Limit theorems for lattice group-valued k-triangular set functions

A. Boccuto and X. Dimitriou

Abstract. Using sliding hump-type techniques, we prove some Schur, Vitali-Hahn-Saks and Nikodým-type theorems for lattice group-valued k-triangular set functions.

Let R be a Dedekind complete lattice group, G be an infinite set, Σ be a σ-algebra of subsets of G, $m: \Sigma \to R$ be a bounded set function, $v: \Sigma \to \mathbb{R} \cup \{+\infty\}$ be a non-negative and monotone set function, k be a fixed positive integer and let the involved intervals and halflines be intended in $\mathbb{N}$.

Definitions 1. (a) A sequence $(p_n)_n$ in R is an (O)-*sequence* iff it is decreasing and $\bigwedge_n \; p_n = 0$.
(b) A bounded double sequence $(a_{t,l})_{t,l}$ in R is a (D)-*sequence* or a *regulator* iff $(a_{t,l})_l$ is an (O)-sequence for any $t \in \mathbb{N}$.
(c) We say that R is *weakly* σ-*distributive* iff $\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}}(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)}) = 0$ for any (D)-sequence $(a_{t,l})_{t,l}$.
(d) A sequence $(x_n)_n$ in R is (D)-*convergent* to x iff there is a (D)-sequence $(a_{t,l})_{t,l}$ in R such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $n^* \in \mathbb{N}$ with $|x_n - x| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ whenever $n \ge n^*$, and in this case we write $(D) \lim_n \; x_n = x$.
(e) We call *sum* of a series $\sum_{n=1}^{\infty} x_n$ in R the limit $(D) \lim_n \sum_{r=1}^{n} x_r$, if it exists in R.
(f) The *semivariation* of m is defined by $v(m)(A) := \bigvee\{\,|m(B)| : B \in \Sigma,\, B \subset A\}$.
(g) We say that m is k-*triangular* on Σ iff $m(A) - k\, m(B) \le m(A \cup B) \le m(A) + k\, m(B)$ whenever $A, B \in \Sigma,\; A \cap B = \emptyset$ and $0 = m(\emptyset) \le m(A)$ for each $A \in \Sigma$.

Proposition 2. *Let $m: \Sigma \to R$ be a k-triangular set function. Then $v(m)$ is k-triangular too. Moreover for any $n \in \mathbb{N},\; n \ge 2$, and for every pairwise disjoint sets $E_1,\; E_2, \dots, E_n \in \Sigma$ we get*

$$m(E_1) - k \sum_{q=2}^{n} m\,(E_q) \le m(\bigcup_{q=1}^{n} E_q) \le m(E_1) + k \sum_{q=2}^{n} m\,(E_q).$$

From now on we assume that R is a Dedekind complete and weakly σ-distributive lattice group.

Proposition 3. *Assume that $m: \mathcal{P}(\mathbb{N}) \to R$ be a k-triangular set function, such that $(D)\, lim_n\, v\,(m)(]n, +\infty[) = 0$. Then it is*

$$(D)\, \underset{n}{lim}(\bigvee_{A \subset \mathbb{N}} |\, m(A) - m(A \cap [1, n])|) = 0$$

and $(D)\, lim_n\, m\,(A \cap [1, n]) = m(A)$ for each $A \subset \mathbb{N}$.

Definitions 4. (a) Given a set function $m: \Sigma \to R$ and an algebra $\mathcal{L} \subset \Sigma$, the *semivariation of m with respect to $\mathcal{L}$* is defined by $v_{\mathcal{L}}(m)(A) = \bigvee\{\,|m(B)| : B \in \mathcal{L},\; B \subset A\}$.
(b) A set function $m: \Sigma \to R$ is said to be *continuous from above at* $\emptyset$ iff for every decreasing sequence $(H_n)_n$ in Σ with $\bigcap_{n=1}^{\infty} H_n = \emptyset$ we get $(D) \lim_n v_{\mathcal{L}}\,(m)(H_n) = \bigwedge_n v_{\mathcal{L}}\,(m)(H_n) = 0$, where $\mathcal{L}$ is the σ-algebra generated by $(H_n)_n$ in H_1.
(c) The set functions $m_j: \Sigma \to R,\; j \in \mathbb{N}$, are *uniformly continuous from above at* $\emptyset$ iff

$$(D) \lim_n (\bigvee_j v_{\mathcal{L}}\,(m_j)(H_n)) = \bigwedge_n (\bigvee_j v_{\mathcal{L}}\,(m_j)(H_n)) = 0$$

for each decreasing sequence $(H_n)_n$ in Σ with $\bigcap_{n=1}^{\infty} H_n = \emptyset$.
(d) A set function $m: \Sigma \to R$ is said to be v-*absolutely continuous* on Σ iff for each decreasing sequence $(H_n)_n$ in Σ, with $\lim_n v\,(H_n) = 0$, it is

$$(D)\lim_n v_{\mathcal{L}}(m)(H_n) = \bigwedge_n v_{\mathcal{L}}(m)(H_n) = 0.$$

(e) The set functions $m_j : \Sigma \to R$, $j \in \mathbb{N}$, are *uniformly* ν*-absolutely continuous* on Σ iff

$$(D)\lim_n(\bigvee_j v_{\mathcal{L}}(m_j)(H_n)) = \bigwedge_n(\bigvee_j v_{\mathcal{L}}(m_j)(H_n)) = 0$$

whenever $(H_n)_n$ is a decreasing sequence in Σ such that $\lim_n \nu(H_n) = \emptyset$.
(f) The set functions $m_j : \Sigma \to R$, $j \in \mathbb{N}$, are *equibounded* iff there is an element $u \in R$ with $|m_j(A)| \leq u$ for all $j \in \mathbb{N}$ and $A \subset \Sigma$.
(g) Given a sequence of set functions $m_j : \Sigma \to R$, $j \in \mathbb{N}$, we say that $(m_j)_j$ (RD)*-converges* (or *converges pointwise with respect to a single regulator*) to m_0, or $(RD)\lim_j m_j = m_0$, iff there is a (D)-sequence $(b_{t,l})_{t,l}$ such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $A \in \Sigma$ there is $j_0 \in \mathbb{N}$ with $|m_j(A) - m_0(A)| \leq \bigvee_{t=1}^{\infty} b_{t,\varphi(t)}$ for all $j \geq j_0$.
(h) We say that $(D)\lim_j m_j(A) = m_0(A)$ *uniformly with respect to* $A \in \Sigma$, or $(UD)\lim_j m_j = m_0$, iff there is a (D)-sequence $(c_{t,l})_{t,l}$ with the property that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $j_0 \in \mathbb{N}$ with $|m_j(A) - m_0(A)| \leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}$ for every $A \in \Sigma$ and $j \geq j_0$.

Lemma 5. *Let* $m : \Sigma \to R$ *be a* k*-triangular set function,* $(H_n)_n$ *be any decreasing sequence from* Σ*, put* $H = \bigcap_{n=1}^{\infty} H_n$ *and assume that* $m(H) = 0$*. Set* $B_n = H_n \setminus H_{n+1}$ *for all* $n \in \mathbb{N}$*, and denote by* $\mathcal{K}$ *and* $\mathcal{L}$ *the* σ*-algebras generated by the* B_n*'s in* H_1 *and by the* H_n*'s in* H_1 *respectively. Then we get* $v_{\mathcal{L}}(m)(H_n) \leq k\, v_{\mathcal{K}}(m)(\bigcup_{l=n}^{\infty} B_l)$ *for every* $n \in \mathbb{N}$.

We now give a characterization of continuity from above at $\emptyset$ for lattice group-valued set functions defined on $\mathcal{P}(\mathbb{N})$. Here, $v(m) = v_{\mathcal{P}(\mathbb{N})}(m)$.

Proposition 6. *An* R*-valued set function* m*, defined on* $\mathcal{P}(\mathbb{N})$*, is continuous from above at* $\emptyset$ *if and only if* $(D)\, lim_n\, v(m)(]n, +\infty[) = \bigwedge_n v(m)(]n, +\infty[) = 0$.

Remark 7. Observe that an analogous version of Proposition 6 holds also for set functions, which are uniformly continuous from above at $\emptyset$.

Lemma 8. *Let* $m_j : \mathcal{P}(\mathbb{N}) \to R$ *,* $j \in \mathbb{N}$ *, be a sequence of continuous from above at* $\emptyset$ *and equibounded* k*-triangular set functions, with* $(RD)\, lim_j\, m_j = 0$*. Then there is a* (D)*-sequence* $(d_{t,l})_{t,l}$ *in* R *such that, for every* $\varphi \in \mathbb{N}^{\mathbb{N}}$ *and for each sequence* $(j_s)_s$ *in* $\mathbb{N}$ *with* $j_s \geq s$ *for any* $s \in \mathbb{N}$*, there exists* $s_0 \in \mathbb{N}$ *with* $m_{j_s}(A) \leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)}$ *whenever* $s \geq s_0$ *and* $A \subset \mathbb{N}$.

We now give our version of the Schur theorem for k-triangular lattice group-valued set functions.

Theorem 9. *Let* R*,* m_j*,* $j \in \mathbb{N}$*, be as in Lemma* 8*. Then it is*

$$(D)\, lim_j(\sum_{n=1}^{\infty} m_j(\{n\})) = 0.$$

Furthermore, $(UD)\, lim_j\, m_j = 0$ *and the set functions* $m_j, j \in \mathbb{N}$*, are uniformly continuous from above at* $\emptyset$.

Theorem 10. (Vitali-Hahn-Saks theorem) *Let* $m_j : \Sigma \to R$*,* $j \in \mathbb{N}$*, be a sequence of equibounded,* ν*-absolutely continuous and* k*-triangular set functions, with* $(RD)\, lim_j\, m_j = 0$*. Then the* m_j*'s are uniformly* ν*-absolutely continuous.*
Theorem 11. (Nikodým convergence theorem) *Let* $m_j : \Sigma \to R$ *,* $j \in \mathbb{N}$ *, be a sequence of equibounded set functions, continuous from above at* $\emptyset$*, with* $(RD)\, lim_j\, m_j = 0$*. Then the* m_j*'s are uniformly continuous from above at* $\emptyset$.