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Non-numeric labels and constrained focal elements

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Abstract: *The theory of belief functions allows to build a large family of combination operators, based mostly on intersections and unions between the focal elements expressed by the experts, and multiplications and additions on the masses affected to these focal elements. This chapter explores some algebraic structures where these operators behave differently, masses being linguistic labels, or focal elements being more, or less, than an union of singletons of a discernment space. In some cases, it will be necessary to forget this space and the notion of singleton to work within a space of possible focal elements. We propose five new definitions of labels, with the corresponding algebras, to replace the masses of* $[0, 1]$*. Adaptations of the theory of belief functions to six constrained spaces for the focal elements expressed by the experts are presented.*

9.1 Introduction

The theory of belief functions, also called theory of evidence or Dempster-Shafer theory [4, 20], relies on the definition of basic belief assignments. A large family of combination operators provides information fusion capabilities. An overview of their behaviour, based on the decisions they induce, has been made with some analysis of similarity tools [18]. This chapter extends some previous works of the authors [19].

However, an automatic process is likely to express a mass between 0 and 1, but a human expert may find it difficult. Interpreting the result can even be a bit more difficult. The operations applied on the focal elements, particularly the union, may lead to elements that cannot be interpreted, nor expressed in the expert's syntax. Therefore, the objective of providing a meaningful basic belief assignment implies the ability to constrain its focal elements within an extension of the discernment space that is restricted to acceptable elements. Dempster and Shafer propose to build this extension by closing the discernment space by the union operator, while Dezert and Smarandache [21] close it by union and intersection (and even by complementation in [23]), and define an equivalence class of the empty set to restrict the hyper-power set obtained. An aim of this chapter is to explore the question of the other closures of the discernment space, with an algebraic point of view, and some algorithmic complexity concerns.

The interpretability and robustness of the values given to the masses in a man-machine interaction is another of the topics of this chapter, and we propose some algebraic constructions to address both the formulas of the theory of belief functions and the human processes of decision making.

The section 9.2 browses the most common definitions, functions and operators of the theory of belief functions. They are classified by their needs of algebraic structures, considering the operators appearing in their definitions – a list of their structures used in the chapter is given in appendix. The section 9.3 proposes three new types of linguistic labels: auto-indenting labels, unfinite auto-indenting labels and soft auto-indenting labels, test their algebraic properties, and illustrates their differences on an example, including a $\max - \min$ algebra reference. Two other extensions of the $[0, 1]$ real segment are proposed. The section 9.4 shows how to adapt the theory of belief functions to six situations where the properties of the space containing the focal elements of the basic belief assignment are more constrained than a power set or a hyper-power set. Next to the list of the compatible operators, one should keep an eye to the combinatorial complexity they require.

9.2 Theories of belief functions

9.2.1 Basic belief assignments

On a finite or discrete set Θ, called the *discernment space*, it allows to provide mass on any subset of Θ instead of its singletons. Such a mass repartition is called a *basic belief assignment* (bba) m:

$$\sum_{X \in 2^\Theta} m(X) = 1 \tag{9.1}$$

$$\forall X \in 2^\Theta \;\; m(X) \geqslant 0 \tag{9.2}$$

The hypothesis of *closed world* [20] can be added to this definition:

$$m(\emptyset) = 0 \tag{9.3}$$

It is equivalent to allow an *open world*, or add a special element to Θ, receiving the mass $\emptyset$, and use the properties of a closed world.

If A is an element of 2^Θ with a non-zero mass, it is called a *focal element*. As a possible bearer of mass, Θ is the *ignorance*. We will call $\mathcal{F}(m)$ the set of the focal elements of m, the *focal set* of m. The notation n will be reserved for the cardinal of Θ.

The equation (9.1) can be extended to the hyper-power set D^Θ, closure of Θ under union and intersection operators. Therefore the exclusivity between elements of Θ is not necessary, and one can put some mass on their intersection:

$$\sum_{X \in D^\Theta} m(X) = 1 \tag{9.4}$$

The set of the possible focal elements will be called the *extension* of Θ, noted $E(\Theta)$. It may be 2^Θ (the power set), D^Θ (the hyper-power set), S^Θ (the super-power set), or another closure of Θ.

9.2.2 Decision-aid functions

Belief (Bel), plausibility (Pl) and pignistic probability (BetP) can be used to build increasing monotonic functions on 2^Θ : $A \subset B$ implies $f(A) \leqslant f(B)$. As $\mathrm{Bel}(A) \leqslant \mathrm{BetP}(A) \leqslant \mathrm{Pl}(A)$, the pignisitic probability is often considered as an interesting compromise. For any $X \in E(\theta)$, these functions are defined by:

$$\mathrm{Bel}(X) = \sum_{Y \in E(\Theta), Y \subset X, Y \neq \emptyset} m(Y) \tag{9.5}$$

$$\mathrm{Pl}(X) = \sum_{Y \in E(\Theta), Y \cap X \neq \emptyset} m(Y) \tag{9.6}$$

$$\mathrm{BetP}(X) = \sum_{Y \in E(\Theta), Y \neq \emptyset} \frac{|X \cap Y|}{|Y|} \frac{m(Y)}{1 - m(\emptyset)} \tag{9.7}$$

To take a decision, one can choose the maximum of mass, the maximum of belief, the maximum of plausibility or the maximum of pignistic probability. As the three last functions are increasing, their maximum is reached for the ignorance Θ. They will be used for decision-making after selecting a subset of $E(\Theta)$, where all elements are pair-wise incomparable, by example by fixing the cardinal of a possible decision, generally limiting it to the singletons. It is also possible to use a discounting method to deceive the larger elements of $E(\Theta)$.

The cardinal $|X|$ is the number of singletons of Θ included in X when $E(\Theta)$ is 2^Θ, and is defined by the number of regions of the Venn diagram of Θ included in X when $E(\Theta)$ is D^Θ [5]. Many other decision functions have been proposed on 2^Θ, most recently by Cuzzolin [3] and Sudano [27] and adapted to D^Θ and qualitative labels by Dezert and Smarandache [7].

9.2.3 Usual combination operators

A combination operator takes two or more bba's to build another bba. It is an inner operation (Bel, Pl or BetP are not).

The *mean* operator is the simpler one. Its focal set is the union of the focal sets of the input bba's.

$$\mathrm{Mean}(m_1, \ldots, m_N)(X) = \frac{1}{N} \sum_{i=1}^{N} m_i(X) \tag{9.8}$$

The *conjunctive operator*, proposed by Smets [25] for two input bba's m_1 and m_2, is given by the equation (9.9). It puts the mass $m_1(A)m_2(B)$ on the set $A \cap B$. It is an associative operator, so it is useless to write its expression for N input bba's[1]. Dempster [4] prefers a normalized version, multiplying all terms by $\frac{1}{1-m(\emptyset)}$: it has the same associativity property. Yager transfers the conflicting mass $m(\emptyset)$ on ignorance $m(\Theta)$, loosing associativity.

$$m_{\mathrm{Conj}}(X) = \sum_{A \cap B = X} m_1(A) m_2(B) \tag{9.9}$$

The *disjunctive operator* tranfers the mass $m_1(A)m_2(B)$ on the set $A \cup B$. It is usually seen as insufficiently informative, as it transfers mass on a local ignorance in case of distinct focal elements; it preserves the closed world hypothesis. Like the conjunctive operator, it is associative.

$$m_{\mathrm{Dis}}(X) = \sum_{A \cup B = X} m_1(A) m_2(B) \tag{9.10}$$

[1] Fusing N bbas of p focal elements, such an expression would decline into a $\mathcal{O}(p^N)$ algorithm, but the associativity can lead to an algorithm in $\mathcal{O}(npN)$ operations, if the number of possible focal elements is linearly limited to $n = |\Theta|$.

The Dubois & Prade combination operator [8] is an interesting compromise between the conjunctive and the disjunctive ones. It puts the mass $m_1(A)m_2(B)$ on $A \cap B$ if $A \cap B$ is not empty, and on $A \cup B$ if it is. It respects the closed world hypothesis, adds information like the conjunctive rule (and even better, as local ignorance should be preferred to conflict), but is not associative. The extension of this rule, called hybrid DSm rule (DSmH) in Dezert-Smarandache Theory (DSmT) framework for dealing with dynamic frames of discernment with non existential integrity constraints has been proposed in [21].

$$m_{\mathrm{DP}}(X) = \sum_{A \cap B = X} m_1(A)m_2(B) + \sum_{\substack{A \cap B = \emptyset \\ A \cup B = X}} m_1(A)m_2(B) \tag{9.11}$$

$$m_{\mathrm{DP}}(\emptyset) = 0 \tag{9.12}$$

A panel of conflict redistributing rules have been proposed [12, 14, 22, 23]; none is associative. The most used is the PCR5/6 combination operator[2] which is defined for two bba's by.

$$m_{\mathrm{PCR5/6}}(X) = m_{\mathrm{Conj}}(X) + \sum_{\substack{Y \subset \Theta, \\ X \cap Y = \emptyset}} \left(\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X) + m_1(Y)} \right) \tag{9.13}$$

9.2.4 Enough operators available?

To define a basic belief assignment, to compute its belief, plausibility and pignistic probability, to apply combination operators, we need to have access to many operators on the masses and the focal elements:

- **M**asses: For most operators, they are multiplied and added. For the Mean combination, they are multiplied by a real number. For the normalization procedure of Demspter, the PCR5/6 operator or the pignistic probability, it is necessary to divide by a mass.

- **E**lements: They pass through intersection and union operators. They are also compared with $\emptyset$ and with a given element of $E(\Theta)$.

The methods exposed in section 9.2.5 need that the masses are expressed in a $\mathbb{R}$-algebra (addition, inner invertible multiplication, external multiplication). They need that the focal elements are expressed in a lattice where Θ and $\emptyset$ are extremum. The appendix provides a list of definitions for all the algebraic structures presented in this chapter.

[2]PCR5 and PCR6 are identical when combining two bba's, and differ for more.

In this chapter, we are interested in the following question:

Q: What does happen if the masses and focal elements live in poorer algebraic structures?

9.2.5 Operators for the usual bba operations

As 2^Θ is built by taking all the possible unions of elements of Θ, it is a semi-lattice. If there is no union operator, but a meet operator $\vee$, one gets the closure of Θ by $\vee$.

The lattice $(2^\Theta, \cap, \cup)$ (power set of Θ) is obtained by closing Θ by the operator $\cup$. Its bottom is $\emptyset$, its top is Θ.

The lattice $(D^\Theta, \cap, \cup)$ (hyper-power set of Θ), used as basis of DSmT [21], is obtained by closing Θ by the operators $\cap$ and $\cup$. If $\Theta = \{\theta_1, \ldots, \theta_n\}$. Its bottom is $\cap_{i=1}^{n}\theta_i$, its top is Θ.

Adding constraints on intersections and unions to build an equivalence class for $\emptyset$ corresponds to an anti-chain in the more general lattice. The anti-chain cuts the lower part of the lattice, and its bottom becomes $\emptyset$, as an efficient element of the equivalence class.

The section 9.4 explores some of the lattices that can also be used to express focal elements.

As a mass is usually a real number, the term *label* will be used when the values assigned to focal elements are not necessarily in a field.

To define a basic belief assignment, the normalization condition (9.1) implies the labels can be added, and a constant value takes the role of "1". The fact 1 is the neutral element for the multiplication operator, which eliminates the normalization step for the Conj, Dis, DP or PCR combination operators.

This normalization condition may have to be relaxed in an other label algebra, if the "addition" operator cannot have these comfortable properties.

Calculating the plausibility $\mathrm{Pl}(A)$ (9.6) requires an inner addition for the labels (semi-group structure), and determines if an intersection between A and another element of $E(\Theta)$ is empty.

Calculating the belief $\mathrm{Bel}(A)$ (9.5) requires an inner addition for the labels, and determine if an element X of $E(\Theta)$ is included in A. This can be extended to any partial order $\leqslant$ on $E(\Theta)$:

$$\mathrm{Bel}(A) = \sum_{X \leqslant A} m(X) \tag{9.14}$$

Calculating the pignistic probability $\mathrm{Bel}(A)$ (9.7) requires an inner addition, an inner multiplication and a scalar multiplication – an algebra over $\mathbb{R}$ or $\mathbb{Q}$ – for the labels. In an open world hypothesis ($m(\emptyset)$ can be nonzero) the inner multiplication operator must be invertible.

Non-numeric labels will hardly support a pignistic transformation, but in using the DSm Field and Linear Algebra of Refined Labels (FLARL) proposed in this volume. On $E(\Theta)$, an intersection and a cardinal are required.

For the Mean operator (9.8), the only requirements are an inner addition and a scalar multiplication. Labels can be elements of a vector space.

For the Dis operator (9.10), the requirement is an union operator on $E(\Theta)$. It can be the same operator that the one used to extend Θ to $E(\Theta)$: we get 2^Θ. Then $E(\Theta)$ only needs to be a semi-lattice $(\Theta, \vee)$. Labels live in a ring.

For the Conj operator (9.9), the requirement is an intersection operator, distinct of the one used to extend Θ to $E(\Theta)$. So we need a complete lattice structure on Θ. If $E(\Theta)$ exists without any reference to Θ, a semi-lattice $(E(\Theta), \wedge)$ can be sufficient. Labels must be in a ring too. The DP combination operator (9.11) and Yager's rule have the same constraints.

The normalized Dempster rule needs the multiplication and the addition on the labels to be invertible, because of the multiplication by $1/(1 - m(\emptyset))$.

The PCR5/6 operator, like the pignistic transformation, needs the labels to be expressed in a $\mathbb{R}$-algebra, with an invertible inner multiplication. An intersection operator is needed, but not the cardinal. That makes the hyper-power set D^Θ a convenient lattice for this operator.

9.3 Extending the definition of labels

Smarandache and Dezert proposed in this volume and in [13, 24] a field structure for linguistic labels, allowing all the combination operators and functions described in the sections 9.2.2 and 9.2.3. Their approach requires a hypothesis of equi-repartition of the linguistic labels which may be hard to fit with human experts' outputs. The normalization condition (9.1) is hard to satisfy, as the value 1 should be reached after at least one integer approximation step, that's why we explore here other algebraic structures.

The section proposes five algebraic structures to associate a belief level with a focal element. None of them is concerned by an equi-repartition hypothesis: the intervals bear the repartition information, where a discretization just give an element of all the admissible values, possibly the center of them. The four other ones just take into account an order or a lattice, and are not concerned with the repartition question, bound to field structure.

The first three structures are variations around the $\max - \min$ algebra on a finite total order, which contains all the possible labels. So, the finite linguistic set L is predetermined ordered set. Some structures allows it to evolve: the soft auto-indenting labels is such an example.

The fourth structure concerns an extension to any lattice for the labels. It is illustrated by a partial order on semantic labels, but placing the labels in the $[0,1]^3$ cube would fell in this algebra too.

The interval structure extends real numbers of $[0,1]$ to real intervals of $[0,1]$. Therefore the normalization condition becomes $1 \in \sum_{X\in\mathcal{F}(m)} m(X)$; a drawback of this system is that what the information on the focal elements is refined by the fusion operators, the information on the labels is diluted. It approaches the behaviour of a Galois lattice.

9.3.1 Discrete and totally ordered labels

The Conj, Dis and DP combination operators are based on a ring structure over the labels: $(L, +, \times)$. These operators can be replaced to get some other rings: $(L, +, \max)$ or $(L, \max, \min)$.

In the first case, they form a structure equivalent to $\mathbb{N}$: one can take a positive non-zero element of L, and define the successor of an element ℓ of L by $\ell + x$. So L either is not finite, and therefore inadequate for linguistic labels, either there is an element whose successor is zero, and it is impossible to define an order on L (that's why $\mathbb{Z}/n\mathbb{Z}$ is not useful for semantic labels).

As L is a finite ordered set, $s(x)$ denotes the successor of an element x: $x \leqslant s(x)$, $x \neq s(x)$, and if $x \neq y$ and $x \leqslant y$, then $s(x) \leqslant y$. the minimum of L is noted 0_L, and M_L its maximum. An element of $E(\Theta)$ with a label 0_L is not a focal element.

9.3.2 Max-Min algebra

The max and min operators effectively fulfill the distribution property, and define a ring on L:

$$\min(a, \max(b, c)) = \max(\min(a, b), \min(a, c))$$

Note that this ring[3] is also a lattice.

For any elements a and b of an ordered set L, $\min(a, b) \in L$ and $\max(a, b) \in L$. So, the result of any expression involving elements of L, min and max is still in L. If

[3] As this ring is compatible with the multiplication by a positive number, it is usually called an algebra. Here, of course, its ring properties are the only useful ones.

such an expression involves elements a_1, …, a_k, then

$$f(a_1, \ldots, a_k) \leqslant \max(a_1, \ldots, a_k) \tag{9.15}$$

Therefore, if $\top$ is the top of the lattice or semi-lattice $E(\Theta)$,

$$\mathrm{Pl}(\top) = \max_{A \in \mathcal{F}(m)} m(A) \tag{9.16}$$

A normalization condition can be that at least one label of m is M_L. This condition is robust to the Conj, Dis and DP combination operators.

However, this algebra lacks a useful property of the usual combination operators of the theory of belief functions: many small amounts of evidence cannot make a high amount of evidence.

In the two following structure, min is kept as a replacement for the inner multiplication, but the max operator is slightly transformed.

9.3.3 Auto-indenting labels

The operators for the auto-indenting labels (AIL) treat differently the case of equality for the max.

$$x \oplus_{\mathrm{AIL}} y = \begin{cases} \max(x, y) & \text{if } x \neq y \\ s(x) & \text{if } x = y, x \neq 0_L, x \neq M_L \\ 0_L & \text{if } x = y = 0_L \\ M_L & \text{if } x = y = M_L \end{cases} \tag{9.17}$$

The second condition allows to enforce a focal element receiving many similar labels. The third condition guarantees that O_L is a neutral element for $\oplus$. The fourth condition guarantees that M_L remains the higher possible label. If it is removed, new labels over M_L are allowed. This defines an other operator, which creates unfinite auto-indenting labels, but appearing more slowly than in a $(L, +, \max)$ ring. It is noted AIL_∞:

$$x \oplus_{\mathrm{AIL}_\infty} y = \begin{cases} \max(x, y) & \text{if } x \neq y \\ s(x) & \text{if } x = y, x \neq 0_L, x \neq M_L \\ 0_L & \text{if } x = y = 0_L \end{cases} \tag{9.18}$$

Note that, for AIL and AIL_∞, $x \oplus x \oplus x \oplus x = s(s(x))$. The later example, in the section 9.3.5, corresponds to $x \oplus x \oplus x \oplus x \oplus s(s(x)) = x(s(s(x)))$.

These operators are not distributive over min:

$$\min(1, 2 \oplus 2) = \min(1, 3) = 1 \tag{9.19}$$

$$\min(1, 2) \oplus \min(1, 2) = 1 \oplus 1 = 2 \tag{9.20}$$

So using AIL or AIL_∞ suppresses the associativity property of the Conj and Dis combination operators. AIL respects the normalization property (9.1) through the Conj, Dis and DP combination operators, but AIL_∞ does not.

9.3.4 Soft auto-indenting labels

To distinguish between a label reached by the bba's' information and a label reached by accumulation of lower labels, one should prefer to create intermediary labels than jump to the next one[4]. This new label, taking place between x and $s(x)$, is noted x^+ and called "a bit more than x".

$$x \oplus_{\text{SAIL}} y = \begin{cases} \max(x,y) & \text{if } x \neq y \\ x^+ & \text{if } x = y, x = y^+, \text{ or } x^+ = y \\ 0_L & \text{if } x = y = 0_L \\ M_L & \text{if } x = y = M_L \end{cases} \tag{9.21}$$

The following table gives the value of $x \oplus_{\text{SAIL}} y$ for x and y taking their values in a label set extended from $\{0, 1, 2, M\}$.

$x \backslash y$	0	1	1^+	2	M
0	0	1	1^+	2	M
1	1	1^+	1^+	2	M
1^+	1^+	1^+	1^+	2	M
2	2	2	2	2^+	M
M	M	M	M	M	M

SAIL respects the normalization property (9.1) through the Conj, Dis and DP combination operators.

9.3.5 Example

The set of linguistic labels is $L = \{\text{no, low, mod, high}\}$ (where mod means moderate). The label "no" is the non-focal label 0_L, and the label "high" is the maximum of L, M_L.

In the following table, we consider the labels in a ring or a pseudo-ring $(L, \oplus, \otimes)$. We eventually transform the $\otimes$ operator in a more usual multiplication symbol on

[4]To formalize a debate initiated by Terry Gillian, an African swallow is stronger than an European swallow, but how many European swallows are required to carry as much weight as an African swallow?

the labels ($x.y$ or x^2). The Conj combination operator gives:

$$\begin{aligned}
m(A) &= m_1(A)\otimes m_2(A) \oplus m_1(A)\otimes m_2(A\cup B) \oplus \\
&\quad m_1(A)\otimes m_2(A\cup C) \oplus m_1(A\cup B)\otimes m_2(A) \oplus \\
&\quad m_1(A\cup B)\otimes m_2(A\cup C) \oplus m_1(\Theta)\otimes m_2(A) \\
&= \text{low}^2 \oplus \text{low}^2 \oplus \text{low.high} \oplus \text{high.low} \oplus \\
&\quad \text{high}^2 \oplus \text{mod.low} \\
m(A\cup B) &= m_1(A\cup B)\otimes m_2(A\cup B) \oplus m_1(\Theta)\otimes m_2(A\cup B) \\
&= \text{high.low} \oplus \text{mod.low} \\
m(A\cup C) &= m_1(\Theta)\otimes m_2(A\cup C) = \text{mod.high}
\end{aligned}$$

	A	$A\cup B$	$A\cup C$	Θ
m_1	low	high	no	mod
m_2	low	low	high	no
$\max - \min$	high	low	mod	no
AIL	high	mod	mod	no
AIL_∞	more than high	mod	mod	no
SAIL	high	a bit more than low	mod	no

The label systems AIL, AIL_∞ and SAIL are purely discrete and semantic; they allow a certain form of normalization (at least, the bba's they produce respect a normalization constraint); they allow a decision step by maximizing belief, plausibility or mass; they can't produce a pignistic probability.

9.3.6 Lattice Labels

A projection on a total order – numeric or linguistic – may be insufficiently for modeling a confidence on a piece of information. In the *TTA150*, a French military manual, a confidence on a communication channel should be characterized by its strength (*strong*, *quite strong*, *feeble*, *very feeble*) and its quality (*clear*, *readable*, *deformed*, *with interference*). Therefore, the quality of an information received through such a channel should be characterized by the pair formed by its *strength* and its *quality*. The pairs $(strength, quality)$ live in a lattice, where $(x, y) \leqslant (x', y')$ if $x \leqslant x'$ and $y \leqslant y'$. Therefore, $(x, y) \vee (x', y') = (\max(x, x'), \max(y, y'))$ and $(x, y) \wedge (x', y') = (\min(x, x'), \min(y, y'))$. The top of the lattice is "strong and clear" while its bottom is "very feeble with interferences". A lattice is usually not distributive and this one, unlike the max and min operators of section 9.3.2, is not distributive. So using with more than two input bba's make it depend on the order of the fusion.

In the following example, the strength labels are compressed into (*Str, QS, Fee, VF*) and the quality labels into (*Cl, Read, Def, Int*).

	A	$A \cup B$	$A \cup C$	Θ
m_1	Str, Cl	Str, Def		Fee, Int
m_2		QS, Def	Fee, Read	VF, Read
m_{Conj}	QS, Read	Fee, Def	Fee, Int	VF, Int

The label on A is obtained through:

$$\begin{aligned} m(A) &= m_1(A) \wedge m_2(\Theta) \vee m_1(A) \wedge m_2(A \cup B) \vee m_1(A) \wedge m_2(A \cup C) \\ &\quad \vee m_1(A \cup B) \wedge m_2(A \cup C) \\ &= (VF, Read) \vee (QS, Def) \vee (Fee, Read) \vee (Fee, Def) \\ &= (QS, Read) \end{aligned}$$

9.3.7 Interval masses

A way to make easier the numerical expression of a human expert is to allow him to give a lower and an upper bound for each mass he commits to a focal element. Therefore the label on X is an interval $[x] = [\underline{x}, \overline{x}]$ where $0 \leqslant \underline{x} \leqslant \overline{x} \leqslant 1$. X is a focal element unless $\underline{x} = \overline{x} = 0$. This idea for working with imprecise mass of belief has been proposed and extended to unions of intervals by Dezert and Smarandache [6]; this section focus on the algebraic properties of such extension.

Interval arithmetic [10] does not have a true distributivity property, but only $[x] \times ([y] + [z]) \subset [x] \times [y] + [x] \times [z]$, it is better to factorize the expression obtained before calculation of the upper and lower bounds. But if the lower bounds of $[x]$, $[y]$ and $[z]$ are positive, the equality is reached. Therefore, the context of bba's brings the distributivity property.

Considering $[x]$ and $[y]$ two intervals of $[0, 1]$, the operators $+$ and $\times$ are defined as follows. The last line is only valid if $0 \notin [x]$, and the second one cease to be valid if $\underline{x} < 0$ or $\underline{y} < 0$.

$$\begin{aligned} [x] + [y] &= [\underline{x} + \underline{y}, \overline{x} + \overline{y}] \\ [x] \times [y] &= [\underline{x} \times \underline{y}, \overline{x} \times \overline{y}] \\ 1/[x] &= [1/\overline{x}, 1/\underline{x}] \end{aligned}$$

A real x is also the interval $[x, x]$, as stated by the bba 1 for the focal element A. As all the $m(A)$ are intervals, $\sum_{A \in E(\Theta)} m(A)$ is also an interval and one can verify that both bba's are valid toward the relaxed normalization rule:

$$1 \in \sum_{A \in E(\Theta)} m(A) \tag{9.22}$$

Example:

	m_1	m_2	m_{Conj}	m_{DS}	BetP
$\emptyset$			$[0.12, 0.16]$	0	0
A	0.2	0	$[0.22, 0.26]$	$[0.25, 0.31]$	$[0.29, 0.524]$
B	0	0	$[0.18, 0.41]$	$[0.205, 0.488]$	$[0.381, 1]$
C	0	0	0	0	$[0.137, 0.333]$
$A \cup B$	$[0.3, 0.5]$	$[0.6, 0.8]$	$[0.07, 0.36]$	$[0.08, 0.429]$	0
$B \cup C$	0	$[0.1, 0.3]$	$[0.24, 0.56]$	$[0.273, 0.667]$	0
$A \cup B \cup C$	$[0.4, 0.7]$	0	0	0	0

As the normalization step between the conjunctive fusion operator and the Dempster-Shafer operator is also a part of the pignistic transformation, we obtain the same pignistic probability from any of the two bba's obtained by fusion of bba's 1 and 2. One should verify that m_{Conj}, m_{DS} and BetP still satisfy the relaxed normalization rule, but the width of $\sum_{A \in E(\Theta)} m(A)$ increases.

The calculus of $\text{BetP}(B)$ through the interval arithmetic should provide $[0.381, 1.036]$, but it can be truncated without any information loss, if the further treatments do not resume the intervals by their centers[5].

9.4 Constraints over the focal elements

What does happen if the lattice $E(\Theta)$ uses operators different of $\cap$ and $\cup$? These operators may create focal elements incompatible with the model elements that should appear in the bba produced by the experts (human or artificial). The following examples browse some of these situations, from an order set to a formalism near the natural language, through some classification models. It is possible that the "singletons of Θ" are difficult to exhibit; in this case, $E(\Theta)$ should be considered as the set of interest, as browsing the singletons is interesting only if they are privileged by the experts, or are necessary to calculate a pignistic probability.

9.4.1 Ordered set

If Θ is an ordered set, a subset A of Θ is *connected* if, for any x and y in A, $x \leqslant y$ brings $x \leqslant z \leqslant y$ implies $z \in A$ (resp. $y \leqslant x$ brings $y \leqslant z \leqslant x$ implies $z \in A$). Disconnected subsets do not have any signification in the context of an ordered set (Θ can be a discretization of some real value : $\{36, 39, 42, 45, 48, 51\}$). Therefore, the elements of $E(\Theta)$ should be the intervals of $\leqslant$, noted $[\![x, y]\!]$. If $x = y$ the interval is a singleton. If $y < x$, the interval is $\emptyset$.

[5]This procedure, anyway, should not lead to a probability, as the the sum of the centers is not expected to be 1.

To remain within $E(\Theta)$, the $\cap$ operator is convenient, as it preserves the connectiveness of its operands, but $\cup$ is not: $\{36\} \cup \{42\} = \{36, 42\} \notin E(\Theta)$, but $\{36, 39, 42\}$ is. The hull of the operands gives the smallest interval:

$$[\![x_1, y_1]\!] \vee_{\leqslant} [\![x_2, y_2]\!] = [\![\min(x_1, x_2), \max(x_2, y_2)]\!] \tag{9.23}$$

The cardinal of an element of $E(\Theta)$ holds as usual, so all the combination operators and decision-aid functions of section 9.2.5 can be used.

As the cardinal of $\mathcal{F}(m)$ is bounded by $\frac{(n+2)(n+1)}{2}$, where 2^Θ has a size of 2^n, the constraint of a total order on Θ can limit the combinatorial explosion inherent to most combinatorial operators.

9.4.2 Intervals of $\mathbb{R}^N$

In a context of interval analysis [10, 16], the manipulated objects and the results are intervals of $\mathbb{R}^N$. If the theory guarantees some non-void intersections when manipulating the solutions of an equation, its application in an information fusion system with unpredicted events may lead to conflicting situations.

Here an *interval* $[\mathbf{x}, \mathbf{y}]$ corresponds to a Cartesian product $[x_1, y_1] \times \ldots \times [x_N, y_N]$. The intersection works as usual, giving an join operator $\wedge_I$ for the lattice $E(\Theta)$:

$$[\mathbf{x^1}, \mathbf{y^1}] \wedge_I [\mathbf{x^2}, \mathbf{y^2}] = \mathop{\times}_{i=1}^{n} [\max(x_i^1, x_i^2), \min(y_i^1, y_i^2)] \tag{9.24}$$

If for some dimension i, $\max(x_i^1, x_i^2) > \min(y_i^1, y_i^2)$, then

$$[\mathbf{x^1}, \mathbf{y^1}] \wedge_I [\mathbf{x^2}, \mathbf{y^2}] = \emptyset \tag{9.25}$$

For the $\vee_I$ operator, the smallest interval of $\mathbb{R}^N$ containing the operands is taken:

$$[\mathbf{x^1}, \mathbf{y^1}] \vee_I [\mathbf{x^2}, \mathbf{y^2}] = \mathop{\times}_{i=1}^{N} [\min(x_i^1, x_i^2), \max(y_i^1, y_i^2)] \tag{9.26}$$

The measure of Lebesgue gives the cardinal of interval:

$$\mu([\mathbf{x}, \mathbf{y}]) = \prod_{i=1}^{N} (y_i - x_i) \tag{9.27}$$

Therefore, all the combination operators and decision-aid functions of section 9.2.5 can be used in a context of interval calculus.

Unlike the usual $(\cup, \cap)$ lattice or the $(\vee_{\leqslant}, \wedge_{\leqslant})$ lattice on an ordered set, the $(\vee_I, \wedge_I)$ lattice is just a lattice, not a ring: $\vee_I$ does not distribute over $\wedge_I$. On figure 9.1 the intervals are $A = [0, 1] \times [2, 3]$, $B = [2, 3] \times [4, 5]$, $C = [2, 3] \times [0, 1]$, $D = [4, 5] \times [2, 3]$. So:

$$(A \vee_I B) \wedge_I (C \vee_I D) = [2, 3] \times [2, 3]$$

$$(A \wedge_I C) \vee_I (A \wedge_I D) \vee_I (B \wedge_I C) \vee_I (B \wedge_I D) = \emptyset$$

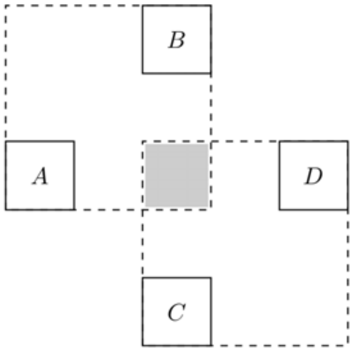

Figure 9.1: Non-distributivity.

9.4.3 Partitions

A set $\mathcal{P}$ of subsets of Θ is a *partition* if for any pair A, B of elements of $\mathcal{P}$, either $A = B$ or $A \cap B = \emptyset$, and $\bigcup\{A \in \mathcal{P}\} = \Theta$. This structure is popular for unsupervised classification problems; a vast family algorithms, around K-means [15], produce results within this model. The intersection between two partitions, $\mathcal{P}_1$ and $\mathcal{P}_2$, can be easily defined:

$$\mathcal{P}_1 \wedge_P \mathcal{P}_2 = \{A \cap B \,|\, A \in \mathcal{P}_1, B \in \mathcal{P}_2\} \tag{9.28}$$

However, replacing $\cap$ by $\cup$ in (9.28) does not produce a partition. An operator $\vee_P$ should be constructed by considering the connected parts of the hyper-graph $\mathcal{P}_1 \cup \mathcal{P}_2$, but this tends to give a degenerated partition even if $\mathcal{P}_1$ and $\mathcal{P}_2$ differs only slightly. See Guénoche and Garreta [9] for robust methods of comparison between partitions. It is necessary to limit the operators to a closed semi-lattice, whose bottom $\perp$ is a partition in n singletons and top $\top$ is a partition containing only Θ.

The cardinal of a partition $n - |\mathcal{P}|$ where n is the cardinal of Θ and $|\mathcal{P}|$ the number of subsets of Θ in $\mathcal{P}$. So

$$\begin{aligned}
\text{Card}\,(\perp) &= 0 \\
\text{Card}\,(\top) &= n - 1 \\
\text{Card}\,(\mathcal{P}_1 \wedge_P \mathcal{P}_2) &\leqslant \min(\text{Card}\,(\mathcal{P}_1)\,, \text{Card}\,(\mathcal{P}_1))
\end{aligned} \tag{9.29}$$

The partitions on Θ can be used as focal elements for bba's, and use them for all the decision-aid functions, including BetP, and for the Conj and PCR5/6 combination operators.

9.4.4 Hierarchies

A *hierarchy* on Θ is a set $\mathcal{H}$ of subsets of Θ such that:

- for any $x \in \Theta$, $\{x\} \in \mathcal{H}$;
- $\Theta \in \mathcal{H}$;
- for any A and B in $\mathcal{H}$, $A \cap B \in \{A, B, \emptyset\}$.

This structure is also very popular for unsupervised classification. It is produced by Ward's algorithm, single linkage, complete linkage and many others [11]. A merging operation between two hierarchies $\mathcal{H}_1$ and $\mathcal{H}_2$ can be defined by $A \in (\mathcal{H}_1 \odot \mathcal{H}_2)$ if

$$A = \bigcap \{X \in \mathcal{H}_1 \text{ or } X \in \mathcal{H}_2 \mid A \cap X \neq X\} \tag{9.30}$$

This operation is not associative, but it is idempotent, and admits $\mathcal{H}_0$, the hierarchy containing only the singletons and Θ, as a neutral element. The structure defined is only a pseudo-semi-lattice using this operator $\odot$. Its bottom is $\mathcal{H}_0$. It has no unique top, but the complete hierarchies (containing exactly $2n{-}1$ subsets of Θ) have no dominating hierarchy: if $\mathcal{H}$ and $\mathcal{H}'$ are complete, $\mathcal{H}' \odot \mathcal{H} = \mathcal{H}$ implies that $\mathcal{H}' = \mathcal{H}$.

The usual intersection operator works on hierarchies. However, it destroys information instead of making it sharper when possible. The usual union of two hierarchies is not necessarily a hierarchy. So a semi-lattice $(\mathbf{H}, \cap)$ is obtained, whose bottom is $\mathcal{H}_0$, and whose greatest elements are the complete hierarchies.

$$A \in (\mathcal{H}_1 \cap \mathcal{H}_2) \text{ iff } A \in \mathcal{H}_1 \text{ and } A \in \mathcal{H}_2 \tag{9.31}$$

Taking the number of subsets of Θ the hierarchy content is efficient to identify the complete hierarchies, but it is not decreasing with $\wedge_H$. Other definitions are hardly constant on the complete hierarchies.

The pseudo-semi-lattice defined by $\odot$ can be used to model bba's in a hierarchy space, apply on them the decision-aid functions Bel and Pl, and combine them through the PCR5/6 and Conj operators. However, in this latter case, the associativity of the operator is lost.

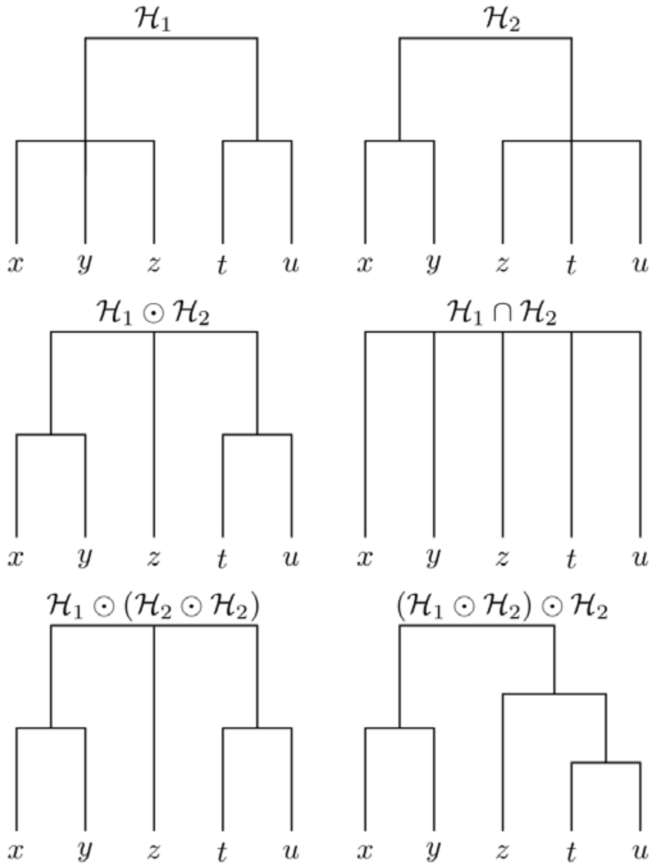

Figure 9.2: Dealing with hierarchies.

9.4.5 Binary clustering systems

A *binary clustering system* [1] on Θ is a set $\mathcal{E}$ of subsets of Θ (called *clusters*) such that, for any x and y in Θ, the set $\mathcal{E}(x,y) = \{A \in \mathcal{E} \mid x \in A, y \in A\}$ admits an unique smallest element, called $A(x,y)$. It is said proper if $A(x,x) = \{x\}$. Hierarchies defined in section 9.4.4 are binary clustering systems; partitions defined in section 9.4.3 are non-proper binary clustering systems (add singletons and Θ for a better system); the closure by intersection of set of intervals of a complete order (see section 9.4.1) is a binary clustering system. $\mathcal{E}$ can be seen as a *hyper-graph*, its elements are its *hyper-edges* and its vertex set is Θ [2].

Therefore, the definition of $\mathcal{E}$ can be restricted to clusters that can be built as a smallest cluster containing a pair of elements of Θ. So the size of $\mathcal{E}$ is bounded by $\mathcal{O}(n^2)$, and the restriction of the intersection of two binary clustering systems on Θ has the same size, and can be calculated in $\mathcal{O}(n^4)$ operations. A join operator $\wedge_E$ can be defined by :

$$\mathcal{E}_1 \wedge_E \mathcal{E}_2 = \bigcup_{x\in\Theta, y\in\Theta} \{A_1(x,y) \cap A_2(x,y)\} \tag{9.32}$$

It defines a semi-lattice whose bottom is $\mathcal{E}_\perp$, a hyper-graph containing all the possible hyper-edges with 1 or 2 vertices. Its non-proper version is the complete graph whose vertex set is Θ. Its top is the hyper-graph $\mathcal{E}_\top$ whose only hyper-edge is Θ. A cardinal function can be defined by:

$$\text{Card}\,(\mathcal{E}) = \sum_{\{x,y\}\subset\Theta} (|A(x,y)| - 2) \tag{9.33}$$

So $\text{Card}\,(\mathcal{E}_\perp) = 0$, $\text{Card}\,(\mathcal{E}_\top) = \frac{1}{2}n(n+1)(n-2)$, and $\mathcal{E} \leqslant \mathcal{E}'$ (see section 9.7.2 for the definition of $\leqslant$ in a lattice) brings $\text{Card}\,(\mathcal{E}) \leqslant \text{Card}\,(\mathcal{E}')$.

So the binary clustering systems on Θ can be used as focal elements for bba's, and they can feed all the decision-aid functions, including BetP, and the Conj and PCR5/6 combination operators.

9.4.6 Semantic assertions

Semantic assertions can be modelized by *conceptual graphs* [26] or ontologies. Meet and join operators can be defined, but these operations can lead to NP-hard problems in a general case. However, some sub-classes of conceptual graphs avoid this combinatorial problem [17].

With the same restrictions on the shape of graphs than the other operations on them, to avoid combinatorial explosion, they can be combined by Conj and Dis operators. As obtaining $\emptyset$ by conjunction of two conceptual graphs is very unlikely, the

PCR5/6 and DP combination operators should not be used: they are nearly equivalent to Conj.

The usual ways to calculate the cardinal of a graph (number of edges or number or vertices) is not compatible with the meet operator, and does not make sense with the specialization of labels. The decision-aid functions should be limited to Pl and Bel.

Then, the conjunctive combination operator, receiving bba's containing "*Johann have seen a Leclerc*" and "*A man have seen a tank near the river*" can put a mass (or a label) on the assertion "*Johann have seen a Leclerc tank near the river*".

9.5 Conclusion

The combination operators of the theory of belief functions are often heavy to manipulate: cumbersome equations[6], data ill-adapted to matrix calculus under popular scientific software, real risks of combinatorial explosion, by example. However, they are more likely than Bayesian approaches or fuzzy sets to be adapted to many forms of symbolic data. In this chapter, we have shown that the link between the theory of belief functions and the probabilities, the pignistic transformation, relies on "difficult" operations: scalar multiplication and cardinal calculus. Dropping only this link and keeping most of their properties allows bba's to explore many facets of experts' opinions, and build a fused information from them, while other theories only deal with a projection of the experts' assertions on a too small space.

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9.7 Appendix: algebras

All the operators defined on the following structures are *inner* operators : $x, y \in \Theta$ brings $x \Diamond y \in \Theta$ for the inner operator $\diamond$. As most of the algebraic properties of this chapter concern the set of possible focal elements $E(\Theta)$, the symbol Θ is used for the algebraic structures. Obviously, dealing with the space of the labels is not different.

9.7.1 Orders and partial orders

$(\Theta, \leqslant)$ is a partially ordered set (*poset*), if for any a, b, and c in Θ, we have that:

reflexivity : $a \leqslant a$,

antisymmetry : $a \leqslant b$ and $b \leqslant a$ implies $a = b$,

transitivity : $a \leqslant b$ and $b \leqslant c$ implies $a \leqslant c$.

If $x \leqslant y$ and $x \neq y$, we will note $x < y$.

If for any a and b of Θ we have either $a \leqslant b$ or $b \leqslant a$ then Θ is called a *totally ordered set* or simply an *ordered set*.

Expressing a label in a ordered or partially ordered set is easier for an human expert than expressing a significant mass in $[0, 1]$.

9.7.2 Lattice

$(\Theta, \vee, \wedge)$ is a *lattice* if the *join* operator $\vee$ and the *meet* operator $\wedge$ satisfy for any a, b and c of Θ:

commutativity : $a \vee b = b \vee a$, $a \wedge b = b \wedge a$,

associativity : $(a \vee b) \vee c = a \vee (b \vee c)$, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$,

idempotence : $a \vee a = a$, $a \wedge a = a$,

absorbtion : $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a$.

The lattice is *closed* if it has a smallest element $\perp$ (its *bottom*) and a greatest element $\top$ (its *top*): for any $x \in \Theta$, $\perp \wedge x = \perp$, $\perp \vee x = x$, $\top \wedge x = x$, $\top \vee x = \top$.

If we just define a join operator, we get a *semi-lattice*.

For any set Θ, $(2^{\Theta}, \cap, \cup)$ is a lattice.

Defining the relation $\leqslant$ by $a \leqslant b$ iff $a \vee b = b$, makes $(\Theta, \leqslant)$ a poset.

9.7.3 Ring and semi-ring

$(\Theta, +, \times)$ is a *ring* if

- the *addition* operator is invertible, associative and commutative, and has a neutral element, called 0_{Θ} or 0 if not ambiguous;
- the *multiplication* operator is associative, commutative, and has a neutral element, called 1_{Θ} or 1 if not ambiguous;;
- the multiplication *distributes* over the addition:

$$a \times (b + c) = (a \times b) + (a \times c)$$

This implies $(\Theta, +)$ is a *group*, and $(\Theta, \times)$ is a *monoid*.

For most of the fusion applications, the addition does not need to be invertible, so we just need $(\Theta, +)$ be a *semi-group*. So a *semi-ring* structure for $(\Theta, +, \times)$ is sufficient.

9.7.4 Field

A *field* $(\Theta, +, \times)$ is a ring where the multiplication is invertible on $\Theta \backslash \{0\}$. The set of real numbers $\mathbb{R}$ with the usual addition and multiplication is a field.

It is because the real interval $[0,1]$ is a part of a field that we can use all of the operators and functions of the section 9.2.5, but they are not inner operations for this segment $[0,1]$: taking x and y in $[0,1]$ can lead to $x^{-1} \notin [0,1]$ and $x+y \notin [0,1]$.

9.7.5 Vector space

$(\Theta, +, \bullet)$ is a *vector space* over a field $\mathbb{F}$ if $+$ is an inner operator which is invertible, associative and commutative, and has a neutral element 0. The operator $\bullet$ is the external multiplication, also called scalar multiplication, by an element of $\mathbb{F}$, satisfying, for any a and b in $\mathbb{F}$ and x and y in Θ:

- Distributivity of $\bullet$ over $+$: $a \bullet (x+y) = a \bullet x + a \bullet y$,
- Distributivity of addition over $\bullet$: $(a +_{\mathbb{F}} b) \bullet x = a \bullet x + b \bullet x$,
- Associativity of multiplications: $(a \times_{\mathbb{F}} b)x = a \bullet (b \bullet x)$.

9.7.6 Algebra over a field

$(\Theta, +, \times, \bullet)$ is an *algebra* over a field $\mathbb{F}$ if $(\Theta, +, \times)$ is a ring, $(\Theta, +, \bullet)$ is vector space over $\mathbb{F}$, and for any a and b in $\mathbb{F}$ and x and y in Θ we have $(a \bullet x) \times (b \bullet y) = (ab) \bullet (x \times y)$. The multiplication between a and b in (ab) is the multiplication defined in the field $\mathbb{F}$: $ab = a \times_{\mathbb{F}} b$.

A field can be seen as an algebra over itself, indentifying the inner multiplication and the scalar multiplication: $(\mathbb{R}, +, \bullet, \bullet)$.

No richer algebraic structure will be considered for the fusion applications presented in this chapter.