1\right)\left(K_2u_+ + u_- / K_2\right)=K_1K_2u_+ + u_- / \left(K_1K_2\right)\Rightarrow K=K_1K_2,$$

which is the exact formula from the special theory of relativity and it is analogous to velocities addition rule.

It goes without saying that the geometric product gave us the possibility of writing "relativistic" formulas **without the use of the Minkowski space**. If Einstein knew that …

The special theory of relativity

The reader could take an advantage of the previous chapter and apply it to multivectors of form $t+\boldsymbol{x}$ (*paravectors*) and so immediately get necessary formulas. But anyway, we have a lot to comment.

The Special Theory of Relativity (SR), in its classic form, is the theory of transformations of coordinates and especially important is the concept of the velocity. Geometric algebra does not substantially depend on specific coordinates, which gives us the opportunity to consider general geometric relationships, not only relations between coordinates, which is certainly desirable because physical processes do not depend on coordinate systems in which they are formulated. Unfortunately, many authors who use geometric algebra cannot resist to use coordinates, and that makes formulas non-transparent and blurs the geometric content. It's hard to get rid of old habits. There are many texts and comments about SR, there is a lot of opponents too, which often only show a lack of understanding of the theory. So, for example, they say that Einstein "wrote nonsense" because in formulas he uses the "speed of photon" as c and $c\pm v$, not realizing important and simple fact that the speed of a photon is c in any inertial reference system. But if we want to find the time photon needs to reach the wall of the rail car that runs away from the photon (viewed from the rails system, the *collision time*) we must use $c+v$. Why? Because it is the relative velocity of the photon and the wall of the rail car in the rails system. Speed of the photon and the speed of the wall are both measured in the same reference system, so are added simple, without a relativistic addition rule. It is quite another matter when we have the man in the rail car which walks in the direction of movement of the train with the speed u, relative to the train. Velocity of the man as measured in the rails system is $(v+u)/\left(1+uv/\mathrm{c}^2\right)$, but here the speed u is measured in the train system, while the speed v (the speed of the train) is measured in the rails system. So, we use relativistic velocity addition formulas for velocities measured in different frames of reference. Quantities from one a single system of reference we are not to transform, so there is no formulas that arise from transformations (here Lorentz transformations).

Before we proceed it may be useful to clarify some terms. We say that laws of physics need to be *covariant*, meaning that in different reference frames have the same form, so, a formula $A=B$ leads to $A'=B'$. A physical quantity is a *constant* if it does not depend on coordinates, for example, number 3 or the charge of an electron. The speed of light is **not** a constant in that sense, it is an *invariant*. It means that it **depends on coordinates** ($c=\left|d\mathbf{r}/dt\right|$), but has the same value in any inertial reference frame. The speed of light is a constant of nature in the sense that it is limiting speed, but related to Lorentz transformations it is an invariant (scalar).

Another common misconception is about postulates of the special theory of relativity. Let the covariance postulate be the first and invariance of the speed of light postulate the second one. From the first we have, for example, $v = dx/dt$ and $v' = dx'/dt'$. The second postulate is mainly motivated by Maxwell's electromagnetic theory, which predicts the invariance of the speed of light in inertial reference frames. Now, it is important to note that we need **the first postulate only** to derive Lorentz transformations (LT) (it is not hard to find references, so we highly recommend to do it, see [26]). Once we have LT immediately follows the existence of the maximum speed (v_g), invariant one. It means that we don't need the second postulate to have that in the theory. Accordingly, in relativistic formulas we can use v_g instead of c. Einstein simply assumed that $v_g = c$, relaying mainly on Maxwell's theory. However, the existence of the speed limit does not necessarily mean that there must be an object that is moving at such a speed. We think that light is such an object. But we can imagine that the limit speed is 1 *mm/s* larger than c. What experiment could show the difference? But, if that were so, a photon would have to have a mass, no matter how small it was. We could then imagine a reference system that moves along with the photon, so that the photon is at rest in it. But light is a wave too, so, we would see a wave that is not moving. The wave phase would be constant to us (maximum amplitude, for example), so we couldn't see any vibrations. Now, without the change of the electric field in time, there is no magnetic field, so we see an electrostatic field. However, there is no a charge distribution in space that could create such a field (*Einstein*). So, instead of v_g we use c, but that does not mean that the assumption of the invariance of the speed of light is necessary for validity of SR. Our first postulate is certainly deeply natural and typical for Einstein, who was among the first which stressed the importance of symmetries in physics, and this is certainly the question of a symmetry. True, it is easier to make sense of the thought experiments and derive formulas using the postulate of the speed of light. It is done so in almost all textbooks, so students get the impression that there is no the theory without the second postulate. Let us also mention that there are numerous tests that confirm SR, and none (as far as is known to the author) that refutes it, although many are trying to show things differently, even make up stories about a "relativists conspiracy". Let us mention two important facts. First, quantum electromagnetic theory (QED) is deeply based on the special theory of relativity, and it is known that the predictions of QED are in unusually good agreement with experiments. Second, we have the opportunity almost every day to monitor what is happening at speeds comparable to the speed of light, namely, we have particle accelerators. They are built using formulas of the special theory of relativity, and it is really hard to imagine that they would operate if SR was not valid.

There is one more thing to discuss. Usually in textbooks is an inertial coordinate system defined as an "un-accelerated system", but that implies homogeneity, in agreement with the Newton's first law only, **not all Newton laws**, as authors state. To include the third Newton's law we have to introduce the concept of isotropy (of inertia). Why? Consider two protons at rest and let them to move freely. Then we expect that protons move in opposite orientations due to the repulsion, but we also expect that both protons have exactly the same kinematical properties. All orientations in space are equal. Without that we have not the third Newton's law. The isotropy is directly connected to the possibility to synchronize clocks. It is also natural to expect that the light speed is equal in all possible orientations (although this is not so important here, we will not use a light in the derivation of LT and clocks can be synchronized using our protons). Then we have an inertial coordinate systems (ICS) with the homogeneity and isotropy (of inertia) included. The class of inertial coordinate systems (rotated, translated) that are not moving relative to some inertial coordinate system we call the inertial reference frame (IRF). Now, with homogeneity and isotropy included we do not need the light speed postulate, symmetries are enough to obtain Lorentz transformations. Thus light loses the central role in the theory.

Let's see how to do that. Due to the linearity we expect transformations like (v is a relative velocity between systems, measured in one of systems)

$$x' = Ax + Bt \qquad t' = Cx + Dt .$$

For $x' = const$ we have $dx' = 0$, so, $B = -vA$. Inverse transformations are

$$x = \frac{Dx' - Bt'}{AD - BC} \qquad t = \frac{-Cx' + At'}{AD - BC} ,$$

then from $x = const$ we have $B = -vD$, so, $D = A$. If we denote

$$\delta = \sqrt{AD - BC} \text{ and } \lambda = A/\kappa ,\ \kappa = \frac{\lambda^2 - 1}{v^2\lambda^2}$$

we have transformations

$$x' = \delta\lambda(x - vt) \qquad t' = \delta\lambda(t - \kappa vx) ,$$

$$x = \frac{\lambda}{\delta}(x' + vt') \qquad t = \frac{\lambda}{\delta}(t' + \kappa vx') .$$

If we replace v with $-v$ these two transformations should be exchanged (due to isotropy) and we have $\delta = 1$(note that it means that transformation is orthogonal). Now we have

$$x' = \lambda(x - vt) \qquad t' = \lambda(t - \kappa vx) ,$$

$$x = \lambda(x' + vt') \qquad t = \lambda(t' + \kappa vx') .$$

From

$$\lambda = 1/\sqrt{1 - \kappa v^2} ,$$

we get general transformations in the form

$$x' = \frac{x - vt}{\sqrt{1 - \kappa v^2}} \qquad t' = \frac{t - \kappa vx}{\sqrt{1 - \kappa v^2}} .$$

Reader is encouraged to show (using three inertial coordinate systems) that $\kappa(v) = const$. Using appropriate physical units we get only three interesting possibilities for κ: -1, 0, 1. Looks familiar?

For $\kappa = -1$ we have a pure Euclidean rotation in the (x,t) plane, by the angle $\tan^{-1}(v)$. For $\kappa = 0$ we have Galilean transformations. For $\kappa = 1$ we have Lorentz transformations. Experiments in physics teaching us that we have to use $\kappa = 1$, but notice that Galilean relativity is the valid relativity theory, all of this is a consequence of our definition of the ICS. The direct consequence of Lorentz transformations is existence of the maximum speed, but we discussed this already.

Recall that we have already seen numbers -1, 0, 1 here in text, we discussed rotations, dual numbers and hyperbolic numbers obtained from general multivectors in *Cl*3.

P*aravectors* in *Cl*3, like $t + \boldsymbol{x}$ (a multivector with grades 0 and 1), give a paravector again when squared (check it), therefore the module of a paravector is to be defined differently. For complex and hyperbolic numbers (or quaternions) we have a similar obstacle, so we use conjugations. For paravectors we don't need any *ad hoc conjugation* as we already have the Clifford involution, so we define

$$p\bar{p} = (t + \boldsymbol{x})(t - \boldsymbol{x}) = t^2 - x^2 \in \Re ,$$

which is exactly the desired form of an invariant interval required in the special theory of relativity. Recall that the Clifford involution is combination of *grade* involution and *reverse* involution, so we can try to interpret it geometrically in $\Re^3$, namely, the grade involution means a space inversion, while for the reverse involution we have seen that it is related to the fact that Pauli matrices are Hermitian.

To be clear, if we specify the paravector $\alpha+\beta e_1$, with $e_1^2=1$ we have a natural „hyperbolic unit". It follows

$$\left(\alpha+\beta e_1\right)^2=\alpha^2+\beta^2+\alpha\beta e_1,$$

so, we have a paravector again, with the same direction of vector, but

$$\left(\alpha+\beta e_1\right)\left(\alpha-\beta e_1\right)=\alpha^2-\beta^2\in\Re.$$

Notice that with the Clifford involution there is no need for a negative signature (Minkowski). According to the Minkowski formulation of SR we can define the unit vector „in the time direction" $e_0,\quad e_0^2=1$ and three space vectors $e_i,\quad e_i^2=-1$, which means that we have a negative signature (1, -1, -1, -1). Such an approach is possible in geometric algebra, too, we have STA (space-time algebra, *Hestenes*). But, everything you can do with STA you can do in *Cl*3 also, without the negative signature (*Sobczyk, Baylis*). Those who argue that the negative signature is necessary in SR are maybe wrong. Some authors write sentences like: „*The principle of relativity* ***force us*** *to consider the scalar product with negative square of vectors*", forgetting that their definition of norm of elements prejudice such a result (*Witte*: *Classical Physics with geometric algebra*). Yet, it is possible to describe a geometry in one space using formalism of higher space, so we can say that the Minkowski geometry formulation of SR is a 3D problem described in 4D. But in *Cl*3, all we need are three orthonormal vectors and one involution. Time is not a fourth dimension any more, it is just a real parameter (as is in the quantum mechanics). If there is a fourth dimension of time how it is that we cannot move through the time as we move through the space? There are other interesting arguments in favor of the 3D space, for example, gravitational and electrostatic forces depend on the square of the distance. And what about definition of velocity (we use it also in the theory of relativity): dx/dt? If there is a time dimension then time is a vector, which means that the speed is naturally a bivector, like a magnetic field, not a vector. It does not matter if we use a proper time to define the four-velocity vector, the space velocity is still defined by the previous formula, up to a factor. Minkowski gave us a nice mathematical theory, but his conclusion about the fourth time dimension was pure mathematical abstraction, widely accepted among physicist. At that time, geometric ideas of Grassmann, Hamilton and Clifford were largely suppressed. This begs us to question what would Einstein choose if he knew that? At the beginning of the 20th century an another important theory was developing, the quantum mechanics, where Pauli introduces his matrices to formulate the half spin, we already commented it. Dirac's matrices are also representation of one Clifford algebra, but again, Dirac's theory has a nice formulation in *Cl*3 (*Baylis*), as well as the minimal standard model in *Cl*7 (*Baylis*) ... It is not without grounds to question the merits of introducing time as a fourth dimension. A usual argument is one that Minkowski gave, in fact, this is not an argument, it is just the observation that in the special theory of relativity an invariant interval is not dt^2+dx^2 but dt^2-dx^2. But we see that the invariant interval dt^2-dx^2 is easy to get in *Cl*3, with completely natural requirements for a multiplication of vectors. Minkowski has introduced a fourth dimension *ad hoc*. If his formalism was undoubtedly the **only possible** to formulate the special theory of relativity then there would be a solid base to believe that indeed there must be a fourth dimension of time. Thus, without that condition, with the knowledge that there is a natural way to formulate the theory without the fourth dimension, it is difficult to avoid the impression that this widely accepted mantra of fourth dimension does not have a solid foundation. According to some authors, one of the stumbling blocks in the theory of quantum gravity is probably the existence of a fourth dimension of time in the formalism. Here we develop a formalism using paravectors which define the 4D linear space, but time is identified as a real scalar, we say that time is a real parameter. It would be interesting to investigate whether there is any experiment that would unambiguously prove the existence of a fourth dimension of time. Probably, there is no such an experiment. Therefore, it is difficult to avoid the impression how physicists are binding a ritual cat during the meditation. But the future will show, perhaps the time dimension does exist, maybe more of them (if time exists). In any case, it is not true that the Minkowski space is the only correct framework for the formulation of SR. Especially, it is **not true** that in SR we **must** introduce vectors whose square is negative.

We'll use a system of physical units in which is $c=1$. In geometric algebra we are combining different geometric objects which may have different physical units. Therefore we always choose the system of units such that all is reduced to the same physical unit (usually the length). So we study geometric relationships, and that is the goal here. In an application to a particular situation (experiment) physical units are converted (analysis of physical units), so that there is no problem here.

Starting from the invariant interval in SR $t^2-x^2=\tau^2$, where τ is the invariant *proper time* in the particle rest frame, it follows

$$t^2-x^2=t^2\left(1-v^2\right)=\tau^2 \Rightarrow t^2/\tau^2=1/\left(1-v^2\right)=\gamma^2,$$

where γ is well known relativistic factor. Now, instead of the four-velocity vector, we define the *proper velocity* (paravector) $u\equiv\gamma\left(1+\boldsymbol{v}\right)$ which is simply $u_0=1$ in the rest frame. Notice that the proper velocity is not a list of coordinates, like a four-velocity vector, but plays the same role. Obviously, $u\bar{u}=1$. Let us imagine that a body initially at rest we want to analyze in a new reference frame in which the body has a velocity $\boldsymbol{v}$ (boost). Recipe is very simple: just make geometric product of two proper velocities $u_0\rightarrow u_0u=u$. For the series of boosts we have a series of transformations

$$u_0\rightarrow u_0u_1\rightarrow u_0u_1u_2=u_1u_2.$$

Notice that this is really easy to calculate, and that from the form of the proper velocity paravector we immediately see the relativistic factor γ and the 3D velocity vector $\boldsymbol{v}$. For example, let's specify that all velocity vectors are parallel to e_1, then

$$\gamma_1\left(1+v_1e_1\right)\gamma_2\left(1+v_2e_1\right)=\gamma_1\gamma_2\left(1+v_1v_2+\left(v_1+v_2\right)e_1\right)=\gamma_1\gamma_2\left(1+v_1v_2\right)\left(1+\frac{v_1+v_2}{1+v_1v_2}e_1\right),$$

so, from the form of the paravector (parts are colored in red) we immediately see that

$$\gamma=\gamma_1\gamma_2\left(1+v_1v_2\right),\quad \boldsymbol{v}=\frac{v_1+v_2}{1+v_1v_2}e_1,$$

known results of the special theory of relativity (relativistic velocity addition). Notice how the geometric product makes the derivation of formulas easy and, as stated earlier, obtained formulas are just special cases of general formulas in *Cl*3. So, from the polar form of general multivector

$$M=\rho\left(\cosh\varphi+\boldsymbol{f}\sinh\varphi\right)=\rho\gamma\left(1+\vartheta\boldsymbol{f}\right),\quad \gamma^{-1}=\sqrt{1-\vartheta^2},$$

reducing to the real part of a multivector (paravector) we have

$$\gamma=\cosh\varphi\ ,\ \gamma v=\sinh\varphi,\quad u=\cosh\varphi+\hat{\boldsymbol{v}}\sinh\varphi=\cosh\varphi\left(1+\hat{\boldsymbol{v}}\tanh\varphi\right)=\exp\left(\varphi\hat{\boldsymbol{v}}\right).$$

Using the spectral decomposition we have

$$\gamma\left(1+v\hat{\boldsymbol{v}}\right)=k_+u_++k_-u_-\Rightarrow k_\pm=\gamma\left(1\pm v\right)=\cosh\varphi\pm\sinh\varphi,$$

defining implicitly the factor k (*Bondi factor*) $\varphi\equiv\ln k$ and recalling definitions of hyperbolic sine and cosine we get

$$k^{\pm 1} = \cosh\varphi \pm \sinh\varphi, \quad k = \sqrt{(1+v)/(1-v)}, \quad u = ku_+ + k^{-1}u_-.$$

Our earlier example with two "boosts" parallel to e_1 now has the form

$$u_1 u_2 = (k_1 u_+ + u_- / k_1)(k_2 u_+ + u_- / k_2) = k_1 k_2 u_+ + u_- / (k_1 k_2),$$

i.e. the relativistic velocity addition rule is equivalent to the multiplication of the Bondi factors: $k = k_1 k_2$.

Example: In the referent frame S_1 the starship has velocity v, in the referent frame of the starship another starship has velocity v and so on, all in the same direction. Find the velocity v_n of the *n*-th starship in S_1. Discuss a solution for $n \to \infty$?

Solution:

Let $k_1 = \sqrt{(1+v)/(1-v)}$, then $k_n = \sqrt{(1+v_n)/(1-v_n)} = k_1^n = \left(\sqrt{(1+v)/(1-v)}\right)^n$, whence we find the required velocity v_n.

If velocity vectors do not lie in the same direction, in expressions appears the versor $\boldsymbol{v}_1\boldsymbol{v}_2$, which may seem like a complication, but actually provides new opportunities for elegant research, for example, it is rather easy to get the Thomas precession (see [14]), for some time unnoticed, but the scope of this text seeks to stop here.

Lorentz transformations

We are now ready to comment on restricted Lorentz transformations (LT). Generally, LT consists of „boosts“ B and rotors R. We can write (see [22]), quite generally $L = BR$, $L\bar{L} = 1$ (the *unimodularity condition*). Here we can regard this condition as the *definition* of Lorentz transformations, which is well researched and justified. If we define (see above)

$$B = \cosh(\varphi/2) + \hat{\boldsymbol{v}}\sinh(\varphi/2) = e^{\varphi\hat{\boldsymbol{v}}/2}, \quad R = \cos(\theta/2) - j\hat{\boldsymbol{w}}\sin(\theta/2) = e^{-j\hat{\boldsymbol{w}}\theta/2},$$

(the unit vector $\hat{\boldsymbol{w}}$ defines the rotation axis) we can write LT of some element, say vector, as

$$p' = LpL^\dagger = BRpR^\dagger B^\dagger.$$

There is a possibility to write L as

$$L = e^{\varphi\hat{\boldsymbol{v}}/2 - j\hat{\boldsymbol{w}}\theta/2} \neq e^{\varphi\hat{\boldsymbol{v}}/2} e^{-j\hat{\boldsymbol{w}}\theta/2},$$

where we have to be careful due to a general non-commutativity of vectors in the exponent (see [19]). However, it is always possible to find (using logarithms) vectors $\hat{\boldsymbol{v}}'$ and $\hat{\boldsymbol{w}}'$ that satisfy

$$L = e^{\varphi\hat{\boldsymbol{v}}/2 - j\hat{\boldsymbol{w}}\theta/2} = e^{\varphi\hat{\boldsymbol{v}}'/2} e^{-j\hat{\boldsymbol{w}}'\theta/2}.$$

It is convenient in applications to resolve an element into components parallel and orthogonal to $\hat{\boldsymbol{v}}$ or $\hat{\boldsymbol{w}}$ and take the advantage of commutation properties. For further details see Baylis articles about APS (algebra of physical space, *Cl*3). In [22] you can find a nice chapter about the special theory of relativity.

We see that rotations are natural part of LT, so, geometric algebra formalism can provide a lot of opportunities because of powerful rotor techniques. Later in the text we will discuss some powerful techniques with spinors (*eigenspinors*).

Extended Lorentz transformations. Speed limit?

This chapter is **speculative**, with interesting consequences (**new preserved quantities and a change of the speed limit in nature**). Those faint hearted can take this as just a mathematical exercise.

Earlier we defined MA as

$$M\bar{M} = |M|^2 = t^2 - x^2 + n^2 - b^2 + 2j\left(tb - \boldsymbol{x}\cdot\boldsymbol{n}\right) \in \mathbb{C},$$

and showed its properties. Now we look for a general bilinear transformation $M' = XMY$ that preserves MA (see [11]):

$$M' = XMY \Rightarrow M'\bar{M}' = XMY\bar{Y}\bar{M}\bar{X} = |M|^2 |X|^2 |Y|^2,$$

so we have possibilities

$$|X|^2 = |Y|^2 = \pm 1,$$

which gives

$$X = e^{Z+\boldsymbol{F}} \Rightarrow |X|^2 = e^{Z+\boldsymbol{F}} e^{Z-\boldsymbol{F}} = e^{2Z} = \pm 1$$

and we will choose (for now) the possibility 1 and *Z* = 0, although we could consider $Z = j\pi/2$, too. Now the general transformation is given by

$$M' = XMY = e^{\vec{p}+j\vec{q}} M e^{\vec{r}+j\vec{s}},$$

so we have 12 parameters from four vectors in exponents.

The question is what a motive for the consideration of such transformations we have. Elements of geometric algebra are linear combinations of unit blades of Clifford basis, each of which actually defines a subspace. If we limit ourselves to the real part of multivectors only (paravectors) we put in a privileged position space of real numbers (grade 0) and vectors (grade 1). The idea is that all subspaces we treat equally. In fact, this whole structure is based on a new multiplication of vectors, so, manipulating multivectors we actually manipulate subspaces. Addition of vectors and bivectors is actually an operation that relates subspaces and it is important to understand it well. If subspaces are treated equally, then we must consider all possible transformations of subspaces and all possible symmetries and they are more than what classical (restricted) Lorentz transformations imply. The reader should be able to stop a little and think carefully about this. Remember that symmetries in the flow of time give the law of conservation of energy, the translational invariance gives the law of conservation of momentum, etc. Where we to stop, and why? If we truly accept the naturalness of the new multiplication of vectors we must accept the consequences of such a multiplication, too and they reveal an unusually rich structure of our good old 3D Euclidean space. But true, the final judgement will be given by experiments (hope).

Considering the invariant MA expressed in two reference frames we can compare the real and the imaginary parts

$$t^2 - x^2 + n^2 - b^2 = t'^2 - x'^2 + n'^2 - b'^2,$$

$$tb - \boldsymbol{x}\cdot\boldsymbol{n} = t'b' - \boldsymbol{x}'\cdot\boldsymbol{n}'.$$

Differential of the multivector

$$dX = dt + d\boldsymbol{x} + jd\boldsymbol{n} + jdb$$

gives MA

$$|dX|^2 = dt^2 - dx^2 + dn^2 - db^2 + 2j\left(dbdt - d\boldsymbol{x}\cdot d\boldsymbol{n}\right)$$

and we can try to find conditions for the existence of the **real** proper time. There are many reasons to define a real proper time, for example, makes it easy to define a generalized velocity. Typically, in the special theory of relativity we will chose the rest frame. Here, due to additional elements (except the velocity), it will not be enough, because we want (τ is a proper time)

$$|dX|^2 = dt^2 - dx^2 + dn^2 - db^2 + 2j\left(dbdt - d\boldsymbol{x}\cdot d\boldsymbol{n}\right) = d\tau^2 \in \Re .$$

The first condition, if we want a real proper time, is certainly the disappearance of the imaginary part of the MA in each system of reference (recall that the MA is invariant to our transformations and cannot have an imaginary part in one reference frame and not in the other). This means that in every reference frame must be valid

$$dbdt - d\boldsymbol{x}\cdot d\boldsymbol{n} = dt^2\left(d\dot{b} - d\dot{\boldsymbol{x}}\cdot d\dot{\boldsymbol{n}}\right) = dt^2\left(h - d\dot{\boldsymbol{x}}\cdot d\dot{\boldsymbol{n}}\right) = 0 \Rightarrow h = d\dot{\boldsymbol{x}}\cdot d\dot{\boldsymbol{n}}, \quad d\dot{b} \equiv h,$$

with a common designation $dx/dt \equiv \dot{x}$, which implies $h' = d\dot{\boldsymbol{x}}'\cdot d\dot{\boldsymbol{n}}'$. If we define $d\dot{\boldsymbol{x}} \equiv \boldsymbol{v}$, $\dot{\boldsymbol{n}} \equiv \boldsymbol{w}$, it follows $h = \boldsymbol{w}\cdot\boldsymbol{v}$. The vector $\boldsymbol{w}$ comes from the bivector part of the multivector, so we expect it to be related to angular momentum-like quantities, then h could be a flow of such a quantity, much like flow is defined for the flowing of a liquid through a tube. The difference is that here bivectors do not transform as surfaces (see [11]).

Considering the invariance of MA and the proper time as a real number we have

$$|dX|^2 = |dX'|^2 = d\tau^2 = dt^2 - dx^2 + dn^2 - db^2 \Rightarrow$$

$$1 = \frac{dt^2}{d\tau^2}\left(1 - \frac{dx^2}{dt^2} + \frac{dn^2}{dt^2} - \frac{db^2}{dt^2}\right) = \gamma^2\left(1 - v^2 + w^2 - h^2\right),$$

$$\gamma = 1/\sqrt{1 - v^2 + w^2 - \left(\boldsymbol{w}\cdot\boldsymbol{v}\right)^2} = 1/\sqrt{1 - v^2 + w^2 - w^2v^2\cos^2\alpha} .$$

Note that our relativistic factor γ now has contributions from all subspaces. It would be natural to require that the „rest frame“ (with the condition $v = 0$) be replaced by $\gamma = 1$, which would mean that there is no resting particles, but

$$-v^2 + w^2 - w^2v^2\cos^2\alpha = 0 \Rightarrow v = w/\sqrt{1 + w^2\cos^2\alpha} .$$

It is not so difficult to accept this, because what if the velocity of the particle may not be zero? For example, how to reconcile principles of quantum mechanics and the idea of completely peaceful electrons? Including all subspaces and all quantities related to them it follows that a „rest frame“ becomes something like a „center of energy-impulse-angular momentum, etc. frame“.

The relativistic factor γ is defined as the ratio of two real times, so it must be a real number, which gives us condition

$$1 - v^2 + w^2 - w^2v^2\cos^2\alpha > 0 \Rightarrow v_{\max} < \sqrt{\frac{1 + w^2}{1 + w^2\cos^2\alpha}} .$$

This is a completely new result: the limit speed is 1 for $w = 0$ or $\cos\alpha = \pm 1$, otherwise it is greater than 1. This result is not new in geometric algebra (*Pavšić*, using C-algebras, but the author got this result independently, in the comment on the article [12]). What could be the physical meaning of this? Consider an electron, it certainly isn't a „small ball“ (recall the great Ruđer Bošković and his points as the source of force), namely, it has the spin, and the spin is just „like angular momentum“ quantity.

Can we treat the relativistic electron as an Einstein's relativistic train? Probably not! In the eyes of geometry it is hard to accept that electron is just a very little ball with the spin packed in it, like a train with passengers in it. Relativistic formulas for a train do not depend on the type of cargo in it. But spin is probably not just a “cargo”, rather, it is a geometric property, so it should be a part of transformations. **If** just derived formulas are applicable to the electron then its (limit) speed would depend on the orientation of the spin relative to the velocity vector. And more, the speed for a given energy would depend on the orientation of the spin, so electrons with the same energy are supposed to arrive at the target with different times. Maybe someone in the future will carry out such an experiment, a positive result would certainly significantly change our current understanding of the relativity. Especially interesting would be to see how electrons behave in quantum tunneling, because there are suggestions of some authors that an electron might be moving with speeds exceeding 1. This is sometime formulated by introducing complex numbers, making the whole philosophy about it, although it is likely the matter of an inappropriate mathematics. But, it is hard to be sure.

Now that we have defined (invariant, real) proper time we can define the multivector of generalized velocity

$$V=\frac{dX}{d\tau}=\frac{dt}{d\tau}+\frac{d\boldsymbol{x}}{dt}\frac{dt}{d\tau}+j\frac{d\boldsymbol{n}}{dt}\frac{dt}{d\tau}+j\frac{db}{dt}\frac{dt}{d\tau}=\gamma\left(1+\boldsymbol{v}+j\boldsymbol{w}+jh\right),$$

with the invariant amplitude

$$V\bar{V}=1\Rightarrow\frac{d\left(V\bar{V}\right)}{d\tau}=\frac{dV}{d\tau}\bar{V}+V\frac{d\bar{V}}{d\tau}=0,$$

which is a kind of expression of the orthogonality of the generalized velocity and the generalized acceleration. Multiplying the generalized velocity with a mass we get the generalized momentum

$$P=mV=E+\boldsymbol{p}+j\boldsymbol{l}+jH,\quad P\bar{P}=E^2-p^2+l^2-H^2=m^2.$$

This is very different from the usual formula for energy-momentum $E^2-p^2=m^2$. Two additional conserved quantities appear, the last of which (H) is a brand new ([11]). Under the terms of our derivation must be $H=\gamma mh=\boldsymbol{l}\cdot\boldsymbol{v}$, so the new conserved quantity has a form of *flow*, and we have finally

$$P\bar{P}=E^2-p^2+l^2-\left(\boldsymbol{l}\cdot\boldsymbol{v}\right)^2=m^2.$$

If this is physical, the motion of particles with a spin should satisfy the law of conservation of flow, an idea that had been already presented by some authors (unfortunately, I forgot the source!). The speed of a particle will be generally higher with $\boldsymbol{w}$ than without it for a given energy, which can be deduced from the previous formula: adding a positive term to the negative square of momentum gives the possibility to increase the speed. Let's make sure about it directly

$$\gamma=E/m=1/\sqrt{1-v^2+w^2-\left(\boldsymbol{w}\cdot\boldsymbol{v}\right)^2}\Rightarrow$$

$$v=\sqrt{\frac{1+w^2-m^2/E^2}{1+w^2\cos^2\alpha}}=\sqrt{\frac{1+\left(l/E\right)^2-m^2/E^2}{1+\left(l/E\right)^2\cos^2\alpha}}\geq\sqrt{1-m^2/E^2}.$$

We should note at the end that just discussed formalism reduces to the usual special theory of relativity, it is sufficient to reject the imaginary part of multivectors and keep the real one, i.e. paravectors. All “strange” implications just disappear. Of course, there is a plenty of possibilities to treat a time differently, but we will stop here (see ([11] and ([12] for some interesting discussions).

Electromagnetic field in geometric algebra

Here we will not describe the entire EM theory in *Cl*3 (see *Hestenes*, *Baylis*, *Chappell*, *Jancewicz*), we only comment on a few ideas. In the geometric algebra formalism, for electromagnetic (EM) wave in vacuum we define

$$E = B,\ \boldsymbol{E} \perp \boldsymbol{B},\ \ \boldsymbol{F} = \boldsymbol{E} + j\boldsymbol{B} \Rightarrow \boldsymbol{F}^2 = 0,\quad c = 1,$$

so the complex vector $\boldsymbol{F}$ is a *nilpotent*. Note that the term with magnetic field is a bivector. It is useful to expand the magnetic field bivector in an orthonormal basis, so, if we start with the magnetic field vector

$$\boldsymbol{B} = B_1 e_1 + B_2 e_2 + B_3 e_3$$

we get the bivector

$$j\boldsymbol{B} = j\left(B_1 e_1 + B_2 e_2 + B_3 e_3\right) = B_3 e_1 e_2 + B_2 e_3 e_1 + B_1 e_2 e_3\,.$$

The reader should check this simple expression and try to create a mental image of it. Also, we can represent bivectors using parallelograms, it is straightforward to see how to add them graphically (figure next to the main title). The *GAViewer* can help here. Although this may seem like "just a neat trick", here we are going to try to show that the bivector, as a geometric object, is fully adequate for the description of the magnetic field, actually, physical properties of the magnetic field **require** to be treated as a bivector. In any formalism that does not imply the existence of bivectors (as for the Gibbs vector formalism in the language of the scalar and cross products) problematic situations must necessarily occur. Here we will discuss the issue of the Maxwell's theory and the mirror symmetry, as an example. If we use a coordinate approach then in 3D we can define a richer structure by introducing tensors. Let's look at a quote from the article by Jancewicz: *A system of units compatible with geometry*, 2004:

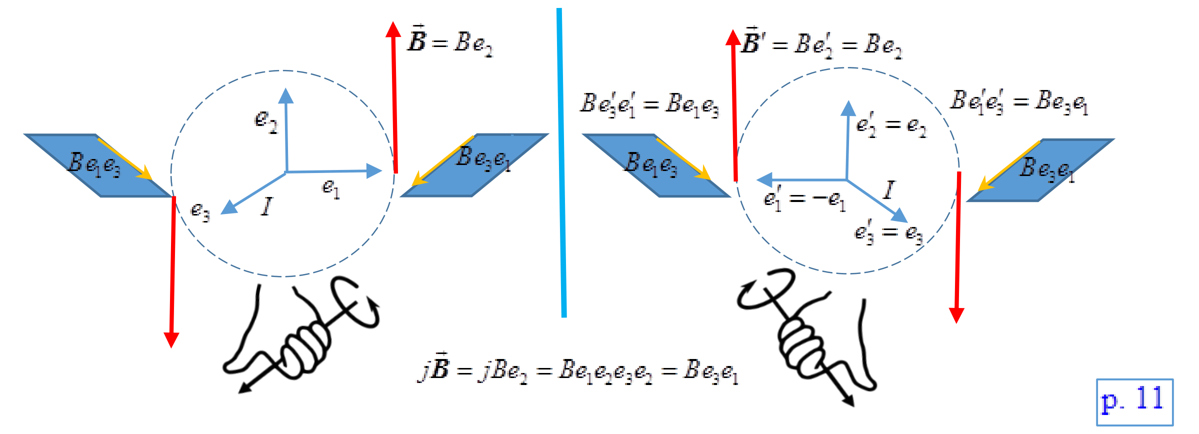
„For the three-dimensional description, antisymmetric contravariant tensors are needed of ranks from zero to three (they are known as multivectors) as well as antisymmetric covariant tensors of ranks from zero to three (called exterior forms). It makes altogether eight types of directed quantities.“

So, for example, axial vectors (like a cross product) become antisymmetric tensors of the second rank. This whole geometric structure becomes simple and intuitive in geometric algebra, without the need for introducing any coordinates (here we often introduce a basis, but it is solely for the purpose of easier understanding of the text and it is not necessary). Due to the independence of the basis, ceases to be important, for example, if we work in the right or the left coordinate system, geometric product takes care of everything. For two vectors we could have expressions like $\boldsymbol{ab} \pm \boldsymbol{ba}$ and we do not need to use any basis to conclude that it is a scalar or a bivector, in any dimension.

The Maxwell's electromagnetic theory is the first physical theory which initially met the postulate(s) of the special theory of relativity. It is therefore no wonder that both theories fit perfectly in *Cl*3. Let's look at some interesting facts related to the theory in the language of geometric algebra. For example, we can visualize solutions for an electromagnetic wave in vacuum ([22]) by a simple and interesting picture, namely, a wave vector $\boldsymbol{k}$ is parallel to the direction vector of the nilpotent $\boldsymbol{F}$, so solution can be written (for $\boldsymbol{k} \parallel \boldsymbol{x}$) as

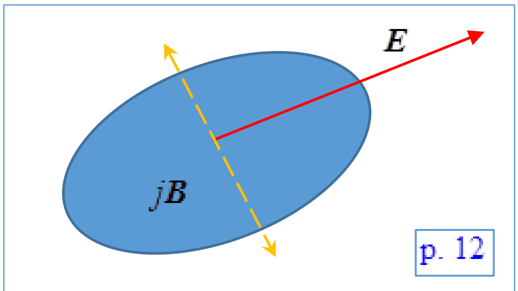
$$\boldsymbol{F}_0 e^{\pm jkx} e^{j\omega t},\quad \boldsymbol{F}_0 = \boldsymbol{E}_0 + j\boldsymbol{B}_0\,.$$

We can imagine the spatial part of the wave $F_0 e^{\pm jkx}$ as spatial periodic **steady** spiral which extends along the direction of the wave propagation. This "spiral" is the nilpotent here because it is proportional to F_0. It turns out (see below) that rotation of a nilpotent around his direction can be achieved by multiplying it by a complex phase, like $e^{j\omega t}$, so, we have a spiral in space which rotates around the direction of propagation of the wave. The bivector part $j\boldsymbol{B}$ defines the plane orthogonal to $\boldsymbol{B}$, so the vector $\boldsymbol{E}$ belongs to that plane. The bivector $j\boldsymbol{B}$ provides an opportunity for consideration of electromagnetic phenomena more complete and more elegant than a (axial) vector $\boldsymbol{B}$.



p. 11

Let's look at some more properties of EM fields in vacuum. Maxwell's equations are completely mirror-symmetric in the language of geometric algebra, as well as their solutions. When we use the cross product we immediately need to introduce the right hand rule, and we see that we have the left hand rule in the mirror. If we set the figures (the original and the one in the mirror, p. 11) one over another they do not match, vectors are standing wrong. However, if vectors (axial) of the magnetic field are replaced by bivectors, images exactly match. And, of course, we don’t need the right hand rule, as stated, the geometric product takes care of everything. It is clear that for those who have thought for a long time in the language "vectors-right hand rule" will be difficult to accept a new paradigm. We're taught to imagine arrows, so we have yet to develop an intuition for objects that are not vectors or scalars. Geometry is the language of physics more than we dreamed of.



p. 12

The vector $\boldsymbol{E}$ belongs to the plane defined by bivector $j\boldsymbol{B}$. The area of the circle which presents the bivector is B, so its radius must be $\sqrt{B/\pi} \approx 0.56\sqrt{B}$. The direction and possible orientations of the wave propagation are plotted in p. 12.

This image is rotated about the wave propagation axis by an angle dependent on the position ($\boldsymbol{k}\cdot\boldsymbol{x}$), which gives a static image in space. It has been said, this whole spatial image rotates in time, depending on the frequency of the wave (ω). This last assertion reader can check itself if he takes a simple nilpotent $e_1 + je_2$ and multiply it by a complex phase $e^{j\varphi}$. Immediately we get the matrix of the rotation around the z-axis. We can rotate elegantly the whole picture, so this particular example is not special in any way.

For any complex vector $\boldsymbol{F} = \boldsymbol{v} + j\boldsymbol{w}$ in *Cl*3 we have

$$FF^{\dagger} = (\boldsymbol{v} + j\boldsymbol{w})(\boldsymbol{v} - j\boldsymbol{w}) = v^2 + w^2 - 2j\boldsymbol{v} \wedge \boldsymbol{w} = v^2 + w^2 + 2\boldsymbol{v} \times \boldsymbol{w},$$

so if we use the complex vector of the electromagnetic field in vacuum (nilpotent) $\boldsymbol{F} = \boldsymbol{E} + jc\boldsymbol{B}$, where we use SI system of units for a moment, we get

$$FF^{\dagger} = E^2 + c^2B^2 + 2c\boldsymbol{E} \times \boldsymbol{B}.$$

Now we have

$$\frac{1}{2}\varepsilon_0 c FF^{\dagger} = c\xi + \boldsymbol{S},$$

with

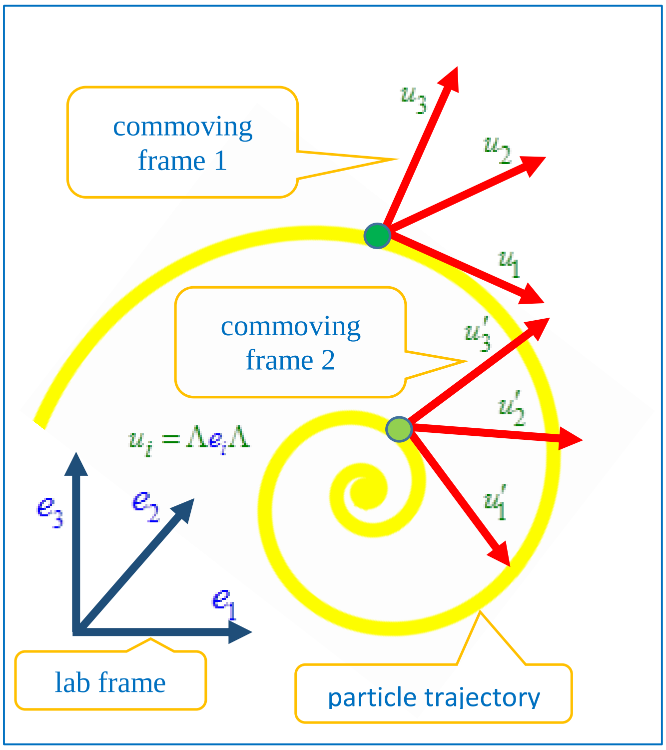
$$\xi = \frac{1}{2}\varepsilon_0\left(E^2 + c^2B^2\right),$$
$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B} / \mu_0.$$

Here ξ is the energy density and $\boldsymbol{S}$ is the energy-current density (energy flow), known as *Poynting vector*. So, the Poynting vector is proportional to the nilpotent direction vector. Note also that generally

$$\boldsymbol{F}^2 = E^2 - c^2B^2 + 2jc\boldsymbol{E} \cdot \boldsymbol{B}$$

is a complex scalar which we can use to classify fields (it is zero in vacuum).

Eigenspinors



Let's look at one rather elegant and powerful way to describe a motion of relativistic particles ([5], [6], [7]). Imagine the laboratory reference frame and the frame that is fixed to the particle in motion under the influence of, for example, an electromagnetic field. We shall consider here paravectors and restricted Lorentz transformations only. At any moment we can find a transformation which transforms elements from the lab frame to the inertial frame of reference that coincides with the particle movement (*commoving frame*). The proper velocity in the lab frame is $u_0 \equiv e_0 = 1$, so, for that very instant of time we can get the proper velocity of the particle as

$$u = \Lambda e_0 \Lambda^{\dagger},$$

where Λ is the Lorentz transformation, named *eigenspinor* due to the special choice of the reference frame. Let us mention passing that applying such a transformation to orthonormal basis vectors $u_\mu = \Lambda e_\mu \Lambda^{\dagger}$, $\mu = 0, 1, 2, 3$ we get so-called *Frenet tetrad*. Recall, for

Lorentz transformations we have $\Lambda\bar{\Lambda}=1$ (unimodularity). If an eigenspinor Λ is known at every instant of time we have all information needed to describe the particle movement. Eigenspinors are changing in time, so we need the first time derivative

$$\dot{\Lambda}=\dot{\Lambda}\bar{\Lambda}\Lambda\equiv\boldsymbol{\Omega}\Lambda/2,\quad \boldsymbol{\Omega}=2\dot{\Lambda}\bar{\Lambda}.$$

This all seems like a trivial relation, but it is not. We have

$$\frac{d}{dt}\left(\Lambda\bar{\Lambda}\right)=0=\dot{\Lambda}\bar{\Lambda}+\Lambda\dot{\bar{\Lambda}},$$

and using $\overline{\dot{\Lambda}\bar{\Lambda}}=\Lambda\dot{\bar{\Lambda}}=-\dot{\Lambda}\bar{\Lambda}$ we see that $\boldsymbol{\Omega}$ is a complex vector (so it is in the bold format). For the first time derivative of the proper velocity we have

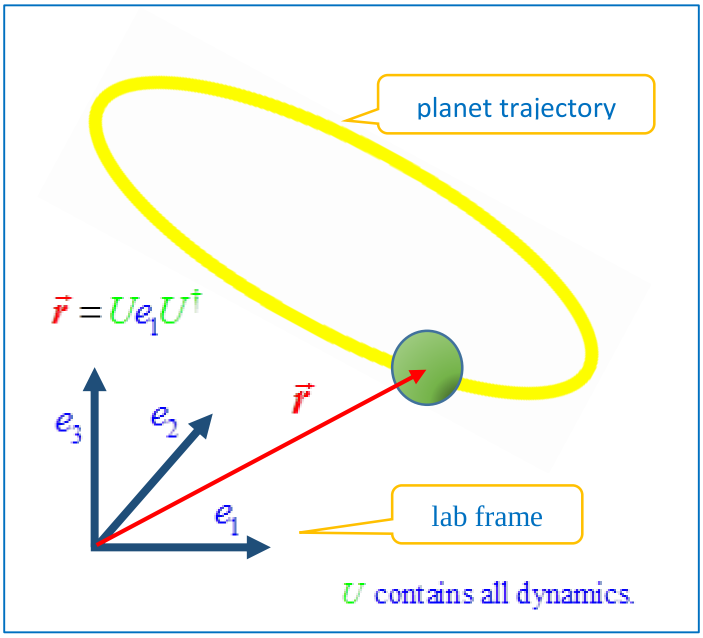
$$\dot{u}=\dot{\Lambda}e_0\Lambda^{\dagger}+\Lambda e_0\dot{\Lambda}^{\dagger}=\dot{\Lambda}\bar{\Lambda}\Lambda e_0\Lambda^{\dagger}+\left(\dot{\Lambda}\bar{\Lambda}\Lambda e_0\Lambda^{\dagger}\right)^{\dagger}=\frac{\boldsymbol{\Omega}u+\left(\boldsymbol{\Omega}u\right)^{\dagger}}{2}=\left\langle\boldsymbol{\Omega}u\right\rangle_R,$$

which is a paravector. The Lorentz force (see [8]) is now

$$\dot{p}=m\dot{u}=e\left\langle\boldsymbol{F}u\right\rangle_R,\ \boldsymbol{F}=\boldsymbol{E}+j\boldsymbol{B},$$

so we see how just defined $\boldsymbol{\Omega}$ gets a physical meaning: it is proportional to the complex vector of the electromagnetic field $\boldsymbol{F}$. It is surprising how the electromagnetic theory simply and naturally formulates in *Cl*3. And this is not an isolated example. The geometric product makes the geometry of our 3D world a natural framework for physics. Someone who knows the geometric algebra well, but knows nothing about the electromagnetism, could probably discover an electromagnetic field as a purely geometric object. Gibbs scalar and cross products and then the whole apparatus of theoretical physics with coordinates, matrices, tensors ... blurred the whole picture, a lot.

This brief review on eigenspinors should point out on a powerful and elegant technique that is widely applicable, except in electromagnetism, for example, in quantum mechanics.



Spinorial equations

Having a particle radius-vector in 2D we can write

$$\boldsymbol{r}=e_1x+e_2y=e_1r\left(\frac{x}{r}+\frac{y}{r}e_1e_2\right)$$
$$=e_1r\exp\left(\varphi e_1e_2\right).$$

We have seen how this expression cannot be generalized to higher dimensions, but we can do that in the "sandwich" form

$$\boldsymbol{r}=Ue_1U^{\dagger}.$$

What we wrote? In 2D, U is a complex number with the imaginary unit e_1e_2, but it generally can be treated as a *spinor* (for the definition of spinors see literature, it is enough here to use the term „spinor“ as an element of the even part of the *Cl*3, or just a rotor with a dilatation). Note that starting with the unit vector e_1 we can get any vector in the plane defined by the bivector e_1e_2, where the special choice of the vector and the bivector is not important. These relations are easy to generalize to higher dimensions. If the spinor U depends on time we have all dynamics contained in the spinor. This is a really powerful technique to describe various types of movement, that we will see below, informative only. It turns out that equations, as a rule, are much easier to solve if they are expressed in terms of U instead of $\boldsymbol{r}$.

Note that for a complex number U we get a complex conjugate as $U^\dagger$. The module of $\boldsymbol{r}$ is just $r=\sqrt{\boldsymbol{r}^2}$, i.e. $r=UU^\dagger$. The time rate of the vector $\boldsymbol{r}$ is the first derivation of $\boldsymbol{r}=Ue_1U^\dagger$, what is generally the correct approach. In 2D, for simplicity, we will take the derivative of U^2e_1, i.e.

$$\dot{\boldsymbol{r}}=2\dot{U}Ue_1 \Rightarrow \dot{\boldsymbol{r}}e_1=2\dot{U}U \Rightarrow \dot{\boldsymbol{r}}e_1U^\dagger=2r\dot{U}.$$

Introducing a new variable

$$\frac{d}{ds}=r\frac{d}{dt}, \quad \frac{dt}{ds}=r$$

and using $\frac{dU}{ds}=U'$, we get a new equation for U

$$2U'=\dot{\boldsymbol{r}}Ue_1,$$

or, deriving once more

$$2U''=r\ddot{\boldsymbol{r}}Ue_1+\dot{\boldsymbol{r}}U'e_1=U\left(\ddot{\boldsymbol{r}}\boldsymbol{r}+\dot{\boldsymbol{r}}^2/2\right).$$

For the particular problem, let's look at the motion of the body under the action of a central force (the *Kepler's problem*)

$$\mu\,\ddot{\boldsymbol{r}}=-k\boldsymbol{r}r^{-3},$$

where μ is a reduced mass and k is a constant. The equation for U now becomes

$$U''=\frac{1}{2\mu}U\left(\frac{\mu\dot{\boldsymbol{r}}^2}{2}-\frac{k}{r}\right)=\frac{E}{2\mu}U \Rightarrow$$
$$U''=\kappa U, \quad \kappa=const,$$

where we have introduced the total energy E. This is a well-known and relatively simple equation, which for bound states ($E<0$) takes the form of the equation for a harmonic oscillator. The advantages of this approach are numerous, like ease of solving specific equations, better stability of solutions (no singularities for $r=0$) and observe that the equation is linear, which has a great advantages in the perturbation approach (a better stability).

*Cl*3 and quantum mechanics

We have already shown that orthonormal basis vectors in *Cl*3 could be represented by Pauli matrices. Now we are to develop this idea a little further in order to get a sense of how quantum mechanics fits nicely in the formalism of geometric algebra.

In the "standard" quantum mechanics formulation a wave function of an electron has the form

$$|\psi\rangle=\alpha|\uparrow\rangle+\beta|\downarrow\rangle, \quad \alpha,\ \beta\in\mathbb{C},$$

so that such a quantum state is usually given in the form of a spinor (see [18])

$$\left|\psi\right\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

If we set the direction of the *z*-axis in the direction of the state $\left|\uparrow\right\rangle$ we get spin operators in the form

$$\hat{s}_k = \frac{1}{2}\hbar\hat{\sigma}_k ,$$

where $\hat{\sigma}_k$ are earlier defined Pauli matrices. We can now look for observables in the form

$$s_k = \frac{1}{2}\hbar n_k = \left\langle\psi\right|\hat{s}_k\left|\psi\right\rangle, \quad n_k = \left\langle\psi\right|\hat{\sigma}_k\left|\psi\right\rangle,$$

where components are given as

$$n_1 = \alpha\beta^* + \alpha^*\beta, \quad n_2 = i\left(\alpha\beta^* - \alpha^*\beta\right), \quad n_3 = \alpha\alpha^* - \beta\beta^* .$$

We have

$$\left|\boldsymbol{n}\right|^2 = \left\langle\psi\middle|\psi\right\rangle^2 = \left(\left|\alpha\right|^2 + \left|\beta\right|^2\right)^2 ,$$

so we can take the advantage of this relation and normalize the vector $\boldsymbol{n}$ to be $\left|\boldsymbol{n}\right| = 1$. By introducing spherical coordinates we can write

$$n_1 = \sin\theta\cos\varphi, \quad n_2 = \cos\theta\sin\varphi, \quad n_3 = \cos\theta ,$$

or

$$\alpha = \cos\left(\theta/2\right)e^{i\gamma}, \quad \beta = \sin\left(\theta/2\right)e^{i\delta}, \quad \delta - \gamma = \varphi ,$$

which gives for the spinor

$$\left|\psi\right\rangle = \begin{pmatrix} \cos\left(\theta/2\right)e^{-i\varphi/2} \\ \sin\left(\theta/2\right)e^{i\varphi/2} \end{pmatrix} e^{i(\gamma+\delta)/2} .$$

We can neglect the overall phase $\exp\left(i\left(\gamma+\delta\right)/2\right)$. We see the dependence on half angles, suggesting a link to rotors in *Cl*3. Let us introduce now a common designation in *Cl*3: $e_i \to \sigma_i$, it follows the relation to rotors

$$\boldsymbol{n} = \sum_{k=1}^{3} n_k\sigma_k = \sin\theta\left(\sigma_1\cos\varphi + \sigma_2\sin\varphi\right) + \sigma_3\cos\theta \equiv R\sigma_3 R^\dagger, \quad R = e^{-j\varphi\sigma_3/2}e^{-j\theta\sigma_2/2} .$$

If we now introduce a spinor in *Cl*3, which by analogy we denote by ψ, we will seek a general form for this new object by comparison

$$\left|\psi\right\rangle = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \leftrightarrow \psi = a^0 + a^k j\sigma_k ,$$

where the summation over k is understood. We see immediately that $\left|\uparrow\right\rangle \leftrightarrow 1, \quad \left|\downarrow\right\rangle \leftrightarrow -j\sigma_2$ and that appropriate vectors of the observable have components (0, 0, ±1). For operators we can find the relationship

$$\hat{\sigma}_k\left|\psi\right\rangle \leftrightarrow \sigma_k\psi\sigma_3 ,$$

where σ_3 is included to ensure belonging to the even part of the algebra. This choice is, of course, a consequence of the initial choice of the *z*-axis and does not affect the generality of the expression. The choice of the *z*-axis usually has a physical background, for example, the direction of an external magnetic field. What we get if we multiply all three Pauli matrices? We can establish an analogy with the multiplication by the imaginary unit as

$$i|\psi\rangle \leftrightarrow \psi j\sigma_3 .$$

Suggestive is that we have a multiplication by the bivector $j\sigma_3$, for it is to expect. Namely, we get vectors of observables just by rotation of the vector σ_3 which is invariant to rotations in the $j\sigma_3$ plane, which gives a geometric picture of the phase invariance. Notice here the role of pseudoscalars and bivectors, which, unlike the ordinary imaginary unit, immediately give a clear geometric meaning to quantities in the theory. This is definitely a good motivation for the study of quantum mechanics in this new language. Instead of non-intuitive matrices over complex numbers we have now elements of geometric algebra, which always introduce clarity. And we were just a little peek into the area …

Now let's look at observables in the Pauli theory. We will assume that we can separate spatial and spin components. The *inner* product in quantum mechanics is defined as

$$\langle\psi|\phi\rangle = \left(\psi_1^*, \psi_2^*\right)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \psi_1^*\phi_1 + \psi_2^*\phi_2 .$$

The real part can now be found as

$$\operatorname{Re}\langle\psi|\phi\rangle \leftrightarrow \langle\psi^\dagger\phi\rangle ,$$

for example

$$\operatorname{Re}\langle\psi|\psi\rangle \leftrightarrow \langle\psi^\dagger\psi\rangle = \left\langle\left(a^0 - a^j j\sigma_j\right)\left(a^0 + a^j j\sigma_j\right)\right\rangle = \sum_{k=0}^{3} a^k a^k .$$

We have

$$\langle\psi|\phi\rangle = \operatorname{Re}\langle\psi|\phi\rangle - i\operatorname{Re}\langle\psi|i\phi\rangle ,$$

so we can find the analogy (do not confuse $\langle a|b\rangle$ with $\langle ab\rangle$, a grade 0)

$$\langle\psi|\phi\rangle \leftrightarrow \langle\psi^\dagger\phi\rangle - \langle\psi^\dagger\phi j\sigma_3\rangle j\sigma_3 .$$

Here we have $\langle\psi^\dagger\phi\rangle$, the grade 0 of the product $\psi^\dagger\phi$, as well as $\langle\psi^\dagger\phi j\sigma_3\rangle j\sigma_3$, the projection of the product $\psi^\dagger\phi$ to the plane $j\sigma_3$.

Let's look for the expected value of the spin $\langle\psi|\hat{s}_k|\psi\rangle$. We demand

$$\langle\psi|\hat{\sigma}_k|\psi\rangle \leftrightarrow \langle\psi^\dagger\sigma_k\psi\sigma_3\rangle - \langle\psi^\dagger\sigma_k\psi j\rangle j\sigma_3 .$$

If we take the advantage of the reverse involution it follows

$$\left(\psi^\dagger\sigma_k\psi j\right)^\dagger = j^\dagger\psi^\dagger\sigma_k\psi = -\psi^\dagger\sigma_k\psi j ,$$

which means that there is no grades 0 or 1, so must be $\langle\psi^\dagger\sigma_k\psi j\rangle = 0$, what we expect because $\hat{\sigma}_k$ are Hermitian operators. The element $\psi^\dagger\sigma_k\psi$ has odd grades only and it is equal to its reverse, so, it is a vector. Using that, we define the *spin vector* as

$$\boldsymbol{s} \equiv \frac{1}{2}\hbar\,\psi\sigma_3\psi^{\dagger}.$$

The expected value is now

$$\left\langle \psi \middle| \hat{s}_k \middle| \psi \right\rangle = \frac{1}{2}\hbar\left\langle \sigma_k\psi\sigma_3\psi^{\dagger}\right\rangle = \sigma_k \cdot \boldsymbol{s}\,.$$

This expression is different from what we are accustomed in quantum mechanics. Instead of calculating the expected value of the operator here we have a simple projection of the spin vector on the desired direction in space. This immediately raises the question of a co-existence of all three components of the spin vector. The problem does not really exist, reader is referred to the article *Doran et al*, 1996b.

We can use our form of spinors and define the scalar $\rho = \psi\psi^{\dagger}$, then if we define

$$R = \rho^{-1/2}\psi$$

we see that $RR^{\dagger} = 1$, so we have a rotor. According to that, spinors here are just rotors with dilatation, and the spin vector is

$$\boldsymbol{s} = \frac{1}{2}\hbar\rho R\sigma_3 R^{\dagger}.$$

It follows that the form of the expected value is just instruction for the rotation of the fixed vector σ_3 in the direction of the spin vector, followed by its dilatation. Note again a clear geometric meaning, which is not so easy to achieve in quantum theory as is usually formulated.

Let us imagine now that we want to rotate the spin vector, so let's introduce a transformation $\boldsymbol{s} \to R_0\boldsymbol{s}R_0^{\dagger}$. In doing so, the spinor must transform as $\psi \to R_0\psi$ (show that), what is often taken as a way to identify an object as a spinor. A similar property we have for already mentioned *eigenspinors* in the special theory of relativity, which under the general Lorentz transformation transform as $\Lambda \to L\Lambda$, i.e. not a "sandwich" form, but a „spinor" form of transformation. We leave to the reader, taking into account just shown property of transformation, to show that spinors change sign after rotation by 2π. This is a result that is also clear in "ordinary" quantum theory, but here we see that there is nothing "quantum" in this phenomenon, it's actually the property of our 3D space. This is certainly not an insignificant conclusion, one could say that we have good reason to re-examine the fundamentals (and philosophy, if you like) of the quantum theory. And again, all that is just due to the new product of vectors. So, if you want, this can also be support to grounds for the new multiplication of vectors.

Differentiation and integration

Here we will only briefly comment on this area, reader is referred to the literature. Geometric algebra contains a powerful apparatus of differential and integral calculus. It should be no surprise that here we also have a significant improvement over the traditional approach. In particular, there is a *fundamental theorem of calculus* which combines and expands many well-known theorems of classical integral calculus. In addition, all elements of the algebra (including full multivectors) can be on equal footing included in the calculus, so we can derive in the "direction" of the multivector, a bivector for example. It is, in fact, very nice feature! What essentially distinguishes the classical theory of geometric algebra is reflected in several elements.

First, in the differential calculus we use various "operators" containing elements of the algebra, so, due to the property of non-commutativity, this provide us new opportunities. Let's look at an example of such an "operator"

$$\nabla = e^k \partial_k, \quad \partial_k \equiv \frac{\partial}{\partial x^k},$$

the Einstein summation convention is understood. Here we introduced vectors of reciprocal basis again, which in an orthonormal systems are equal to base vectors, but this is convenient here for the Einstein summation convention and for a possible generalization. Notice that $\nabla \equiv e^k \partial_k$ has the grade 1, i.e. acts as a vector. In *Cl*3 is often used the operator

$$\partial = \partial_t - \nabla,$$

which has the form of a paravector. We can derive from left and from right, however, in this we have geometric products of basis vectors which is not commutative. Operators of derivation, as elements of the algebra, can be inverted. For example, Maxwell's equations can without much effort be written in the form

$$\partial \boldsymbol{F} = \boldsymbol{J},$$

and this makes it possible to find the inverse of the operator ∂ using Green's functions. In this way, this simple mathematical form of the equation is not just a "neat trick" but actually provides features that without the geometric product would not exist (or it would be difficult to achieve). Note again that "operators" are also elements of the algebra, therefore here we do not consider them as operators. For interesting examples of the power of geometric algebra in electromagnetism see [10], or [2].

Second, in the integral calculus we encounter with the *measure*, objects like $dxdy$. In geometric algebra such objects have the orientation (like blades), which gives many possibilities. For example, unification of all the important theorems of classical integral calculus (Stokes, Gauss, Green, Cauchy ...) in one is a great and inspiring achievement of geometric algebra. We refer the reader ready to learn this beautiful, but nontrivial topic, to the literature [18], [20] (on the internet you can search for the phrase "*geometric calculus*").

Geometric models

It is known that the geometry of the 3D space can be formulated inserting 3D vector space in a space of higher dimension. Geometric algebra is an ideal framework for an investigation of such models. Here we will only briefly discuss one of them, conformal, developed and patented by *Hestenes*. The idea of the conformal model is that the n-dimensional Euclidean vector space is modeled in n+2 - dimensional Minkowski space. For $n=3$ apart from the usual unit vectors let's introduce two more: e, $e^2=1$ and $\bar{e}$, $\bar{e}^2=-1$, so we have the basis

$$\{e, e_1, e_2, e_3, \bar{e}\}.$$

This is an orthonormal basis of the 5D vector space $\Re^{4,1}$. Two added unit vectors define the 2D subspace $\Re^{1,1}$ in which we will introduce a new basis $\{o, \infty\}$ (character ∞ we use here as the designation of the vector that represents the point at infinity, while character *o* represents the origin)

$$o = (e + \bar{e})/2, \qquad \infty = \bar{e} - e.$$

Factor ½ does not play an important role, may be dropped, but then the rest of formulas will have a little different form. Show that o and ∞ are nilpotents and that $e \wedge \overline{e} = o \wedge \infty$, $o \cdot \infty = \infty \cdot o = -1$, $o \cdot o = \infty \cdot \infty = 0$. Now we have a new basis

$$\{o, e_1, e_2, e_3, \infty\}$$

in which geometric elements such as lines or circles have a simple form.

If we have two points in $\Re^3$ and two vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ coming out of the common origin and end up in our points, the squared distance of points is given as $(\boldsymbol{p}-\boldsymbol{q}) \cdot (\boldsymbol{p}-\boldsymbol{q})$. The idea is to find vectors p and q in this algebra which inner product will give us the distance of 3D points (up to a factor), i.e. $p \cdot q \sim (\boldsymbol{p}-\boldsymbol{q}) \cdot (\boldsymbol{p}-\boldsymbol{q})$. In this case, it should be $p \cdot p = 0$, because the distance is zero, so such vectors are called *null-vectors*. Accordingly, points are represented by null-vectors in this model, so it can be shown that for the 3D vector $\boldsymbol{p}$ the corresponding null vector is given by

$$p = o + \boldsymbol{p} + \boldsymbol{p}^2 \infty / 2,$$

where $p \cdot p = 0$ (check it). Find $p \cdot q$. In the conformal model points can have a *weight*, but here we will not deal with it, except for the note that weight has a geometric meaning, for example, the weight can show the way in which a straight line and a plane are intersecting. Vectors of the model that are not points (are not the null-vectors) can represent a variety of geometric elements. Take the vector $\pi = \boldsymbol{n} + \lambda \infty$ as an example, and if we want to find all points x that would belong to such an object we have to write the condition $x \cdot \pi = 0$, which means that the distance between the point represented by x and the point represented by π is zero. We have

$$x \cdot \pi = \left(o + \boldsymbol{x} + \boldsymbol{x}^2 \infty / 2\right) \cdot (\boldsymbol{n} + \lambda \infty) = \boldsymbol{x} \cdot \boldsymbol{n} - \lambda = 0,$$

so we see that we have the equation of the plane perpendicular to the vector $\boldsymbol{n}$, with the distance from the origin λ / n. If we recall that a circle in 3D is defined by three points we could appreciate the fact that a circle in this model we get easily: make the outer product of their null-vectors. If one of the points is ∞ we get a straight line. It cannot be easier than that. It is particularly important that transformations of elements can be implemented using a single formalism, thus, for example, the same formalism operate rotations and translations. Interested reader can find the beautifully exposed theory in [19], where you can take advantage of the software that accompanies the book: *GAViewer*. Everything is available free on the Internet.

Appendix

A1. Some properties of Pauli matrices

Let's look at Pauli matrices and some of their properties. For linear combinations of Pauli matrices

$$a = \sum_{i=1}^{3} a_i \hat{\sigma}_i \quad \text{and} \quad b = \sum_{i=1}^{3} b_i \hat{\sigma}_i , \quad a_i ,\; b_i \in \Re ,$$

we have

$$a^2 = aa = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sum_{i=1}^{3} a_i^2 ,$$

so a behaves like a vector. Also we have

$$\frac{ab + ba}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sum_{i=1}^{3} a_i b_i ,$$

which means that Pauli matrices could be interpreted as unit vectors (we have a scalar product of vectors $\vec{a} = \sum_{i=1}^{3} a_i e_i$ and $\vec{b} = \sum_{i=1}^{3} b_i e_i$). Of course, it means that products of unit vectors e_i should be anti-commutative (as for matrices). If we find the antisymmetric part

$$\frac{ab - ba}{2}$$

and multiply it by matrix

$$\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$$

we obtain coefficients of the cross product (show that). From $\hat{\sigma}_i \hat{\sigma}_j = -\hat{\sigma}_j \hat{\sigma}_i$ follows

$$\left(\hat{\sigma}_i \hat{\sigma}_j \right)^2 = \hat{\sigma}_i \hat{\sigma}_j \hat{\sigma}_i \hat{\sigma}_j = -\hat{\sigma}_i \hat{\sigma}_i \hat{\sigma}_j \hat{\sigma}_j = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So, we have objects with the negative square, they are not vectors, obviously. It means that matrix $\hat{\sigma}_i \hat{\sigma}_j$ does not represent a vector. Here we have a problem of geometric interpretation. But, with unit vectors we have $e_i e_j$ which clearly gives us geometric meaning (an oriented parallelogram), defining a plane along the way. Similarly, $\hat{\sigma}_1 \hat{\sigma}_2 \hat{\sigma}_3$ is just the product of the unit matrix and imaginary unit, but $e_1 e_2 e_3$ is the oriented volume that squares to -1 and commutes with all vectors, which means that we have again an „imaginary unit“, but this time with a clear geometric interpretation. Finally, if we seek for the 2D matrix representation of *Cl*3 we get Pauli matrices as a solution. The very existence of the matrix representation proves that *Cl*3 is a well-defined algebra.

A2. Everything is a "boost"

For the complex vector $\boldsymbol{F}=\vec{\boldsymbol{v}}+j\vec{\boldsymbol{w}}$ we have $W=\sqrt{\boldsymbol{F}^2}\in\mathbb{C}$ or $\sqrt{\boldsymbol{F}^2}=\boldsymbol{F}$, so for $\boldsymbol{F}\neq\boldsymbol{N}$, $\boldsymbol{N}^2=0$ we define $\boldsymbol{F}/W=\boldsymbol{f}$, $\boldsymbol{f}^2=1$ and $\boldsymbol{F}/\sqrt{-\boldsymbol{F}^2}=\boldsymbol{I}=-j\boldsymbol{f}$, $\boldsymbol{I}^2=-1$. Suppose we have the exponential form $\exp(\varphi\boldsymbol{f})$, defining $\tanh\varphi=W$, $\Gamma=1/\sqrt{1-W^2}$, $\kappa=\sqrt{(1+W)/(1-W)}=\Gamma(1+W)$ (the generalized *Bondi factor*, $\varphi=\ln\kappa$) and idempotents $\boldsymbol{f}_\pm=(1\pm\boldsymbol{f})/2$, $\boldsymbol{f}_+\boldsymbol{f}_-=0$ we have

$$\mathrm{e}^{\varphi\boldsymbol{f}}=\cosh\varphi+\boldsymbol{f}\sinh\varphi=\Gamma(1+W\boldsymbol{f})=\kappa\boldsymbol{f}_++\kappa^{-1}\boldsymbol{f}_-.$$

Now we can read the "speed" as W and it is easy to find successive "boosts" as

$$\mathrm{e}^{\varphi_1\boldsymbol{f}}\mathrm{e}^{\varphi_2\boldsymbol{f}}=\mathrm{e}^{(\varphi_1+\varphi_2)\boldsymbol{f}}=\Gamma_1\Gamma_2(1+W_1\boldsymbol{f})(1+W_2\boldsymbol{f})=\Gamma_1\Gamma_2(1+W_1W_2)\left(1+\frac{W_1+W_2}{1+W_1W_2}\boldsymbol{f}\right),$$

$$\Gamma=\Gamma_1\Gamma_2(1+W_1W_2),\qquad W=\frac{W_1+W_2}{1+W_1W_2},$$

or

$$\mathrm{e}^{\varphi_1\boldsymbol{f}}\mathrm{e}^{\varphi_2\boldsymbol{f}}=\left(\kappa_1\boldsymbol{f}_++\kappa_1^{-1}\boldsymbol{f}_-\right)\left(\kappa_2\boldsymbol{f}_++\kappa_2^{-1}\boldsymbol{f}_-\right)=\kappa_1\kappa_2\boldsymbol{f}_++\kappa_1^{-1}\kappa_2^{-1}\boldsymbol{f}_-\Rightarrow\kappa=\kappa_1\kappa_2.$$

Generally we have a complex scalar $\varphi=\ln\kappa=\varphi_{\mathrm{R}}+j\varphi_{\mathrm{I}}$ (explicit formulae for φ_{R} and φ_{I} are rather cumbersome, one can use *Mathematica* and $j\to i$, where *i* is the ordinary imaginary unit) which leads to $\exp(\varphi\boldsymbol{f})=\exp(\varphi_{\mathrm{R}}\boldsymbol{f})\exp(\varphi_{\mathrm{I}}j\boldsymbol{f})$.

For $\boldsymbol{F}=\vec{\boldsymbol{v}}+j\vec{\boldsymbol{w}}$, $W=\sqrt{(\vec{\boldsymbol{v}}+j\vec{\boldsymbol{w}})^2}=\sqrt{v^2-w^2+2j\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{w}}}$, for $\vec{\boldsymbol{w}}=0$ we have well-known relations for boosts in the restricted special relativity.

For $\boldsymbol{F}=j\vec{\boldsymbol{w}}$ we have $W=\sqrt{(j\vec{\boldsymbol{w}})^2}=\sqrt{-w^2}=jw$, $\boldsymbol{f}=j\vec{\boldsymbol{w}}/jw=\hat{\boldsymbol{w}}$, $\Gamma=1/\sqrt{1+w^2}$, $\kappa=\sqrt{(1+jw)/(1-jw)}$, $\varphi=\log\kappa=j\,\mathrm{atan}\,w$, $\exp(\varphi\hat{\boldsymbol{w}})=\Gamma(1+jw\hat{\boldsymbol{w}})$ and for successive transformations we have

$$\Gamma=\Gamma_1\Gamma_2(1-w_1w_2),\qquad w=(w_1+w_2)/(1-w_1w_2).$$

It is an interesting possibility to interpret such transformations like "boosts", defining new "rotating" frames of reference with "time" $t=\Gamma\tau$ (τ is a "proper time"), introducing thus such "rotating" frames as an analog to inertial frames. Regarding the invariance of MA instigates to reexamine the paradigm an "inertial frame of reference".

For well-known pure rotations $\exp\left(\theta j\hat{\boldsymbol{n}}\right)$ we have $\theta j\hat{\boldsymbol{n}} = j\theta\left(j\theta\hat{\boldsymbol{n}}\right)/j\theta$, $\varphi = j\theta$, $W = \tanh\left(j\theta\right) = j\tan\theta$, $\Gamma = 1/\sqrt{1+\tan^2\theta}$, $\kappa = \sqrt{\left(1+j\tan\theta\right)/\left(1-j\tan\theta\right)}$, $\boldsymbol{f}_{\pm} = \left(1\pm\hat{\boldsymbol{n}}\right)/2$ and so

$$\mathrm{e}^{\theta j\hat{n}} = \Gamma\left(1+j\hat{\boldsymbol{n}}\tan\theta\right),$$

$$\Gamma = \Gamma_1\Gamma_2\left(1-\tan\theta_1\tan\theta_2\right),$$

$$\tan\theta = \left(\tan\theta_1+\tan\theta_2\right)/\left(1-\tan\theta_1\tan\theta_2\right) = \tan\left(\theta_1+\theta_2\right).$$

Acknowledgement

I would like to express the gratitude to Eckhard Hitzer for kind words and support.

Literature

[1] Baez: *Division Algebras and Quantum Theory*, arXiv:1101.5690v3

[2] Baylis et al: *Motion of charges in crossed and equal E and B fields*, AJP

[3] Baylis, *Geometry of Paravector Space with Applications to Relativistic Physics*, Kluwer Academic Publishers, 2004

[4] Baylis, Hadi: *Rotations in n dimensions as spherical vectors*, Applications of Geometric Algebra in Computer Science and Engineering

[5] Baylis: *Classical Eigenspinors and the Dirac Equation*, Ph. Rev. A, 45, 7, 1992

[6] Baylis: *Eigenspinors in Electrodynamics*, in *Clifford (geometric) algebras with applications to physics, mathematics, and engineering*, ch 18

[7] Baylis: *Eigenspinors in Quantum Theory*, in *Clifford (geometric) algebras with applications to physics, mathematics, and engineering*, ch 19

[8] Baylis: *Electrodynamics: A Modern Geometric Approach*, Progress in Mathematical Physics

[9] Baylis, Sobczyk: *Relativity in Clifford's Geometric Algebras of Space and Spacetime*, arXiv:math-ph/0405026v1

[10] Baylis: *Surprising Symmetries in Relativistic Charge Dynamics*, arXiv:physics/0410197v1

[11] Chappell, Hartnett, Iannella, Abbott: *Deriving time from the geometry of space*, arXiv:1501.04857v2

[12] Chappell, Hartnett, Iannella, Abbott, Iqbal: *Exploring the origin of Minkowski spacetime*, arXiv:1501.04857v3

[13] Chappell, Iqbal, Gunn, Abbott: *Functions of multivector variables*, arXiv:1409.6252v1

[14] Chappell, Iqbal, Iannella, Abbott: *Generalizing the Lorentz transformations*, https://www.researchgate.net/publication/309765990

[15] Chappell, Iqbal, Iannella, Abbott: *Revisiting special relativity: A natural algebraic alternative to Minkowski spacetime*, PLoS ONE 7(12)

[16] Doran: *Geometric Algebra and its Application to Mathematical Physics*, thesis

[17] Doran, Lasenby, Gull: *Gravity as a gauge theory in the spacetime algebra*, Fundamental Theories of Physics, 55, 1993

[18] Doran, Lasenby: *Geometric Algebra for Physicists*, Cambridge University Press

[19] Dorst, Fontijne, Mann: *Geometric Algebra for Computer Science*, http://www.geometricalgebra.net

[20] Hestenes, Sobczyk: *Clifford Algebra to Geometric Calculus*, Springer

[21] Hestenes: *Mathematical Viruses*, in Micali et al., *Clifford Algebras and their Applications in Mathematical Physics*, Kluwer Academic Publishers 1992

[22] Hestenes: *New Foundations for Classical Mechanics*, Kluwer Academic Publishers

[23] Hitzer, Helmstetter, Ablamowicz: *Square Roots of -1 in Real Clifford Algebras*, arXiv:1204.4576v2

[24] Jancewicz: *Multivectors And Clifford Algebra In Electrodynamics*, World Scientific, Singapore (1988)

[25] Lasenby, Doran, Gull: *Gravity, Gauge Theories and Geometric Algebra*, gr-qc/0405033v1

[26] Mermin: *Relativity without light*, Am. J. Phys. 52, 119 (1984)

[27] Mornev: *Idempotents and nilpotents of Clifford algebra* (russian), Гиперкомплексные числа в геометрии и физике, 2(12), том 6, 2009

[28] Sabbata, Datta: *Geometric algebra and applications in physics*, Taylor&Francis

[29] Sobczyk: *Geometric matrix algebra*, Linear Algebra and its Applications 429 (2008) 1163–1173

[30] Sobczyk: *New Foundations in Mathematics*: *The Geometric Concept of Number*, Birkhäuser

[31] Sobczyk: *Special relativity in complex vector algebra*, arXiv:0710.0084v1

[32] Sobczyk: *Vector Analysis of Spinors*, http://www.garretstar.com

[33] Sobczyk: *The Missing Spectral Basis in Algebra and Number Theory*, the American Mathematical Monthly, Vol. 108, No. 4. (Apr., 2001), pp. 336-346

[34] Trayling, Baylis: *The Cl7 approach to the standard model*, Progress in Mathematical Physics 34 pp 547-558

[35] Vince: *Geometric algebra for computer graphics*, Springer, 2008

Some more interesting texts

Bouma, Timaeus: *Invertible Homogeneous Versors are Blades*, 2001, http://www.science.uva.nl/ga/publications

Bromborsky, Alan: *An Introduction to Geometric Algebra and Calculus*, http://www2.montgomerycollege.edu/departments/planet/planet/Numerical_Relativity/bookGA.pdf

Chisholm J.S.R., Common A.K.: *Clifford Algebras and their Applications in Mathematical Physics*, Reidel Publishing Company, 1985

Hildenbrand, Dietmar: *Foundations of Geometric Algebra Computing*, Springer

Hitzer, Eckhard: *Introduction to Clifford's Geometric Algebra*, arXiv:1306.1660v1

Jones, George Llewellyn: *The Pauli algebra approach to relativity*, thesis, University of Windsor

Kanatani, Kenichi: *Understanding Geometric Algebra*, CRC Press

Kosokowsky, David E.: *Dirac theory in the Pauli algebra*, thesis, University of Windsor

Macdonald, Alan: *A Survey of Geometric Algebra and Geometric Calculus*, http://faculty.luther.edu/~macdonal/GA&GC.pdf

Macdonald, Alan: *Linear and Geometric Algebra*, http://faculty.luther.edu/~macdonal/laga/

Macdonald, Alan: *Vector and Geometric Calculus*, http://faculty.luther.edu/~macdonal/vagc/

Meinrenken, Eckhard: *Clifford algebras and Lie groups* , http://isites.harvard.edu/fs/docs/icb.topic1048774.files/clif_mein.pdf

Perwass, Christian: *Geometric Algebra with Applications in Engineering*, Springer, 2009

Wei, Jiansu: *Quantum mechanics in the real Pauli algebra*, thesis, University of Windsor

Yao, Yuan: *New relativistic solutions for classical charges in an electromagnetic field*, thesis, University of Windsor

A small amount of literature is offered intentionally, but, due to current possibilities of obtaining information, we will give a list of names of people active in this area, so the reader can easily obtain the necessary articles, books, etc.

3D geometry

http://geocalc.clas.asu.edu/GA_Primer/GA_Primer/index.html

Some web resources

https://gaupdate.wordpress.com

http://geocalc.clas.asu.edu

https://staff.science.uva.nl/l.dorst/clifford/

Software

Cinderella

https://www.cinderella.de/tiki-index.php

clifford.m, Mathematica package, https://arxiv.org/abs/0810.2412

http://www.fata.unam.mx/investigacion/departamentos/nanotec/aragon/software/clifford.m

CLIFFORD

http://math.tntech.edu/rafal/

Clifford algebra for CAS Maxima

https://github.com/dprodanov/clifford

Clifford Multivector Toolbox for MATLAB

http://clifford-multivector-toolbox.sourceforge.net/

CLUCalc/CLUViz

http://www.clucalc.info/

GA20 and GA30 (Formerly called pauliGA)

https://github.com/peeterjoot/gapauli

Gaalet

https://sourceforge.net/projects/gaalet/

Gaalop

http://www.gaalop.de/

GABLE

https://staff.fnwi.uva.nl/l.dorst/GABLE/index.html

Gaigen

https://sourceforge.net/projects/g25/

GAlgebra

https://github.com/brombo/galgebra

GA Sandbox

https://sourceforge.net/projects/gasandbox/

GAViewer, http://www.geometricalgebra.net/downloads.html, this nice tool is recommended with the text. You can manipulate the images to some extent.

GluCat

https://sourceforge.net/projects/glucat/

GMac

https://gacomputing.info/gmac-info/

SpaceGroupVisualizer

http://spacegroup.info/

The GrassmannAlgebra package

https://sites.google.com/site/grassmannalgebra/thegrassmannalgebrapackage

Versor

http://versor.mat.ucsb.edu/

Some important details can be found at https://gacomputing.info/ga-software/.

List of names of people in the field of geometric algebra

(in alphabetical order)

Those I did not remember, or those who I do not know for, forgive me, list will be expanding.

Abbott, Derek

Ablamowicz, Rafal

Aharonov, Yakir

Almeida, José

Aragón-Camarasa, Gerardo

Aragón, José Luis

Aristidou, Andreas

Artin, Emil

Arthur, John

Artūras, Acus

Augello, Agnese

Babalic, Elena-Mirela

Barbosa, Afonso Manuel

Bajguz, Wieslaw

Bartsch, Thomas

Baugh, James

Baylis, William

Bayro-Corrochano, Eduardo

Benger, Werner

Bengtsson, Ingemar

Blanchfield, Kate

Bouma, Timaeus

Brackx, Freddy

Brini, Andrea

Bromborsky, Alan

Browne, John

Buchholz, Sven

Cameron, Jonathan

Campbell, Earl

Castro, Carlos

Challinor, Anthony

Chapell, James

Chisholm, John Stephen Roy

Chisolm, Eric

Cibura, Carsten

Clifford, William Kingdon

Colapinto, Pablo

Conte, Elio

Cory, David G.

Dargys, Adolfas

Denker, John

Dixon, Geoffrey

Doran, Chris

Dorst, Leo

Dresden, Max

Dress, Andreas

Eid, Ahmad Hosny

Falcón, Luis Eduardo

Farach, Horacio A.

Farouki, Rida T.

Fernandes , Leandro

Fernandez, Virginia

Figueroa-O'Farrill, José

Finkelstein, David Ritz

Fontijne, Daniel

Franchini, Silvia

Francis, Matthew

Galiautdinov, Andrei

Gentile, Antonio

Goldman, Ronald

Grassmann, Hermann (first ideas)

Gull, Stephen

Gunn, Charles

Gunn, Lachlan

Halma, Arvid

Havel, Timothy F.

Henselder, Peter

Hestenes, David

Hilley, Basil

Hildenbrand, Dietmar

Hitzer, Eckhard

Horn, Martin Erik

Howard, Mark

Ichikawa, Daisuke

Iqbal, Azhar

Ivezić, Tomislav

Jancewicz, Bernard

Jones, George Liewellyn

Joot, Peeter

Kanatani, Kenichi

Klawitter , Daniel

Kocik, Jerzy

Kosokowsky, David E.

Kumar, Datta Bidyut

Kuroe, Yasuaki

Kwasniewski, Andrzej Krzysztof

Lachièze-Rey, Marc

Lasenby, Anthony

Lasenby, Joan

Laurinolli, Teuvo

Lazaroiu, Calin Luliu

Leopardi, Paul

Lewis, Antony

Li, Hongbo

Lounesto, Pertti

Luca, Redaelli

Lundholm, Douglas

Lynn, Ben

Mann, Stephen

Matos, Sérgio

Matzke, Douglas

Mawardi, Bahri

Macdonald, Alan

Meinrenken, Eckhar

Miller, Richard Alan

Mornev, Oleg

Moya, Antonio Manuel

Naeve, Ambjorn

Najfeld, Igor

Nesterov, Michael M.

Nitta, Tohru

Oliveira, Manuel

Paiva, Carlos

Parkin, Spencer T.

Parra, Josep

Pavšič, Matej

Perwass, Christian

Pesonen, Janne

Pezzaglia, William

Poole, Charles P. Jr

Porteous, Ian R.

Pozo, Jose Maria

Regonati, Francesco

Renaud, Peter

Riesz, Marcel

Rocha, Marlene Lucete Matias

Rockwood, Alyn

Rodrigues, Waldyr

Salingaros, Nikos

Setiawan, Sandi

Seybold, Florian

Schönberg, Mário

Shirokov, Dmitry

Snygg, John

Sobczyk, Garrett Eugene

Soiguine, Alexander

Somaroo, Shyamal

Sommen, Frank

Sommer, Gerald

Sorbello, Filippo

Sugon, Quirino

Suter, Jaap

Svenson, Lars

Tachibana, Kanta

Tarkhanov, Victor

Teolis, Antonio

Tisza, Laszlo

Trayling, Greg

Vargas, José

Varlamov, Vadim

Vassallo, Giorgio

Velebil, Jirka

Vugt, Floris T. van

Zamora, Julio

Zhang, Hua

Zhaoyuan, Yu

Yao , Yuan

Wang, Dongming

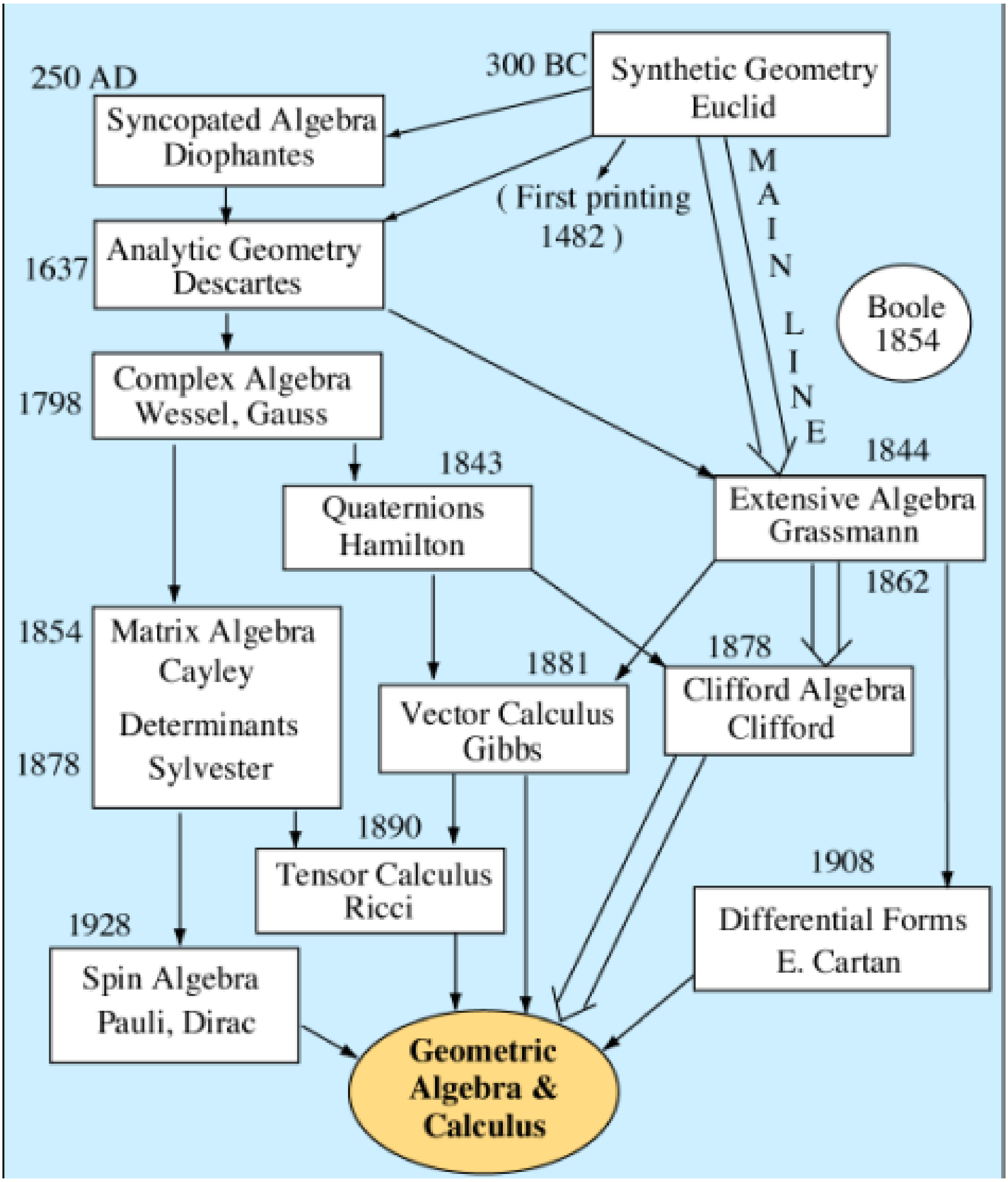
Wareham, Rich

Wei, Jiansu

Wei, Lu

Witte, Frank

Wu, Dimin



250 AD
300 BC
Synthetic Geometry
Euclid
Syncopated Algebra
Diophantes
(First printing
1482)
MAIN LINE
1637
Analytic Geometry
Descartes
Boole
1854
1798
Complex Algebra
Wessel, Gauss
1843
1844
Quaternions
Hamilton
Extensive Algebra
Grassmann
1862
1854
Matrix Algebra
Cayley
Determinants
Sylvester
1878
1881
1878
Vector Calculus
Gibbs
Clifford Algebra
Clifford
1890
Tensor Calculus
Ricci
1908
Differential Forms
E. Cartan
1928
Spin Algebra
Pauli, Dirac
Geometric
Algebra &
Calculus