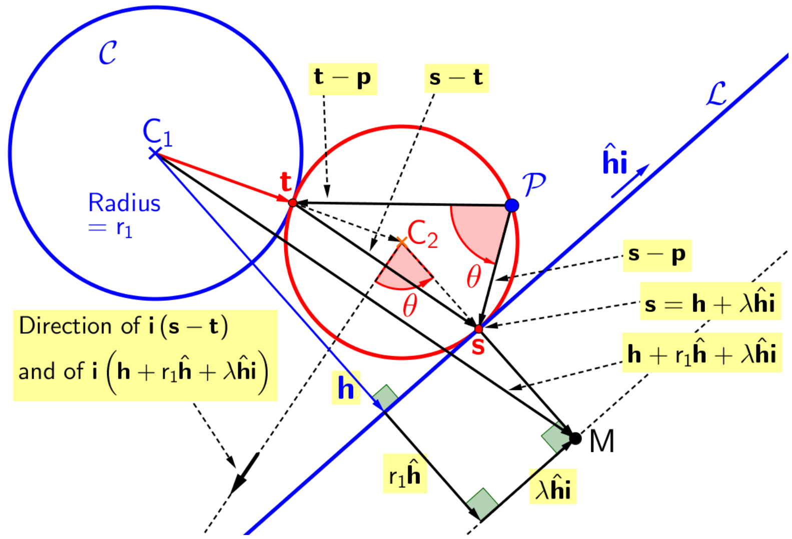
Solution of the Special Case "CLP" of the Problem of Apollonius

via Vector Rotations using Geometric Algebra



Geometric-Algebra Formulas for Plane (2D) Geometry

The Geometric Product, and Relations Derived from It

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,
$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
$\boldsymbol{b} \wedge \boldsymbol{a} = -\boldsymbol{a} \wedge \boldsymbol{b}$
$\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b}$
$\boldsymbol{ba} = \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{a} \wedge \boldsymbol{b}$

$\boldsymbol{ab} + \boldsymbol{ba} = 2\boldsymbol{a} \cdot \boldsymbol{b}$
$\boldsymbol{ab} - \boldsymbol{ba} = 2\boldsymbol{a} \wedge \boldsymbol{b}$
$\boldsymbol{ab} = 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{ba}$
$\boldsymbol{ab} = 2\boldsymbol{a} \wedge \boldsymbol{b} - \boldsymbol{ba}$

Definitions of Inner and Outer Products (Macdonald A. 2010 p. 101.)

The inner product
The inner product of a j-vector A and a k-vector B is
$A \cdot B = \langle AB \rangle_{k-j}$. Note that if $j>k$, then the inner product doesn't exist.
However, in such a case $B \cdot A = \langle BA \rangle_{j-k}$ does exist.

The outer product
The outer product of a j-vector A and a k-vector B is
$A \wedge B = \langle AB \rangle_{k+j}$.

Relations Involving the Outer Product and the Unit Bivector, i.

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,
$\boldsymbol{ia} = -\boldsymbol{ai}$
$\boldsymbol{a} \wedge \boldsymbol{b} = [(\boldsymbol{ai}) \cdot \boldsymbol{b}]\, \boldsymbol{i} = -[\boldsymbol{a} \cdot (\boldsymbol{bi})]\, \boldsymbol{i} = -\boldsymbol{b} \wedge \boldsymbol{a}$

Equality of Multivectors

For any two multivectors $\mathcal{M}$ and $\mathcal{N}$,
$\mathcal{M} = \mathcal{N}$ if and only if for all k, $\langle \mathcal{M} \rangle_k = \langle \mathcal{N} \rangle_k$.

Formulas Derived from Projections of Vectors and Equality of Multivectors

Any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written in the form of "Fourier expansions" with respect to a third vector, $\boldsymbol{v}$:

$\boldsymbol{a} = (\boldsymbol{a} \cdot \hat{\boldsymbol{v}})\, \hat{\boldsymbol{v}} + [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\, \hat{\boldsymbol{v}}\boldsymbol{i}$ and $\boldsymbol{b} = (\boldsymbol{b} \cdot \hat{\boldsymbol{v}})\, \hat{\boldsymbol{v}} + [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\, \hat{\boldsymbol{v}}\boldsymbol{i}$.

Using these expansions,

$$\boldsymbol{ab} = \{(\boldsymbol{a} \cdot \hat{\boldsymbol{v}})\, \hat{\boldsymbol{v}} + [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\, \hat{\boldsymbol{v}}\boldsymbol{i}\}\, \{(\boldsymbol{b} \cdot \hat{\boldsymbol{v}})\, \hat{\boldsymbol{v}} + [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\, \hat{\boldsymbol{v}}\boldsymbol{i}\}$$

Equating the scalar parts of both sides of that equation,

$\boldsymbol{a} \cdot \boldsymbol{b} = [\boldsymbol{a} \cdot \hat{\boldsymbol{v}}]\,[\boldsymbol{b} \cdot \hat{\boldsymbol{v}}] + [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\,[\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]$, and
$\boldsymbol{a} \wedge \boldsymbol{b} = \{[\boldsymbol{a} \cdot \hat{\boldsymbol{v}}]\,[\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})] - [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\,[\boldsymbol{b} \cdot \hat{\boldsymbol{v}}]\}\,\boldsymbol{i}$.

Also, $a^2 = [\boldsymbol{a} \cdot \hat{\boldsymbol{v}}]^2 + [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]^2$, and $b^2 = [\boldsymbol{b} \cdot \hat{\boldsymbol{v}}]^2 + [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]^2$.

Reflections of Vectors, Geometric Products, and Rotation operators
For any vector $\boldsymbol{a}$, the product $\hat{\boldsymbol{v}}\boldsymbol{a}\hat{\boldsymbol{v}}$ is the reflection of $\boldsymbol{a}$ with respect to the direction $\hat{\boldsymbol{v}}$.
For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\hat{\boldsymbol{v}}\boldsymbol{a}\boldsymbol{b}\hat{\boldsymbol{v}} = \boldsymbol{b}\boldsymbol{a}$, and $\boldsymbol{v}\boldsymbol{a}\boldsymbol{b}\boldsymbol{v} = v^2\boldsymbol{b}\boldsymbol{a}$.
Therefore, $\hat{\boldsymbol{v}}e^{\theta\boldsymbol{i}}\hat{\boldsymbol{v}} = e^{-\theta\boldsymbol{i}}$, and $\boldsymbol{v}e^{\theta\boldsymbol{i}}\boldsymbol{v} = v^2e^{-\theta\boldsymbol{i}}$.

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\hat{\boldsymbol{v}}\boldsymbol{a}\boldsymbol{b}\hat{\boldsymbol{v}} = \boldsymbol{b}\boldsymbol{a}$, and $\boldsymbol{v}\boldsymbol{a}\boldsymbol{b}\boldsymbol{v} = v^2\boldsymbol{b}\boldsymbol{a}$. Therefore, $\hat{\boldsymbol{v}}e^{\theta\boldsymbol{i}}\hat{\boldsymbol{v}} = e^{-\theta\boldsymbol{i}}$, and $\boldsymbol{v}e^{\theta\boldsymbol{i}}\boldsymbol{v} = v^2e^{-\theta\boldsymbol{i}}$.

A useful relationship that is valid only in plane geometry: $\boldsymbol{abc} = \boldsymbol{cba}$.
Here is a brief proof:

$$
\begin{aligned}
\boldsymbol{abc} &= \{\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b}\}\,\boldsymbol{c} \\
&= \{\boldsymbol{a} \cdot \boldsymbol{b} + [(\boldsymbol{ai}) \cdot \boldsymbol{b}]\,\boldsymbol{i}\}\,\boldsymbol{c} \\
&= (\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c} + [(\boldsymbol{ai}) \cdot \boldsymbol{b}]\,\boldsymbol{ic} \\
&= \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{b}) - \boldsymbol{c}\,[(\boldsymbol{ai}) \cdot \boldsymbol{b}]\,\boldsymbol{i} \\
&= \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{b}) + \boldsymbol{c}\,[\boldsymbol{a} \cdot (\boldsymbol{bi})]\,\boldsymbol{i} \\
&= \boldsymbol{c}\,(\boldsymbol{b} \cdot \boldsymbol{a}) + \boldsymbol{c}\,[(\boldsymbol{bi}) \cdot \boldsymbol{a}]\,\boldsymbol{i} \\
&= \boldsymbol{c}\,\{\boldsymbol{b} \cdot \boldsymbol{a} + [(\boldsymbol{bi}) \cdot \boldsymbol{a}]\,\boldsymbol{i}\} \\
&= \boldsymbol{c}\,\{\boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{a}\} \\
&= \boldsymbol{cba}.
\end{aligned}
$$

Additional Solutions of the Special Case "CLP" of the Problem of Apollonius via Vector Rotations using Geometric Algebra

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Contents

ABSTRACT

This document adds to the collection of solved problems presented in [1]-[6]. The solutions presented herein are not as efficient as those in [6], but they give additional insight into ways in which GA can be used to solve this problem. After reviewing, briefly, how reflections and rotations can be expressed and manipulated via GA, it solves the CLP limiting case of the Problem of Apollonius in three ways, some of which identify the the solution circles' points of tangency with the given circle, and others of which identify the solution circles' points of tangency with the given line. For comparison, the solutions that were developed in [1] are presented in an Appendix.

1 Introduction

This document has been prepared for two very different audiences: for my fellow students of GA, and for experts who are preparing materials for us, and need to know which GA concepts we understand and apply readily, and which ones we do not. In some sense, the solutions presented herein are "leftovers" from work that was done to find more-efficient solutions, so the document is still in somewhat of a rough condition, with a certain amount of repetition.

Readers are encouraged to study the following documents, GeoGebra worksheets, and videos before beginning:

"Rotations of Vectors via Geometric Algebra: Explanation, and Usage in Solving Classic Geometric "Construction" Problems"
https://drive.google.com/file/d/0B2C4TqxB32RRdE5KejhQTzMtN3M/view?usp=sharing

"Answering Two Common Objections to Geometric Algebra"
As GeoGebra worksheet
As YouTube video.

"Geometric Algebra: Find unknown vector from two dot products"
As GeoGebra worksheet
As YouTube video

For an more-complete treatment of rotations in plane geometry, be sure to read Hestenes D. 1999, pp. 78-92. His section on circles (pp. 87-89) is especially relevant to the present document. Macdonald A. 2010 is invaluable in many respects, and González Calvet R. 2001, Treatise of Plane Geometry through Geometric Algebra is a must-read.

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2 A brief review of reflections and rotations in 2D GA

2.1 Review of reflections

2.1.1 Reflections of a single vector

For any two vectors $\hat{\boldsymbol{u}}$ and $\boldsymbol{v}$, the product $\hat{\boldsymbol{u}}\boldsymbol{v}\hat{\boldsymbol{u}}$ is

$$\begin{aligned}
\hat{\boldsymbol{u}}\boldsymbol{v}\hat{\boldsymbol{u}} &= \{2\hat{\boldsymbol{u}} \wedge \boldsymbol{v} + \boldsymbol{v}\hat{\boldsymbol{u}}\}\,\hat{\boldsymbol{u}} && (2.1)\\
&= \boldsymbol{v} + 2\,[(\hat{\boldsymbol{u}}\boldsymbol{i}) \cdot \boldsymbol{v}]\,\boldsymbol{i}\hat{\boldsymbol{u}} && (2.2)\\
&= \boldsymbol{v} - 2\,[\boldsymbol{v} \cdot (\hat{\boldsymbol{u}}\boldsymbol{i})]\,\hat{\boldsymbol{u}}\boldsymbol{i}, && (2.3)
\end{aligned}$$

which evaluates to the reflection of the reflection of $\boldsymbol{v}$ with respect to $\hat{\boldsymbol{u}}$ (Fig. 2.1).

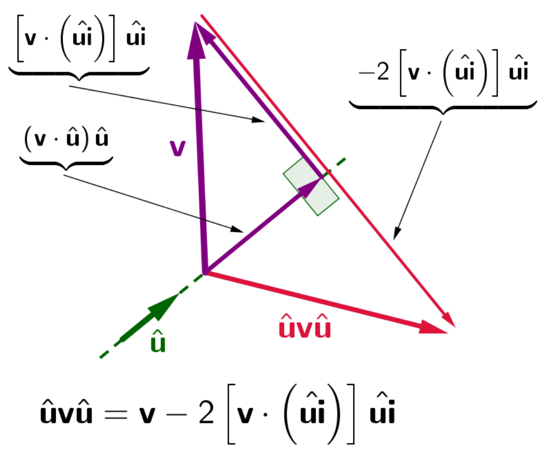

Figure 2.1: Geometric interpretation of $\hat{\boldsymbol{u}}\boldsymbol{v}\hat{\boldsymbol{u}}$, showing why it evaluates to the reflection of $\boldsymbol{v}$ with respect to $\hat{\boldsymbol{u}}$.

We also note that because $\boldsymbol{u} = |\boldsymbol{u}|\,\hat{\boldsymbol{u}}$,

$$\boldsymbol{u}\boldsymbol{v}\boldsymbol{u} = u^2\,(\hat{\boldsymbol{u}}\boldsymbol{v}\hat{\boldsymbol{u}}) = \boldsymbol{u}^2\boldsymbol{v} - 2\,[\boldsymbol{v} \cdot (\boldsymbol{u}\boldsymbol{i})]\,\boldsymbol{u}\boldsymbol{i}. \qquad (2.4)$$

2.1.2 Reflections of a bivector, and of a geometric product of two vectors

The product $\hat{\boldsymbol{u}}\boldsymbol{v}\boldsymbol{w}\hat{\boldsymbol{u}}$ is

$$\begin{aligned}
\hat{\boldsymbol{u}}\boldsymbol{v}\boldsymbol{w}\hat{\boldsymbol{u}} &= \hat{\boldsymbol{u}}\left(\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{v}\wedge\boldsymbol{w}\right)\hat{\boldsymbol{u}} \\
&= \hat{\boldsymbol{u}}\left(\boldsymbol{v}\cdot\boldsymbol{w}\right)\hat{\boldsymbol{u}}+\hat{\boldsymbol{u}}\left(\boldsymbol{v}\wedge\boldsymbol{w}\right)\hat{\boldsymbol{u}} \\
&= \hat{\boldsymbol{u}}^2\left(\boldsymbol{v}\cdot\boldsymbol{w}\right)+\hat{\boldsymbol{u}}\left[(\boldsymbol{v}\boldsymbol{i})\cdot\boldsymbol{w}\right]\boldsymbol{i}\hat{\boldsymbol{u}} \\
&= \boldsymbol{v}\cdot\boldsymbol{w}+\hat{\boldsymbol{u}}\left[-\boldsymbol{v}\cdot(\boldsymbol{w}\boldsymbol{i})\right]\left(-\hat{\boldsymbol{u}}\boldsymbol{i}\right) \\
&= \boldsymbol{v}\cdot\boldsymbol{w}+\hat{\boldsymbol{u}}^2\left[(\boldsymbol{w}\boldsymbol{i})\cdot\boldsymbol{v}\right]\boldsymbol{i} \\
&= \boldsymbol{w}\cdot\boldsymbol{v}+\boldsymbol{w}\wedge\boldsymbol{v} \\
&= \boldsymbol{w}\boldsymbol{v}.
\end{aligned}$$

In other words, the reflection of the geometric product $\boldsymbol{v}\boldsymbol{w}$ is $\boldsymbol{w}\boldsymbol{v}$, and does not depend on the direction of the vector with respect to which it is reflected. We saw that the scalar part of $\boldsymbol{v}\boldsymbol{w}$ was unaffected by the reflection, but the bivector part was reversed.

Further to that point, the reflection of geometric product of $\boldsymbol{v}$ and $\boldsymbol{w}$ is equal to the geometric product of the two vectors' reflections:

$$\begin{aligned}
\hat{\boldsymbol{u}}\boldsymbol{v}\boldsymbol{w}\hat{\boldsymbol{u}} &= \hat{\boldsymbol{u}}\boldsymbol{v}\left(\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}\right)\boldsymbol{w}\hat{\boldsymbol{u}} \\
&= \left(\hat{\boldsymbol{u}}\boldsymbol{v}\hat{\boldsymbol{u}}\right)\left(\hat{\boldsymbol{u}}\boldsymbol{w}\hat{\boldsymbol{u}}\right).
\end{aligned}$$

That observation provides a geometric interpretation (Fig. 2.2) of why reflecting a bivector changes its sign: the direction of the turn from $\boldsymbol{v}$ to $\boldsymbol{w}$ reverses.

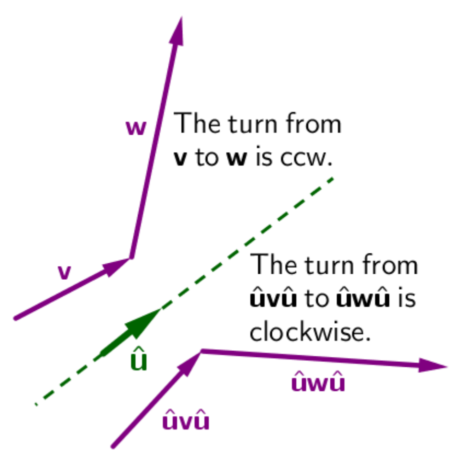

Figure 2.2: Geometric interpretation of $\hat{\boldsymbol{u}}\boldsymbol{v}\boldsymbol{w}\hat{\boldsymbol{u}}$, showing why it evaluates to the reflection of $\boldsymbol{v}$ with respect to $\hat{\boldsymbol{u}}$. Note that $\hat{\boldsymbol{u}}\boldsymbol{v}\boldsymbol{w}\hat{\boldsymbol{u}} = \hat{\boldsymbol{u}}\boldsymbol{v}\left(\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}\right)\boldsymbol{w}\hat{\boldsymbol{u}} = \left(\hat{\boldsymbol{u}}\boldsymbol{v}\hat{\boldsymbol{u}}\right)\left(\hat{\boldsymbol{u}}\boldsymbol{w}\hat{\boldsymbol{u}}\right)$.

2.2 Review of rotations

One of the most-important rotations—for our purposes—is the one that is produced when a vector is multiplied by the unit bivector, $\boldsymbol{i}$, for the plane: $\boldsymbol{vi}$ is $\boldsymbol{v}$'s 90-degree counter-clockwise rotation, while $\boldsymbol{iv}$ is $\boldsymbol{v}$'s 90-degree clockwise rotation.

Every geometric product $\boldsymbol{bc}$ is a rotation operator, whether we use it as such or not:

$$\boldsymbol{bc} = \|\boldsymbol{b}\|\|\boldsymbol{c}\|e^{\theta \boldsymbol{i}}\,.$$

where θ is the angle of rotation from $\boldsymbol{b}$ to $\boldsymbol{c}$. From that equation, we obtain the identity

$$e^{\theta \boldsymbol{i}} = \frac{\boldsymbol{bc}}{\|\boldsymbol{b}\|\|\boldsymbol{c}\|} = \left[\frac{\boldsymbol{b}}{\|\hat{\boldsymbol{b}}\|}\right]\left[\frac{\boldsymbol{c}}{\|\hat{\boldsymbol{c}}\|}\right] = \hat{\boldsymbol{b}}\hat{\boldsymbol{c}}\,.$$

Note that $\boldsymbol{a}\,[\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}]$ evaluates to the rotation of $\boldsymbol{a}$ by the angle from $\hat{\boldsymbol{u}}$ to $\hat{\boldsymbol{v}}$, while $[\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}]\,\boldsymbol{a}$ evaluates to the rotation of $\boldsymbol{a}$ by the angle from $\hat{\boldsymbol{v}}$ to $\hat{\boldsymbol{u}}$.

A useful corollary of the foregoing is that any product of an odd number of vectors evaluates to a vector, while the product of an odd number of vectors evaluates to the sum of a scalar and a bivector.

An interesting example of a rotation is the product $\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}\boldsymbol{ab}\hat{\boldsymbol{v}}\hat{\boldsymbol{u}}$. Writing that product as $(\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}\boldsymbol{a})\,(\boldsymbol{b}\hat{\boldsymbol{v}}\hat{\boldsymbol{u}})$, and recalling that any geometric product of two vectors acts as a rotation operator in 2D (2.2), we can see why $\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}\boldsymbol{ab}\hat{\boldsymbol{v}}\hat{\boldsymbol{u}} = \boldsymbol{ab}$ (Fig. 2.3).

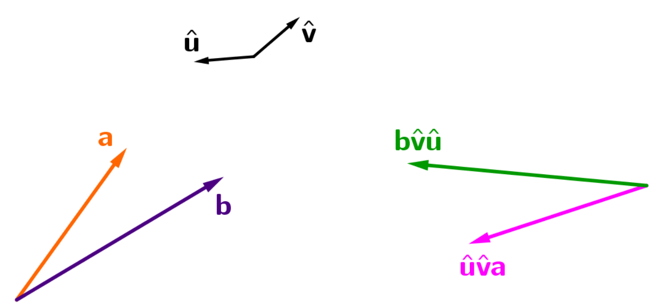

Figure 2.3: Left-multiplying $\boldsymbol{a}$ by the geometric product $\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}$ rotates $\boldsymbol{a}$ by an angle equal (in sign as well as magnitude) to that from $\hat{\boldsymbol{v}}$ to $\hat{\boldsymbol{u}}$. Right-multiplying $\boldsymbol{b}$ by the geometric product $\hat{\boldsymbol{v}}\hat{\boldsymbol{u}}$ rotates $\boldsymbol{b}$ by that same angle. The orientations of $\boldsymbol{a}$ and $\boldsymbol{b}$ with respect to each other are the same after the rotation, and the magnitudes of $\boldsymbol{a}$ and $\boldsymbol{b}$ are unaffected. Therefore, the geometric products $\boldsymbol{ab}$ and $(\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}\boldsymbol{a})\,(\boldsymbol{b}\hat{\boldsymbol{v}}\hat{\boldsymbol{u}})$ are equal.

We can also write $\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}\boldsymbol{ab}\hat{\boldsymbol{v}}\hat{\boldsymbol{u}}$ as $\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}\,[\boldsymbol{ab}]\,\hat{\boldsymbol{v}}\hat{\boldsymbol{u}}$, giving it the form of a rotation of the geometric product $\boldsymbol{ab}$. Considered in this way, the result $\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}\boldsymbol{ab}\hat{\boldsymbol{v}}\hat{\boldsymbol{u}} = \boldsymbol{ab}$ is a special case of an important theorem: rotations preserve geometric products [?].

3 The Problem of Apollonius, and Its CLP Limiting Case

The famous "Problem of Apollonius", in plane geometry, is to construct all of circles that are tangent, simultaneously, to three given circles. In one variant of that problem, one of the circles has infinite radius (i.e., it's a line). The Wikipedia article that's current as of this writing has an extensive description of the problem's history, and of methods that have been used to solve it. As described in that article, one of the methods reduces the "two circles and a line" variant to the so-called "Circle-Line-Point" (CLP) limiting case:

> *Given a circle $\mathcal{C}$, a line $\mathcal{L}$, and a point $\mathcal{P}$, construct the circles that are tangent to $\mathcal{C}$ and $\mathcal{L}$, and pass through $\mathcal{P}$.*

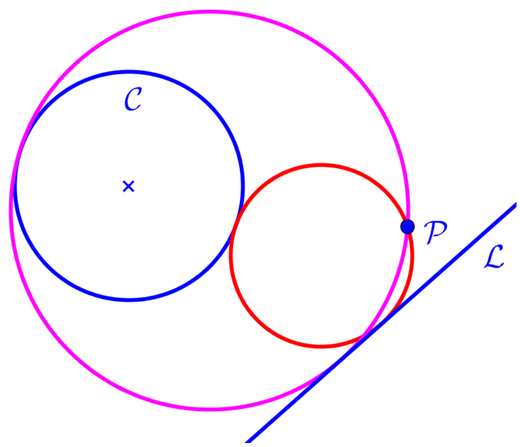

Figure 3.1: The CLP Limiting Case of the Problem of Apollonius: *Given a circle $\mathcal{C}$, a line $\mathcal{L}$, and a point $\mathcal{P}$, construct the circles that are tangent to $\mathcal{C}$ and $\mathcal{L}$, and pass through $\mathcal{P}$.*

3.1 Observations, and Potentially Useful Elements of the Problem

From the figure presented in the statement of the problem, we can see that there are two types of solutions. That is, two types of circles that satisfy the stated conditions:

- Circles that enclose $\mathcal{C}$;
- Circles that do not enclose $\mathcal{C}$.

As we will see later, there are actually two circles that do enclose the given circle, and two that do not. We'll begin by discussing circles that do not enclose it. Most of our observations about that type will also apply, with little modification, to circles that do enclose it.

Based upon our experience in solving other "construction problems" involving tangency, a reasonable choice of elements for capturing the geometric content of the problem is as shown below:

- Use the center point of the given circle as the origin;

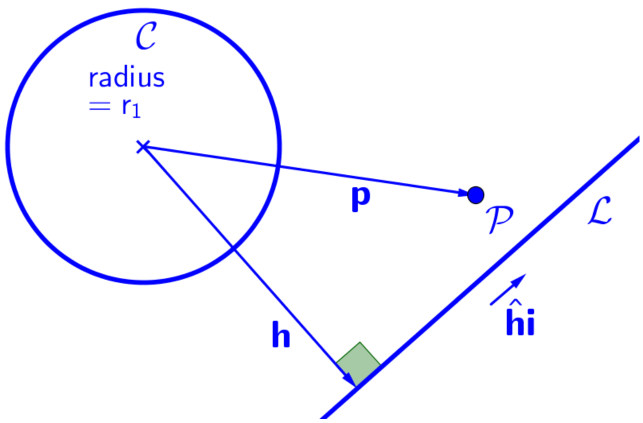

Figure 3.2: Initial step in choosing elements for capturing the geometric content of the problem. (See text for explanation.)

- Capture the perpendicular distance from $\boldsymbol{c_1}$'s center to the given line in the vector $\boldsymbol{h}$;

- Express the direction of the given line as $\pm\hat{\boldsymbol{h}}\boldsymbol{i}$.

- Label the solution circle's radius and its points of tangency with $\mathcal{C}$ and $\mathcal{L}$ as shown in Fig. 3.3:

In deriving our solution, we'll use the same symbol —for example, $\boldsymbol{t}$ —to denote both a point and the vector to that point from the origin. We'll rely upon context to tell the reader whether the symbol is being used to refer to the point, or to the vector.

Now, we'll express key features of the problem in terms of the elements that we've chosen. First, we know that we can write the vector $\boldsymbol{s}$ as $\boldsymbol{s} = \boldsymbol{h} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}$, where λ is some scalar. We can even identify that scalar, specifically, as $\left[\hat{\boldsymbol{t}} \cdot \left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\right]$. We also know that the points of tangency $\boldsymbol{t}$ and $\boldsymbol{s}$ are equidistant (by r_2) from the center point of the solution circle. Combining those observations, we can equate three expressions for the vector $\boldsymbol{s}$:

$$\boldsymbol{s} = (r_1 + r_2)\,\hat{\boldsymbol{t}} + r_2\hat{\boldsymbol{h}} = \boldsymbol{h} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i} = \boldsymbol{h} + \left[\hat{\boldsymbol{t}} \cdot \left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\right]\hat{\boldsymbol{h}}\boldsymbol{i}. \tag{3.1}$$

Examining that equation, we observe that we can obtain an expression for

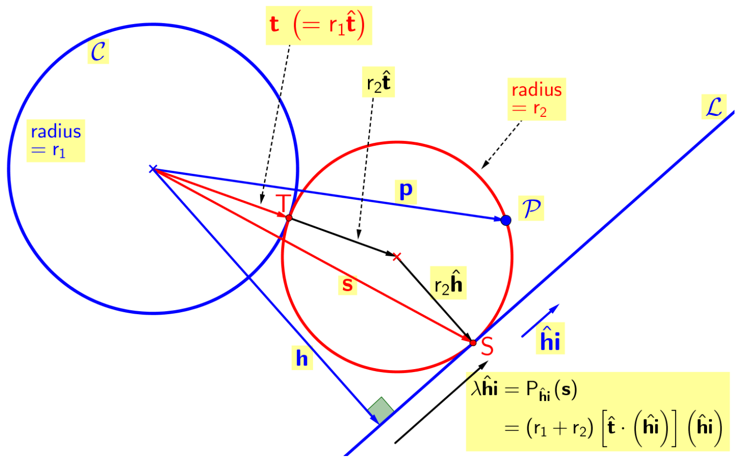

Figure 3.3: Further steps in choosing elements for capturing the geometric content of the problem. (See text for explanation.)

r_2 in terms of known quantities by "dotting" both sides with $\hat{\boldsymbol{h}}$:

$$\begin{aligned}
\left[(r_1 + r_2)\,\hat{\boldsymbol{t}} + r_2\hat{\boldsymbol{h}}\right] \cdot \hat{\boldsymbol{h}} &= \left[\boldsymbol{h} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right] \cdot \hat{\boldsymbol{h}} \\
(r_1 + r_2)\,\hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{h}} + r_2\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{h}} &= \boldsymbol{h} \cdot \hat{\boldsymbol{h}} + \lambda\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{h}} \\
(r_1 + r_2)\,\hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{h}} + r_2 &= |\boldsymbol{h}| + 0; \\
\therefore r_2 &= \frac{|\boldsymbol{h}| - r_1\hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{h}}}{1 + \hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{h}}}.
\end{aligned} \tag{3.2}$$

The denominator of the expression on the right-hand side might catch our attention now because one of our two expressions for the vector $\boldsymbol{s}$, namely

$$\boldsymbol{s} = (r_1 + r_2)\,\hat{\boldsymbol{t}} + r_2\hat{\boldsymbol{h}}$$

can be rewritten as

$$\boldsymbol{s} = r_1\hat{\boldsymbol{t}} + r_2\left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}}\right).$$

That fact becomes useful (at least potentially) when we recognize that $\left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}}\right)^2 = 2\left(1 + \hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{h}}\right)$. Therefore, if we wish, we can rewrite Eq. (3.2) as

$$r_2 = 2\left[\frac{|\boldsymbol{h}| - r_1\hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{h}}}{\left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}}\right)^2}\right].$$

Those results indicate that we should be alert to opportunities to simplify expressions via appropriate substitutions invoving $\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}}$ and $1 + \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}$

As a final observation, we note that when a circle is tangent to other objects, there will be many angles that are equal to each other. For example, the angles whose measures are given as θ in Figs. 3.4 and 3.5:

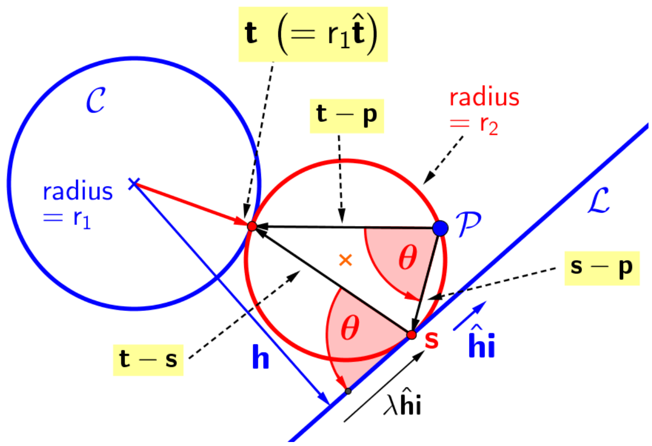

Figure 3.4: A pair of equal angles that might provide a basis for solving the problem via rotations.)

We also see in Fig. 3.5 that points T and S are reflections of each other with respect to a line that passes through the center of the solution circle, and has the direction of $(\boldsymbol{t}-\boldsymbol{s})\,\boldsymbol{i}$. References [1]-[6] showed how GA enables us to make use of the reflections and rotations that we've noted.

Many of the ideas that we'll employ here will also be used when we treat solution circles that do enclose $\mathcal{C}$.

3.1.1 Formulating a Strategy

Now, let's combine our observations about the problem in a way that might lead us to a solution. Our previous experiences in solving problems via vector rotations suggest that we should equate two expressions for the rotation $e^{\theta \boldsymbol{i}}$:

$$\begin{aligned}\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{|\boldsymbol{t}-\boldsymbol{p}|}\right]\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{|\boldsymbol{s}-\boldsymbol{p}|}\right] &= \left[\frac{\boldsymbol{t}-\boldsymbol{s}}{|\boldsymbol{t}-\boldsymbol{s}|}\right]\left[-\hat{\boldsymbol{h}}\boldsymbol{i}\right] \\ &= \left[\frac{\boldsymbol{s}-\boldsymbol{t}}{|\boldsymbol{s}-\boldsymbol{t}|}\right]\left[\hat{\boldsymbol{h}}\boldsymbol{i}\right]. \end{aligned} \tag{3.3}$$

We've seen elsewhere that we will probably want to transform that equation into one in which some product of vectors involving our unknowns $\boldsymbol{t}$ and $\boldsymbol{s}$ is equal either to a pure scalar, or a pure bivector. By doing so, we may find some way of identifying either $\boldsymbol{t}$ or $\boldsymbol{s}$.

We'll keep in mind that although Eq. (3.3) has two unknowns (the vectors $\boldsymbol{t}$ and $\boldsymbol{s}$), our expression for r_2 (Eq. (3.2)) enables us to write the vector $\boldsymbol{s}$ in terms of the vector $\hat{\boldsymbol{t}}$.

Therefore, our strategy is to

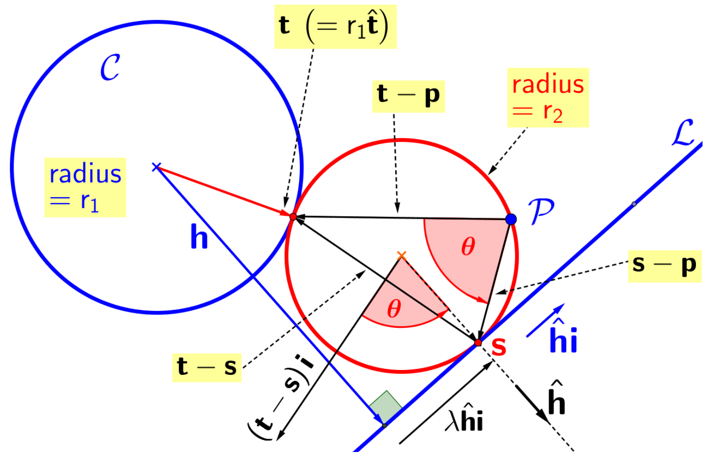

Figure 3.5: Another pair of equal angles that might provide a basis for solving the problem via rotations. Note, too, that the points $\boldsymbol{T}$ and $\boldsymbol{s}$ are reflections of each other with respect to the vector $(\boldsymbol{t}-\boldsymbol{s})\,\boldsymbol{i}$.

- Equate two expressions, in terms of the unknown vectors $\boldsymbol{t}$ and $\boldsymbol{s}$, for the rotation $e^{\theta\boldsymbol{i}}$;
- Transform that equation into one in which on side is either a pure scalar or a pure bivector;
- Watch for opportunities to simplify equations by substituting for r_2; and
- Solve for our unknowns.

4 Solving the Problem

Our first solution, of which two variants are presented, uses rotations alone. The second uses a combination of reflections and rotations. Reference [1] can be studied for comparison.

4.1 Solution via Rotations Alone

4.1.1 Solving for the Solution Circles that Do Not Enclose the Given Circle

We'll start by examining Fig. 4.1, which is based upon Fig. 3.5. Using ideas presented in Section 3.1, we can write

$$\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{\|\boldsymbol{s}-\boldsymbol{p}\|}\right]\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{\|\boldsymbol{t}-\boldsymbol{p}\|}\right]=e^{\phi\boldsymbol{i}}=\hat{\boldsymbol{h}}\left[\frac{\boldsymbol{i}\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)}{\|\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\|}\right],$$

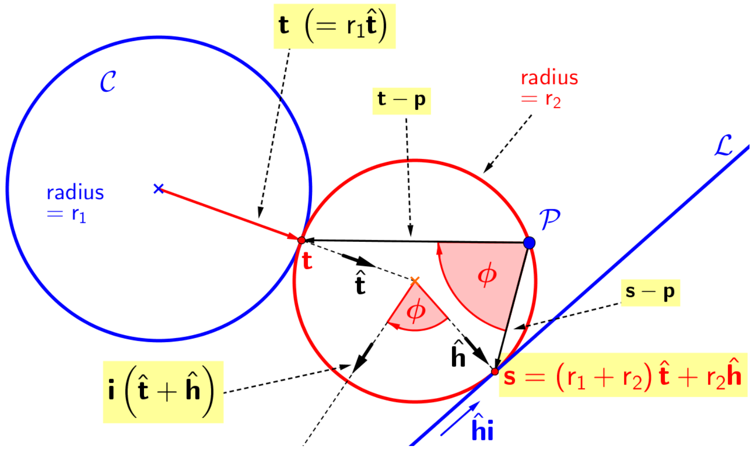

Figure 4.1: Elements used in finding the solution circles that do not enclose the given one. The vector $\boldsymbol{s}$ has been written as $(r_1+r_2)\,\hat{\boldsymbol{t}}+r_2\hat{\boldsymbol{h}}$, and we've made use of the fact that $\boldsymbol{i}\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)$ has the same direction as $\boldsymbol{i}\,(\boldsymbol{s}-\boldsymbol{t})$. (Compare Fig. 3.5.)

from which

$$\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{\|\boldsymbol{s}-\boldsymbol{p}\|}\right]\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{\|\boldsymbol{t}-\boldsymbol{p}\|}\right] = \hat{\boldsymbol{h}}\left[\frac{\boldsymbol{i}\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)}{\|\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\|}\right],$$

$$\hat{\boldsymbol{h}}\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{\|\boldsymbol{s}-\boldsymbol{p}\|}\right]\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{\|\boldsymbol{t}-\boldsymbol{p}\|}\right] = \left[\frac{\boldsymbol{i}\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)}{\|\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\|}\right],$$

$$\hat{\boldsymbol{h}}\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{\|\boldsymbol{s}-\boldsymbol{p}\|}\right]\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{\|\boldsymbol{t}-\boldsymbol{p}\|}\right] = \left[\frac{\boldsymbol{i}\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)}{\|\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\|}\right],$$

$$\hat{\boldsymbol{h}}\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{\|\boldsymbol{s}-\boldsymbol{p}\|}\right]\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{\|\boldsymbol{t}-\boldsymbol{p}\|}\right]\left[\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right] = \boldsymbol{i}\|\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\|, \text{ and}$$

$$\hat{\boldsymbol{h}}\,[\boldsymbol{s}-\boldsymbol{p}]\,[\boldsymbol{t}-\boldsymbol{p}]\left[\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right] = \boldsymbol{i}\|\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\|\|\boldsymbol{s}-\boldsymbol{p}\|\|\boldsymbol{t}-\boldsymbol{p}\|,$$

the right-hand side of which is a bivector. Therefore,

$$\langle\hat{\boldsymbol{h}}\,[\boldsymbol{s}-\boldsymbol{p}]\,[\boldsymbol{t}-\boldsymbol{p}]\left[\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right]\rangle_0 = 0.$$

We can save ourselves some work by expanding the left-hand side in a way that let's us take advantage of what we've learned about the form in which GA can express reflections:

$$\langle\hat{\boldsymbol{h}}\,[\boldsymbol{s}-\boldsymbol{p}]\,[\boldsymbol{t}-\boldsymbol{p}]\,\hat{\boldsymbol{t}}\rangle_0 + \langle\hat{\boldsymbol{h}}\,[\boldsymbol{s}-\boldsymbol{p}]\,[\boldsymbol{t}-\boldsymbol{p}]\,\hat{\boldsymbol{h}}\rangle_0 = 0. \tag{4.1}$$

Note that the product $\hat{\boldsymbol{h}}\,[\boldsymbol{s}-\boldsymbol{p}]\,[\boldsymbol{t}-\boldsymbol{p}]\,\hat{\boldsymbol{h}}$ is just the reflection of $[\boldsymbol{s}-\boldsymbol{p}]\,[\boldsymbol{t}-\boldsymbol{p}]$ with respect to $\hat{\boldsymbol{h}}$, and is therefore equal to $[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}]$. Let's find the scalar part of that product, then return to the first term in 4.1. We'll use the substitutions $\boldsymbol{t}=r_1\hat{\boldsymbol{t}}$; $r_2 = \dfrac{\|h\|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}$; $r_1+r_2 = \dfrac{r_1+\|h\|}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}$; and $\boldsymbol{s}=(r_1+r_2)\,\hat{\boldsymbol{t}}+r_2\hat{\boldsymbol{h}}$

to write

$$
\begin{aligned}
[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}] &= \left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\left[\left(\frac{r_1+\|h\|}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right)\hat{\boldsymbol{t}}+\left(\frac{\|h\|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right)\hat{\boldsymbol{h}}-\boldsymbol{p}\right] \\
&= \left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\left[\frac{(r_1+\|h\|)\,\hat{\boldsymbol{t}}+\left(\|h\|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)\hat{\boldsymbol{h}}-\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)\boldsymbol{p}}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right].
\end{aligned}
$$

That last step—putting everything in the last factor over the common denominator $1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}$ —is quite a natural one to take, but turns out to be unnecessary. By taking it, I complicated solutions in other problems needlessly, by forming awkward products like $\left(\boldsymbol{p}\cdot\hat{\boldsymbol{t}}\right)\left(\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)$. Later in this solution we'll see why that step is unnecessary, but for now, let's go back to

$$
[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}] = \left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\left[\left(\frac{r_1+\|h\|}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right)\hat{\boldsymbol{t}}+\left(\frac{\|h\|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right)\hat{\boldsymbol{h}}-\boldsymbol{p}\right],
$$

then continue to expand the right-hand side:

$$
\begin{aligned}
[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}] &= \left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\left[\left(\frac{r_1+\|h\|}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right)\hat{\boldsymbol{t}}+\left(\frac{\|h\|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right)\hat{\boldsymbol{h}}-\boldsymbol{p}\right] \\
&= \left[\frac{r_1\,(r_1+\|h\|)}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right]\hat{\boldsymbol{t}}\hat{\boldsymbol{t}}+\left[\frac{r_1\left(\|h\|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right]\hat{\boldsymbol{t}}\hat{\boldsymbol{h}} \\
&\quad -r_1\hat{\boldsymbol{t}}\boldsymbol{p}-\left[\frac{r_1+\|h\|}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right]\boldsymbol{p}\hat{\boldsymbol{t}}-\left[\frac{\|h\|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right]\boldsymbol{p}\hat{\boldsymbol{h}}+p^2,
\end{aligned}
$$

the scalar part of which is

$$
\begin{aligned}
\langle[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}]\rangle_0 &= \left[\frac{r_1\,(r_1+\|h\|)}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right]+\left[\frac{r_1\left(\|h\|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right]\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}} \\
&\quad -r_1\boldsymbol{p}\cdot\hat{\boldsymbol{t}}-\left[\frac{r_1+\|h\|}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right]\boldsymbol{p}\cdot\hat{\boldsymbol{t}}-\left[\frac{\|h\|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right]\boldsymbol{p}\cdot\hat{\boldsymbol{h}}+p^2.
\end{aligned}
$$

Now, let's find $\langle \hat{\boldsymbol{h}} [\boldsymbol{s}-\boldsymbol{p}] [\boldsymbol{t}-\boldsymbol{p}] \hat{\boldsymbol{t}} \rangle_0$ from 4.1:

$$
\begin{aligned}
\hat{\boldsymbol{h}} [\boldsymbol{s}-\boldsymbol{p}] [\boldsymbol{t}-\boldsymbol{p}] \hat{\boldsymbol{t}} &= \hat{\boldsymbol{h}} \left[\left(\frac{r_1 + \|h\|}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}} \right) \hat{\boldsymbol{t}} + \left(\frac{\|h\| - r_1 \hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}} \right) \hat{\boldsymbol{h}} - \boldsymbol{p} \right] \left[r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right] \hat{\boldsymbol{t}} \\
&= \left[\frac{r_1 (r_1 + \|h\|)}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}} \right] \hat{\boldsymbol{h}}\hat{\boldsymbol{t}}\hat{\boldsymbol{t}}\hat{\boldsymbol{t}} - \left[\frac{r_1 + \|h\|}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}} \right] \hat{\boldsymbol{h}}\hat{\boldsymbol{t}}\boldsymbol{p}\hat{\boldsymbol{t}} \\
&\quad + \left[\frac{r_1 \left(\|h\| - r_1 \hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}} \right)}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}} \right] \hat{\boldsymbol{h}}\hat{\boldsymbol{h}}\hat{\boldsymbol{t}}\hat{\boldsymbol{t}} - \left[\frac{r_1 \left(\|h\| - r_1 \hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}} \right)}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}} \right] \hat{\boldsymbol{h}}\hat{\boldsymbol{h}}\boldsymbol{p}\hat{\boldsymbol{t}} \\
&\quad - r_1 \hat{\boldsymbol{h}}\boldsymbol{p}\hat{\boldsymbol{t}}\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}}\boldsymbol{p}\boldsymbol{p}\hat{\boldsymbol{t}} \\
&= \left[\frac{r_1 (r_1 + \|h\|)}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}} \right] \hat{\boldsymbol{h}}\hat{\boldsymbol{t}} - \left[\frac{r_1 + \|h\|}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}} \right] \hat{\boldsymbol{h}}\hat{\boldsymbol{t}}\boldsymbol{p}\hat{\boldsymbol{t}} \\
&\quad + \left[\frac{r_1 \left(\|h\| - r_1 \hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}} \right)}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}} \right] - \left[\frac{r_1 \left(\|h\| - r_1 \hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}} \right)}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}} \right] \boldsymbol{p}\hat{\boldsymbol{t}} \\
&\quad - r_1 \hat{\boldsymbol{h}}\boldsymbol{p} + p^2 \hat{\boldsymbol{h}}\hat{\boldsymbol{t}}.
\end{aligned}
$$

The scalar part of $\hat{\boldsymbol{h}}\hat{\boldsymbol{t}}\boldsymbol{p}\hat{\boldsymbol{t}}$ is $\langle \hat{\boldsymbol{h}}\hat{\boldsymbol{t}} \left(2\boldsymbol{p}\cdot\hat{\boldsymbol{t}} - \hat{\boldsymbol{t}}\boldsymbol{p} \right) \rangle_0 = 2 \left(\boldsymbol{p}\cdot\hat{\boldsymbol{t}} \right) \left(\hat{\boldsymbol{h}}\cdot\boldsymbol{p} \right) - \hat{\boldsymbol{h}}\cdot\boldsymbol{p}$. Therefore,

$$
\begin{aligned}
\langle \hat{\boldsymbol{h}} [\boldsymbol{s}-\boldsymbol{p}] [\boldsymbol{t}-\boldsymbol{p}] \hat{\boldsymbol{t}} \rangle_0 &= \left[\frac{r_1 (r_1 + \|h\|)}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}} \right] \hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}} \\
&\quad - \left[\frac{r_1 + \|h\|}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}} \right] \left[2 \left(\boldsymbol{p}\cdot\hat{\boldsymbol{t}} \right) \left(\hat{\boldsymbol{h}}\cdot\boldsymbol{p} \right) - \hat{\boldsymbol{h}}\cdot\boldsymbol{p} \right] \\
&\quad + \left[\frac{r_1 \left(\|h\| - r_1 \hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}} \right)}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}} \right] - \left[\frac{r_1 \left(\|h\| - r_1 \hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}} \right)}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}} \right] \boldsymbol{p}\cdot\hat{\boldsymbol{t}} \\
&\quad - r_1 \hat{\boldsymbol{h}}\cdot\boldsymbol{p} + p^2 \hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}.
\end{aligned}
$$

As we noted in 6.1,

$$
\langle \hat{\boldsymbol{h}} [\boldsymbol{s}-\boldsymbol{p}] [\boldsymbol{t}-\boldsymbol{p}] \hat{\boldsymbol{t}} \rangle_0 + \langle \hat{\boldsymbol{h}} [\boldsymbol{s}-\boldsymbol{p}] [\boldsymbol{t}-\boldsymbol{p}] \hat{\boldsymbol{h}} \rangle_0 = 0.
$$

Adding our the expressions that we've just developed for $\langle \hat{\boldsymbol{h}} [\boldsymbol{s}-\boldsymbol{p}] [\boldsymbol{t}-\boldsymbol{p}] \hat{\boldsymbol{t}} \rangle_0$

and $\langle \hat{\boldsymbol{h}}\,[\boldsymbol{s}-\boldsymbol{p}]\,[\boldsymbol{t}-\boldsymbol{p}]\,\hat{\boldsymbol{h}}\rangle_0$, we obtain

$$
\begin{aligned}
\left[\frac{r_1\left(r_1+\|h\|\right)}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right]+\left[\frac{r_1\left(\|h\|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right]\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}&\\
-r_1\boldsymbol{p}\cdot\hat{\boldsymbol{t}}-\left[\frac{r_1+\|h\|}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right]\boldsymbol{p}\cdot\hat{\boldsymbol{t}}-\left[\frac{\|h\|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right]\boldsymbol{p}\cdot\hat{\boldsymbol{h}}+p^2&\\
+\left[\frac{r_1\left(r_1+\|h\|\right)}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right]\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}-\left[\frac{r_1+\|h\|}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right]\left[2\left(\boldsymbol{p}\cdot\hat{\boldsymbol{t}}\right)\left(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right)-\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right]&\\
+\left[\frac{r_1\left(\|h\|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right]-\left[\frac{r_1\left(\|h\|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right]\boldsymbol{p}\cdot\hat{\boldsymbol{t}}&\\
-r_1\hat{\boldsymbol{h}}\cdot\boldsymbol{p}+p^2\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}=0.& \qquad (4.2)
\end{aligned}
$$

Note the pairs of expressions in red and blue: those pairs sum, respectively, to $r_1\left(r_1+\|h\|\right)$ and $r_1\left(\|h\|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)$. That fact is one of the reasons why we didn't need to put all of the terms over the common denominator $1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}$. (See the discussion that follows 4.1.)

After simplification, 4.2 reduces to

$$2\left(r_1+\|h\|\right)\boldsymbol{p}\cdot\hat{\boldsymbol{t}}-\left(p^2-{r_1}^2\right)\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}=2\|h\|r_1+{r_1}^2+p^2.$$

We saw equations like this last one many times in Smith J A 2016. There, we learned to solve those equations by grouping the dot products that involve $\boldsymbol{t}$ into a dot product of $\boldsymbol{t}$ with a linear combination of known vectors:

$$\underbrace{\left[2\left(r_1+|\boldsymbol{h}|\right)\boldsymbol{p}-\left(p^2-{r_1}^2\right)\hat{\boldsymbol{h}}\right]}_{\text{A linear combination of }\hat{\boldsymbol{h}}\text{ and }\boldsymbol{p}}\cdot\hat{\boldsymbol{t}}=2\left|\boldsymbol{h}\right|r_1+{r_1}^2+p^2. \qquad (4.3)$$

The geometric interpretation of Eq. (4.3) is that $2\left|\boldsymbol{h}\right|r_1+{r_1}^2+p^2$ is the projection of the vector $2\left(r_1+|\boldsymbol{h}|\right)\boldsymbol{p}-\left(p^2-{r_1}^2\right)\hat{\boldsymbol{h}}$ upon $\hat{\boldsymbol{t}}$. Because we want to find $\boldsymbol{t}$, and know $2\left(r_1+|\boldsymbol{h}|\right)\boldsymbol{p}-\left(p^2-{r_1}^2\right)\hat{\boldsymbol{h}}$, we'll transform Eq. (4.3) into a version that tells us the projection of the vector $\boldsymbol{t}$ upon $2\left(r_1+|\boldsymbol{h}|\right)\boldsymbol{p}-\left(p^2-{r_1}^2\right)\hat{\boldsymbol{h}}$.

First, just for convenience, we'll multiply both sides of Eq. (4.3) by $r_1\left|\boldsymbol{h}\right|$:

$$\left[2\left(r_1\left|\boldsymbol{h}\right|+h^2\right)\boldsymbol{p}-\left(p^2-{r_1}^2\right)\boldsymbol{h}\right]\cdot\boldsymbol{t}=2h^2{r_1}^2+r_1\left|\boldsymbol{h}\right|\left({r_1}^2+p^2\right).$$

Next, we'll use the symbol "$\boldsymbol{w}$" for the vector $\left[2\left(r_1\left|\boldsymbol{h}\right|+h^2\right)\boldsymbol{p}-\left(p^2-{r_1}^2\right)\boldsymbol{h}\right]$, and write

$$\boldsymbol{w}\cdot\boldsymbol{t}=2h^2{r_1}^2+r_1\left|\boldsymbol{h}\right|\left({r_1}^2+p^2\right).$$

Finally, because $\boldsymbol{P_w}(\boldsymbol{t})$, the projection of the vector $\boldsymbol{t}$ upon $\boldsymbol{w}$ is $(\boldsymbol{t}\cdot\hat{\boldsymbol{w}})\,\hat{\boldsymbol{w}}$, we have

$$\boldsymbol{P_w}(\boldsymbol{t}) = \left[\frac{2h^2{r_1}^2 + r_1\,|\boldsymbol{h}|\left({r_1}^2 + p^2\right)}{|\boldsymbol{w}|}\right]\hat{\boldsymbol{w}}. \tag{4.4}$$

As we learned in Smith J A 2016, Eq. (4.4) tells us that Eq (4.3) has two solutions. That is, there are two circles that are tangent to $\mathcal{L}$ and pass through the point $\mathcal{P}$, and are also tangent to $\mathcal{C}$ without enclosing it:

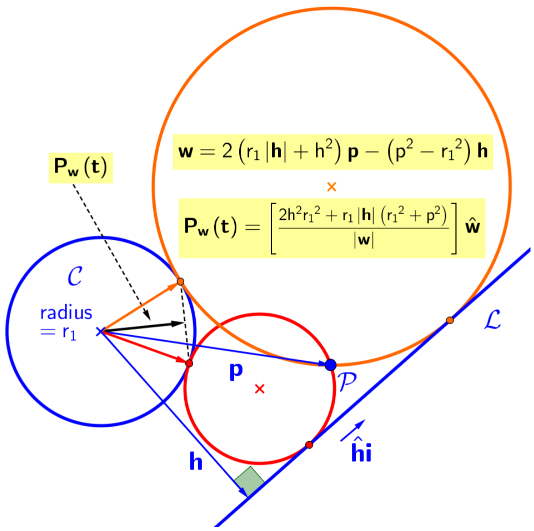

Figure 4.2: The two solution circles that do not enclose the given circle, $\mathcal{C}$.

Having identified $\boldsymbol{P_w}(\boldsymbol{t})$, the points of tangency with $\mathcal{C}$ and $\mathcal{L}$ can be determined using methods shown in Smith J A 2016, as can the equations for the corresponding solution circles.

To round off our treatment of solution circles that don't enclose $\mathcal{C}$, we should note that we derived our solution starting from equations that express the relationship between $\mathcal{C}$, $\mathcal{L}$, $\mathcal{P}$, and the smaller of the two solution circles. You may have noticed that the larger solution circle does not bear quite the same relationship to $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{C}$ as the smaller one. To understand in what way those relationships differ, please see the discussion in 6.

4.1.2 Solving for the Solution Circles that Do Enclose the Given Circle

To introduce some new ideas and show how to employ useful GA identities, we'll follow a somewhat different route in this section than we did when finding the solution circles that don't enclose the given one. Instead of expressing $\boldsymbol{s}$ —the point of tangency with the given line—as a linear combination of $\hat{\boldsymbol{t}}$ and

$\hat{\boldsymbol{h}}$, we'll express it as a linear combination of $\boldsymbol{h}$ and $\hat{\boldsymbol{h}}\boldsymbol{i}$: $\boldsymbol{s} = \boldsymbol{h} + \lambda\hat{\boldsymbol{h}}$, where λ is a scalar. Because $\lambda\hat{\boldsymbol{h}}\boldsymbol{i}$ is the projection of $(r_1 - r_3)\,\hat{\boldsymbol{t}}$ upon $\hat{\boldsymbol{h}}\boldsymbol{i}$ (Fig. 4.3), we can identify λ, specifically, as $(r_1 - r_3)\left[\hat{\boldsymbol{t}} \cdot \left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\right]$.

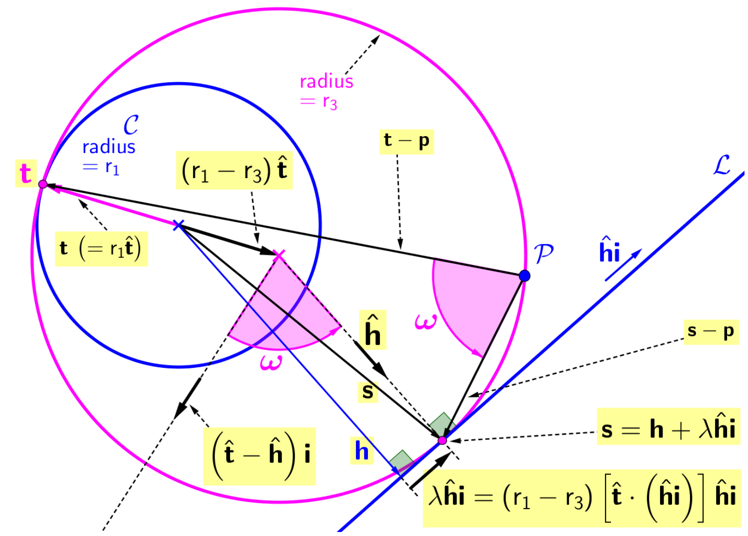

Figure 4.3: The two solution circles that do enclose the given circle, $\mathcal{C}$. The vector $\left(\hat{\boldsymbol{t}} - \hat{\boldsymbol{h}}\right)\boldsymbol{i}$ has the same direction as $(\boldsymbol{t} - \boldsymbol{s})\,\boldsymbol{i}$. Because $\lambda\hat{\boldsymbol{h}}\boldsymbol{i}$ is the projection of $(r_1 - r_3)\,\hat{\boldsymbol{t}}$ upon $\hat{\boldsymbol{h}}\boldsymbol{i}$, we can identify λ, specifically, as $(r_1 - r_3)\left[\hat{\boldsymbol{t}} \cdot \left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\right]$.

At this point, we can foresee needing to derive an expression for r_3 that's analogous to that which we presented in Eq. 3.2 for r_2. We'll do so by equating two expressions for $\boldsymbol{s}$, then "dotting" with $\hat{\boldsymbol{h}}$:

$$
\begin{aligned}
(r_1 - r_3)\,\hat{\boldsymbol{t}} + r_3\hat{\boldsymbol{h}} &= \boldsymbol{h} + \lambda\hat{\boldsymbol{h}} \\
\left[(r_1 - r_3)\,\hat{\boldsymbol{t}} + r_3\hat{\boldsymbol{h}}\right] \cdot \hat{\boldsymbol{h}} &= \left[\boldsymbol{h} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right] \cdot \hat{\boldsymbol{h}} \\
(r_1 - r_3)\,\hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{h}} + r_3 &= \|h \\
\therefore \quad r_3 = \frac{\|h\| - r_1}{1 - \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}}, \quad \text{and} \quad r_1 - r_3 &= \frac{r_1 - \|h\|}{1 - \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}}.
\end{aligned}
\tag{4.5}
$$

Now we're ready to begin to solve for $\boldsymbol{t}$. We could start by equating two expressions for $e^{\omega \boldsymbol{i}}$, but just to be different we'll note that the product $\left[\dfrac{\boldsymbol{t} - \boldsymbol{p}}{\|\boldsymbol{t} - \boldsymbol{p}\|}\right]\left[\dfrac{\boldsymbol{s} - \boldsymbol{p}}{\|\boldsymbol{s} - \boldsymbol{p}\|}\right]$, which is equal to $e^{\omega \boldsymbol{i}}$, is a rotation operator that brings $\left(\hat{\boldsymbol{t}} - \hat{\boldsymbol{h}}\right)\boldsymbol{i}$ into alignment with $\hat{\boldsymbol{h}}$:

$$
\left[\frac{\left(\hat{\boldsymbol{t}} - \hat{\boldsymbol{h}}\right)\boldsymbol{i}}{\|\hat{\boldsymbol{t}} - \hat{\boldsymbol{h}}\|}\right]\left\{\left[\frac{\boldsymbol{t} - \boldsymbol{p}}{\|\boldsymbol{t} - \boldsymbol{p}\|}\right]\left[\frac{\boldsymbol{s} - \boldsymbol{p}}{\|\boldsymbol{s} - \boldsymbol{p}\|}\right]\right\} = \hat{\boldsymbol{h}},
$$

which we then transform via

$$\left[\frac{\left(\hat{\boldsymbol{t}}-\hat{\boldsymbol{h}}\right)\boldsymbol{i}}{\|\hat{\boldsymbol{t}}-\hat{\boldsymbol{h}}\|}\right]\left\{\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{\|\boldsymbol{t}-\boldsymbol{p}\|}\right]\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{\|\boldsymbol{s}-\boldsymbol{p}\|}\right]\right\}=\hat{\boldsymbol{h}},$$

$$\boldsymbol{i}\left[\frac{\hat{\boldsymbol{t}}-\hat{\boldsymbol{h}}}{\|\hat{\boldsymbol{t}}-\hat{\boldsymbol{h}}\|}\right]\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{\|\boldsymbol{t}-\boldsymbol{p}\|}\right]\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{\|\boldsymbol{s}-\boldsymbol{p}\|}\right]=-\hat{\boldsymbol{h}},$$

$$\left[\frac{\hat{\boldsymbol{t}}-\hat{\boldsymbol{h}}}{\|\hat{\boldsymbol{t}}-\hat{\boldsymbol{h}}\|}\right]\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{\|\boldsymbol{t}-\boldsymbol{p}\|}\right]\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{\|\boldsymbol{s}-\boldsymbol{p}\|}\right]=\boldsymbol{i}\hat{\boldsymbol{h}},$$

$$\left[\frac{\hat{\boldsymbol{t}}-\hat{\boldsymbol{h}}}{\|\hat{\boldsymbol{t}}-\hat{\boldsymbol{h}}\|}\right]\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{\|\boldsymbol{t}-\boldsymbol{p}\|}\right]\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{\|\boldsymbol{s}-\boldsymbol{p}\|}\right]\hat{\boldsymbol{h}}=\boldsymbol{i},$$

$$\langle\left[\hat{\boldsymbol{t}}-\hat{\boldsymbol{h}}\right][\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}]\,\hat{\boldsymbol{h}}\rangle_0=0. \tag{4.6}$$

In addition, we'll use the Fourier expansion $\hat{\boldsymbol{t}}=\left(\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}\right)\hat{\boldsymbol{h}}+\left[\hat{\boldsymbol{t}}\cdot\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\right]\hat{\boldsymbol{h}}\boldsymbol{i}$ to transform the factor $\hat{\boldsymbol{t}}-\hat{\boldsymbol{h}}$:

$$\begin{aligned}\hat{\boldsymbol{t}}-\hat{\boldsymbol{h}}&=\left(\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}\right)\hat{\boldsymbol{h}}+\left[\hat{\boldsymbol{t}}\cdot\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\right]\hat{\boldsymbol{h}}\boldsymbol{i}\\&=\left(\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}-1\right)\hat{\boldsymbol{h}}+\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{t}}\right]\hat{\boldsymbol{h}}\boldsymbol{i}.\end{aligned}$$

Now, we'll substitute that expression in (4.6), and make an initial expansion:

$$\langle\left[\hat{\boldsymbol{t}}-\hat{\boldsymbol{h}}\right][\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}]\,\hat{\boldsymbol{h}}\rangle_0=0,$$

$$\langle\left\{\left(\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}-1\right)\hat{\boldsymbol{h}}+\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{t}}\right]\hat{\boldsymbol{h}}\boldsymbol{i}\right\}[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}]\,\hat{\boldsymbol{h}}\rangle_0=0,$$

from which

$$\left(\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}-1\right)\langle\hat{\boldsymbol{h}}\,[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}]\,\hat{\boldsymbol{h}}\rangle_0$$
$$+\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{t}}\right]\langle\left[\hat{\boldsymbol{h}}\boldsymbol{i}\right][\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}]\,\hat{\boldsymbol{h}}\rangle_0=0$$

and

$$\left(\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}-1\right)\langle\hat{\boldsymbol{h}}\,[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}]\,\hat{\boldsymbol{h}}\rangle_0$$
$$-\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{t}}\right]\langle\left\{\hat{\boldsymbol{h}}\,[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}]\,\hat{\boldsymbol{h}}\right\}\boldsymbol{i}\rangle_0=0.$$

Both of the terms within the angle brackets are reflections of the product $[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}]$ with respect to $\hat{\boldsymbol{h}}$. Therefore,

$$\left(\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}-1\right)\langle[\boldsymbol{s}-\boldsymbol{p}]\,[\boldsymbol{t}-\boldsymbol{p}]\rangle_0$$
$$-\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{t}}\right]\langle\{[\boldsymbol{s}-\boldsymbol{p}]\,[\boldsymbol{t}-\boldsymbol{p}]\}\,\boldsymbol{i}\rangle_0=0,$$

from which

$$\left(\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}-1\right)[\boldsymbol{s}-\boldsymbol{p}]\cdot[\boldsymbol{t}-\boldsymbol{p}]$$
$$-\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{t}}\right]\{[\boldsymbol{s}-\boldsymbol{p}]\wedge[\boldsymbol{t}-\boldsymbol{p}]\}\,\boldsymbol{i}=0. \tag{4.7}$$

We'll treat each of those terms separately. From (4.5), and from $\boldsymbol{s} = \boldsymbol{h} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i} = \boldsymbol{h} + (r_1 - r_3)\left[\hat{\boldsymbol{t}} \cdot \left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\right]\hat{\boldsymbol{h}}\boldsymbol{i}$, $\left(\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} - 1\right)[\boldsymbol{s} - \boldsymbol{p}] \cdot [\boldsymbol{t} - \boldsymbol{p}]$ is

$$\left(\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} - 1\right)\left\{\boldsymbol{h} + \left(\frac{r_1 - \|h\|}{1 - \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}}\right)\left[\hat{\boldsymbol{t}} \cdot \left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\right]\hat{\boldsymbol{h}}\boldsymbol{i} - \boldsymbol{p}\right\} \cdot \left\{r_1\left(\hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{h}}\right)\hat{\boldsymbol{h}} + r_1\left[\hat{\boldsymbol{t}} \cdot \left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\right]\hat{\boldsymbol{h}}\boldsymbol{i} - \boldsymbol{p}\right\}.$$

We needn't put the terms in curly brackets over the common denominator $1 - \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}$, because that denominator is canceled by the factor $\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} - 1$, leaving -1.

After expansion, that result simplifies to

$$\begin{aligned}
\left(\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} - 1\right)[\boldsymbol{s} - \boldsymbol{p}] \cdot [\boldsymbol{t} - \boldsymbol{p}] = & -\|h\|\left(\hat{\boldsymbol{h}} \cdot \boldsymbol{p}\right)\left(\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right) - \|h\|\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \boldsymbol{p}\right]\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right] \\
& + r_1\|h\|\left(\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right)^2 + r_1\|h\|\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right]^2 \\
& + 2r_1\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \boldsymbol{p}\right]\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right] - r_1\|h\|\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} \\
& + \|h\|\hat{\boldsymbol{h}} \cdot \boldsymbol{p} - {r_1}^2\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right]^2 - r_1\left(\hat{\boldsymbol{h}} \cdot \boldsymbol{p}\right)\left(\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right)^2 \\
& + r_1\left(\hat{\boldsymbol{h}} \cdot \boldsymbol{p}\right)\left(\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right) - r_1\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \boldsymbol{p}\right]\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right]\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} \\
& + p^2\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} - p^2.
\end{aligned}$$

We could simplify even further by recognizing that

$$\left(\hat{\boldsymbol{h}} \cdot \boldsymbol{p}\right)\left(\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right) + \left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \boldsymbol{p}\right]\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right] = \boldsymbol{p} \cdot \hat{\boldsymbol{t}},$$

so that the terms in red reduce to $-\|h\|\boldsymbol{p} \cdot \hat{\boldsymbol{t}}$. However, we will not do so because our goal is to obtain a solution in the form of a linear combination of $\hat{\boldsymbol{h}}$ and $\hat{\boldsymbol{h}}\boldsymbol{i}$. The terms in blue are a different story: they sum to $r_1\|h\|$ because $\left(\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right)^2 + \left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right]^2 = \hat{\boldsymbol{t}}^2 = 1$. Therefore,

$$\begin{aligned}
\left(\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} - 1\right)[\boldsymbol{s} - \boldsymbol{p}] \cdot [\boldsymbol{t} - \boldsymbol{p}] = & -\|h\|\left(\hat{\boldsymbol{h}} \cdot \boldsymbol{p}\right)\left(\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right) - \|h\|\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \boldsymbol{p}\right]\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right] \\
& + r_1\|h\| + 2r_1\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \boldsymbol{p}\right]\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right] - r_1\|h\|\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} \\
& + \|h\|\hat{\boldsymbol{h}} \cdot \boldsymbol{p} - {r_1}^2\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right]^2 - r_1\left(\hat{\boldsymbol{h}} \cdot \boldsymbol{p}\right)\left(\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right)^2 \\
& + r_1\left(\hat{\boldsymbol{h}} \cdot \boldsymbol{p}\right)\left(\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right) - r_1\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \boldsymbol{p}\right]\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right]\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} \\
& + p^2\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} - p^2. \qquad (4.8)
\end{aligned}$$

Now, we'll treat the term $\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right]\{[\boldsymbol{s} - \boldsymbol{p}] \wedge [\boldsymbol{t} - \boldsymbol{p}]\}\boldsymbol{i}$ from Eq. (4.7). I'll go into some detail so that you may see the interesting uses of GA identities that arise.

Using the same substitutions for r_3 and λ as we did when deriving (4.8), $\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right]\{[\boldsymbol{s} - \boldsymbol{p}] \wedge [\boldsymbol{t} - \boldsymbol{p}]\}\boldsymbol{i}$ is

$$\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right]\left\{\boldsymbol{h} + \left(\frac{r_1 - \|h\|}{1 - \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}}\right)\left[\hat{\boldsymbol{t}} \cdot \left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\right]\hat{\boldsymbol{h}}\boldsymbol{i} - \boldsymbol{p}\right\} \wedge \left[r_1\hat{\boldsymbol{t}} - \boldsymbol{p}\right].$$

Expanding that product using the identity $\boldsymbol{a} \wedge \boldsymbol{b} \equiv [(\boldsymbol{a}\boldsymbol{i}) \cdot \boldsymbol{b}]\, \boldsymbol{i} \equiv -[\boldsymbol{a} \cdot (\boldsymbol{b}\boldsymbol{i})]\, \boldsymbol{i}$, we find that

$$\begin{aligned}
\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right] \{[\boldsymbol{s}-\boldsymbol{p}] \wedge [\boldsymbol{t}-\boldsymbol{p}]\}\, \boldsymbol{i} = & \left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right] \Big\{ r_1 \|\hat{\boldsymbol{h}}\| \left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right] \boldsymbol{i} - \|\hat{\boldsymbol{h}}\| \left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \boldsymbol{p}\right] \boldsymbol{i} \\
& - \frac{r_1 \left(r_1 - \|\boldsymbol{h}\|\right) \left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right] \left[\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right] \boldsymbol{i}}{1 - \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}} \\
& - \frac{\left(r_1 - \|\boldsymbol{h}\|\right) \left[\left(\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \boldsymbol{i}\right) \cdot \boldsymbol{p}\right] \left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right] \boldsymbol{i}}{1 - \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}} \\
& - r_1 \left[\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right] \left[(\boldsymbol{p}\boldsymbol{i}) \cdot \hat{\boldsymbol{h}}\right] \boldsymbol{i} \\
& - r_1 \left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right] \left[(\boldsymbol{p}\boldsymbol{i}) \cdot \left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\right] \boldsymbol{i} \Big\}\, \boldsymbol{i}. \qquad (4.9)
\end{aligned}$$

As was the case when we expanded and simplified $\left(\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} - 1\right) [\boldsymbol{s}-\boldsymbol{p}] \cdot [\boldsymbol{t}-\boldsymbol{p}]$, the presence of $1 - \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}$ in the denominators of some of the terms is not a problem because the only terms with that denominator contain the factor $\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right]$ in their numerators. Thus, when we multiply through by the factor $\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right]$ that's outside the curly brackets, the numerators of those terms will contain the factor $\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right]^2$, which is equal to $1 - \left[\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right]^2$, and therefore to $\left(1 - \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right)\left(1 + \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right)$. The factor $\left(1 - \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right)$ in that product will cancel with the $\left(1 - \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right)$ in the denominator, leaving $\left(1 + \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right)$ in the numerator.

Making use of that observation about the cancellation of the denominators in Eq. (4.9), we can further expand and simplify terms in that equation to obtain

$$\begin{aligned}
\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right] \{[\boldsymbol{s}-\boldsymbol{p}] \wedge [\boldsymbol{t}-\boldsymbol{p}]\} = & - \Big\{ r_1 \|\boldsymbol{h}\| \left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right]^2 - \|\boldsymbol{h}\| \left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \boldsymbol{p}\right] \left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right] \\
& - r_1 \left(r_1 - \|\boldsymbol{h}\|\right) \left[\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right] - r_1 \left(r_1 - \|\boldsymbol{h}\|\right) \left[\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right]^2 \\
& + \left(r_1 - \|\boldsymbol{h}\|\right) \left[\hat{\boldsymbol{h}} \cdot \boldsymbol{p}\right] + \left(r_1 - \|\boldsymbol{h}\|\right) \left[\hat{\boldsymbol{h}} \cdot \boldsymbol{p}\right] \left[\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right] \\
& - r_1 \left[(\boldsymbol{p}\boldsymbol{i}) \cdot \hat{\boldsymbol{h}}\right] \left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right] \left[\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}\right] \\
& - r_1 \left[\hat{\boldsymbol{h}} \cdot \boldsymbol{p}\right] \left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right]^2 \Big\}. \qquad (4.10)
\end{aligned}$$

In (Eq. (4.7)), we noted that

$$\left(\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} - 1\right) [\boldsymbol{s}-\boldsymbol{p}] \cdot [\boldsymbol{t}-\boldsymbol{p}] - \left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{t}}\right] \{[\boldsymbol{s}-\boldsymbol{p}] \wedge [\boldsymbol{t}-\boldsymbol{p}]\}\, \boldsymbol{i} = 0.$$

Substituting in that equation the expressions for $\left(\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} - 1\right) [\boldsymbol{s}-\boldsymbol{p}] \cdot [\boldsymbol{t}-\boldsymbol{p}]$ and

$\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{t}}\right]\{[\boldsymbol{s}-\boldsymbol{p}]\wedge[\boldsymbol{t}-\boldsymbol{p}]\}\,\boldsymbol{i}$ in Eqs. (4.8) and (4.10),

$$\begin{aligned}
&\|h\|\left(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right)\left(\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)-\|h\|\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\boldsymbol{p}\right]\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{t}}\right]\\
&+r_1\|h\|+2r_1\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\boldsymbol{p}\right]\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{t}}\right]-r_1\|h\|\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\\
&+\|h\|\hat{\boldsymbol{h}}\cdot\boldsymbol{p}-{r_1}^2\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{t}}\right]^2-r_1\left(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right)\left(\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)^2\\
&+r_1\left(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right)\left(\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)-r_1\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\boldsymbol{p}\right]\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{t}}\right]\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\\
&\qquad+p^2\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}-p^2\\
&+r_1\|\boldsymbol{h}\|\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{t}}\right]^2-\|\boldsymbol{h}\|\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\boldsymbol{p}\right]\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{t}}\right]\\
&-r_1\left(r_1-\|\boldsymbol{h}\|\right)\left[\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right]-r_1\left(r_1-\|\boldsymbol{h}\|\right)\left[\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right]^2\\
&+\left(r_1-\|\boldsymbol{h}\|\right)\left[\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right]+\left(r_1-\|\boldsymbol{h}\|\right)\left[\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right]\left[\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right]\\
&-r_1\left[(\boldsymbol{p}\boldsymbol{i})\cdot\hat{\boldsymbol{h}}\right]\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{t}}\right]\left[\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right]-r_1\left[\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right]\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{t}}\right]^2=0.
\end{aligned}$$

In simplifying that equation, we again make use of the identity $\left(\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)^2+\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{t}}\right]^2=\hat{\boldsymbol{t}}^2=1$. The result is

$$\begin{aligned}
&2r_1\|\boldsymbol{h}\|-\|\boldsymbol{h}\|\left[\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right]\left[\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right]-\|\boldsymbol{h}\|\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\boldsymbol{p}\right]\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{t}}\right]-{r_1}^2\\
&+\left(2r_1-\|\boldsymbol{h}\|\right)\left[\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right]\left[\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right]+\left(2r_1-\|\boldsymbol{h}\|\right)\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\boldsymbol{p}\right]\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{t}}\right]\\
&\qquad+p^2\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}-p^2-{r_1}^2\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}=0.
\end{aligned}$$

As we did for the solution circles that don't enclose the given, we transform the equation that we've just obtained into one involving the dot product of $\boldsymbol{t}$ with a linear combination of known vectors:

$$\begin{aligned}
&\left\{\left[2\left(\|\boldsymbol{h}\|-r_1\right)\hat{\boldsymbol{h}}\cdot\boldsymbol{p}+{r_1}^2-p^2\right]\hat{\boldsymbol{h}}\right.\\
&\quad\left.+\left[2\left(\|\boldsymbol{h}\|-r_1\right)\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\boldsymbol{p}\right]\hat{\boldsymbol{h}}\boldsymbol{i}\right\}\cdot\boldsymbol{t}=2r_1\|\boldsymbol{h}\|-p^2-{r_1}^2.
\end{aligned}\tag{4.11}$$

4.2 Solution via a Combination of Reflections and Rotations

For more information on the ideas used in this solution, please see [2]. The key ideas that we'll use, with reference to Fig. 4.2, are:

- The vector $\boldsymbol{s}$ can be expressed as the sum of $\boldsymbol{h}$ and a scalar multiple (λ) of $\hat{\boldsymbol{h}}\boldsymbol{i}$;
- Our goal will be to determine the values of λ for the four solution circles;
- The points $\boldsymbol{s}$ and $\boldsymbol{t}$ are reflections of each other with respect to the vector $\boldsymbol{i}\left(\boldsymbol{s}-\boldsymbol{t}\right)$, which is the mediatrix of the base of the isosceles triangle $\triangle\boldsymbol{t}C_2\boldsymbol{s}$;

- The isosceles triangle $\triangle C_1C_2M$ is similar to $\triangle \boldsymbol{t}C_2\boldsymbol{s}$. Therefore, the points $\boldsymbol{s}$ and $\boldsymbol{t}$ are reflections of each other with respect to the vector $\boldsymbol{i}\left(\boldsymbol{h}+r_1\hat{\boldsymbol{h}}+\lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right)$. The advantage of using that vector (as opposed to $\boldsymbol{s}-\boldsymbol{t}$) for expressing reflections is that it is written in terms of λ, and has the origin (C_1) as its starting point.

- Similarly, the vectors $\hat{\boldsymbol{t}}$ and $-\hat{\boldsymbol{h}}$ are reflections of each other with respect to $\boldsymbol{i}\left(\boldsymbol{h}+r_1\hat{\boldsymbol{h}}+\lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right)$.

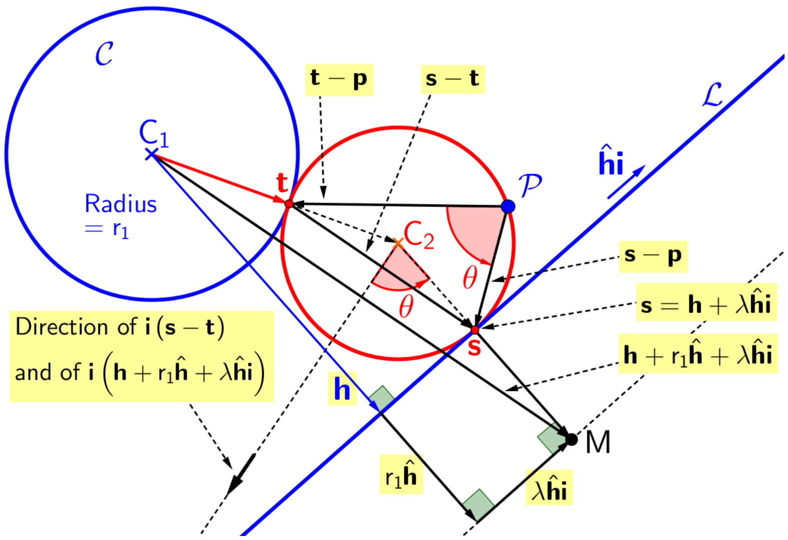

Figure 4.4: Elements used in using a combination of reflections and rotations to identify the two solution circles that do not enclose the given circle, $\mathcal{C}$.

We'll incorporate those observations in our usual technique of equating two expressions for the same rotation, which in this case will be $e^{\theta\boldsymbol{i}}$:

$$\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{\|\boldsymbol{t}-\boldsymbol{p}\|}\right]\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{\|\boldsymbol{s}-\boldsymbol{p}\|}\right]=\left[\frac{\boldsymbol{i}\left(\boldsymbol{h}+r_1\hat{\boldsymbol{h}}+\lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right)}{\|\boldsymbol{h}+r_1\hat{\boldsymbol{h}}+\lambda\hat{\boldsymbol{h}}\boldsymbol{i}\|}\right]\left[\hat{\boldsymbol{h}}\right];$$

$$\therefore\;[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}]\left[\hat{\boldsymbol{h}}\right]\left[\boldsymbol{h}+r_1\hat{\boldsymbol{h}}+\lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]=\text{ a bivector, and}$$
$$\left\langle\left(\boldsymbol{ts}-\boldsymbol{tp}-\boldsymbol{ps}+p^2\right)\left[\hat{\boldsymbol{h}}\right]\left[\boldsymbol{h}+r_1\hat{\boldsymbol{h}}+\lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\right\rangle_0=0.$$

We'll expand that result, initially, as

$$\begin{aligned}&\left\langle(\boldsymbol{ts})\left[\hat{\boldsymbol{h}}\right]\left[\boldsymbol{h}+r_1\hat{\boldsymbol{h}}+\lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\right\rangle_0\\&-\left\langle(\boldsymbol{tp})\left[\hat{\boldsymbol{h}}\right]\left[\boldsymbol{h}+r_1\hat{\boldsymbol{h}}+\lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\right\rangle_0\\&-\left\langle(\boldsymbol{ps})\left[\hat{\boldsymbol{h}}\right]\left[\boldsymbol{h}+r_1\hat{\boldsymbol{h}}+\lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\right\rangle_0\\&+\left\langle p^2\left[\hat{\boldsymbol{h}}\right]\left[\boldsymbol{h}+r_1\hat{\boldsymbol{h}}+\lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\right\rangle_0=0.\end{aligned}\tag{4.12}$$

To expand each of those four terms, we´ll write $\boldsymbol{s}$ as $\boldsymbol{s} = \boldsymbol{h} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i} = \|\boldsymbol{h}\|\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}$, and $\left(\boldsymbol{h} + r_1\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right)$ as $(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}$. In addition, we'll write $\hat{\boldsymbol{t}}$ as the reflection of $-\hat{\boldsymbol{h}}$ with respect to $\boldsymbol{i}\left(\boldsymbol{h} + r_1\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right)$:

$$\begin{aligned}\hat{\boldsymbol{t}} &= \left[\frac{\boldsymbol{i}\left(\boldsymbol{h} + r_1\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right)}{\|\ \|\boldsymbol{h}\|\hat{\boldsymbol{h}} + r_1\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\|}\right]\left[-\hat{\boldsymbol{h}}\right]\left[\frac{\boldsymbol{i}\left(\boldsymbol{h} + r_1\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right)}{\|\ \|\boldsymbol{h}\|\hat{\boldsymbol{h}} + r_1\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\|}\right] \\ &= \frac{\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\left[\hat{\boldsymbol{h}}\right]\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]}{(r_1 + \|\boldsymbol{h}\|)^2 + \lambda^2},\end{aligned}$$

from which

$$\boldsymbol{t} = \frac{r_1\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\left[\hat{\boldsymbol{h}}\right]\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]}{(r_1 + \|\boldsymbol{h}\|)^2 + \lambda^2}. \tag{4.13}$$

Now, for the expansions of each of the four terms in (4.12).

The factors in red constitute a rotation operator. See 2.3.

$$\begin{aligned}&(\boldsymbol{ts})\left[\hat{\boldsymbol{h}}\right]\left[\boldsymbol{h} + r_1\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right] \\ &= \left\{\frac{r_1\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\left[\hat{\boldsymbol{h}}\right]\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]}{(r_1 + \|\boldsymbol{h}\|)^2 + \lambda^2}\right\}\left[\|\boldsymbol{h}\|\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\left[\hat{\boldsymbol{h}}\right]\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right] \\ &= \left\{\frac{r_1\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\left[\hat{\boldsymbol{h}}\right]\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\left[\|\boldsymbol{h}\|\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\left[\hat{\boldsymbol{h}}\right]\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]}{(r_1 + \|\boldsymbol{h}\|)^2 + \lambda^2}\right\} \\ &= r_1\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\left[\|\boldsymbol{h}\|\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right] \\ &\therefore\ \langle(\boldsymbol{ts})\left[\hat{\boldsymbol{h}}\right]\hat{\boldsymbol{h}}\left[\boldsymbol{h} + r_1\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\rangle_0 = {r_1}^2\|\boldsymbol{h}\| + r_1h^2 + r_1\lambda^2.\end{aligned} \tag{4.14}$$

The factors in red constitute a rotation operator. See 2.3.

$$\begin{aligned}&(\boldsymbol{tp})\left[\hat{\boldsymbol{h}}\right]\left[\boldsymbol{h} + r_1\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right] \\ &= \left\{\frac{r_1\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\left[\hat{\boldsymbol{h}}\right]\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]}{(r_1 + \|\boldsymbol{h}\|)^2 + \lambda^2}\right\}[\boldsymbol{p}]\left[\hat{\boldsymbol{h}}\right]\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right] \\ &= \left\{\frac{r_1\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\left[\hat{\boldsymbol{h}}\right]\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right][\boldsymbol{p}]\left[\hat{\boldsymbol{h}}\right]\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]}{(r_1 + \|\boldsymbol{h}\|)^2 + \lambda^2}\right\} \\ &= r_1\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right][\boldsymbol{p}] \\ &\therefore\ \langle(\boldsymbol{tp})\left[\hat{\boldsymbol{h}}\right]\left[\boldsymbol{h} + r_1\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\rangle_0 = r_1(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}}\cdot\boldsymbol{p} - \lambda r_1\left[\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\boldsymbol{p}\right].\end{aligned} \tag{4.15}$$

$$\begin{aligned}&(\boldsymbol{ps})\left[\hat{\boldsymbol{h}}\right]\left[\boldsymbol{h} + r_1\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right] \\ &= \left\{[\boldsymbol{p}]\left[\|\boldsymbol{h}\|\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\right\}\left[\hat{\boldsymbol{h}}\right]\left[(r_1 + \|\boldsymbol{h}\|)\,\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right] \\ &\therefore\ \langle(\boldsymbol{ps})\left[\hat{\boldsymbol{h}}\right]\left[\boldsymbol{h} + r_1\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\rangle_0 = \left(r_1\|\boldsymbol{h}\| + h^2\right)\boldsymbol{p}\cdot\hat{\boldsymbol{h}} - \lambda\left(r_1 + 2\|\boldsymbol{h}\|\right)\left[\boldsymbol{p}\cdot\left(\hat{\boldsymbol{h}}\right)\right] + \lambda^2\boldsymbol{p}\cdot\hat{\boldsymbol{h}}.\end{aligned} \tag{4.16}$$

$$
\begin{aligned}
&p^2 \left[\hat{\boldsymbol{h}}\right] \left[\boldsymbol{h} + r_1\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right] = \\
&\left(r_1 + \|\boldsymbol{h}\|\right) p^2 + \lambda p^2 \boldsymbol{i} \\
&\therefore\ \langle p^2 \left[\hat{\boldsymbol{h}}\right] \left[\boldsymbol{h} + r_1\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\rangle_0 = \left(r_1 + \|\boldsymbol{h}\|\right) p^2. \qquad (4.17)
\end{aligned}
$$

Using the scalar parts that we identified in Eqs. (4.14)-(4.17), we find that (4.12) reduces to the following quadratic in λ:

$$
\begin{aligned}
&\left(r_1 + \boldsymbol{p}\cdot\hat{\boldsymbol{h}}\right)\lambda^2 - 2\left\{\left(r_1 + \|\boldsymbol{h}\|\right)\left[\boldsymbol{p}\cdot\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\right]\right\}\lambda \\
&+ \left(r_1 + \|\boldsymbol{h}\|\right)\left(p^2 + r_1\|\boldsymbol{h}\|\right) - \left(r_1 + \|\boldsymbol{h}\|\right)^2 \boldsymbol{p}\cdot\hat{h} = 0.
\end{aligned}
$$

Its solutions are

$$
\lambda = \frac{\left(r_1 + \|\boldsymbol{h}\|\right)\left[\boldsymbol{p}\cdot\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\right] \pm \sqrt{\left(p^2 - {r_1}^2\right)\left(r_1 + \|\boldsymbol{h}\|\right)\left[\left(\boldsymbol{h} - \boldsymbol{p}\right)\right]\cdot\hat{\boldsymbol{h}}}}{r_1 + \boldsymbol{p}\cdot\hat{\boldsymbol{h}}}. \qquad (4.18)
$$

The interpretation of that result is that there are two solution circles that don't enclose $\mathcal{C}$, and each of those circles is tangent to $\mathcal{L}$ at its own distance λ from the point at which $\boldsymbol{h}$ intersects $\mathcal{L}$.

To find the solution circles that do enclose $\mathcal{C}$, we start from Fig. 4.5.

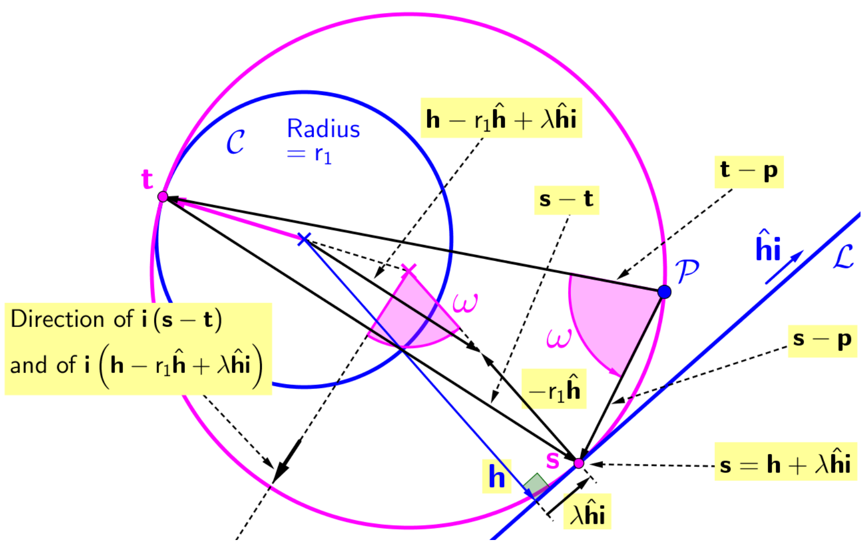

Figure 4.5: Elements used in using a combination of reflections and rotations to identify the solution circles that enclose the given circle, $\mathcal{C}$.

This time, we must express $\hat{\boldsymbol{t}}$ as a reflection of $\hat{\boldsymbol{h}}$ with respect to $\boldsymbol{i}\left(\boldsymbol{h} - r_1\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right)$. From there, the procedure is the same as we used to find the solution circles that do not enclose $\mathcal{C}$. But note: $\hat{\boldsymbol{t}}$ for solution circles that do enclose $\mathcal{C}$ is the reflection of $+\hat{\boldsymbol{h}}$ with respect to $\boldsymbol{i}\left(\boldsymbol{h} - r_1\hat{\boldsymbol{h}} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right)$, whereas $\hat{\boldsymbol{t}}$ for solution circles that don't enclose $\mathcal{C}$ is the reflection of $-\hat{\boldsymbol{h}}$ with respect to $\boldsymbol{i}\left(\boldsymbol{h}+r_1\hat{\boldsymbol{h}}+\lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right)$. The result, for solution circles that do enclose $\mathcal{C}$, is

$$\lambda = \frac{(r_1 - \|\boldsymbol{h}\|)\left[\boldsymbol{p}\cdot\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\right] \pm \sqrt{(p^2 - {r_1}^2)\left(\|\boldsymbol{h}\| - r_1\right)\left[(\boldsymbol{h}-\boldsymbol{p})\right]\cdot\hat{\boldsymbol{h}}}}{r_1 - \boldsymbol{p}\cdot\hat{\boldsymbol{h}}}. \tag{4.19}$$

5 Literature Cited

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"Geometric Algebra: Find unknown vector from two dot products"
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http://tube.geogebra.org/material/simple/id/1481375
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6 Appendix: The Rotations-Only Method Used in the Original Version of this Document

6.1 Identifying the Solution Circles that Don't Enclose $\mathcal{C}$

Many of the ideas that we'll employ here will also be used when we treat solution circles that do enclose $\mathcal{C}$.

6.1.1 Formulating a Strategy

Now, let's combine our observations about the problem in a way that might lead us to a solution. Our previous experiences in solving problems via vector rotations suggest that we should equate two expressions for the rotation $e^{\theta \boldsymbol{i}}$:

$$\begin{aligned}\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{|\boldsymbol{t}-\boldsymbol{p}|}\right]\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{|\boldsymbol{s}-\boldsymbol{p}|}\right] &= \left[\frac{\boldsymbol{t}-\boldsymbol{s}}{|\boldsymbol{t}-\boldsymbol{s}|}\right]\left[-\hat{\boldsymbol{h}}\boldsymbol{i}\right] \\ &= \left[\frac{\boldsymbol{s}-\boldsymbol{t}}{|\boldsymbol{s}-\boldsymbol{t}|}\right]\left[\hat{\boldsymbol{h}}\boldsymbol{i}\right].\end{aligned} \tag{6.1}$$

We've seen elsewhere that we will probably want to transform that equation into one in which some product of vectors involving our unknowns $\boldsymbol{t}$ and $\boldsymbol{s}$ is equal either to a pure scalar, or a pure bivector. By doing so, we may find some way of identifying either $\boldsymbol{t}$ or $\boldsymbol{s}$.

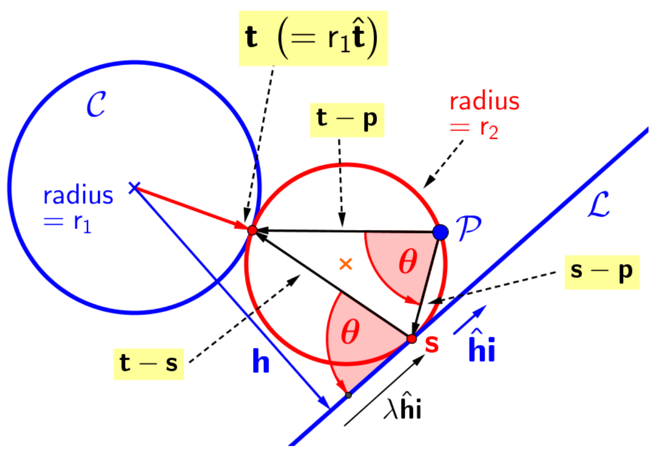
We'll keep in mind that although Eq. (6.1) has two unknowns (the vectors $\boldsymbol{t}$ and $\boldsymbol{s}$), our expression for r_2 (Eq. (3.2)) enables us to write the vector $\boldsymbol{s}$ in terms of the vector $\hat{\boldsymbol{t}}$.

Therefore, our strategy is to

- Equate two expressions, in terms of the unknown vectors $\boldsymbol{t}$ and $\boldsymbol{s}$, for the rotation $e^{\theta \boldsymbol{i}}$;
- Transform that equation into one in which on side is either a pure scalar or a pure bivector;
- Watch for opportunities to simplify equations by substituting for r_2; and
- Solve for our unknowns.

6.1.2 Transforming and Solving the Equations that Resulted from Our Observations and Strategizing

For convenience, we'll present our earlier figure again:



By examining that figure, we identified and equated two expressions for the rotation $e^{\theta \boldsymbol{i}}$, thereby obtaining Eq. (6.1):

$$\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{|\boldsymbol{t}-\boldsymbol{p}|}\right]\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{|\boldsymbol{s}-\boldsymbol{p}|}\right]=\left[\frac{\boldsymbol{s}-\boldsymbol{t}}{|\boldsymbol{s}-\boldsymbol{t}|}\right]\left[\hat{\boldsymbol{h}}\boldsymbol{i}\right].$$

We noted that we might wish at some point to make the substitution

$$\begin{aligned}\boldsymbol{s} &= [r_1+r_2]\,\hat{\boldsymbol{t}}+r_2\hat{\boldsymbol{h}}\\ &= \left[r_1+\frac{|\boldsymbol{h}|-r_1\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}}{1+\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}}\right]\hat{\boldsymbol{t}}+\left[\frac{|\boldsymbol{h}|-r_1\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}}{1+\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}}\right]\hat{\boldsymbol{h}}. \end{aligned} \tag{6.2}$$

We also noted that we'll want to transform Eq. (6.1) into one in which one side is either a pure scalar or a pure bivector. We should probably do that transformation before making the substitution for $\boldsymbol{s}$. One way to effect the transformation is by left-multiplying both sides of Eq. (6.1) by $\boldsymbol{s}-\boldsymbol{t}$, then by $\hat{\boldsymbol{h}}$, and then rearranging the result to obtain

$$\hat{\boldsymbol{h}}\,[\boldsymbol{s}-\boldsymbol{t}]\,[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}] = |\boldsymbol{s}-\boldsymbol{t}|\,|\boldsymbol{t}-\boldsymbol{p}|\,|\boldsymbol{s}-\boldsymbol{p}|\,\boldsymbol{i} \tag{6.3}$$

This is the equation that we sought to obtain, so that we could now write

$$\langle\hat{\boldsymbol{h}}\,[\boldsymbol{s}-\boldsymbol{t}]\,[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}]\rangle_0 = 0. \tag{6.4}$$

Next, we need to expand the products on the left-hand side, but we'll want to examine the benefits of making a substitution for $\boldsymbol{s}$ first. We still won't, as yet, write $\boldsymbol{s}$ in terms of $\hat{\boldsymbol{t}}$. In hopes of keeping our equations simple enough for us to identify useful simplifications easily at this early stage, we'll make the substitution

$$\boldsymbol{s} = (r_1+r_2)\,\hat{\boldsymbol{t}} + r_2\hat{\boldsymbol{h}},$$

rather than making the additional substitution (Eq. (3.2)) for r_2. Now, we can see that $\boldsymbol{s}-\boldsymbol{t} = r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)$. Using this result, and $\boldsymbol{t} = r_1\hat{\boldsymbol{t}}$, Eq. (6.4) becomes

$$\langle\hat{\boldsymbol{h}}\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\right]\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\left[(r_1+r_2)\,\hat{\boldsymbol{t}}+r_2\hat{\boldsymbol{h}}-\boldsymbol{p}\right]\rangle_0 = 0.$$

Now here is where I caused myself a great deal of unnecessary work in previous versions of the solution by plunging in and expanding the product that's inside the $\langle\rangle_0$ without examining it carefully. Look carefully at the last factor in that product. Do you see that we can rearrange it to give the following?

$$\langle\hat{\boldsymbol{h}}\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\right]\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\underbrace{\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)+r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]}_{After\ rearrangement}\rangle_0 = 0.$$

That result is interesting, but is it truly useful to us? To answer that question, let's consider different ways in which we might expand the product, then find its scalar part.

If we effect the multiplications in order, from left to right, we're likely to end up with a confusing mess. However, what if we multiply the last three factors together? Those three factors, together, compose a product of the form $\boldsymbol{ab}\,[\boldsymbol{a}+\boldsymbol{b}]$:

$$\underbrace{\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\right]}_{\boldsymbol{a}}\underbrace{\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]}_{\boldsymbol{b}}\left[\underbrace{r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)}_{\boldsymbol{a}}+\underbrace{r_1\hat{\boldsymbol{t}}-\boldsymbol{p}}_{\boldsymbol{b}}\right].$$

The expansion of $\boldsymbol{ab}\,[\boldsymbol{a}+\boldsymbol{b}]$ is

$$\begin{aligned}\boldsymbol{ab}\,[\boldsymbol{a}+\boldsymbol{b}] &= \boldsymbol{aba}+b^2\boldsymbol{a}\\ &= 2\,(\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{a}-a^2\boldsymbol{b}+b^2\boldsymbol{a} \quad \text{(among other possibilites).}\end{aligned}$$

That expansion evaluates to a vector, of course. Having obtained the corresponding expansion of the product $\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\right]\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)+r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]$, we'd then "dot" the result with $\hat{\boldsymbol{h}}$ to obtain $\langle\hat{\boldsymbol{h}}\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\right]\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)+r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\rangle_0$. We know, from the solutions to Problem 6 in Smith J A 2016 , that such a maneuver can work out quite favorably. So, let's try it.

Expanding $\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\right]\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)+r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]$ according to the identity $\boldsymbol{ab}\left[\boldsymbol{a}+\boldsymbol{b}\right]=2\left(\boldsymbol{a}\cdot\boldsymbol{b}\right)\boldsymbol{a}-a^2\boldsymbol{b}+b^2\boldsymbol{a}$, we obtain, initially,

$$2\left\{{r_2}^2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\cdot\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\right\}\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)-{r_2}^2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)^2\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]+r_2\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]^2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)$$

When we've completed our expansion and dotted it with $\hat{\boldsymbol{h}}$, we'll set the result to zero, so let's divide out the common factor r_2 now:

$$2\left\{r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\cdot\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\right\}\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)-r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)^2\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]+\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]^2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)$$

Recalling that $\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)^2=2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)$, the preceding becomes

$$2\left\{r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\cdot\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\right\}\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)-2r_2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]+\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]^2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right).$$

This is the form that we'll dot with $\hat{\boldsymbol{h}}$. Having done so, the factor $\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}$ becomes $1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}$. Then, as planned, we set the resulting expression equal to zero:

$$2\left\{r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\cdot\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\right\}\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)-2r_2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)\left[r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}-\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right]+\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]^2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)=0.$$

Next, we'll rearrange that equation to take advantage of the relation $r_2=\frac{|\boldsymbol{h}|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}$ (see Eq. (3.2)). We'll show the steps in some detail:

$$\begin{aligned}
2\left\{r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\cdot\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\right\}\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)-2r_2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)\left[r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}-\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right]+\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]^2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)&=0.\\
2r_2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)\left\{\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\cdot\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right\}+\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]^2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)&=0\\
2\left[\frac{|\boldsymbol{h}|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right]\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)\left\{\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\cdot\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right\}+\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]^2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)&=0\\
2\left[|\boldsymbol{h}|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right]\left\{\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\cdot\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right\}+\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]^2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)&=0\\
2\left[|\boldsymbol{h}|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right]\left\{r_1-\boldsymbol{p}\cdot\hat{\boldsymbol{t}}+r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}-\hat{\boldsymbol{h}}\cdot\boldsymbol{p}-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right\}+\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]^2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)&=0\\
2\left[|\boldsymbol{h}|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right]\left\{r_1-\boldsymbol{p}\cdot\hat{\boldsymbol{t}}\right\}+\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]^2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)&=0.
\end{aligned}$$

Now that the dust has settled from the r_2 substitution, we'll expand $\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]^2$, then simplify further:

$$\begin{aligned}
2\left[|\boldsymbol{h}|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right]\left\{r_1-\boldsymbol{p}\cdot\hat{\boldsymbol{t}}\right\}+\left[{r_1}^2-2r_1\boldsymbol{p}\cdot\hat{\boldsymbol{t}}+p^2\right]\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)&=0\\
\left[-2{r_1}^2+2r_1\boldsymbol{p}\cdot\hat{\boldsymbol{t}}+{r_1}^2-2r_1\hat{\boldsymbol{t}}\cdot\boldsymbol{p}+p^2\right]\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}-2\left|\boldsymbol{h}\right|\boldsymbol{p}\cdot\hat{\boldsymbol{t}}+2\left|\boldsymbol{h}\right|r_1+{r_1}^2-2r_1\boldsymbol{p}\cdot\hat{\boldsymbol{t}}+p^2&=0\\
\left(p^2-{r_1}^2\right)\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}-2\left(r_1+|\boldsymbol{h}|\right)\boldsymbol{p}\cdot\hat{\boldsymbol{t}}+{r_1}^2+p^2+2\left|\boldsymbol{h}\right|r_1&=0.
\end{aligned}$$

We saw equations like this last one many times in Smith J A 2016. There, we learned to solve those equations by grouping the dot products that involve $\boldsymbol{t}$ into a dot product of $\boldsymbol{t}$ with a linear combination of known vectors:

$$\underbrace{\left[2\left(r_1+|\boldsymbol{h}|\right)\boldsymbol{p}-\left(p^2-{r_1}^2\right)\hat{\boldsymbol{h}}\right]}_{\text{A linear combination of } \hat{\boldsymbol{h}} \text{ and } \boldsymbol{p}}\cdot\hat{\boldsymbol{t}}=2\left|\boldsymbol{h}\right|r_1+{r_1}^2+p^2. \tag{6.5}$$

The geometric interpretation of Eq. (6.5) is that $2\,|\boldsymbol{h}|\,r_1 + {r_1}^2 + p^2$ is the projection of the vector $2\,(r_1 + |\boldsymbol{h}|)\,\boldsymbol{p} - \left(p^2 - {r_1}^2\right)\hat{\boldsymbol{h}}$ upon $\hat{\boldsymbol{t}}$. Because we want to find $\boldsymbol{t}$, and know $2\,(r_1 + |\boldsymbol{h}|)\,\boldsymbol{p} - \left(p^2 - {r_1}^2\right)\hat{\boldsymbol{h}}$, we'll transform Eq. (6.5) into a version that tells us the projection of the vector $\boldsymbol{t}$ upon $2\,(r_1 + |\boldsymbol{h}|)\,\boldsymbol{p} - \left(p^2 - {r_1}^2\right)\hat{\boldsymbol{h}}$.

First, just for convenience, we'll multiply both sides of Eq. (6.5) by $r_1\,|\boldsymbol{h}|$:

$$\left[2\left(r_1\,|\boldsymbol{h}| + h^2\right)\boldsymbol{p} - \left(p^2 - {r_1}^2\right)\boldsymbol{h}\right]\cdot\boldsymbol{t} = 2h^2{r_1}^2 + r_1\,|\boldsymbol{h}|\left({r_1}^2 + p^2\right).$$

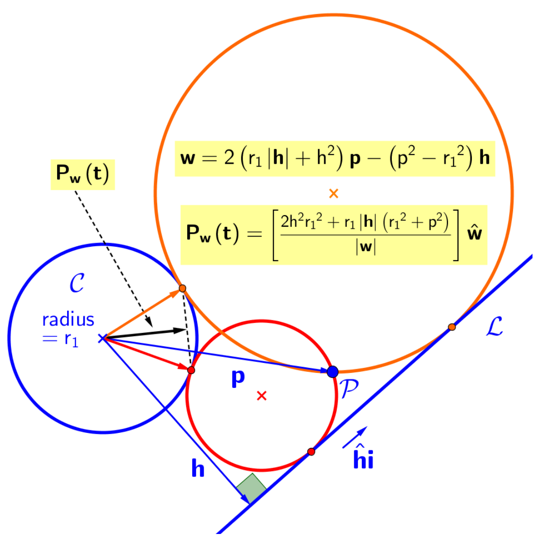
Next, we'll use the symbol "$\boldsymbol{w}$" for the vector $\left[2\left(r_1\,|\boldsymbol{h}| + h^2\right)\boldsymbol{p} - \left(p^2 - {r_1}^2\right)\boldsymbol{h}\right]$, and write

$$\boldsymbol{w}\cdot\boldsymbol{t} = 2h^2{r_1}^2 + r_1\,|\boldsymbol{h}|\left({r_1}^2 + p^2\right).$$

Finally, because $\boldsymbol{P_w}(\boldsymbol{t})$, the projection of the vector $\boldsymbol{t}$ upon $\boldsymbol{w}$ is $(\boldsymbol{t}\cdot\hat{\boldsymbol{w}})\,\hat{\boldsymbol{w}}$, we have

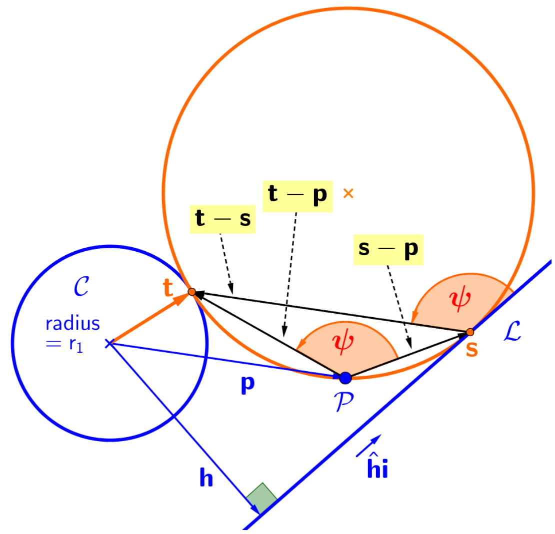
$$\boldsymbol{P_w}(\boldsymbol{t}) = \left[\frac{2h^2{r_1}^2 + r_1\,|\boldsymbol{h}|\left({r_1}^2 + p^2\right)}{|\boldsymbol{w}|}\right]\hat{\boldsymbol{w}}. \tag{6.6}$$

As we learned in Smith J A 2016, Eq. (6.6) tells us that Eq (6.5) has two solutions. That is, there are two circles that are tangent to $\mathcal{L}$ and pass through the point $\mathcal{P}$, and are also tangent to $\mathcal{C}$ without enclosing it:



Having identified $\boldsymbol{P_w}(\boldsymbol{t})$, the points of tangency with $\mathcal{C}$ and $\mathcal{L}$ can be determined using methods shown in Smith J A 2016, as can the equations for the corresponding solution circles.

To round off our treatment of solution circles that don't enclose $\mathcal{C}$, we should note that we derived our solution starting from equations that express the relationship between $\mathcal{C}$, $\mathcal{L}$, $\mathcal{P}$, and the smaller of the two solution circles. You may have noticed that the larger solution circle does not bear quite the same relationship to $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{C}$ as the smaller one. To understand in what way those relationships differ, let's examine the following figure.



By equating two expressions for the rotation $e^{\psi \boldsymbol{i}}$, we'd find that

$$\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{|\boldsymbol{s}-\boldsymbol{p}|}\right]\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{|\boldsymbol{t}-\boldsymbol{p}|}\right]=\left[\hat{\boldsymbol{h}}\boldsymbol{i}\right]\left[\frac{\boldsymbol{t}-\boldsymbol{s}}{|\boldsymbol{t}-\boldsymbol{s}|}\right].$$

Compare that result to the corresponding equation for the smaller of the solution circles:

$$\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{|\boldsymbol{t}-\boldsymbol{p}|}\right]\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{|\boldsymbol{s}-\boldsymbol{p}|}\right]=\left[\frac{\boldsymbol{s}-\boldsymbol{t}}{|\boldsymbol{s}-\boldsymbol{t}|}\right]\left[\hat{\boldsymbol{h}}\boldsymbol{i}\right].$$

We followed up on that equation by transforming it into one in which $\hat{\boldsymbol{h}}$ was at one end of the product on the left-hand side. The result was Eq. (6.3):

$$\hat{\boldsymbol{h}}\left[\boldsymbol{s}-\boldsymbol{t}\right]\left[\boldsymbol{t}-\boldsymbol{p}\right]\left[\boldsymbol{s}-\boldsymbol{p}\right]=|\boldsymbol{s}-\boldsymbol{t}|\,|\boldsymbol{t}-\boldsymbol{p}|\,|\boldsymbol{s}-\boldsymbol{p}|\,\boldsymbol{i}.$$

We saw the advantages of that arrangement when we proceeded to solve for $\boldsymbol{t}$. All we had to do in order to procure that arrangement was to left-multiply both sides of the equation

$$\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{|\boldsymbol{t}-\boldsymbol{p}|}\right]\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{|\boldsymbol{s}-\boldsymbol{p}|}\right]=\left[\frac{\boldsymbol{s}-\boldsymbol{t}}{|\boldsymbol{s}-\boldsymbol{t}|}\right]\left[\hat{\boldsymbol{h}}\boldsymbol{i}\right]$$

by $\boldsymbol{s}-\boldsymbol{t}$, and then by $\hat{\boldsymbol{h}}$.

To procure a similar arrangement starting from the equation that we wrote for the larger circle, using the angle ψ ,

$$\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{|\boldsymbol{s}-\boldsymbol{p}|}\right]\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{|\boldsymbol{t}-\boldsymbol{p}|}\right]=\left[\hat{\boldsymbol{h}}\boldsymbol{i}\right]\left[\frac{\boldsymbol{t}-\boldsymbol{s}}{|\boldsymbol{t}-\boldsymbol{s}|}\right],$$

we could left-multiply both sides of the equation by $\hat{\boldsymbol{h}}$, then right-multiply by $\boldsymbol{s}-\boldsymbol{t}$, giving

$$\hat{\boldsymbol{h}}\left[\boldsymbol{s}-\boldsymbol{p}\right]\left[\boldsymbol{t}-\boldsymbol{p}\right]\left[\boldsymbol{t}-\boldsymbol{s}\right]=\left|\boldsymbol{s}-\boldsymbol{p}\right|\left|\boldsymbol{t}-\boldsymbol{p}\right|\left|\boldsymbol{t}-\boldsymbol{s}\right|\boldsymbol{i}.$$

Using the same ideas and transformations as for the smaller circle, we'd then transform the product $\left[\boldsymbol{s}-\boldsymbol{p}\right]\left[\boldsymbol{t}-\boldsymbol{p}\right]\left[\boldsymbol{t}-\boldsymbol{s}\right]$ into

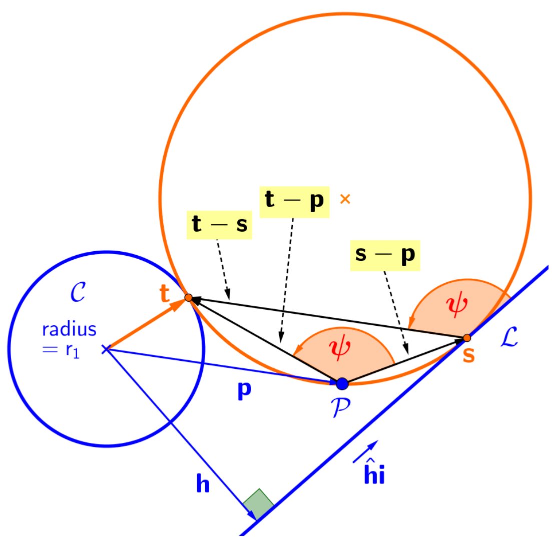
$$\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)+r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\right].$$

By comparison, the product that we obtained for the smaller circle was

$$\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\right]\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)+r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right].$$

In both cases, the products that result from the expansion have the forms $\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\left(r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right)\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)$ and $\left(r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right)^2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)$, so the same simplifications work in both, and give the same results when the result is finally dotted with $\hat{\boldsymbol{h}}$ and set to zero.

The above having been said, let's look at another way of obtaining an equation, for the larger circle, that has the same form as Eq. (6.3). For convenience, we'll present the figure for the larger circle again:

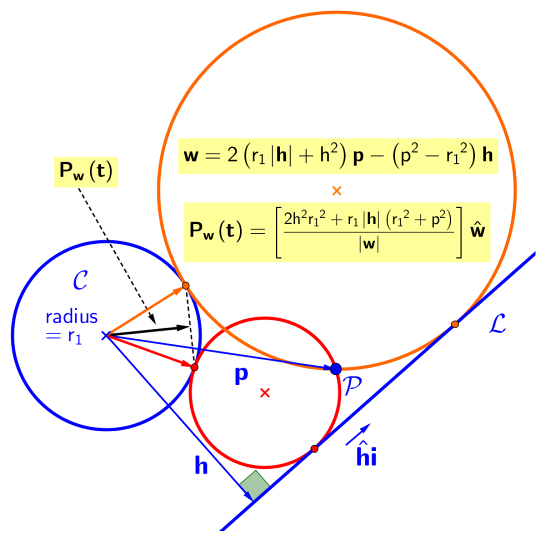


Instead of beginning by equating two expressions for the rotation $e^{\psi\boldsymbol{i}}$, we'll equate two expressions for $e^{-\psi\boldsymbol{i}}$:

$$\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{\left|\boldsymbol{t}-\boldsymbol{p}\right|}\right]\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{\left|\boldsymbol{s}-\boldsymbol{p}\right|}\right]=\left[\frac{\boldsymbol{t}-\boldsymbol{s}}{\left|\boldsymbol{t}-\boldsymbol{s}\right|}\right]\left[\hat{\boldsymbol{h}}\boldsymbol{i}\right].$$

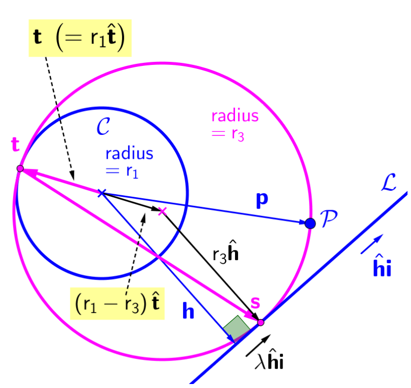
After left-multiplying both sides by $\boldsymbol{t}-\boldsymbol{s}$, then by $\hat{\boldsymbol{h}}$, and rearranging, the result would be identical to Eq. (6.3), except for the algebraic sign of the right-had side, which —because it's a bivector—would drop out when we took the scalar part of both sides.

However, the difference in the sign of that bivector captures the geometric nature of the difference between the relationships of the large and small circles to $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{C}$. That difference in sign is also reflected in the positions, with respect to the vector $\boldsymbol{w}$, of the solution circles' points of tangency $\boldsymbol{t}$:



6.2 Identifying the Solution Circles that Enclose $\mathcal{C}$

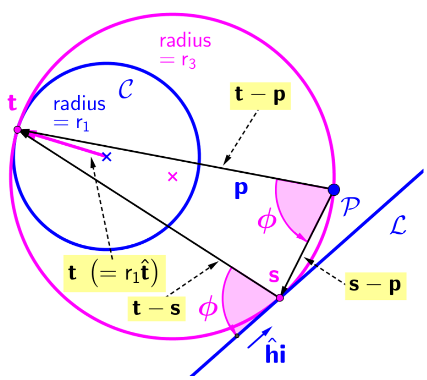
These solution circles can be found by modifying, slightly, the ideas that we used for finding solution circles that don't enclose $\mathcal{C}$. Here, too, we'll want to express the radius r_3 of the solution circle in terms of the vector $\boldsymbol{t}$. Examining the next figure,



we see that $\boldsymbol{s} = (r_1 - r_3)\,\hat{\boldsymbol{t}} + r_3\hat{\boldsymbol{h}}$, and also that $\boldsymbol{s} = \boldsymbol{h} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}$. By equating those two expressions, dotting both sides with $\hat{\boldsymbol{h}}$, and then solving for r_3, we find that

$$r_3 = \frac{|\boldsymbol{h}| - r_1\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}}{1 - \hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}}.$$

As was the case when we found solution circles that didn't enclose $\mathcal{C}$, we'll want to equate expressions for two rotations that involve the unknown points of tangency $\boldsymbol{t}$ and $\boldsymbol{s}$. For example, through the angles labeled ϕ, below:



$$\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{|\boldsymbol{t}-\boldsymbol{p}|}\right]\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{|\boldsymbol{s}-\boldsymbol{p}|}\right] = \left[\frac{\boldsymbol{t}-\boldsymbol{s}}{|\boldsymbol{t}-\boldsymbol{s}|}\right]\left[-\hat{\boldsymbol{h}}\boldsymbol{i}\right]$$
$$= \left[\frac{\boldsymbol{s}-\boldsymbol{t}}{|\boldsymbol{s}-\boldsymbol{t}|}\right]\left[\hat{\boldsymbol{h}}\boldsymbol{i}\right],$$

Left-multiplying that result by $\boldsymbol{s} - \boldsymbol{t}$, and then by $\hat{\boldsymbol{h}}$,

$$\hat{\boldsymbol{h}}\,[\boldsymbol{s}-\boldsymbol{t}]\,[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}] = |\boldsymbol{s}-\boldsymbol{t}|\,|\boldsymbol{t}-\boldsymbol{p}|\,|\boldsymbol{s}-\boldsymbol{p}|\,\boldsymbol{i}$$
$$\therefore\ \langle\hat{\boldsymbol{h}}\,[\boldsymbol{s}-\boldsymbol{t}]\,[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}]\rangle_0 = 0,$$

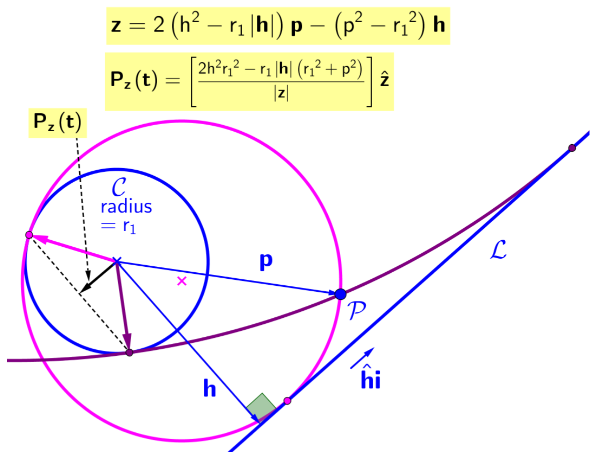
which is identical to Eq. (6.4). To solve for $\boldsymbol{t}$, we use exactly the same technique that we did when we identified the solution circles that don't surround $\mathcal{C}$, with r_3 and $1 - \hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}$ taking the place of r_2 and $1 - \hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}$, respectively. The result is

$$\boldsymbol{z}\cdot\boldsymbol{t} = 2h^2{r_1}^2 - r_1\,|\boldsymbol{h}|\left({r_1}^2 + p^2\right),$$

where $\boldsymbol{z} = \left[2\left(h^2 - r_1\,|\boldsymbol{h}|\right)\boldsymbol{p} - \left(p^2 - {r_1}^2\right)\boldsymbol{h}\right]$. Thus,

$$\boldsymbol{P}_{\boldsymbol{z}}\left(\boldsymbol{t}\right) = \left[\frac{2h^2{r_1}^2 - r_1\,|\boldsymbol{h}|\left({r_1}^2 + p^2\right)}{|\boldsymbol{z}|}\right]\hat{\boldsymbol{z}}. \qquad (6.7)$$

Again, there are two solution circles of this type:



6.3 The Complete Solution, and Comments

There are four solution circles: two that enclose $\mathcal{C}$, and two that don't: