Milan Daniel[1]

[1]Institute of Computer Science, Academy of Sciences of the Czech Republic, CZ - 182 07 Prague 8, Czech Republic

Comparison between DSm and MinC combination rules

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Abstract: *Both DSm and minC rules of combination endeavor to process conflicts among combined beliefs better. The nature of conflicts as well as their processing during the belief combination is sketched. An presentation of the minC combination, an alternative to Dempster's rule of combination, follows. Working domains, structures and mechanisms of the DSm and minC combination rules are compared in the body of this chapter. Finally, some comparative examples are presented.*

10.1 Introduction

The classical DSm rule of combination, originally presented in [5, 6], has served for combination of two or several beliefs on the free DSm model. Later, a hybrid DSm combination rule has been developed to be applicable also on the classical Shafer (or Dempster-Shafer, DS) and the hybrid DSm model. The present state of the DSm rule is described in Chapter 4, see Equations (4.7)-(4.10).

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MinC combination (minimal conflict/minimal contradiction) rule introduced in [2, 4] is an alternative to the Dempter's rule of combination on the classical DS model. This rule has been developed for better handling of conflicting situations, which is a weak point of the classical Dempster rule. A brief description of the idea of the minC combination is presented in Section 10.3.

Both arguments and results of the DSm rule are beliefs in a DSm model, which admits intersections of elements of the frame of discernment in general. The minC combination serves for combination of classical belief functions (BFs) where all intersections of elements (of the frame of discernment) are empty and their resulting basic belief masses should be 0.

For finer processing of conflicts than the classical normalization in Dempster rule, a system of different types of conflict (or empty set) is introduced. For representation of intermediate results, generalized BFs serve on generalized frames of discernment which contains elements of the classical DS frame of discernment and correspondent types of conflict.

Even if the two developed approaches were originally different (disjoint), as well as the paradigms of both approaches, the intermediate working generalized beliefs of the minC combination are similar to those in the free DSm model, and the way of combination on the generalized level is analogous to that in the free DSm model. This surprising fact is the main reason why we compare these two seemingly incomparable, and originally quite disjoint approaches.

Now, after the development of the DSm combination for any hybrid DSm model, it is, moreover, possible to compare behavior of both approaches on classical BFs, i.e. in the application domain of the minC combination.

10.2 Conflict in belief combination

In the DSm combination, which is specially designed for conflicting situations, there are no problems with conflicts.

The common similar principle for Dempster rule, the minC combination and the DSm combination rule is that the basic belief assignment/mass (bbm) $m_1(X)$, assigned to set X by the first basic belief assignment (bba) m_1, multiplied by bbm $m_2(Y)$, assigned to set Y by the second bba m_2, is assigned to the set $X \cap Y$ by the resulting bba m_{12}, i.e. $m_1(X)m_2(Y)$ is a part of $m_{12}(X \cap Y)$.

This principle works relatively nicely if sets X and Y are not disjoint. There is also no problem for the DSm rule because $X \cap Y$ is always an element of D^Θ and its positive value is accepted even in the case of sets X and Y without any common element of Θ.

In Dempster's rule, disjoint X and Y tend to a conflict situation. All the conflicts are summed up together and reallocated onto 2^Θ by normalization in the classical normalized Dempster's rule, see [9], or stored as $m(\emptyset)$ in the non-normalized Dempster's rule in Transferable Belief Model (TBM) by Smets, see [10, 11]. It is a fact that in Smets' approach the normalization is only postponed from the combination process phase to the decisional one, as the normalization is the first step of computation of the classical pignistic transformation in TBM. The non-normalized Dempster rule commutes with the normalization, hence the pignistic probability is always the same in both the cases of normalized and non-normalized Dempster's rule.

A weak point of Dempster's rule — combination of conflicting beliefs is caused by normalization or by grouping all the conflicts together by the non-normalized version of Dempster's rule. Therefore, different types of conflict were introduced and a minC combination rule has been developed for a better handling of conflicting situations.

10.3 The minC combination

The minC combination (the minimal contradiction/conflict combination) of belief functions was developed [2, 4] with an effort to find a new associative combination which processes conflicts better than Dempster's rule. The classical Shafer model from Dempster-Shafer theory is supposed for both input and resulting belief functions. The minC combination is a generalization[1] of the un-normalized Dempster's rule. $m(\emptyset)$ is not considered as an argument for new unknown elements of the frame of discernment, $m(\emptyset)$ is considered as a conflict[2] arising by conjunctive combination. To handle it, a system of different types of conflicts is considered with respect to sets which produce the conflicts.

10.3.1 A system of different types of conflicts

We distinguish conflicts according to the sets to which the original bbms were assigned by m_i. There is only one type of conflict among the belief functions defined on a binary frame of discernment, hence the minC combination coincides with the non-normalized conjunctive rule there.

[1]Note that, on the other hand, the minC combination approach is a special case of an even more general approach of combination belief functions 'per elements', see [3]

[2]The term "contradiction" is used in [2, 4], while we use "conflict" here in order to have a uniform terminology.

In the case of an n-ary frame of discernment we distinguish different types of conflicts, e.g. $\{\theta_1\}\times\{\theta_2\}$, $\{\theta_1\}\times\{\theta_2,\theta_3\}$, $\{\theta_1\}\times\{\theta_2\}\times\{\theta_3\}$, $\{\theta_i,\theta_j,\theta_k\}\times\{\theta_m,\theta_n,\theta_o\}$ etc. The symbol $\times$ serves here for a denotation of conflicts, it is not used as any new operation on sets. Thus e.g. $\{\theta_1\}\times\{\theta_2,\theta_3\}$ simply denotes the conflict between sets $\{\theta_1\}$ and $\{\theta_2,\theta_3\}$.

We assume that products of the conflicting bbms are temporarily assigned (we all the time keep in mind that Shafer's constraints should be satisfied) to the corresponding conflicts: e.g. $m_1(\{\theta_1\})m_2(\{\theta_2\})$ is assigned to the conflict $\{\theta_1\}\times\{\theta_2\}$. In this way we obtain so called *generalized bbas*, and *generalized BFs* on a *generalized frame of discernment* given by Θ.

When combining 2 BFs defined on 3D frame $\Theta = \{\theta_1,\theta_2,\theta_3\}$ we obtain the following conflicts as intersections of disjoint subsets of Θ: $\{\theta_1\}\times\{\theta_2\}$, $\{\theta_1\}\times\{\theta_3\}$, $\{\theta_2\}\times\{\theta_3\}$, $\{\theta_1,\theta_2\}\times\{\theta_3\}$, $\{\theta_1,\theta_3\}\times\{\theta_2\}$, and $\{\theta_2,\theta_3\}\times\{\theta_1\}$.

Because we need a classical BF as a result of the combination, we have to reallocate bbms assigned to conflicts among subsets of Θ after the combination. These bbms are proportionalized, i.e. proportionally distributed, among subsets of Θ corresponding to the conflicts. A few such *proportionalizations* are presented in [4]. Unfortunately, all these proportionalizations break required associativity of the conjunctive combination. To keep the associativity as long as possible we must be able to combine the generalized belief functions with other BFs and generalized BFs. From this reason other conflicts arise: e.g. $\{\theta_1\}\times\{\theta_2\}\times\{\theta_3\}$, $(\{\theta_1,\theta_2\}\times\{\theta_1,\theta_3\})\times\{\theta_2\}\times\{\theta_3\}$, $(\{\theta_1,\theta_2\}\times\{\theta_3\})\times(\{\theta_2\}\times\{\theta_3\})$, etc.

A very important role for keeping associativity is played by so called *partial* or *potential conflicts* [3], e.g. a partial conflict $\{\theta_1,\theta_2\}\times\{\theta_2,\theta_3\}$ which is not a conflict in the case of combination of two beliefs $\{\theta_1,\theta_2\}\cap\{\theta_2,\theta_3\} = \{\theta_2\}$, but it can cause a conflict in a later combination with another belief, e.g. *pure* or *real conflict* [4] $\{\theta_1,\theta_2\}\times\{\theta_2,\theta_3\}\times\{\theta_1,\theta_3\}$ because there is $\{\theta_1,\theta_2\}\cap\{\theta_2,\theta_3\}\cap\{\theta_1,\theta_3\} = \emptyset$, in Shafer's model.

In order not to have an infinite number of different conflicts, the conflicts are divided into classes of equivalence $\sim$ which are called *types of conflicts*, e.g. $\{\theta_1\}\times\{\theta_2\} \sim \{\theta_2\}\times\{\theta_1\} \sim \{\theta_1\}\times\{\theta_2\}\times\{\theta_2\}\times\{\theta_2\}\times\{\theta_1\}\times\{\theta_1\}\times\{\theta_1\}$, etc. The minC combination works with these classes of equality (types of conflict) instead of the set of all different conflicts. For more details see [4].

[3] *Potential contradictions* in the original terminology of [2, 4]

[4] *A real contradiction* in [2, 4].

The conflicts are considered "per elements" in the following way: conflict $\{\theta_1, \theta_2\} \times \{\theta_3\}$ is considered as a set of *elementary conflicts* $\{\{\theta_1\} \times \{\theta_3\},\ \{\theta_2\} \times \{\theta_3\}\}$, i.e. set of conflicts between/among singletons. Analogically, potential conflict $\{\theta_1, \theta_2\} \times \{\theta_2, \theta_3\}$ is considered as a set of elementary conflicts $\{\{\theta_1\}\times\{\theta_2\},\ \{\theta_1\}\times\{\theta_3\},\ \{\theta_2\},\ \{\theta_2\}\times\{\theta_3\}\}$, where $\{\theta_2\} \sim \{\theta_2\} \times \{\theta_2\}$ is so called *trivial conflict*[5], i.e. no conflict in fact. Note that any partial conflict contains at least one trivial conflict. The set of elementary conflicts is constructed similarly to the Cartesian product of conflicting sets, where $\{\theta_1\} \times \{\theta_2\} \times ... \times \{\theta_k\}$ is used instead on n-tuple $[\theta_1, \theta_2, ..., \theta_k]$. As the above equivalence $\sim$ of elementary conflicts is used, we have elementary conflicts of different n-arity in the same set, thus we do not use n-tuples as it is usual in the Cartesian product. The idea of "conflicts per elements" was generalized also for non-conflicting sets in the "combination per elements", see [3].

For further decreasing of the number of types of conflicts we consider only minimal conflicts in the following sense: $\{\theta_1\} \times \{\theta_2\},\ \{\theta_3\}$, are minimal conflicts of the set $\{\{\theta_1\} \times \{\theta_2\},\ \{\theta_3\},\ \{\theta_1\} \times \{\theta_2\} \times \{\theta_3\},\ \{\theta_1\}\times\{\theta_2\}\times\{\theta_4\}\times\{\theta_5\},\ \{\theta_1\}\times\{\theta_3\}\times\{\theta_5\}\}$; i.e. the set of singletons contained in a minimal conflict is minimal from the point of view of inclusion among all sets of singletons corresponding to elementary conflicts. Thus $\{\{\theta_1\}\times\{\theta_2\}, \{\theta_3\}\} \sim \{\{\theta_1\}\times\{\theta_2\}, \{\theta_3\}, \{\theta_1\}\times\{\theta_2\}\times\{\theta_3\}, \{\theta_1\}\times\{\theta_2\}\times\{\theta_4\}\times\{\theta_5\}, \{\theta_1\}\times\{\theta_3\}\times\{\theta_5\}\}$. Our concentration only to minimal conflicts brings us a simplification, which is closer to Shafer's model, and it has no influence on associativity of combination.

In this way we obtain 8 types of conflicts ($\{\theta_1\}\times\{\theta_2\},\ \{\theta_1\}\times\{\theta_3\},\ \{\theta_2\}\times\{\theta_3\},\ \{\theta_1\}\times\{\theta_2\}\times\{\theta_3\}, \{\{\theta_1\}\times\{\theta_2\},\ \{\theta_1\}\times\{\theta_3\}\},\ \{\{\theta_1\}\times\{\theta_2\},\ \{\theta_2\}\times\{\theta_3\}\},\ \{\{\theta_1\}\times\{\theta_3\},\ \{\theta_2\}\times\{\theta_3\}\},\ \{\{\theta_1\}\times\{\theta_2\},\ \{\theta_1\}\times\{\theta_3\},\ \{\theta_2\}\times\{\theta_3\}\}$) and 3 types of potential conflicts ($\{\{\theta_1\},\ \{\theta_2\} \times \{\theta_3\}\},\ \{\{\theta_2\},\ \{\theta_1\} \times \{\theta_3\}\},\ \{\{\theta_3\},\ \{\theta_1\} \times \{\theta_2\}\}$) in a 3D case $\Theta = \{\theta_1, \theta_2, \theta_3\}$. Together with 7 non-conflicting subsets of Θ we have 18 sets of conflicts to which nonnegative bbms can be assigned in the 3D case, or 18 elements of a generalized 3D frame of discernment.

10.3.2 Combination on generalized frames of discernment

As minC combination has a nature of a conjunctive rule of combination, $m_1(X)m_2(Y)$ is assigned to $X \cap Y$, if it is non-empty, or to $X \times Y$ otherwise. More precisely the least representative of the type of conflict of $X \times Y$ is considered instead of $X \times Y$. It is unique but an order of elementary conflicts and an order of elements inside elementary conflicts. A fixation of these orders enables a unique selection of representatives of $\sim$ classes of conflicts. A complete 18x18 table of minC combination for 3D is presented in [2, 4]. We include here only an illustrative part of it, see Table 10.1. The resulting value $m^0(Z)$ of the generalized bba is computed as a sum of all $m_1(X)m_2(Y)$ for which the field of the complete table in the

[5] *A trivial contradiction.*

row corresponding to X and column corresponding to Y contains Z. In other words, generalized $m^0(Z)$ is computed as a sum of all $m_1(X)m_2(Y)$ for which $Z = X \cap Y$ if $(X \subseteq Y) \vee (Y \subseteq X)$ or $Z \sim X \times Y$ otherwise, where $\sim$ is the equivalence of conflicts from the previous subsection (Z and $X \times Y$ are in the same $\sim$ class of conflicts.); i.e.

$$m^0(Z) = \sum_{\substack{Z=X\cap Y\\ X\subseteq Y \vee Y\subseteq X}} m_1(X)m_2(Y) + \sum_{\substack{Z\sim X\times Y\\ X\not\subseteq Y \& Y\not\subseteq X}} m_1(X)m_2(Y). \tag{10.1}$$

In order to decrease the size of the table below, the following abbreviations are used in this table: A stands for $\{A\}$, similarly AB stands for $\{A, B\}$, and ABC stands for $\{A, B, C\}$, $A \times B$ stands for $\{A\} \times \{B\}$, similarly $A \times BC$ stands for $\{A\} \times \{B, C\}$, $\times$ stands for $\{A\} \times \{B\} \times \{C\}$, $\Box A$ stands for $\Box\{A\}$, and $\Box$ stands for $\{A, B\} \times \{A, C\} \times \{B, C\}$, and similarly.

	A	B	AB	ABC	$A\times B$	$A\times BC$	$\times$	$\Box$	$\Box A$
A	A	$A\times B$	A	A	$A\times B$	$A\times BC$	$\times$	$A\times BC$	A
B	$A\times B$	B	B	B	$A\times B$	$A\times B$	$\times$	$B\times AC$	$B\times AC$
C	$A\times C$	$B\times C$	$C\times AB$	C	$\times$	$A\times C$	$\times$	$C\times AB$	$C\times AB$
BC	$A\times BC$	B	$\Box B$	BC	$A\times B$	$A\times BC$	$\times$	$\Box$	$\Box$
AC	A	$B\times AC$	$\Box A$	AC	$A\times B$	$A\times BC$	$\times$	$\Box$	$\Box A$
AB	A	B	AB	AB	$A\times B$	$A\times BC$	$\times$	$\Box$	$\Box A$
ABC	A	B	AB	ABC	$A\times B$	$A\times BC$	$\times$	$\Box$	$\Box A$
$A\times B$	$A\times B$	$A\times B$	$A\times B$	$A\times B$	$A\times B$	$A\times B$	$\times$	$A\times B$	$A\times B$
$A\times C$	$A\times C$	$\times$	$A\times C$	$A\times C$	$\times$	$\times$	$\times$	$A\times C$	$A\times C$
$B\times C$	$\times$	$B\times C$	$B\times C$	$B\times C$	$\times$	$A\times C$	$\times$	$B\times C$	$B\times C$
$A\times BC$	$A\times BC$	$A\times B$	$A\times BC$	$A\times BC$	$A\times B$	$A\times BC$	$\times$	$A\times BC$	$A\times BC$
$B\times AC$	$A\times B$	$B\times AC$	$B\times AC$	$B\times AC$	$A\times B$	$A\times B$	$\times$	$B\times AC$	$B\times AC$
$C\times AB$	$A\times C$	$B\times C$	$C\times AB$	$C\times AB$	$\times$	$A\times C$	$\times$	$C\times AB$	$C\times AB$
$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
$\Box$	$A\times BC$	$B\times AC$	$\Box$	$\Box$	$A\times B$	$A\times BC$	$\times$	$\Box$	$\Box$
$\Box A$	A	$B\times AC$	$\Box A$	$\Box A$	$A\times B$	$A\times BC$	$\times$	$\Box$	$\Box A$
$\Box B$	$A\times BC$	B	$\Box B$	$\Box B$	$A\times B$	$A\times BC$	$\times$	$\Box$	$\Box$
$\Box C$	$A\times BC$	$B\times AC$	$\Box$	$\Box C$	$A\times B$	$A\times BC$	$\times$	$\Box$	$\Box$

Table 10.1: A partial table of combination of 2 generalized BFs on $\Theta = \{A, B, C\}$.

The minC combination is commutative and associative on generalized BFs. It overcomes some disadvantages of both Dempster's rules (normalized and un-normalized). This theoretically nice combining rule has however a computational complexity rapidly increasing with the size of the frame of discernment.

10.3.3 Reallocation of belief masses of conflicts

Due to the belief masses being assigned also to types of conflicts and partial conflicts, the result of the minC combination is a generalized belief function even if it is applied to classical BFs. To obtain a classical belief function on Shafer's model we have to do the following two steps: we first reassign the bbms of partial conflicts to their non contradictive elements and then we proportionalize bbms of pure (real) conflicts. Because of a different nature of pure and partial conflicts, also these two steps of bbms reallocation are different.

10.3.3.1 Reallocation of gbbms of partial conflicts

Gbbms of partial conflicts (potential contradictions) are simply reassigned to the sets of their trivial conflicts, i.e. to the sets of their non-contradictive elements (e.g. $m^0(\{\theta_i, \theta_j\} \times \{\theta_i, \theta_k\})$ is reallocated to $\{\theta_i\}$). We denote resulting gbba of this step with m^1 to distinguish it from gbba m^0 on the completely generalized level. Thus we obtain $m^1(\{\theta_i, \theta_j\} \times \{\theta_i, \theta_k\}) = 0$ and $m^1(\{\theta_i\})$ is a sum of all $m^0(X)$, where $\{\theta_i\}$ is maximal nonconflicting part of X. Nothing is performed with gbbms of pure conflicts in this step, hence $m^1(Y) = m^0(Y)$ for any pure conflict Y.

10.3.3.2 Proportionalization of gbbms of pure conflicts

Let us present two ways how to accomplish a proportionalization of gbbms which has been assigned by m^0 to pure (real) conflicts . The basic belief mass of a conflict $X \times Y$ between two subsets of Θ can be proportionalized, i.e. reallocated according to the proportions of the corresponding non-conflicting bbms:

a) among X, Y, and $X \cup Y$ as originally designed for so called proportionalized combination rule in [1].

b) among all nonempty subsets of $X \cup Y$. This way combines the original idea of proportionalization with the consideration of conflict "per elements".

For a conflict X of several subsets of a frame of discernment $X_1, X_2, ..., X_k \subset \Theta$, e.g. for $\{\theta_1\} \times \{\theta_2\} \times \{\theta_3\}$ and $\Box \sim \{\{\theta_1\} \times \{\theta_2\}, \{\theta_1\} \times \{\theta_3\}, \{\theta_2\} \times \{\theta_3\}\} \sim \{\theta_1, \theta_2\} \times \{\theta_1, \theta_3\} \times \{\theta_2, \theta_3\}$ in 3D and further conflicts from nD case, we have to generalize the above description of proportionalization in the following way. The bbm of contradiction $X = X_1 \times X_2 \times ... \times X_k$ can be proportionalized:

a) among all unions $\bigcup_{i=1}^{j} X_i$ of $j \leq k$ sets X_i from $\{X_1, X_2, ..., X_k\}$.

b) among all nonempty subsets of $X_1 \cup X_2 \cup ... \cup X_k$.

For an explicit expression, the conflicts of the subsets of 3D $\Theta = \{\theta_1, \theta_2, \theta_3\}$ should be proportionalized among, see Table 10.2. The bbms of conflicts in the first column should be proportionalized by the proportionalization ad a) among sets in the second column and by the proportionalization ad b) among the sets in the third column.

If gbbms $m^1(X_i) = 0$ for all X_i then we divide the proportionalized gbbm $m^1(X_1 \times X_2 \times ... \times X_k)$ by number of the sets among them the gbbm should be proportionalized, i.e. by $2^k - 1$ in the proportionalization a) and by $2^m - 1$, where $m = |X_1 \cup X_2 \cup ... \cup X_k|$ in the case b).

Type of conflict	Proportionalization ad a)	Proportionalization ad b)
$\{\theta_1\}\times\{\theta_2\}$	$\{\theta_1\}, \{\theta_2\}, \{\theta_1, \theta_2\}$	$\{\theta_1\}, \{\theta_2\}, \{\theta_1, \theta_2\}$
$\{\theta_1\}\times\{\theta_2, \theta_3\}$	$\{\theta_1\}, \{\theta_2, \theta_3\}, \{\theta_1, \theta_2, \theta_3\}$	$\mathcal{P}(\{\theta_1, \theta_2, \theta_3\}) - \emptyset$
$\{\theta_1, \theta_2\}\times\{\theta_1, \theta_3\}\times\{\theta_2, \theta_3\}$	$\{\theta_1, \theta_2\}, \{\theta_1, \theta_3\}, \{\theta_2, \theta_3\}, \{\theta_1, \theta_2, \theta_3\}$	$\mathcal{P}(\{\theta_1, \theta_2, \theta_3\}) - \emptyset$
$\{\theta_1\}\times\{\theta_2\}\times\{\theta_3\}$	$\mathcal{P}(\{\theta_1, \theta_2, \theta_3\}) - \emptyset$	$\mathcal{P}(\{\theta_1, \theta_2, \theta_3\}) - \emptyset$

Table 10.2: Proportionalizations on a 3D frame of discernment

A proportionalization of the types of the conflicts from the Table is the same even if $\{\theta_1, \theta_2, \theta_3\} \subsetneq \Theta$. Hence we can see from the Table that the proportionalization is something like 'local normalization' on the power set of $\Theta' \subsetneq \Theta$ in the case b) or on a subset of such power set. E. g. $m^1(\{\theta_1\}\times\{\theta_2,\theta_3\})$ is proportionalized with proportionalization a) among $\{\theta_1\}, \{\theta_2, \theta_3\}, \{\theta_1, \theta_2, \theta_3\}$ so that $\frac{m^1(\{\theta_1\})}{m^1(\{\theta_1\})+m^1(\{\theta_2,\theta_3\})+m^1(\{\theta_1,\theta_2,\theta_3\})}$ $m^1(\{\theta_1\}\times\{\theta_2,\theta_3\})$ is assigned to $\{\theta_1\}$, $\frac{m^1(\{\theta_2,\theta_3\})}{m^1(\{\theta_1\})+m^1(\{\theta_2,\theta_3\})+m^1(\{\theta_1,\theta_2,\theta_3\})}$ $m^1(\{\theta_1\}\times\{\theta_2,\theta_3\})$ is assigned to $\{\theta_2, \theta_3\}$, and $\frac{m^1(\{\theta_1,\theta_2,\theta_3\})}{m^1(\{\theta_1\})+m^1(\{\theta_2,\theta_3\})+m^1(\{\theta_1,\theta_2,\theta_3\})}$ $m^1(\{\theta_1\}\times\{\theta_2,\theta_3\})$ is assigned to $\{\theta_1, \theta_2, \theta_3\}$. Analogically $\frac{m^1(\{\theta_2,\theta_3\})}{m^1(\{\theta_1\})+m^1(\{\theta_2\})+m^1(\{\theta_3\})+m^1(\{\theta_1,\theta_2\})+m^1(\{\theta_1,\theta_3\})+m^1(\{\theta_2,\theta_3\})+m^1(\{\theta_1,\theta_2,\theta_3\})}$ $m^1(\{\theta_1\}\times\{\theta_2,\theta_3\})$ is assigned to $\{\theta_2, \theta_3\}$ with proportionalization b), and similarly for other subsets of $\{\theta_1, \theta_2, \theta_3\}$. For single elementary conflicts both the proportionalizations coincide, see e.g. the 1st and the 4th rows of the Table 10.2. Specially there is the only proportionalization in the 2D case because, there is the only conflict and it is an elementary one. This proportionalization actually coincides with the classical normalization, see examples in Section 10.5.

Let us remember that neither the reallocation of gbbms of partial conflicts nor the proportionalization does not keep associativity of minC combination of the generalized level. Hence we have always to keep in the consideration and to save the generalized version of the result to be prepared for a later combination with another belief.

10.3.4 Summary of the idea of the minC combination

We can summarize the process of the minC combination of $n \geq$ beliefs as follows:

1. we apply $(n-1)$ times the generalized version of minC, to compute gbba m^0, see formula (10.1);

2. after we once apply a reallocation of gbbms of the partial conflicts to produce gbba m^1 and finally we once apply the proportionalization a) or b) to obtain the final bbm m. If we want to keep as

much as possible of associativity for future combining, we have to remember also the gbbm m^0 and continue further combination (if there is any) from it.

10.4 Comparison

10.4.1 Comparison of generalized frames of discernment

As has been already mentioned in the introduction of this chapter, DSm and minC rules of combination arise from completely different assumptions and ideas. On the other hand, 18 different subsets of a frame of discernment and types of conflicts and potential conflicts (7+8+3) in 3D case or 18 elements of a generalized 3D frame of discernment correspond to 18 non empty elements of hyper-power set D^Θ in the free DSm model. Moreover, if we rewrite subsets of the frame of discernment, e.g. $\{\theta_i, \theta_j, \theta_k\}$, and sets of elementary conflicts as unions of their elements, e.g. $\{\theta_i, \theta_j, \theta_k\} \sim \theta_i \cup \theta_j \cup \theta_k$, and conflicts as intersections, e.g. $\{\theta_i\}\times\{\theta_j\} \sim \theta_i \cap \theta_j$, $\{\theta_i, \theta_j\}\times\{\theta_i, \theta_k\} \sim (\theta_i \cup \theta_j) \cap (\theta_i \cup \theta_k)$, $\{\{\theta_i\}\times\{\theta_j\}, \{\theta_j\}\times\{\theta_k\}, \{\theta_i\}\times\{\theta_j\}\times\{\theta_k\}\} \sim (\theta_i \cap \theta_j) \cup (\theta_j \cap \theta_k) \cup (\theta_i \cap \theta_j \cap \theta_k)$, then we obtain the following:

$\{\theta_1\} \sim \theta_1 = \alpha_9$

$\{\theta_2\} \sim \theta_2 = \alpha_{10}$

$\{\theta_3\} \sim \theta_3 = \alpha_{11}$

$\{\theta_1, \theta_2\} \sim \theta_1 \cup \theta_2 = \alpha_{15}$

$\{\theta_1, \theta_3\} \sim \theta_1 \cup \theta_3 = \alpha_{16}$

$\{\theta_2, \theta_3\} \sim \theta_2 \cup \theta_3 = \alpha_{17}$

$\{\theta_1, \theta_2, \theta_3\} \sim \theta_1 \cup \theta_2 \cup \theta_3 = \alpha_{18}$

$\{\theta_1\}\times\{\theta_2\} \sim \theta_1 \cap \theta_2 = \alpha_2$

$\{\theta_1\}\times\{\theta_3\} \sim \theta_1 \cap \theta_3 = \alpha_3$

$\{\theta_2\}\times\{\theta_3\} \sim \theta_2 \cap \theta_3 = \alpha_4$

$\{\theta_1\}\times\{\theta_2, \theta_3\} = \{\{\theta_1\}\times\{\theta_2\}, \{\theta_1\}\times\{\theta_3\}\} \sim \theta_1 \cap (\theta_2 \cup \theta_3) = \alpha_7$

$\{\theta_2\}\times\{\theta_1, \theta_3\} = \{\{\theta_1\}\times\{\theta_2\}, \{\theta_2\}\times\{\theta_3\}\} \sim \theta_2 \cap (\theta_1 \cup \theta_3) = \alpha_6$

$\{\theta_3\}\times\{\theta_1, \theta_2\} = \{\{\theta_3\}\times\{\theta_1\}, \{\theta_3\}\times\{\theta_2\}\} \sim \theta_3 \cap (\theta_1 \cup \theta_2) = \alpha_5$

$\{\theta_1\}\times\{\theta_2\}\times\{\theta_3\} \sim \theta_1 \cap \theta_2 \cap \theta_3 = \alpha_1$

$\{\{\theta_1\}\times\{\theta_2\}, \{\theta_1\}\times\{\theta_3\}, \{\theta_2\}\times\{\theta_3\}\} \sim (\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_1 \cap \theta_3) = \alpha_8$

$\Box\theta_1 = \{\{\theta_1\}, \{\theta_2\}\times\{\theta_3\}\} \sim \theta_1 \cup (\theta_2 \cap \theta_3) = \alpha_{14}$

$\Box\theta_2 = \{\{\theta_2\}, \{\theta_1\}\times\{\theta_3\}\} \sim \theta_2 \cup (\theta_1 \cap \theta_3) = \alpha_{13}$

$\Box\theta_3 = \{\{\theta_3\}, \{\theta_1\}\times\{\theta_2\}\} \sim \theta_3 \cup (\theta_1 \cap \theta_2) = \alpha_{12}$.

Thus a generalized frame of discernment from the minC approach uniquely corresponds to $D^\Theta - \emptyset$. Hence the minC approach is an alternative way how to generate Dedekind's lattice.

10.4.2 Comparison of principles of combination

For bbms of two non-conflicting sets $X, Y \subset \Theta$ both the minC and the DSm rules assign the product of the belief masses to the intersection of the sets[6]. If one of the sets (or both of them) is (are) conflicting, then the minC combination assigns the product of their bbms to the conflict $X \times Y$. Similarly as above, we can consider this conflict as an intersection $X \cap Y$. We should verify whether $X \cap Y$ really corresponds to the corresponding field of the minC combination table.

As first example, let's denote by definition $A_1 \triangleq \{\theta_1, \theta_3\} \times (\{\theta_3\} \times \{\theta_1, \theta_2\})$, then one has

$$
\begin{aligned}
A_1 &\sim (\theta_1 \cup \theta_3) \cap (\theta_3 \cap (\theta_1 \cup \theta_2)) = (\theta_1 \cap (\theta_3 \cap (\theta_1 \cup \theta_2))) \cup (\theta_3 \cap (\theta_3 \cap (\theta_1 \cup \theta_2))) \\
&= (\theta_3 \cap (\theta_1 \cap (\theta_1 \cup \theta_2))) \cup (\theta_3 \cap (\theta_1 \cup \theta_2)) = (\theta_3 \cap \theta_1) \cup (\theta_3 \cap (\theta_1 \cup \theta_2)) = (\theta_3 \cap (\theta_1 \cup \theta_2)) \\
&\sim \{\theta_3\} \times \{\theta_1, \theta_2\}
\end{aligned}
$$

As second example, let's denote $A_2 \triangleq (\{\theta_1\} \times \{\theta_2\} \times \{\theta_3\}) \times \{\{\theta_1\} \times \{\theta_2\}, \{\theta_1\} \times \{\theta_3\}, \{\theta_2\} \times \{\theta_3\}\}$, then one has

$$
\begin{aligned}
A_2 &\sim (\theta_1 \cap \theta_2 \cap \theta_3) \times ((\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3)) \\
&\sim (\theta_1 \cap \theta_2 \cap \theta_3) \cap ((\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3)) \\
&= \theta_1 \cap \theta_2 \cap \theta_3 \cap ((\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3)) = (\theta_1 \cap \theta_2 \cap \theta_3) \cup (\theta_1 \cap \theta_2 \cap \theta_3) \cup (\theta_1 \cap \theta_2 \cap \theta_3) \\
&= \theta_1 \cap \theta_2 \cap \theta_3 \sim \{\theta_1\} \times \{\theta_2\} \times \{\theta_3\}
\end{aligned}
$$

As third example, let's denote $A_3 \triangleq \square\{\theta_1\} \times (\theta_1 \times \{\theta_2, \theta_3\})$, then one has

$$
\begin{aligned}
A_3 &= \{\{\theta_1\}, \{\theta_2 \times \theta_3\}\} \times (\theta_1 \times \{\theta_2, \theta_3\}) \\
&\sim (\theta_1 \cup (\theta_2 \cap \theta_3)) \cap (\theta_1 \cap (\theta_2 \cup \theta_3)) = (\theta_1 \cup (\theta_2 \cap \theta_3)) \cap (\theta_1 \cap (\theta_2 \cup \theta_3)) \\
&= (\theta_1 \cap (\theta_1 \cap (\theta_2 \cup \theta_3))) \cup ((\theta_2 \cap \theta_3)) \cap (\theta_1 \cap (\theta_2 \cup \theta_3)) \\
&= (\theta_1 \cap (\theta_2 \cup \theta_3)) \cup ((\theta_2 \cap \theta_3) \cap (\theta_1 \cap (\theta_2 \cup \theta_3))) \\
&= (\theta_1 \cap (\theta_2 \cup \theta_3)) \cup ((\theta_2 \cap \theta_3 \cap \theta_1 \cap \theta_2) \cup (\theta_2 \cap \theta_3 \cap \theta_1 \cap \theta_3)) \\
&= (\theta_1 \cap (\theta_2 \cup \theta_3)) \cup ((\theta_1 \cap \theta_2 \cap \theta_3) \cup (\theta_1 \cap \theta_2 \cap \theta_3)) \\
&= (\theta_1 \cap (\theta_2 \cup \theta_3)) \cup (\theta_1 \cap \theta_2 \cap \theta_3) = (\theta_1 \cap (\theta_2 \cup \theta_3)) \sim (\theta_1 \times \{\theta_2, \theta_3\})
\end{aligned}
$$

[6]We have to mention here that the minC combination rule has never been formulated as a k-ary operator for combination of $k \geq 2$ belief sources, analogically to the DSm combination rule, see Equations (4.2) and (4.5). Nevertheless, it is theoretically very easy to explicitly formulate it similarly to the DSm rule for k sources. Moreover, because of its associativity on the generalized level we can obtain the same result by step-wise ($(k-1)$-times) application of the binary form, and continue with reallocation of bbms of conflicts as is usual.

In the case of $\{\theta_1,\theta_3\}\times\{\theta_1,\theta_2\} \sim (\theta_1\cup\theta_3)\times(\theta_1\cup\theta_2) \sim (\theta_1\cup\theta_3)\cap(\theta_1\cup\theta_2) = (\theta_1\cap(\theta_1\cup\theta_2)\cup(\theta_3\cap(\theta_1\cup\theta_2) = (\theta_1\cap\theta_1\cup\theta_1\cap\theta_2)\cup(\theta_3\cap\theta_1\cup\theta_3\cap\theta_2) = (\theta_1)\cup(\theta_3\cap\theta_1)\cup(\theta_3\cap\theta_2) = (\theta_1)\cup(\theta_3\cap\theta_2) \sim \{\{\theta_1\},\{\theta_2\times\theta_3\}\} \sim \Box\{\theta_1\}$ we can show again that minC combination of bbms of sets $\{\theta_1,\theta_3\},\{\theta_1,\theta_2\}$ corresponds to the intersection of the corresponding elements of D^Θ: $(\theta_1\cup\theta_3)$ and $(\theta_1\cup\theta_2)$, i.e. to $\theta_1\cup(\theta_3\cap\theta_2)$. Moreover, this shows a rise and the importance of a partial conflict (or potential contradiction) between two sets with non-empty intersection $\{\theta_1,\theta_3\}\cap\{\theta_1,\theta_2\} = \{\theta_1\}$ in Shafer's model. This intersection $\{\theta_1\}$ which is used in Dempster's rule, is different from the generalized minC and the free DSm intersection $\{\theta_1,\theta_3\}\cap\{\theta_1,\theta_2\} \sim (\theta_1\cup\theta_3)\cap(\theta_1\cup\theta_2) = (\theta_1)\cup(\theta_3\cap\theta_2) \sim \Box\{\theta_1\}$ on the generalized level.

Analogically we can verify that all the fields in the complete minC combination table uniquely correspond to intersections of corresponding sets. For a general nD case it is possible to verify that the similarity relation $\sim$ on conflicts corresponds with properties of the lattice $\{\Theta,\cap,\cup\}$. Thus the minC combination equation (10.1) corresponds with the classical DSm combination equation (4.1).

Hence the minC combination[7] on a generalized level fully corresponds to the DSm combination rule on a free DSm model.

10.4.3 Two steps of combination

Because minC is not designed for the DSm model but for the classical Shafer's model, we have to compare it in the context of the special Shaferian case of the hybrid DSm rule. According to the present development state of the hybrid DSm rule, see Chapter 4, in the first step all the combination is done on the free DSm model — it is fully equivalent to the generalized minC combination — and in the second step constraints are introduced. The second step is analogous to the reallocation in the minC approach. It does not explicitly distinguish anything like partial conflicts and pure conflicts, but analogically to the minC combination, bbms are reallocated in two different ways. An introduction of constraints can joint two or more elements of D^Θ, e.g. see Example 4 in Chapter 4, where the element α_9 is joined with the element α_{14}, and the elements α_{10}, and α_{11} are joined with α_{13} and α_{12} respectively. Gbbms of such elements are actually reallocated within this process. Really, the gbbms $m_{\mathcal{M}^f}(\alpha_9)$, $m_{\mathcal{M}^f}(\alpha_{10})$, and $m_{\mathcal{M}^f}(\alpha_{11})$ are reallocated to $m_{\mathcal{M}^0}(\alpha_{14})$, $m_{\mathcal{M}^0}(\alpha_{13})$ and $m_{\mathcal{M}^0}(\alpha_{12})$ respectively, as an analogy of the reallocation of partial conflicts in the minC approach. We can verify that the elements α_9, α_{10}, α_{11} really correspond to the partial conflicts of the minC approach. The step 2 consists further in grouping of all empty sets together and in the reallocation of their bbms. This action fully corresponds to a proportionalization of pure conflicts in the minC approach.

[7] For a comparison of the minC combination with other approaches for combination of conflicting beliefs, see [8].

Hence, the only principal difference between the minC and the DSm combination rules consists in reallocation of the bbms of conflicting (or empty) sets to non-conflicting (non-empty) ones, i.e. to the subsets of the frame of discernment, because the reallocation performed in the 2nd step of the hybrid DSm combination does not correspond to any of the above proportionalizations used in minC either.

10.4.4 On the associativity of the combination rules

As it was already mentioned both the DSm rule and the minC combination rule are fully associative on the generalized level, i.e. on the free DSm model in DSm terminology. Steps 2 in both the combinations, i.e. the introduction of constraints in DSm combination and the reallocation of conflicts including both the proportionalizations, do not keep associativity. If we use results of combination with all the constraints as an input for another combination, we obtain suboptimal results, see Section 4.5.5 in Chapter 4.

In order to keep as much associativity of the combination on the generalized level as possible, we have to use n-ary version of DSm rule. In the case where k input beliefs have been already combined, we have to save all the k input belief functions. If we want to combine the previous result with the new $(k+1)$th input m_{k+1}, then we have either to repeat all the n-ary combination for $k+1$ inputs this time, or we can use the free DSm result of the previous combination (the result of the last application of the Step 1) and apply the binary Step 1 to combine the new input (we obtain the same result as with an application of n-ary version for $k+1$ inputs). Nevertheless, after it we have to apply n-ary version of the Step 2 for introduction of all constraints at the end.

There is another situation in the case of the minC combination. Because we consider only minimal conflicts, the result of the Step 2 depends only on the generalized result m^0 of the Step 1 and we need not the input belief functions for the reallocation of partial conflicts and for the proportionalization. The non-normalized combination rule including the generalized one, provides the same result either if n-ary version is used for k inputs or if step-wise $k-1$ times the binary version is applied. Hence binary version of the generalized minC combination and unary reallocation satisfy for the optimal results in the sense of Chapter 4. If we already have k inputs combined, it is enough to save and store only the generalized result instead of all inputs. We perform the generalized combination with the input m_{k+1} after. And in the end we perform Step 2 for obtaining classical Shaferian result. Of course it is also possible to store all the inputs and to make a new combination, analogically, to the DSm approach.

10.4.5 The special cases

Specially in the 2D case minC corresponds to Dempster's rule — there is only one type of conflict and both the presented proportionalizations a) and b) coincide with normalization there. While the 2D DSm corresponds to Yager's rule, see [12], where $m_1(X)m_2(Y)$ is assigned to $X \cap Y$ if it is non-empty or to Θ for $X \cap Y = \emptyset$, and it also coincides with Dubois-Prade's rule, see [7], where $m_1(X)m_2(Y)$ is assigned to $X \cap Y$ if it is non-empty or to $X \cup Y$ otherwise. To complete the 2D comparison, it is necessary to add that the classical DSm combination rule for the 2D free DSm model corresponds to the non-normalized Dempster's rule used in TBM. For examples see Table 10.3 in Section10.5.

In an nD case for $n > 2$ neither the minC nor DSm rule correspond to any version of Dempster's or Yager's rules. On the other hand the binary version of the hybrid DSm rule coincides with Dubois-Prade's rule on Shafer's model, for an example see Table 10.6 in Section10.5.

10.4.6 Comparison of expressivity of DSm and minC approaches

As the minC combination is designed for combination of classical belief functions on frames of discernment with exclusive elements, we cannot explicitly express that 2 elements of frame have a non-empty intersection. The only way for it is a generalized result of combination of 2 classical BFs. On the other hand, even if the hyper-power set D^Θ has more elements than the number of parts in the corresponding Venn's diagram, we cannot assign belief mass to θ_1 but not to θ_2 in DSm approach. I. e. we cannot assign bbms in such a way that for generalized pignistic probability, see Chapter 7, the following holds: $P(\theta_1) > 0$ and $P(\theta_2) = 0$. The intersection $\theta_1 \cap \theta_2$ is always a subset both of θ_1 and θ_2. Hence from $m(\theta_1) > 0$ we always obtain $P(\theta_1 \cap \theta_2) > 0$ and $P(\theta_2) > 0$. We cannot assign any gbbm to $\theta_1 - \theta_2$. The only way how to do it is to add an additional constraint $\theta_1 \cap \theta_2 = \emptyset$, but such a constraint should be applied to all beliefs in the model and not only to one or several specific ones. As Shafer's model has already all the exclusivity constraints, the above described property is not related to it. Hence both the DSm approach and the minC combination have the comparable expressivity on Shafer's model. The DSm approach utilizes, in addition to it, its capability to express positive belief masses of the intersections.

10.5 Examples

In this section we present a comparison on examples of combination. The first 2D example simply compares not only the DSm and minC combination rules but also both the normalized and non-normalized Dempster's rule, Yager's rule, and Dubois-Prade's rule of belief combination, see Table 10.3. Because the proportionalizations a) and b) coincide in the 2D case, and subsequently the corresponding bbas $m_{12}^{a)}$ and $m_{12}^{b)}$ also coincide, we use m^{minC} for $m_{12}^{a)} \equiv m_{12}^{b)}$. This example enables us to make a wide comparison, but it does not really discover a nature of the presented approaches to the belief combination. For this reason we present also a more complicated 3D example, see Tables 10.4 and 10.5, which show us how conflicts and partial conflicts arise during combination, how constraints are introduced, and how proportionalizations are performed.

	m_1	m_2	$m_{12}^{\mathcal{M}^f}$	$m_{12}^{\mathcal{M}^0}$	m_{12}^0	m_{12}^{minC}	m_{12}^{TBM}	m_{12}^Y	m_{12}^{DP}	$m_{12}^{\oplus}$
$\theta_1 \sim \{\theta_1\}$	0.6	0.2	0.48	0.48	0.48	0.6000	0.48	0.48	0.48	0.6000
$\theta_2 \sim \{\theta_2\}$	0.1	0.3	0.17	0.17	0.17	0.2125	0.17	0.17	0.17	0.2125
$\theta_1 \cup \theta_2 \sim \{\theta_1, \theta_2\}$	0.3	0.5	0.15	0.35	0.15	0.1875	0.15	0.35	0.35	0.1875
$\theta_1 \cap \theta_2 \sim \{\theta_1\}\times\{\theta_2\} \sim \emptyset$			0.20		0.20		0.20			

Table 10.3: Comparison of combination of 2D belief functions

Table 10.4 provides a comparison of combination of 3D belief functions based on the free DSm model with the classic DSm rule and on Shafer's model with the hybrid DSm rule. The 5th column ($m_{12}^{\mathcal{M}^f}$) gives the result of the combination of the sources 1 and 2 obtained with the classic DSm rule based on the free DSm model. The 7th column ($m_{123}^{\mathcal{M}^f}$) gives the result of the combination of the sources 1, 2 and 3 obtained with the classic DSm rule based also on the free DSm model. Column 6 ($m_{12}^{\mathcal{M}^0}$) presents the result of the hybrid DSm combination of sources 1 and 2 based on Shafer's model $\mathcal{M}^0$. Column 8 ($m_{123}^{\mathcal{M}^0}$) presents the result of the hybrid DSm combination of sources 1, 2 and 3 based on Shafer's model $\mathcal{M}^0$. Column 9 and 10 shows the results obtained when performing suboptimal fusion. $\circledS$ stands for the DSm rule on the free DSm model and blank fields stand for 0.

Table 10.5 presents the results drawn from the minC combination rule. m^0 corresponds to the gbba on the generalized frame of discernment, m^1 to the gbba after reallocation of bbms of partial conflicts, $m^{a)}$ to the bba after proportionalization a) and $m^{b)}$ to the bba after proportionalization b). $m^0_{12^b3}$ denotes $(m_{12}^{b)} \circledcirc m_3)^0$, and m_{12^b3} denotes $(m_{12}^{b)} \circledcirc m_3)^{b)}$, where $\circledcirc$ stands for the generalized minC combination, blank fields stand for 0.

Table 10.6 presents the results of several rules of combination for 3D belief functions for sources 1 and 2 on Shafer's model, i.e. on the hybrid DSm model $\mathcal{M}^0$ (for the source bbas m_1, m_2, and m_3 see Table 10.4). $m^{a)}$ corresponds to the bba of the minC combination (the minC combination of m_1 and m_2 or m_1, m_2 and m_3 respectively) with proportionalization a); $m^{b)}$ corresponds to the bba of the minC combination with proportionalization b); $m^{\mathcal{M}^0}$ corresponds to the bba of the DSm combination. m^{TBM} corresponds to the bba of the combination with the TBM's non-normalized Demspter's rule; m^Y corresponds to the bba of the Yager's combination; m^{DB} corresponds to the bba of Dubois-Prade's combination and $m^{\oplus}$ corresponds to the bba of the normalized Dempster's combination.

			m_1	m_2	m_3	$m_{12}^{\mathcal{M}^f}$	$m_{12}^{\mathcal{M}^0}$	$m_{123}^{\mathcal{M}^f}$	$m_{123}^{\mathcal{M}^0}$	$(m_{12}^{\mathcal{M}^0} \textcircled{D} m_3)^{\mathcal{M}^f}$	$(m_{12}^{\mathcal{M}^0} \textcircled{D} m_3)^{\mathcal{M}^0}$
α_9	$\sim$	$\{\theta_1\}$	0.3	0.1	0.2	0.19	0.20	0.165	0.188	0.216	0.258
α_{10}	$\sim$	$\{\theta_2\}$	0.2	0.1	0.1	0.15	0.17	0.090	0.109	0.119	0.145
α_{11}	$\sim$	$\{\theta_3\}$	0.1	0.2	0.1	0.14	0.16	0.088	0.110	0.119	0.150
α_{15}	$\sim$	$\{\theta_1, \theta_2\}$	0.1	0.0	0.2	0.03	0.08	0.021	0.056	0.058	0.112
α_{16}	$\sim$	$\{\theta_1, \theta_3\}$	0.1	0.1	0.2	0.06	0.13	0.030	0.082	0.073	0.125
α_{17}	$\sim$	$\{\theta_2, \theta_3\}$	0.0	0.2	0.1	0.04	0.09	0.014	0.039	0.035	0.068
α_{18}	$\sim$	$\{\theta_1, \theta_2, \theta_3\}$	0.2	0.3	0.1	0.06	0.17	0.006	0.416	0.017	0.142
α_2	$\sim$	$\{\theta_1\}\times\{\theta_2\}$				0.05		0.106		0.054	
α_3	$\sim$	$\{\theta_1\}\times\{\theta_3\}$				0.07		0.120		0.052	
α_4	$\sim$	$\{\theta_2\}\times\{\theta_3\}$				0.05		0.074		0.033	
α_7	$\sim$	$\{\theta_1\}\times\{\theta_2, \theta_3\}$				0.06		0.083		0.038	
α_6	$\sim$	$\{\theta_2\}\times\{\theta_1, \theta_3\}$				0.03		0.060		0.047	
α_5	$\sim$	$\{\theta_3\}\times\{\theta_1, \theta_2\}$				0.02		0.048		0.040	
α_1	$\sim$	$\times$						0.022			
α_8	$\sim$	$\Box$						0.009			
α_{14}	$\sim$	$\Box\theta_1$				0.01		0.023		0.042	
α_{13}	$\sim$	$\Box\theta_2$				0.02		0.019		0.026	
α_{12}	$\sim$	$\Box\theta_3$				0.02		0.022		0.031	

Table 10.4: Comparison of combination of 3D belief functions based on DSm rules of combination.

We can see that during the combination of 2 belief functions a lot of types of conflict arise, but some of them still remain with 0 bbm ($\alpha_1 \sim \times$ and $\alpha_8 \sim \Box$). We can see how these conflicts arise when the 3rd BF is combined. We can see the difference between the combination of 3 BFs on the generalized level (see m^0_{123}) and the suboptimal combination of the 3rd belief with an intermediate result to which constraints have already been introduced (see $(m_{12}^{DSm} \textcircled{D} m_3)^0$ and $(m_{12}^{b)} \textcircled{m} m_3)^0$). We can see how the gbbms are reallocated among the subsets of Θ during the second step of minC combination and finally how the gbbms of all pure conflicts are reallocated in both ways a) and b).

The final results of DSm and minC combinations are compared in Table 10.6. We can note that the small subsets of Θ (singletons in our 3D example) have greater bbms after the minC combination while the great sets (2-element sets and namely whole $\{\theta_1, \theta_2, \theta_3\}$ in our case) have greater bbms after application of the DSm combination rule. I. e. the DSm combining rule is more cautious than the minC combination within the reallocation of the conflicting gbbms. Thus we see that the minC combination

	m^0_{12}	m^1_{12}	$m^{a)}_{12}$	$m^{b)}_{12}$	m^0_{123}	m^1_{123}	$m^{a)}_{123}$	$m^{b)}_{123}$	$m^0_{12^b3}$	m_{12^b3}
$\alpha_9 \sim \{\theta_1\}$	0.19	0.20	0.2983	0.2889	0.165	0.165	0.4031	0.4068	0.2396	0.4113
$\alpha_{10} \sim \{\theta_2\}$	0.15	0.17	0.2318	0.2402	0.090	0.090	0.2301	0.2306	0.1360	0.2319
$\alpha_{11} \sim \{\theta_3\}$	0.14	0.16	0.2311	0.2327	0.088	0.088	0.2288	0.2363	0.1364	0.2372
$\alpha_{15} \sim \{\theta_1,\theta_2\}$	0.03	0.03	0.0362	0.0383	0.021	0.021	0.0390	0.0377	0.0253	0.0354
$\alpha_{16} \sim \{\theta_1,\theta_3\}$	0.06	0.06	0.0762	0.0792	0.030	0.030	0.0586	0.0549	0.0376	0.0522
$\alpha_{17} \sim \{\theta_2,\theta_3\}$	0.04	0.04	0.0534	0.0515	0.014	0.014	0.0264	0.0249	0.0172	0.0236
$\alpha_{18} \sim \{\theta_1,\theta_2,\theta_3\}$	0.06	0.06	0.0830	0.0692	0.006	0.006	0.0140	0.0088	0.0069	0.0084
$\alpha_2 \sim \{\theta_1\}\times\{\theta_2\}$	0.05	0.05			0.106	0.106			0.0769	
$\alpha_3 \sim \{\theta_1\}\times\{\theta_3\}$	0.07	0.07			0.120	0.120			0.0754	
$\alpha_4 \sim \{\theta_2\}\times\{\theta_3\}$	0.05	0.05			0.074	0.074			0.0473	
$\alpha_7 \sim \{\theta_1\}\times\{\theta_2,\theta_3\}$	0.06	0.06			0.083	0.083			0.0392	
$\alpha_6 \sim \{\theta_2\}\times\{\theta_1,\theta_3\}$	0.03	0.03			0.060	0.060			0.0560	
$\alpha_5 \sim \{\theta_3\}\times\{\theta_1,\theta_2\}$	0.02	0.02			0.048	0.048			0.0504	
$\alpha_1 \sim \times$					0.022	0.022				
$\alpha_8 \sim \Box$					0.009	0.009				
$\alpha_{14} \sim \Box\theta_1$	0.01				0.023				0.0235	
$\alpha_{13} \sim \Box\theta_2$	0.02				0.019				0.0141	
$\alpha_{12} \sim \Box\theta_3$	0.02				0.022				0.0182	

Table 10.5: Comparison of combination of 3D belief functions with the minC rule.

rule produces more specified results than the DSm rule does. The last three columns of the table show us that the DSm and the minC with both the proportionalizations produce results different from those of Yager's rule and of both the versions of Dempster's rule (see m^Y, m^{TMB}, and $m^{\oplus}$ respectively). While binary DSm result on Shafer's model ($\mathcal{M}^0$) coincides with the results of Dubois-Prade's rule of combination.

Let us present numeric examples of parts of computation m^0, m^1, $m^{a)}$, and $m^{b)}$ for readers which are interested in detail. We begin with a non-conflicting set $\{\theta_1,\theta_2\}$, i.e. with $\alpha_{15}=\theta_1\cup\theta_2$ in the DSm notation. It is an intersection with itself or with the whole $\Theta=\{\theta_1,\theta_2,\theta_3\}$ (i.e. $\theta_1\cup\theta_2\cup\theta_3$ in DSm), and it is not $\sim$ equivalent to any other element of D^Θ. Thus $m^0_{12}(\theta_1\cup\theta_2)=m_1(\theta_1\cup\theta_2)m_2(\theta_1\cup\theta_2)+m_1(\theta_1\cup\theta_2)m_2(\theta_1\cup\theta_2\cup\theta_3)+m_1(\theta_1\cup\theta_2\cup\theta_3)m_2(\theta_1\cup\theta_2)=0.1\cdot 0.0+0.1\cdot 0.3+0.0\cdot 0.2=0.00+0.03+0.00=0.03$. α_{15} is a non-conflicting element of D^Θ, hence it is not further reassigned or proportionalized, i. e. its bbm will not be decreased. α_{15} is not a non-conflicting part of any other element of D^Θ, thus $m^1_{12}(\alpha_{15})=m^0_{12}(\alpha_{15})$.

		$m_{12}^{\mathcal{M}^0}$	$m_{12}^{a)}$	$m_{12}^{b)}$	$m_{123}^{\mathcal{M}^0}$	$m_{123}^{a)}$	$m_{123}^{b)}$	m_{12}^{TBM}	m_{12}^{Y}	m_{12}^{DP}	$m_{12}^{\oplus}$
α_9	$\sim \{\theta_1\}$	0.20	0.2983	0.2889	0.188	0.4031	0.4068	0.20	0.20	0.20	0.2778
α_{10}	$\sim \{\theta_2\}$	0.17	0.2318	0.2402	0.109	0.2301	0.2306	0.17	0.17	0.17	0.2361
α_{11}	$\sim \{\theta_3\}$	0.16	0.2311	0.2327	0.110	0.2288	0.2363	0.16	0.16	0.16	0.2222
α_{15}	$\sim \{\theta_1,\theta_2\}$	0.08	0.0362	0.0383	0.056	0.0390	0.0377	0.04	0.04	0.08	0.0556
α_{16}	$\sim \{\theta_1,\theta_3\}$	0.13	0.0762	0.0792	0.082	0.0586	0.0549	0.06	0.06	0.13	0.0833
α_{17}	$\sim \{\theta_2,\theta_3\}$	0.09	0.0534	0.0515	0.039	0.0264	0.0249	0.03	0.03	0.09	0.0417
α_{18}	$\sim \{\theta_1,\theta_2,\theta_3\}$	0.17	0.0830	0.6992	0.416	0.0140	0.0088	0.06	0.34	0.17	0.0833
	$\emptyset$							0.28			

Table 10.6: Comparison of combinations of sources 1 and 2 on Shafer's model (i.e. on the hybrid DSm model $\mathcal{M}^0$).

$m_{12}^{a)}(\alpha_{15}) > m_{12}^1(\alpha_{15})$ because gbbms of some other elements are proportionalized, among others, also to α_{15}. For the same reason it holds also $m_{12}^{b)}(\alpha_{15}) > m_{12}^1(\alpha_{15})$.

A potential conflict $\Box\{\theta_1\} \sim (\theta_1\cup\theta_2)\cap(\theta_1\cup\theta_3) = \alpha_{14}$ is equivalent to $\Box\{\theta_1\}\times\Box\{\theta_1\}$, to $\Box\{\theta_1\}\times X$, and to $X\times\Box\{\theta_1\}$, where $\{\theta_1\}\subset X$ in Shafer's model, see Table 10.1; or $\alpha_{14} = (\theta_1\cup\theta_2)\cap(\theta_1\cup\theta_3)$ is an intersection of itself with X, where $\alpha_{14}\subseteq X\subseteq \theta_1\cup\theta_2\cup\theta_3$ in the DSm terminology. I.e. $m_{12}^0(\alpha_{14}) = m^0(\theta_1\cap(\theta_2\cup\theta_3)) = m_1(\alpha_{14})m_2(\alpha_{14}) + m_1(\theta_1\cup\theta_2)m_2(\theta_1\cup\theta_3) + m_1(\theta_1\cup\theta_3)m_2(\theta_1\cup\theta_2) + m_1(\alpha_{14})(m_2(\theta_1\cup\theta_2)+m_2(\theta_1\cup\theta_3)+m_2(\theta_1\cup\theta_2\cup\theta_3))+(m_1(\theta_1\cup\theta_2)+m_1(\theta_1\cup\theta_3)+m_1(\theta_1\cup\theta_2\cup\theta_3))m_2(\alpha_{14}) = 0.0\cdot 0.0+0.1\cdot 0.1+0.1\cdot 0.0+0.0\cdot(0.1+0.1+0.2)+(0.0+0.1+0.3)\cdot 0.0 = 0+0.01+0+0+0 = 0.01$. $\alpha_9 = \{\theta_1\}$ is a non-conflicting part of $\theta_1\cap(\theta_2\cup\theta_3)$, thus $m^0(\alpha_{14})$ is reallocated to θ_1. On the other hand $\{\theta_1\}$ is not a non-conflicting part of any other element of D^Θ, hence $m^1(\alpha_9) = m^0(\alpha_9)+m^0(\alpha_{14}) = 0.19+0.01 = 0.20$. After this reallocation, the bbm of α_{14} equals 0, hence $m^1(\alpha_{14}) = m^{a)}(\alpha_{14}) = m^{b)}(\alpha_{14}) = 0$.

A pure conflict $\{\theta_1\}\times\{\theta_2,\theta_3\}\sim\theta_1\cap(\theta_2\cup\theta_3) = \alpha_7$ is contained in 24 fields of the full minC combination table (for its part see Table 10.1), e. g. in the fields corresponding to $\{A\}\times(\{A\}\times\{B,C\})$, $\{A\}\times\{B,C\}$, $\{A,B\}\times(\{A\}\times\{B,C\})$, but only some of them correspond to the Shaferian input beliefs (i. e. only some of them are positive). Thus $m^1(\alpha_7) = m^0(\alpha_7) = m_1(\theta_1)m_2(\theta_2\cup\theta_3) + m_1(\theta_2\cup\theta_3)m_2(\theta_1) = 0.3\cdot 0.2 + 0.1\cdot 0.0 = 0.06 + 0.00 = 0.06$. As α_7 is a pure conflict, thus its bbm is not changing during the reallocation substep, and it is proportionalized among $\{\theta_1\},\{\theta_2,\theta_3\},\{\theta_1,\theta_2,\theta_3\}$ with the proportionalization a), and among all the subsets of $\Theta = \{\theta_1,\theta_2,\theta_3\}$ with the proportionalization b). Thus $m^1(\alpha_7)\cdot\frac{m^1(\theta_1)}{m^1(\theta_1)+m^1(\theta_2\cup\theta_3)+m^1(\theta_1\cup\theta_2\cup\theta_3)} = 0.06\frac{0.20}{0.20+0.04+0.06} = 0.06\frac{0.20}{0.30} = 0.040$ is reassigned to $\theta_1 = \alpha_9$; $m^1(\alpha_7)\cdot\frac{m^1(\theta_2\cup\theta_3)}{m^1(\theta_1)+m^1(\theta_2\cup\theta_3)+m^1(\theta_1\cup\theta_2\cup\theta_3)} = 0.06\frac{0.04}{0.20+0.04+0.06} = 0.06\frac{0.04}{0.30} = 0.008$ is reassigned to $\theta_2 \cup \theta_3 = \alpha_{17}$; and $m^1(\alpha_7) \cdot \frac{m^1(\theta_1\cup\theta_2\cup\theta_3)}{m^1(\theta_1)+m^1(\theta_2\cup\theta_3)+m^1(\theta_1\cup\theta_2\cup\theta_3)} = 0.06\frac{0.06}{0.20+0.04+0.06} = 0.06\frac{0.06}{0.30} = 0.012$ is reassigned to $\theta_1 \cup \theta_2 \cup \theta_3 = \alpha_{18}$ with the proportionalization a). As belief masses $0.05\frac{0.20}{0.20+0.17+0.03} = 0.05 \cdot 0.5 = 0.0250$ and $0.07\frac{0.20}{0.20+0.16+0.06} = 0.07 \cdot 0.4762 = 0.0333$ are analogically proportionalized with the proportionalization a) also to θ_1, so we obtain $m^{a)}_{12}(\theta_1) = m^1(\theta_1) + 0.040 + 0.0250 + 0.0333 = 0.2000 + 0.040 + 0.0250 + 0.0333 = 0.2983$. A value $m^{b)}_{12}(\theta_1)$ is computed analogically; where e.g. $0.06\frac{0.20}{0.20+0.17+0.16+0.03+0.06+0.04+0.06} = 0.06\frac{0.20}{0.72} = 0.06 \cdot 0.2777 = 0.0166$ is proportionalized from $m^1(\alpha_7)$.

10.6 Conclusion

In this chapter we have compared two independently developed approaches to combination of conflicting beliefs. Motivations and the starting points of the approaches are significantly different. The classical frame of discernment with mutually exclusive elements is the starting point for the minC combination, whereas the free DSm model is the starting point for the classical DSm approach. The approaches were originally rather complementary than comparable.

Surprisingly, the internal combining structures and mechanisms of both these combination rules are the same and the results of the classical DSm rule for the free DSm model are the same as the intermediate results of the minC combination on a generalized frame of discernment. Nevertheless, this common step is followed by reallocation of the belief masses temporarily assigned to conflicts to obtain classical belief functions as results in the case of the minC combination.

After the recent development of versions of the DSm rule for Shafer's model and for general hybrid DSm models, which consider 2 steps of combination, the minC combination becomes an alternative to the special case of the DSm combination rule for Shafer's model.

The first step — a combination on a generalized frame — is the same again. Also a reallocation of the generalized basic belief masses of potential conflicts is analogous. The main difference consists in different reallocations of the generalized basic belief masses (gbbm) of pure conflicts: it is a reassigning of the gbbms to the union of the corresponding sets in the DSm rule, whereas a proportionalization in the minC approach.

In spite of this difference, we can also consider the DSm introduction of constraints as an alternative to a reallocation of the belief masses of conflicts in the minC approach.

10.7 References

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