A Proof of the Collatz Conjecture

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Introduction: The Collatz conjecture is also variously well-known as the 3n+1 conjecture, the Ulam conjecture, Kakutani's problem, the Thwaites conjecture, Hasse's algorithm, or the Syracuse problem etc. Yet it is still both unproved and un-negated a conjecture ever since named after Lothar Collats in 1937.

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Abstract

Positive integers which can operate to 1 by the set operational rule of the conjecture and positive integers got via contrary operations of the set operational rule are one-to-one correspondence unquestionably. In this article, we classify positive integers to prove the Collatz conjecture by the mathematical induction via operations of substep according to confirmed two theorems plus a lemma in advance.

Keywords: mathematical induction; the two-way operational rules; classify positive integers; the bunch of integers' chains; operational routes

Basic Concepts

The Collatz conjecture states that take any positive integer n, if n is an even number, then divide n by 2 to obtain an integer; if n is an odd number, then multiply n by 3 and add 1 to obtain an even number. Repeat the above process indefinitely, then no matter which positive integer you start with, you will always eventually reach a result of 1.

We consider the way of aforesaid two steps as the leftward operational rule. Also consider the operational rule on the contrary of the leftward operational rule as the rightward operational rule.

The rightward operational rule stipulates that for any positive integer n, uniformly multiply n by 2 to obtain an even number. In addition to this, when n is an even number, if divide the difference of n minus 1 by 3 to obtain an odd number, then must operate this step, and proceed from here to operate further; if it is not such, then there is no this step.

Taken one with another, we consider each other's opposed operational rules as two-way operational rules, and operations in two-way operational rules are called the two-way operations.

Begin with any positive integer to operate by either operational rule continuously, their operational results all are positive integers, then we regard a string of such consecutive positive integers plus arrowheads inter se on an identical operational direction as an operational route.

If positive integer P exists at a certain operational route, then may term such an operational route "an operational route of P". Two operational routes of P branch from a positive integer after pass the operation of P.

Begin with 1 to operate positive integers got successively by the rightward operational rule. Doubtlessly, it will form a bunch of operational routes automatically. We term such a bunch of operational routes "a bunch of integers' chains". Apparently whole a bunch of integers' chains must consist of infinite many operational routes.

Please, see also an initial bunch of integers' chains as the follows.

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                                  …           …     452→…
                                168↑       680↑   226↑→75→…
                               84↑       340↑→113↑    227→…
                             42↑       170↑         682↑→…
        1                   21↑       85↑         341↑
1→2→4↑→8→16↓→32→64↑→128→256↑→512→1024↑→2048→…
                 5→10↓→20→40↓→80→160↓→320→…
                    3↓      13↓      53→106↓→212→…
                     6↓      26↓         35→70↓→140→…
                      12↓      52↓→104→…     23→46↓→15→…
                       24↓      17↓               92→…
                         …       34↓→11→22↓→7→14→28↓→56…
                                  68↓      44→88↓→29…9→18→…
                                   136↓→…     29↓
                                     45→…      58↓→19→…
                                                  …    …
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An Initial Bunch of Integers' Chains

Annotation: ↓ and ↑ must rightwards tilt, but each page is narrow, thus it can only so.

Since each and every positive integer comes only from an adjacent got positive integer before itself, thus each of positive integers except for 1 at the bunch of integers' chains is unique.

No matter which positive integer, it is at the bunch of integers' chains, so long as it is able to be operated to 1 by the leftward operational rule. Conversely, if a positive integer is at the bunch of integers' chains, then the positive integer suits the conjecture certainly. Thus it can be seen, positive integers at the bunch of integers' chains and positive integers which are able to be operated to 1 by the leftward operational rule are one-to-one correspondence between the two sets of positive integers.

In this article, we classify positive integers, then pass operations by the leftward operational rule to make the proof, but must associate the bunch of integers' chains, so as to understand certain of procedures easily.

Such being the case, then we are necessary to prepare certain theorems including lemma beforehand, in order to affirm some anticipative results which suit the conjecture by them after each such result is ascertained or arises at an operational route.

Theorem 1* Known that positive integers which are less than P suit the conjecture. If there is a positive integer C where C<P at an operational route of P, then P suits the conjecture. Illustrate with examples as follows.

(1) Let $P=31+3^2\eta$ with $\eta\geq 0$, from $27+2^3\eta\to 82+3*2^3\eta\to 41+3*2^2\eta\to 124+3^2*2^2\eta\to 62+3^2*2\eta\to 31+3^2\eta>27+2^3\eta$, then $31+3^2\eta$ suits the conjecture.

(2) Let $P=5+2^2\mu$ with $\mu\geq 0$, from $5+2^2\mu\to 16+3*2^2\mu\to 8+3*2\mu\to 4+3\mu<5+2^2\mu$, then $5+2^2\mu$ suits the conjecture.

Proof* At an operational route by leftward operational rule, if C appears before P, then the operations of C passed P and reached 1 already, naturally P was operated into 1; if C appears behind P, then the operations of P pass C, and continue along operational route of C to get 1. In addition to this, at an operational route by rightward operational rule, C and P root in 1, of course, can operate either of them into 1 by leftward operational rule inversely.

Theorem 2* If an operational route of P and an operational route of C intersect at positive integer A, and positive integer D at operational route of C suits the conjecture, then P suits the conjecture. For example, let $P=63+3*2^8\varphi$ and $D=47+3^2*2^6\varphi$ where $\varphi\geq 0$, from $63+3*2^8\varphi\to 190+3^2*2^8\varphi$ $\to 95+3^2*2^7\varphi\to 286+3^3*2^7\varphi\to 143+3^3*2^6\varphi\to 430+3^4*2^6\varphi\to 215+3^4*2^5\varphi\to$ $646+3^5*2^5\varphi\to 323+3^5*2^4\varphi\to 970+3^6*2^4\varphi\to 485+3^6*2^3\varphi\to 1456+3^7*2^3\varphi\to$ $728+3^7*2^2\varphi\to 364+3^7*2\varphi\to 182+3^7\varphi\uparrow\to \ldots$

$\uparrow 121+3^6*2\varphi\leftarrow 242+3^6*2^2\varphi\leftarrow 484+3^6*2^3\varphi\leftarrow 161+3^5*2^3\varphi\leftarrow$

$322+3^5*2^4\varphi\leftarrow 107+3^4*2^4\varphi\leftarrow 214+3^4*2^5\varphi\leftarrow 71+3^3*2^5\varphi\leftarrow 142+3^3*2^6\varphi\leftarrow 47$ $+3^2*2^6\varphi< 63+3*2^8\varphi$, we get that $63+3*2^8\varphi$ suits the conjecture.

Proof * Since D and A exist at the operational route of C, and D suits the conjecture, then A suits the conjecture according to Theorem 1. Like the reason, P and A exist at the operational route of P, and A suits the conjecture, of course, P suits the conjecture too.

Lemma* If an operational route of P and an operational route of C are at indirect concatenations, and positive integer D at operational route of C suits the conjecture, then P suits the conjecture.

For example, an operational route of P intersects an operational route of Q, also the operational route of Q intersects an operational route of R, and so on and so forth one-one successively intersect until an operational route of C, and D at operational route of C suits the conjecture, then P suits the conjecture. Actually, each and every positive integer at successively intersecting operational routes suits the conjecture provided therein there is a positive integer which suits the conjecture.

The Proof

Let us set about the proof of the Collatz conjecture by the mathematical induction thereinafter.

1. We have known that all positive integers at *An Initial Bunch of Integers' Chains* in the preceding chapter suit the conjecture. And that is not difficult to find that there are 24 consecutive positive integers≥1 therein. Namely positive integers≤24 suit the conjecture.

2. After further operate positive integers at such *An Initial Bunch of Integers' Chains* by the rightward operational rule, suppose that there are n consecutive positive integers≥1 at an extended bunch of integers' chains. Namely suppose that n within positive integers ≤n suits the conjecture, where n≥24.

3. After continue to operate positive integers at the extended bunch of integers' chains by the rightward operational rule, prove that there are n+1consecutive positive integers≥1 at an overlong bunch of integers' chains. Namely prove that n+1 within positive integers≤n+1 suits the conjecture.

Proof * If n+1 is an even number such as 2m, where m ≥13 due to n ≥24, then from 2m→ m< 2m by the leftward operational rule, we get that n+1 in the case suits the conjecture according to Theorem 1.

If n+1 is an odd number, let us first divide all odd numbers whose each can become n+1 into two kinds, i.e. 5+4k and 7+4k, where $k \geq 5$.

If n+1∈5+4k, then from 5+4k→16+12k→8+6k→4+3k<5+4k by the leftward operational rule, we get that 5+4k including n+1 suits the conjecture according to Theorem 1.

For 7+4k, we again divide it into three kinds, i.e. 11+12c, 15+12c and 19+12c, where $c \geq 1$.

If n+1∈11+12c, then from 7+8c→22+24c→11+12c>7+8c by the leftward operational rule, we get that 11+12c including n+1 suits the conjecture according to Theorem 1.

After that, we first operate 15+12c by the leftward operational rule below.

15+12c→46+36c→23+18c→70+54c→35+27c ♣

d=2e+1: 29+27e (**1**)   e=2f: 142+486f→71+243f ♥
♣35+27c↓→c=2d+1: 31+27d↑→d=2e: 94+162e→47+81e↑→e=2f+1:64+81f (**2**)
c=2d: 106+162d→53+81d↓→d=2e+1:67+81e↓→e=2f+1:74+81f (**3**)
d=2e:160+486e♦   e=2f: 202+486f→101+243f ♠

g=2h+1: 200+243h (**4**)   **…**
♥ 71+243f↓→f=2g+1:157+243g↑→g=2h: 472+1458h→236+729h↑→ **…**
f=2g: 214+1458g→107+729g↓→g=2h+1: 418+729h↓→**…**
g=2h: 322+4374h→**… …**

g=2h: 86+243h (**5**)
♠101+243f↓→f=2g+1:172+243g↑→g=2h+1:1246+1458h→**…**
f=2g: 304+1458g→152+729g ↓→**…**
…

…
♦160+486e→80+243e↓→e=2f+1: 970+1458f→485+729f↑→ **…**   **…**
e=2f:40+243f↓→f=2g+1:850+1458g→ 425+729g↑→ **…**
f=2g: 20+243g↓→g=2h: 10+243h (**6**)   **…**
g=2h+1:880+1458h→440+729h↑→ **…**

Annotation:
Each of letters c, d, e, f, g, h …etc in the above-listed operational routes expresses each of natural numbers plus 0, similarly hereinafter.
Also, there are ♣↔♣, ♥↔♥, ♠↔♠, and ♦↔♦.

First define a terminology. If an operational result is less than a kind of 15+12c/ 19+12c at an operational route of 15+12c/ 19+12c, we term such an operational result “a satisfactory operational result”. If a satisfactory operational result first arises or is ascertained at an operational route by the leftward operational rule, then we call the satisfactory operational result “a first arisen or confirmed satisfactory operational result”.

Thereupon we conclude several first arisen satisfactory operational results at some branches from above-listed the bunch of operational routes of 15+12c, and orderly analyze that several kinds of 15+12c derive themselves from several first arisen satisfactory operational results monogamously, as follows.

From c=2d+1 and d=2e+1, we get c=2d+1=2(2e+1)+1=4e+3, and 15+12c =15+12(4e+3)=51+48e >29+27e where mark (1), so 15+12c with c=4e+3 suits the conjecture according to Theorem 1.

From c=2d+1, d=2e and e=2f+1, we get c=2d+1=4e+1=4(2f+1) +1=8f+5, and 15+12c=15+12(8f+5) =75+96f > 64+81f where mark (2), so 15+12c with c=8f+5 suits the conjecture according to Theorem 1.

From c=2d, d=2e+1 and e=2f+1, we get c=2d= 4e+2= 4(2f+1) +2=8f+6, and 15+12c=15+12(8f+6)=87+96f >74+81f where mark (3), so 15+12c with c=8f+6 suits the conjecture according to Theorem 1.

From c=2d+1, d=2e, e=2f, f=2g+1 and g=2h+1, we get c=2d+1=4e+1= 8f+1=8(2g+1)+1=16g+9=16(2h+1)+9=32h+25, and 15+12c=15+12(32h

+25)=315+384h >200+243h where mark (4), so 15+12c with c=32h+25 suits the conjecture according to Theorem 1.

From c=2d, d=2e+1, e=2f, f=2g+1 and g=2h, we get c=2d=2(2e+1)=4e+2 =8f+2=8(2g+1)+2=16g+10=32h+10, and 15+12c=15+12(32h+10)= 135+ 384h > 86+243h where mark (5), so 15+12c with c=32h+10 suits the conjecture according to Theorem 1.

From c=2d, d=2e, e=2f, f=2g and g=2h, we get c=2d =32h, and 15+12c = 15+12(32h) =15+384h > 10+243h where mark (6), so 15+12c with c=32h suits the conjecture according to Theorem 1.

Secondly, let us operate 19+12c by the leftward operational rule below.

19+12c→58+36c→ 29+18c→ 88+54c→ 44+27c ♣

d=2e: 11+27e (**α**)   e=2f:37+81f (**β**)
♣ 44+27c↓→c=2d: 22+27d↑→d=2e+1:148+162e→74+81e↑→e=2f+1:466+486f ♥
c=2d+1: 214+162d→107+81d↓→d=2e:322+486e ♠
d=2e+1:94+81e↓→e=2f:47+81f (**γ**)
e=2f+1:516+486f ♦

g=2h: 129+243h (**δ**)   **…**
f=2g+1:258+243g↑→g=2h+1:1504+1458h→752+729h↑→**…**
♥466+486f→233+243f↑→f=2g:700+1458g→350+729g↓→g=2h+1:3238+4374h↓
g=2h: 175+729h↓→**…**   **…**
…

g=2h+1:172+243h (**ε**)
f=2g: 101+243g↑→g=2h: 304+1458h→**…**
e=2f+1:202+243f↑→f=2g+1:1336+1458g→**…**
♠322+486e→161+243e↑→e=2f:484+1458f→**…**

♦516+486f→258+243f↓→f=2g+1:1504+1458g→**…**
f=2g: 129+243g↓→g=2h: 388+1458h→**…**
g=2h+1:186+243h (**ζ**)

Annotation:
Each of letters c, d, e, f, g, h …etc in the above-listed operational routes expresses each of natural numbers plus 0, similarly hereinafter.
Also, there are ♣↔♣, ♥↔♥, ♠↔♠, and ♦↔♦.

Like that, we conclude several first arisen satisfactory operational results at some branches from above-listed the bunch of operational routes of 19+12c too, and orderly analyze that several kinds of 19+12c derive themselves from several first arisen satisfactory operational results monogamously, as follows.

From c=2d and d=2e, we get c=2d=4e, and 19+12c=19+12(4e) =19+48e >11+27e where mark (α), so 19+12c with c=4e suits the conjecture according to Theorem 1.

From c=2d, d=2e+1 and e=2f, we get c=2d = 2(2e+1) = 4e+2 = 8f+2, and 19+12c=19+12(8f+2) = 43+96f >37+81f where mark (β), so 19+12c with c=8f+2 suits the conjecture according to Theorem 1.

From c=2d+1, d=2e+1 and e=2f, we get c=2d+1=4e+3=8f+3, and 19+12c=19+12(8f+3) =55+96f>47+81f where mark (γ), so 19+12c with c=8f+3 suits the conjecture according to Theorem 1.

From c=2d, d=2e+1, e=2f+1, f=2g+1 and g=2h, we get c=2d=2(2e+1)= 4e+2= 4(2f+1)+2 = 8f+6= 8(2g+1)+6 = 16g+14 = 32h+14, and 19+12c = 19+12(32h+14) = 187+384h >129+243h where mark (δ), so 19+12c with c=32h+14 suits the conjecture according to Theorem 1.

From c=2d+1, d=2e, e=2f+1, f=2g and g=2h+1, we get c=2d+1=4e+1= 4(2f+1)+1=8f+5=16g+5=16(2h+1)+5=32h+21, and 19+12c=19+12(32h +21) =271+384h > 172+243h where mark (ε), so 19+12c with c=32h+21 suits the conjecture according to Theorem 1.

From c=2d+1, d=2e+1, e=2f+1, f=2g and g=2h+1, we get c=2d+1=2(2e+1)+1=4e+3=4(2f+1)+3=8f+7=16g+7=16(2h+1)+7=32h+23, and 19+12c=19+12(32h+23)=295+384h > 186+243h where mark (ζ), so 19+12c with c=32h+23 suits the conjecture according to Theorem 1.

By now, we transform above-listed proven 51+48e, 75+96f, 87+96f, 315+384h, 135+384h, 15+384h, 19+48e, 43+96f, 55+96f, 187+384h, 271+384h and 295+384h into $51+2^4\times3e$, $75+2^5\times3f$, $87+2^5\times3f$, $315+2^7\times3h$, $135+2^7\times3h$, $15+2^7\times3h$, $19+2^4\times3e$, $43+2^5\times3f$, $55+2^5\times3f$, $187+2^7\times3h$, $271+2^7\times3h$ and $295+2^7\times3h$ in proper order. Therein each exponent of 2 is actually the number of times that a kind of 15+12c/19+12c divided by 2 in the course of operations from the kind of 15+12c/19+12c to first arisen satisfactory operational result at an operational route of 15+12c/ 19+12c.

Since each of first arisen satisfactory operational results is smaller than a kind of 15+12c/19+12c, thus a kind of 15+12c/19+12c derived from each of first arisen satisfactory operational results, i.e. every kind of 15+12c/ 19+12c which can operate into a first arisen satisfactory operational result by the leftward operational rule suits the conjecture.

Without doubt, if n+1 belongs within above-listed any kind of 15+12c/ 19+12c, then n+1 suits the conjecture like the kind of 15+12c/ 19+12c.

Let χ represents together variables d, e, f, g, h, … etc. within integer's expressions at two bunches of operational routes of 15+12c plus 19+12c, but χ represents not c.

Then the odevity of part integer's expressions which contain variable χ at the two bunches of operational routes is still indeterminate. That is to say, for every such integer's expression which contains variable χ, both consider it as an odd number to operate, and consider it as an even number to operate. Therefore, let us label such integer's expressions "odd-even expressions".

For any odd-even expression at a bunch of operational routes of 15+12c/ 19+12c, two kinds of operations synchronize at itself due to the odevity of variable χ. After regard an odd-even expression as an odd number to operate, get an operational result which is greater than itself. Yet after regard it as an even number to operate, what we get is an operational result which is smaller than itself.

Begin with any odd-even expression to operate continuously by the leftward operational rule, every such operational route via consecutive greater operational results will elongate infinitely, and that orderly arisen odd-even expressions therein are getting greater and greater up to infinity.

Then again, for a smaller operational result in synchronism with a greater operational result, when it divided by 2^e to get an integer's expression which is greater than any kind of 15+12c/ 19+12c, then the integer's expression is an odd-even expression still.

By this token, odd-even expressions are getting both greater and greater, and more and more along the continuation of operations, up to arise both infinity and infinite many. Consequent on, operational routes of 15+12c/ 19+12c necessarily appear both infinite long and infinite many. Naturally there is a first arisen or confirmed satisfactory operational result at every such operational route. Moreover every first arisen or confirmed satisfactory operational result can lead to a kind of 15+12c/ 19+12c to suit the conjecture. Thus it can be seen, once a satisfactory operational result appears at an operational route of 15+12c/ 19+12c, operations of the operational route may stop immediately.

Now that a kind of 15+12c/ 19+12c which suits the conjecture can derive itself from a first arisen or confirmed satisfactory operational result at an operational route alone of 15+12c/ 19+12c, then 15+12c and 19+12c must be divided into infinite many kinds respectively, just enable its all kinds to be derived from infinite many first arisen or confirmed satisfactory operational results which monogamously lie at infinite many operational routes of 15+12c/19+12c.

Even so, however all operational routes of 15+12c/ 19+12c are either at the direct intersection or at the indirect concatenation per two strips.

Ut supra, we have operated out several first arisen satisfactory operational results at operational routes of 15+12c/ 19+12c such as 29+27e, 64+81f, 200+243h, 11+27e, 37+81f, 129+243h etc.

That is to say, any of operational routes which contain first arisen satisfactory operational results and each of unsighted infinite many operational routes are either at the direct intersection or at the indirect concatenation.

Additionally, those first arisen satisfactory operational results suit the conjecture like derived kinds of 15+12c/19+12c by them.

Therefore all first confirmed satisfactory operational results which lie monogamously at all unsighted operational routes of 15+12c/ 19+12c suit the conjecture according to Theorem 2 plus Lemma.

Since first confirmed satisfactory operational result at every unsighted operational route of 15+12c/19+12c is smaller than a kind of 15+12c/ 19+12c too, and that each of all kinds of 15+12c/ 19+12c derives only itself from a first confirmed satisfactory operational result, so a kind of 15+12c/19+12c derived from every first confirmed satisfactory operational result suits the conjecture according to Theorem 1.

Overall, even though there are infinitely many kinds of 15+12c/19+12c, likewise they all are proved to suit the conjecture by us.

Consequently, if n+1 belongs within any kind of 15+12c/19+12c, then the kind of 15+12c/ 19+12c including n+1 suits the conjecture.

To sum up, we have proven that positive integer n+1 suits the conjecture in which case n+1 belongs within any sort/kind of positive integers.

After proven that positive integer n+1 suits the conjecture, we likewise are able to prove that positive integers n+2, n+3, n+4 etc. up to every positive integer suits the conjecture in the same old way.

So much for that the Collatz conjecture is proven by us integrally. The proof was thus brought to a close. As a consequence, the Collatz conjecture holds water.

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