TOPOLOGICAL FRAMEWORK FOR BRANE COSMOLOGY

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Our Universe could either lie on a spatial three-dimensional brane embedded in a higher dimensional bulk, or could be flanked by one or more multi-dimensional branes of different possible sizes and shapes. Here we demonstrate, based on novel topological findings, that disconnected strings with matching descriptions moving on a 4-brane (or more) project to a single string on a 3-brane. This means that strings travelling on the bulk or on higher dimensional branes necessarily display a counterpart in our 3D world, and vice versa. Furthermore, based on the concept of homotopy equivalence, we show how it is possible for branes to stitch together to become condensed branes. Our framework allows a topological duality among different brane theories, because it holds for all the types of branes, independent of their hypothetical shape, curvature, size and boundaries. Furthermore, a topological approach allow branes assessment in the general terms of particle trajectories taking place on donut-like manifolds. Indeed, every high-dimensional brane, independent on the subtending cosmological model, can be described in terms of multi-dimensional toruses mapping to branes with lower spatial dimensions.

1. INTRODUCTION

In brane cosmology, a *brane* is a domain of the Universe with number of dimensions lower than the surrounding or bordering space (Baulieu et al., 2001; Burgess and Van Nierop, 2013). Several brane theories have been developed throughout the years (Candelas et al., 1985; Arkani-Hamed et al., 1998; Randall and Sundrum, 1999; Hitchin, 2003; Hashimoto, 2012). Some scientists believe that our visible four-dimensional Universe is a restricted 3-braneworld endowed in a space extending in all directions, called the *bulk*, equipped with any number of spatial dimensions more than three. At least some of the extra-dimensions are extensive (possibly infinite) and branes may be moving through them. Brane-world scenarios might involve more than one, either warped or large, multi-dimensional branes, giving rise to a so-called *multiverse*. A 3-brane can be either the boundary wall of a bulk spanning every dimension both on and off the brane, or a non-boundary domain flanked by one or more branes. Other brane models state that the additional dimensions are compact and our Universe contains the extra-dimensions, so that no reference to the bulk is needed. The concept of *braneworlds* is based on the assumption that some strings are confined to lower-dimensional branes, trapped there by physical laws. They cannot travel in other dimensions and their forces influence just other strings confined to the same brane. Brane-bound forces would spread out only along their brane, and brane-bound particles would be exchanged solely along its dimensions. Not all strings are confined to a single brane, because some of them might be free to travel throughout the bulk. Gravity, for example, is never confined to a brane. This means that, at least via gravity, braneworlds must interact with the bulk, or must be connected to extra-dimensional branes. Interactions with the bulk, and possibly with other branes, can influence our world, introducing effects not detectable by standard cosmological models.
Here we explore the possibility to assess branes, bulks and multiverse bulks in terms of algebraic topology. We demonstrate, based on novel topological findings, that disconnected strings with matching description embedded in a 4-brane (or more) map to a single string on a 3-brane. Therefore, strings moving on higher dimensional brane necessarily display a counterpart in our 3D world, and vice versa. This leads also to novel scenarios, where branes are able to scatter, collide and combine, so that they merge together in an assessable way. Our framework holds for all the types of branes, independent of their hypothetical shape, curvature, size and boundaries. It means that all the hypothesized branes are dual under topological transformation. Here we show how such duality allows branes assessment in the general terms of particles trajectories taking place on donut-like manifolds. Indeed, every high-dimensional branes'

model can be described in guise of multi-dimensional toruses, projecting and mapping to branes which display lower spatial dimensions.

2. TOPOLOGY, STRINGS AND BRANES

Here we assess a geometric structure, e.g., a string (denoted str) or *worldline* (Olive and Landsberg, 1989), which is a region on the surface of either abstract geometric spaces or non-abstract physical spaces. A string displays zero or non-zero width ([30, [6], [13]) and either bounded or unbounded length. A string is the path followed by a particle moving through a brane, or different branes, or the bulk (Olive, 1987). Our 3D brane can be depicted as a n-dimensional normed linear space, while the bulk (or the higher-dimensional brane) as a n-dimensional hypersphere. Various continuous mappings from an n-dimensional hypersphere S^n to a feature space that is an n-dimensional Euclidean space R^n leadto a string-based incarnation of the Borsuk-Ulam theorem (BUT) (Borsuk 1933; Krantz, 2009; Tozzi and Peters, 2016a). It states states that:

Every continuous map $f : S^n \to R^n$ must identify a pair of antipodal points (on S^n).

Points on S^n are *antipodal*, provided they are diametrically opposite (Weisstein, 2016). If we simply evaluate branes or strings instead of *points*, BUT leads naturally to the possibility of a region-based brane geometry. The two opposite points could stand not just for the description of simple topological points (Marsaglia, 1972), but also of brane or strings features, such as spatial or temporal patterns, tensors, signals, particle trajectories (Peters, 2016). Therefore, we can describe branes or strings features as antipodal points on n-manifolds. Signal shapes can be compared, because the two antipodal points can be assessed at higher-dimensional levels of observation, while single points at lower levels (Tozzi and Peters, 2016b). The two points (or regions) do not need necessarily to be antipodal, in order to be described together (Peters, 2016). Indeed, BUT can be generalized also for the assessment of non-antipodal features, provided there are a pair of regions on a *n*-sphere with the same feature value (Peters and Tozzi, 2016a). We are allowed to consider also homotopic regions on an *n*-sphere that are either adjacent or far apart. Despite BUT was originally described just for positive-curvature n-spheres,we are allowed to look for antipodal points also on manifolds equipped with non convex curvatures (Mitroi-Symeonidis, 2015; Tozzi, 2016). Whether branes display concave, convex or flat geometry, it does not count, because we may always find the points with matching description predicted by BUT.Furthermore, a S^n manifold might not map just to a R^{n-1} Euclidean space, but straight to a S^{n-1} manifold, e.g, a lower-dimensional surface of the same S^n. In other words, either the Euclidean space and other manifolds could be omitted in brane theory. We do not need anymore a S^n manifold curving into a dimensional space R^n: we may think the brane just as existing by itself. Indeed, somebrane theoriesdo not requiretwo branes at all (Randall and Sundrum, 1999): by an intrinsic point of view, a single brane might exist in - and on – itself and does not need to lie in a bulk (Weeks, 2002). It means that the mapping of two antipodal points to a single point in a dimension lower becames a projection internal to the same brane. For technical readers, see also: Dodson and Parker (1997), Matousek (2003), Crabb and Jaworowski (2013).

In the evaluations of strings confined to a brane, we take into account antipodal sets instead of antipodal points (Peters and Naimpally, 2012), because in a point-free geometry regions replace points as the primitives (Di Concilio, 2013; Di Concilio and Gerla, 2006). If we assess a string in terms of a spatial regions on the surface of an n-sphere, or in an n-dimensional normedlinear space, the same string can be defined as *antipodal*, provided the regions encompassing the strings belong to disjoint parallel hyperplanes. Put simply, strings have no points in common (**Figure 1**). A region is called a *worldsheet*, or a brane, if everyone of its subregions contains at least one string. In other words, the term worldsheet or brane designates a nonempty region of a space completely covered by strings, in which every member is a string (Peters and Tozzi, 2016b). A 2D plane worldsheet can be rolled up to form the lateral surface of a 3D cylinder, termed a worldsheet cylinder. Further, a worldsheet cylinder maps to a worldsheet torus, formed by bending the former until the ends meet. In sum, a flattened worldsheet maps to a worldsheet cylinder and a flattened worldsheet cylinder maps to worldsheet torus. It means that a bounded worldsheet cylinder is homotopically equivalent to a worldsheet torus, so that every brane can be described in terms of particle movements along the surface of multi-dimensional toruses. Although strings on different wordsheets are antipodal and descriptively near, e.g., they share matching description, there is however a difference between the strings embedded in branes of diverse dimensions. The higher the dimension of the wordsheet, the more the information encompassed in strings on the same region, because the number of coordinates is higher. Strings contain more information than their projections in lower dimensions. It means that BUT allows us to evaluate systems features in higher-dimensions, in order to increase the amount of detectable information. Vice versa, dropping down a dimension means that each region in the lower-dimensional space is simpler. Hence, BUT provides a way to evaluate changes of information among different branes and bulk in a topological, other than thermodynamical, fashion.

Next, consider Brouwer's fixed point theorem (FPT), which states that every continuous function from a n-sphere of every dimension to itself has at least one fixed point (Volovikov and Yu, 2008). FPT applies, for example, to any disk-shaped area, where it guarantees the existence of a fixed point, which behaves like a sort of whirlpool attracting moving particles. Su (1997) gives a coffee cup illustration of the FPT: no matter how you continuously slosh the coffee around in a coffee cup, some point is always in the same position that it was before the sloshing began. And if you move this point out of its original position, you will eventually move some other point in the sloshing coffee back into its original position.In BUT terms, it means that not only we can always find a n-sphere, e.g, a brane, containing a string, but also that every string with a particular shape comes together with another string, termed a *wired friend*. These observations lead to a *wired friend theorem*: every occurrence of a wired friend string with a particular shape on the structure S^n maps to a fixed description, e.g. to another string that belongs to a manifold or Euclidean space with different dimensions. Every wired friend is recognizable by its shape, because the shape of a string is the silhouette of a wired friend string.Therefore, we are allowed to achieve a map of wired friend strings, projecting from the 2D wordsheet to the 3D cylinder and torus wordsheets. The mean achievement is that we can always find a string embedded in a higher-dimensional brane which is the description of another string in a lower dimensional brane, and vice versa.

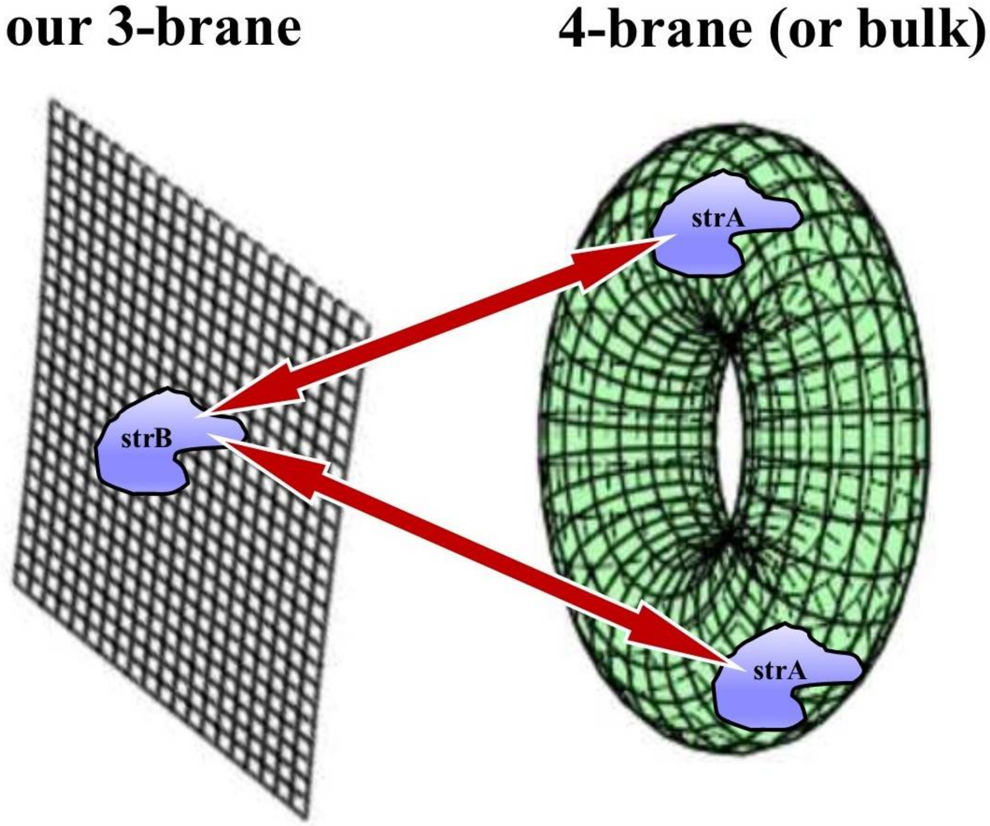

Figure 1. Samples of strings with finite length are represented by the bounded strings strA and strB. Regions strA,strBare examples of antipodal strings. Note that the torus might depict, according to the different brane theories, either a 4-brane adjacent our 3D brane, or a 4D bulk in which our 3-brane is embedded. The branes are depicted, for sake of clarity, one dimension lower than the real ones.

3. BRANES AND HOMOTOPY EQUIVALENCE

In the previous paragraph, we showed why different branes necessarily have at least a few features in common. Here we illustrate how, in topological terms, brane shapes are continually transforming into new homotopically equivalent shapes. Branes might influence each other by scattering, colliding and combining, to create bounded regions into the bulk. Hence, it is possible for branes to stick together to become *condensed branes*, e.g., worldsheets (operationally assessable in terms of the torus described above) which depict the behaviour of a collection of interacting and scattering elements.

Homotopy Equivalence.

Let $f, g : X \rightarrow Y$ be a pair of continuous maps. For example, let $f(X)$ and $g(X)$ be two branes with fixed endpoints. A *homotopy* (Cohen, 1973) between f and g is a continuous map $H : X \times [0,1] \rightarrow Y$ so that $H(x,1) = g(x)$ and $H(x,0) = f(x)$. The interest here is in the possibility of deforming (transforming) one brane with a particular shape into another with a different shape. It means that it the birth of branes, that evolve out of the interaction of initially disjoint branes, is allowed. Let $id_X : X \rightarrow X$ denote an identity map defined by $id_X(x) = x$. Similarly, $id_Y : Y \rightarrow Y$ is defined by $id_Y(t) = y$. The composition $f \circ g(X)$ is defined by $f \circ g(X) = f(g(X))$. Similarly, $g \circ f(X)$ is defined by $g \circ f(X) = g(f(X))$. The sets X and Y are homotopically equivalent, provided there are continuous maps so that $g \circ f \simeq id_X(x)$ and $f \circ g \simeq id_Y(y)$. The sets X and Y are the same homotopy type, provided X and Y are homotopically equivalent (Peters and Inan, 2016). The interest here is in evolving branes that have the same homotopy type. In effect, homotopically equivalent brane shapes have the same homotopy type (Peters and Naimpally, 2012). This leads to a comparison of branes with seemingly varying shapes and sizes that are homotopically equivalent.

Example. Homotopically Equivalent Shapes.

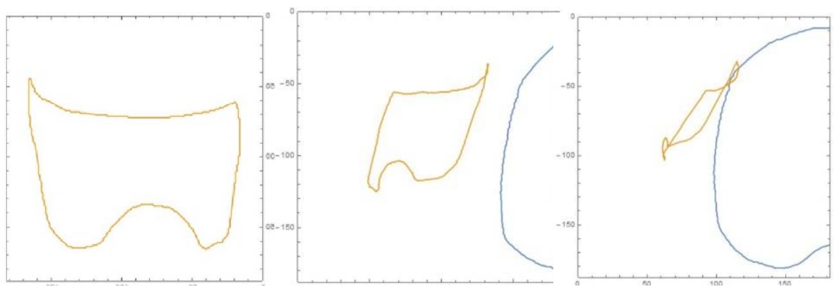

Fig. 2.A 3-Brane Fig. 2.B Two branes Fig. 2.C interacting branes

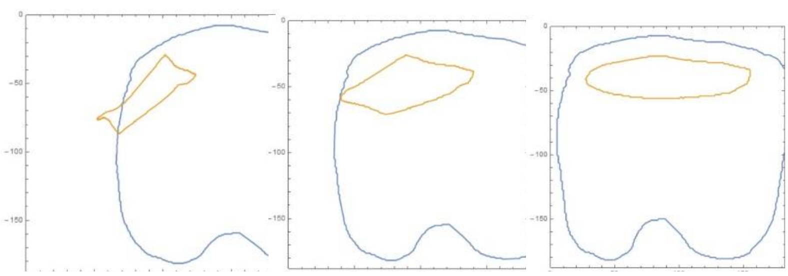

Fig. 2.D dual branes Fig. 2.E concentric branes Fig. 2.F tooth brane

Figure 2. Eventually brane will deform into another, as a result of the collision of a pair of separate branes. Let a brane be represented by the 3-brane in **Fig. 2.A**. This brane evolves over time as it twists and turns through the outer reaches of the bulk. An inkling twisting brane appearing in the neighbourhood of the first one is shown in **Fig. 2.B**. The two branes begin interacting, so that the first now has a region of space in common with the second in **Fig. 2.C**. In effect, as a result of the interaction between the branes, they are partially stitched together in **Fig. 2.C**. The partial absorbtion of one brane in another is shown in **Fig. 2.D**. Here, a very large region of bulk space occupied by the first brane is absorbed by the second and we have the birth of a condensed brane. The two branes become at first concentric in **Fig. 2.E**., then a complete condensed brane is formed with a tooth shape in Fig. **2.F**. When the two branes are completely

transformed into a new one, we have instance of their homotopy of equivalence, with the second that has completely absorbed the first.

A stitching action on a pair of branes is homotopic mapping of one brane into another. This is a further instance of the duality principle in brane cosmology. That is, one brane is the dual of another brane, provided the first can be deformed into the second. The brane in **Fig. 2.E** is an example of a Edelsbrunner-Harer nerve (Peters and Inan, 2016), which is a collection $Nrv\Im$ such that all nonempty subcollections of $Nrv\Im$ have a non-void common intersection, i.e.,

$$Nrv\Im = \left\{ X \in \Im : \bigcap X \neq \varnothing \right\}.$$

Lemma 1. Branes of the same homotopy type are stitched together to form a condensedbrane.
Proof. Let A, B be branes with the same homotopy type. Then brane A can be deformed into brane B. In effect, brane B absorbes brane A. By definition, brane B is a condensed brane. This phenomenon is true in general for homotopically equivalent branes that have the potential to be stitched together. Hence, the desired result follows.

Theorem 1. A condensed brane is an instance of an Edelsbrunner-Harer nerve.
Proof. From Lemma 1, a condensed brane is an instance of an Edelsbrunner-Harer nerve.

Theorem 2. Every brane is a Edelsbrunner-Harer nerve.
Proof. Let A be a brane. Every brane is a collection of strings. Each of the strings have brane A in common. That is, all sub-branes of brane A have nonempty intersection. Consequently, A is an Edelsbrunner-Harer nerve. Hence, the desired result follows.

4. DISCUSSION

Extra dimensional branes can come in different numbers of sizes and shapes and may display either warped or large extra-dimensions. If the bulk separates one brane from another, some strings might be embedded in the bulk, while others are confined to branes. Here we introduced novel topological variants that elucidate the features of different types of extra dimensional setups. We achieve generalizations that allow the assessment of every possible cosmological brane, independent on its shape, size or boundaries. Our results show that sets of strings, equipped either with antipodal or non-antipodal matching description and embedded in d-dimensional branes M^d, map to a single set of strings in M^{d-1}branes, and vice versa. The term matching description means the strings display common feature values, e.g., all the branes, even if embedded in different dimensions, must have features in common. It has been hypothesized that there exist different branes that are too far apart ever to communicate with one another, so that strings bounded on distant branes would never have direct contact. However, our topological investigation reveals that this scenario is unfeasible, because there must be at least one element in common among branes that are very distant one each other too. Further, all the branes and bulks will always have some element in common, not just gravity: branes do not exist in isolation, rather they are part of a larger interconnected whole with which they interact.
In brane cosmology, particle movements are described as functions occurring on manifolds equipped with different possible geometric curvatures. We showed that M may stand for a multi-dimensional brane with any kind of curvature, either concave, convex or flat. M^{d-1} may also be a part of M^d, or in other words, embedded in M^d or lying on its surface. Such rather general BUT dictates encompass almost all the brane models. Indeed, cosmic theories claiming different curvatures of the Universe (from hyperbolic anti-deSitter Cosmos, to inflationary models, to flat Riemannian 4D Universes) and brane theories describing branes with small or large extra-dimensions (from warped scenarios and Calabi-Yau manifolds, to extralarge and infinite dimensions) are DUAL: e.g., their topological description is the same, despite the huge differences in the subtending manifolds. Therefore, the shape is not important in the evaluation of branes, because it is fully interchangeable. This also means that projections between dimensions describe phenomena spanning from the smallest to the highest scales: the distinction in brane size in different theories does not count anymore.
Some branes theories predict the existence of open strings, e.g., strings equipped with two ends lying on diverse branes. For our definition of antipodality, the two strings need to be separated. An appropriate projection mapping shows that, if two strings have matching features (shape, length, width, area), the two strings are the same. This calls for the Brouwer fixed point and wired friend theorems. In sum, two strings with matching description embedded on two

branes of different dimensions are the same string and are not divided. Strings with matching ends (regions) in different dimensions might also help to explain Einstein–Rosen-Podolski bridges.
The fundamental problem with conventional approaches to duality is starting with points instead of regions. It carries over in string theory, where one *stitches* together strings to obtain a new string. This problem is solved in a framework of region-based spacetime geometry (Lentzen, 1939; Disalle, 1995; Henderson, 1996).
Our results also entail that every kind of high-dimensional brane can be described in terms of donut-like structures (Peters and Tozzi, 2016b). We showed that, while strings' features in our 3D Universe can be described in terms of particle trajectories on 3D sheets, strings' features in higher-dimensional branes can be assessed in the generic terms of particle trajectories traveling on multi-dimensional toruses. This methodological advance could be useful in order to uniform the operationalization of the countless theories in brane cosmology.
By way of application of the proposed framework, the closed paths described by BUT and FPT variants represent branes occurring in different dimensions. Every pathway in a brane is a closed string intertwined with strong proximities with other strings in other branes. Indeed, we showed that the branes' components are densely connected with the rest of the Universe. The tight coupling among different branes features gives rise to cosmic systems that are in charge of receiving and interpreting signals from other branes, in closely intertwined relationships.

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