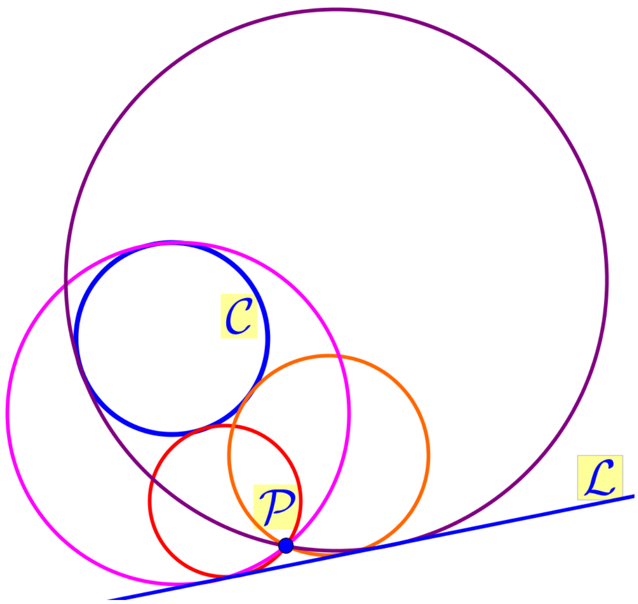
Simplified Solutions of the CLP and CCP Limiting Cases of the Problem of Apollonius

via Vector Rotations using Geometric Algebra



Simplified Solutions of the “CLP” and “CCP” Limiting Cases of the Problem of Apollonius via Vector Rotations using Geometric Algebra

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Contents

ABSTRACT

The new solutions presented herein for the CLP and CCP limiting cases of the Problem of Apollonius are much shorter and more easily understood than those provided by the same author in [1]-[2]. These improvements result from (1) a better selection of angle relationships as a starting point for the solution process; and (2) better use of GA identities to avoid forming troublesome combinations of terms within the resulting equations.

1 Introduction

This document shows how the CLP and CCP limiting cases can be solved more efficiently than in the author's previous work ([1] - [2]). Because that work and [3] discussed the necessary background in detail, the solutions presented here are somewhat abbreviated.

2 Solution of the CLP Limiting Case

For detailed discussions of the ideas used in this solution, please see [1]. We'll show two ways of solving the problem; the second takes advantage of observations made during the first.

The CLP limiting case reads,

> *Given a circle $\mathcal{C}$, a line $\mathcal{L}$, and a point $\mathcal{P}$, construct the circles that are tangent to $\mathcal{C}$ and $\mathcal{L}$, and pass through $\mathcal{P}$.*

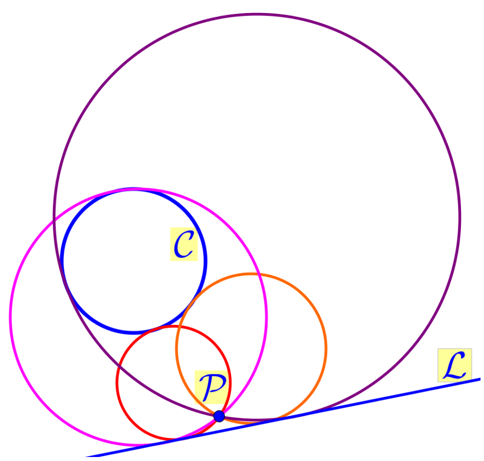

Figure 2.1: The CLP Limiting Case of the Problem of Apollonius: *Given a circle $\mathcal{C}$, a line $\mathcal{L}$, and a point $\mathcal{P}$, construct the circles that are tangent to $\mathcal{C}$ and $\mathcal{L}$, and pass through $\mathcal{P}$.*

The problem has two types of solutions:

- Circles that enclose $\mathcal{C}$;
- Circles that do not enclose $\mathcal{C}$.

There are two solution circles of each type. In this document, we'll treat only those that do not enclose the given circle.

2.1 The First Solution

Fig. 2.2 shows how we will capture the geometric content of the problem. An important improvement, compared to the solution technique presented in [1], is that we will use rotations with respect to the vector from the given point $\mathcal{P}$ to the still-unidentified center point ($\boldsymbol{c}_2$) of the solution circle.

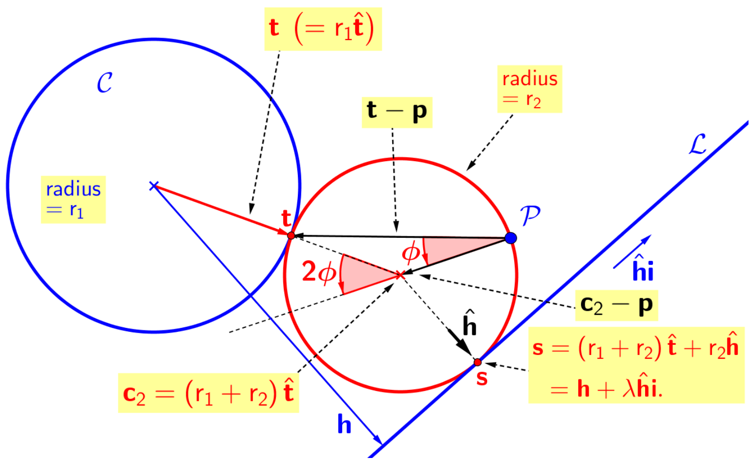

Figure 2.2: Elements used in the first solution of the CLP limiting case.

In deriving our solution, we'll use the same symbol —for example, $\boldsymbol{t}$ —to denote both a point and the vector to that point from the origin. We'll rely upon context to tell the reader whether the symbol is being used to refer to the point, or to the vector.

We'll begin our solution by deriving an expression for r_2 in terms of $\hat{\boldsymbol{t}}$. We´ll do so by equating two independent expressions for $\boldsymbol{s}$, then "dotting" both sides with $\hat{\boldsymbol{h}}$, after which we'll solve for r_2:

$$\begin{aligned}
(r_1+r_2)\,\hat{\boldsymbol{t}} + r_2\hat{\boldsymbol{h}} &= \boldsymbol{h} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i} \\
\left[(r_1+r_2)\,\hat{\boldsymbol{t}} + r_2\hat{\boldsymbol{h}}\right]\cdot\hat{\boldsymbol{h}} &= \left[\boldsymbol{h} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\cdot\hat{\boldsymbol{h}} \\
(r_1+r_2)\,\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}} + r_2\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{h}} &= \boldsymbol{h}\cdot\hat{\boldsymbol{h}} + \lambda\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{h}} \\
(r_1+r_2)\,\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}} + r_2 &= \|\boldsymbol{h}\| + 0;
\end{aligned}$$

$$\therefore r_2 = \frac{\|\boldsymbol{h}\| - r_1\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}}{1+\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}}, \text{ and } \; r_1 + r_2 = \frac{\|\boldsymbol{h}\| + r_1}{1+\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}}. \tag{2.1}$$

Next, we equate two expressions for the rotation $e^{\boldsymbol{i}2\phi}$:

$$\underbrace{\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{\|\boldsymbol{t}-\boldsymbol{p}\|}\right]\left[\frac{\boldsymbol{c}_2-\boldsymbol{p}}{\|\boldsymbol{c}_2-\boldsymbol{p}\|}\right]}_{=e^{\boldsymbol{i}\phi}}\underbrace{\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{\|\boldsymbol{t}-\boldsymbol{p}\|}\right]\left[\frac{\boldsymbol{c}_2-\boldsymbol{p}}{\|\boldsymbol{c}_2-\boldsymbol{p}\|}\right]}_{=e^{\boldsymbol{i}\phi}} = \underbrace{\left[-\hat{\boldsymbol{t}}\right]\left[\frac{\boldsymbol{c}_2-\boldsymbol{p}}{\|\boldsymbol{c}_2-\boldsymbol{p}\|}\right]}_{=e^{\boldsymbol{i}2\phi}},$$

from which

$$\begin{aligned}
\left[\boldsymbol{t}-\boldsymbol{p}\right]\left[\boldsymbol{c}_2-\boldsymbol{p}\right]\left[\boldsymbol{t}-\boldsymbol{p}\right]\left[\hat{\boldsymbol{t}}\right] &= \text{some scalar}, \\
\therefore\ \langle\left[\boldsymbol{t}-\boldsymbol{p}\right]\left[\boldsymbol{c}_2-\boldsymbol{p}\right]\left[\boldsymbol{t}-\boldsymbol{p}\right]\left[\hat{\boldsymbol{t}}\right]\rangle_2 &= 0.
\end{aligned} \tag{2.2}$$

Using the identity $\boldsymbol{ab} \equiv 2\boldsymbol{a} \wedge \boldsymbol{b} + \boldsymbol{ba}$, we rewrite 2.2 as

$$\begin{aligned}
\langle (2\,[\boldsymbol{t}-\boldsymbol{p}] \wedge [\boldsymbol{c}_2-\boldsymbol{p}] + [\boldsymbol{c}_2-\boldsymbol{p}]\,[\boldsymbol{t}-\boldsymbol{p}])\,[\boldsymbol{t}-\boldsymbol{p}]\,[\hat{\boldsymbol{t}}] \rangle_2 &= 0, \\
\langle (2\,[\boldsymbol{t}-\boldsymbol{p}] \wedge [\boldsymbol{c}_2-\boldsymbol{p}])\,[\boldsymbol{t}-\boldsymbol{p}]\,[\hat{\boldsymbol{t}}] + [\boldsymbol{t}-\boldsymbol{p}]^2\,[\boldsymbol{c}_2-\boldsymbol{p}]\,[\hat{\boldsymbol{t}}] \rangle_2 &= 0, \text{ and} \\
\langle (2\,[\boldsymbol{t}-\boldsymbol{p}] \wedge [\boldsymbol{c}_2-\boldsymbol{p}])\,[\boldsymbol{t}-\boldsymbol{p}]\,[\hat{\boldsymbol{t}}] \rangle_2 + \langle [\boldsymbol{t}-\boldsymbol{p}]^2\,[\boldsymbol{c}_2-\boldsymbol{p}]\,[\hat{\boldsymbol{t}}] \rangle_2 &= 0. \qquad (2.3)
\end{aligned}$$

Now, we note that

$$\begin{aligned}
\langle (2\,[\boldsymbol{t}-\boldsymbol{p}] \wedge [\boldsymbol{c}_2-\boldsymbol{p}])\,[\boldsymbol{t}-\boldsymbol{p}]\,[\hat{\boldsymbol{t}}] \rangle_2 &= 2\,([\boldsymbol{t}-\boldsymbol{p}] \wedge [\boldsymbol{c}_2-\boldsymbol{p}])\left([\boldsymbol{t}-\boldsymbol{p}] \cdot [\hat{\boldsymbol{t}}]\right), \\
\text{and } \langle [\boldsymbol{t}-\boldsymbol{p}]^2\,[\boldsymbol{c}_2-\boldsymbol{p}]\,[\hat{\boldsymbol{t}}] \rangle_2 &= [\boldsymbol{t}-\boldsymbol{p}]^2\left([\boldsymbol{c}_2-\boldsymbol{p}] \wedge [\hat{\boldsymbol{t}}]\right).
\end{aligned}$$

Note how the factor $\boldsymbol{p} \wedge \hat{\boldsymbol{t}}$ canceled out in Eq. (2.4). That cancellation suggests an improvement that we'll see in our second solution of the CLP case.

Because $\boldsymbol{t} = r_1\hat{\boldsymbol{t}}$ and $\boldsymbol{c}_2 = (r_1 + r_2)\,\hat{\boldsymbol{t}}$, $\boldsymbol{t} \wedge \boldsymbol{c}_2 = 0$. We can expand $[\boldsymbol{t}-\boldsymbol{p}]^2$ as ${r_1}^2 - 2\boldsymbol{p}\cdot\boldsymbol{t} + p^2$. Using all of these ideas, (2.3) becomes (after simplification)

$$2r_2\left(r_1 - \boldsymbol{p}\cdot\hat{\boldsymbol{t}}\right)\boldsymbol{p} \wedge \hat{\boldsymbol{t}} + \left({r_1}^2 - 2\boldsymbol{p}\cdot\boldsymbol{t} + p^2\right)\boldsymbol{p} \wedge \hat{\boldsymbol{t}} = 0. \qquad (2.4)$$

For $\boldsymbol{p} \wedge \hat{\boldsymbol{t}} \neq 0$, that equation becomes

$$2r_2\left(r_1 - \boldsymbol{p}\cdot\hat{\boldsymbol{t}}\right) + {r_1}^2 - 2\boldsymbol{p}\cdot\boldsymbol{t} + p^2 = 0.$$

Substituting the expression that we derived for r_2 in (2.1), then expanding and simplifying,

$$2\left(\|\boldsymbol{h}\| + r_1\right)\boldsymbol{p}\cdot\hat{\boldsymbol{t}} - \left(p^2 - {r_1}^2\right)\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}} = 2\|\boldsymbol{h}\|r_1 + {r_1}^2 + p^2.$$

Finally, we rearrange that result and multiply both sides by $r_1\|\boldsymbol{h}\|$, giving the equation that we derived in [1]:

$$\left\{2\left(r_1\|\boldsymbol{h}\| + h^2\right)\boldsymbol{p} - \left(p^2 - {r_1}^2\right)\boldsymbol{h}\right\}\cdot\boldsymbol{t} = 2h^2{r_1}^2 + r_1\|\boldsymbol{h}\|\left({r_1}^2 + p^2\right). \qquad (2.5)$$

2.2 The Second Solution: Learning From and Building Upon the First

In Eq. (2.4), we saw how the factor $\boldsymbol{p}\wedge\hat{\boldsymbol{t}}$ canceled out. That cancellation suggests that we might solve the problem more efficiently by expressing rotations with respect to the unknown vector $\hat{\boldsymbol{t}}$, rather than to a vector from $\mathcal{P}$ to $\boldsymbol{c}_2$ (Fig. 2.3).

For this new choice of vectors, our equation relating two expressions for the rotation $e^{\boldsymbol{i}2\phi}$ is:

$$\underbrace{[\hat{\boldsymbol{t}}]\left[\frac{\boldsymbol{p}-\boldsymbol{t}}{\|\boldsymbol{p}-\boldsymbol{t}\|}\right]}_{=e^{\boldsymbol{i}\phi}}\underbrace{[\hat{\boldsymbol{t}}]\left[\frac{\boldsymbol{p}-\boldsymbol{t}}{\|\boldsymbol{p}-\boldsymbol{t}\|}\right]}_{=e^{\boldsymbol{i}\phi}} = \underbrace{[\hat{\boldsymbol{t}}]\left[\frac{\boldsymbol{p}-\boldsymbol{c}_2}{\|\boldsymbol{p}-\boldsymbol{c}_2\|}\right]}_{=e^{\boldsymbol{i}2\phi}},$$

from which

$$\begin{aligned}
[\boldsymbol{p}-\boldsymbol{t}]\,[\hat{\boldsymbol{t}}]\,[\boldsymbol{p}-\boldsymbol{t}]\,[\boldsymbol{p}-\boldsymbol{c}_2] &= \text{some scalar}, \\
\therefore\ \langle [\boldsymbol{p}-\boldsymbol{t}]\,[\hat{\boldsymbol{t}}]\,[\boldsymbol{p}-\boldsymbol{t}]\,[\boldsymbol{p}-\boldsymbol{c}_2] \rangle_2 &= 0. \qquad (2.6)
\end{aligned}$$

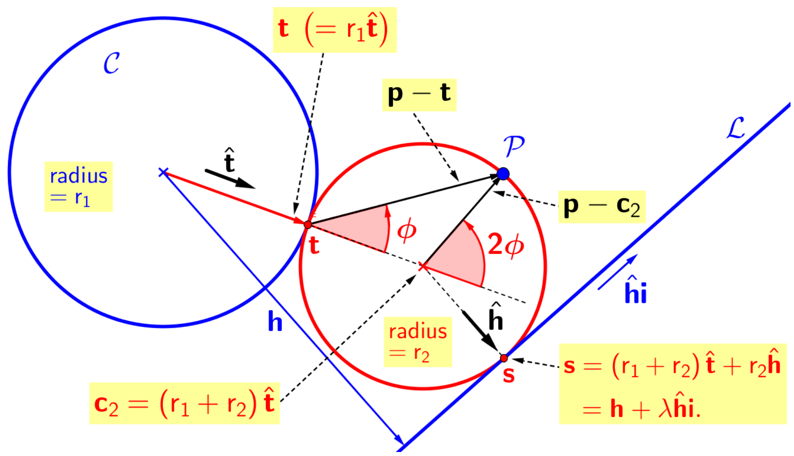

Figure 2.3: Elements used in the second solution of the CLP Limiting Case: rotations are now expressed with respect to the unknown vector $\hat{\boldsymbol{t}}$, rather than to a vector from $\mathcal{P}$ to $\boldsymbol{c}_2$.

Using the identity $\boldsymbol{ab} \equiv 2\boldsymbol{a}\wedge\boldsymbol{b} + \boldsymbol{ba}$, we rewrite 2.6 as

$$\left\langle \left(2\left[\boldsymbol{p}-\boldsymbol{t}\right]\wedge\left[\hat{\boldsymbol{t}}\right] + \left[\hat{\boldsymbol{t}}\right]\left[\boldsymbol{p}-\boldsymbol{t}\right]\right)\left[\boldsymbol{p}-\boldsymbol{t}\right]\left[\boldsymbol{p}-\boldsymbol{c}_2\right]\right\rangle_2 = 0, \text{ and}$$
$$\left\langle \left(2\left[\boldsymbol{p}-\boldsymbol{t}\right]\wedge\left[\hat{\boldsymbol{t}}\right]\right)\left[\boldsymbol{p}-\boldsymbol{t}\right]\left[\boldsymbol{p}-\boldsymbol{c}_2\right]\right\rangle_2 + \left\langle\left[\boldsymbol{p}-\boldsymbol{t}\right]^2\left[\hat{\boldsymbol{t}}\right]\left[\boldsymbol{p}-\boldsymbol{c}_2\right]\right\rangle_2 = 0. \tag{2.7}$$

Now, we note that

$$\left\langle \left(2\left[\boldsymbol{p}-\boldsymbol{t}\right]\wedge\left[\hat{\boldsymbol{t}}\right]\right)\left[\boldsymbol{p}-\boldsymbol{t}\right]\left[\boldsymbol{p}-\boldsymbol{c}_2\right]\right\rangle_2 = \left(2\left[\boldsymbol{p}-\boldsymbol{t}\right]\wedge\left[\hat{\boldsymbol{t}}\right]\right)\left[\boldsymbol{p}-\boldsymbol{t}\right]\cdot\left[\boldsymbol{p}-\boldsymbol{c}_2\right],$$
$$\text{and } \left\langle\left[\boldsymbol{p}-\boldsymbol{t}\right]^2\left[\hat{\boldsymbol{t}}\right]\left[\boldsymbol{p}-\boldsymbol{c}_2\right]\right\rangle_2 = \left[\boldsymbol{p}-\boldsymbol{t}\right]^2\left[\hat{\boldsymbol{t}}\right]\wedge\left[\boldsymbol{p}-\boldsymbol{c}_2\right].$$

Because $\boldsymbol{t} = r_1\hat{\boldsymbol{t}}$ and $\boldsymbol{c}_2 = (r_1+r_2)\,\hat{\boldsymbol{t}}$, $\boldsymbol{t}\wedge\boldsymbol{c}_2 = 0$. We can expand $\left[\boldsymbol{p}-\boldsymbol{t}\right]^2$ as $p^2 - 2\boldsymbol{p}\cdot\boldsymbol{t} + {r_1}^2$. Using all of these ideas, (2.7) becomes (after simplification)

$$2\left(\left[\boldsymbol{p}-\boldsymbol{t}\right]\cdot\left[\boldsymbol{p}-\boldsymbol{c}_2\right]\right)\boldsymbol{p}\wedge\boldsymbol{t} - \left(p^2 - 2\boldsymbol{p}\cdot\boldsymbol{t} + {r_1}^2\right)\boldsymbol{p}\wedge\boldsymbol{t} = 0.$$

For $\boldsymbol{p}\wedge\hat{\boldsymbol{t}} \neq 0$, that equation becomes, after expanding $\left[\boldsymbol{p}-\boldsymbol{t}\right]\cdot\left[\boldsymbol{p}-\boldsymbol{c}_2\right]$ and further simplifications,

$$p^2 - {r_1}^2 - 2\boldsymbol{p}\cdot\boldsymbol{c}_2 + 2\boldsymbol{t}\cdot\boldsymbol{c}_2 = 0.$$

Now, recalling that $\boldsymbol{c}_2 = (r_1+r_2)\,\hat{\boldsymbol{t}}$, we substitute the expression that we derived for $r_1 + r_2$ in (2.1), then expand and simplify to obtain (2.5). This solution process has been a bit shorter than the first because (2.7) was so easy to simplify.

3 Solution of the CCP Limiting Case

In this solution, we'll follow the example of Section 2.2, and use rotations with respect to the vector $\hat{\boldsymbol{t}}$. For detailed discussions of the ideas used in this solution, please see [2].

The CCP limiting case reads,

"Given two circles and a point $\mathcal{P}$, all coplanar, construct the circles that pass through $\mathcal{P}$ and are tangent, simultaneously, to the given circles." (Fig. 3.1).

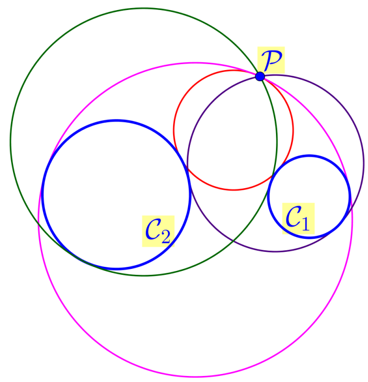

Figure 3.1: The CCP limiting case of the Problem of Apollonius: *"Given two circles ($\mathcal{C}_1$, $\mathcal{C}_2$) and a point $\mathcal{P}$, all coplanar, construct the circles that pass through $\mathcal{P}$ and are tangent, simultaneously, to the given circles."*

We´ll derive the solution for the solution circles that enclose either both of the given ones, or neither. Fig. 3.2 shows how we'll capture the geometric content. As in the CLP case, we can find both solution circles of this type by analyzing the diagram for just one of them.

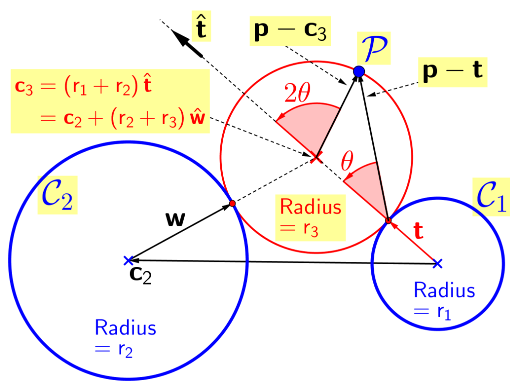

Figure 3.2: Elements used in the solution of the CCP limiting case.

We begin the solution by deriving an expression for r_3 in terms of $\hat{\boldsymbol{t}}$ and the given quantities. From two independent equations for $\boldsymbol{c}_3$,

$$(r_1 + r_3)\,\hat{\boldsymbol{t}} = \boldsymbol{c}_3 = \boldsymbol{c}_2 + (r_2 + r_3)\,\hat{\boldsymbol{w}},$$

we proceed as follows:

$$\begin{aligned}
(r_1 + r_3)\,\hat{\boldsymbol{t}} - \boldsymbol{c}_2 &= (r_2 + r_3)\,\hat{\boldsymbol{w}} \\
\left[(r_1 + r_3)\,\hat{\boldsymbol{t}} - \boldsymbol{c}_2\right]^2 &= \left[(r_2 + r_3)\,\hat{\boldsymbol{w}}\right]^2 \\
(r_1 + r_3)^2 - 2\,(r_1 + r_3)\,\boldsymbol{c}_2 \cdot \hat{\boldsymbol{t}} + {\boldsymbol{c}_2}^2 &= (r_2 + r_3)^2 \\
\therefore\;\; r_3 = \frac{{c_2}^2 + {r_1}^2 - {r_2}^2 - 2 r_1 \boldsymbol{c}_2 \cdot \hat{\boldsymbol{t}}}{2\left(r_2 - r_1 + \boldsymbol{c}_2 \cdot \hat{\boldsymbol{t}}\right)}&, \\
\text{and } r_1 + r_3 = \frac{{c_2}^2 - (r_2 - r_1)^2}{2\left(r_2 - r_1 + \boldsymbol{c}_2 \cdot \hat{\boldsymbol{t}}\right)}&.
\end{aligned} \tag{3.1}$$

Next, we equate two expressions for the rotation $e^{\boldsymbol{i}2\theta}$:

$$\underbrace{\left[\frac{\boldsymbol{p} - \boldsymbol{t}}{\|\boldsymbol{p} - \boldsymbol{t}\|}\right][\hat{\boldsymbol{t}}]}_{=e^{\boldsymbol{i}\theta}}\underbrace{\left[\frac{\boldsymbol{p} - \boldsymbol{t}}{\|\boldsymbol{p} - \boldsymbol{t}\|}\right][\hat{\boldsymbol{t}}]}_{=e^{\boldsymbol{i}\theta}} = \underbrace{\left[\frac{\boldsymbol{p} - \boldsymbol{c}_3}{\|\boldsymbol{p} - \boldsymbol{c}_3\|}\right][\hat{\boldsymbol{t}}]}_{=e^{\boldsymbol{i}2\theta}},$$

from which

$$\begin{aligned}
[\boldsymbol{p} - \boldsymbol{t}]\,[\hat{\boldsymbol{t}}]\,[\boldsymbol{p} - \boldsymbol{t}]\,[\boldsymbol{p} - \boldsymbol{c}_3] &= \text{some scalar}, \\
\therefore\;\; \langle[\boldsymbol{p} - \boldsymbol{t}]\,[\hat{\boldsymbol{t}}]\,[\boldsymbol{p} - \boldsymbol{t}]\,[\boldsymbol{p} - \boldsymbol{c}_3]\rangle_2 &= 0.
\end{aligned} \tag{3.2}$$

Using the identity $\boldsymbol{ab} \equiv 2\boldsymbol{a} \wedge \boldsymbol{b} + \boldsymbol{ba}$, we rewrite 3.2 as

$$\begin{aligned}
\langle\left(2\,[\boldsymbol{p} - \boldsymbol{t}] \wedge [\hat{\boldsymbol{t}}] + [\hat{\boldsymbol{t}}]\,[\boldsymbol{p} - \boldsymbol{t}]\right)[\boldsymbol{p} - \boldsymbol{t}]\,[\boldsymbol{p} - \boldsymbol{c}_3]\rangle_2 &= 0, \text{ and} \\
\langle\left(2\,[\boldsymbol{p} - \boldsymbol{t}] \wedge [\hat{\boldsymbol{t}}]\right)[\boldsymbol{p} - \boldsymbol{t}]\,[\boldsymbol{p} - \boldsymbol{c}_3]\rangle_2 + \langle[\boldsymbol{p} - \boldsymbol{t}]^2\,[\hat{\boldsymbol{t}}]\,[\boldsymbol{p} - \boldsymbol{c}_3]\rangle_2 &= 0.
\end{aligned} \tag{3.3}$$

Now, we note that

$$\begin{aligned}
\langle\left(2\,[\boldsymbol{p} - \boldsymbol{t}] \wedge [\hat{\boldsymbol{t}}]\right)[\boldsymbol{p} - \boldsymbol{t}]\,[\boldsymbol{p} - \boldsymbol{c}_3]\rangle_2 &= \left(2\,[\boldsymbol{p} - \boldsymbol{t}] \wedge [\hat{\boldsymbol{t}}]\right)[\boldsymbol{p} - \boldsymbol{t}] \cdot [\boldsymbol{p} - \boldsymbol{c}_3] \\
\text{and } \langle[\boldsymbol{p} - \boldsymbol{t}]^2\,[\hat{\boldsymbol{t}}]\,[\boldsymbol{p} - \boldsymbol{c}_3]\rangle_2 &= [\boldsymbol{p} - \boldsymbol{t}]^2\,[\hat{\boldsymbol{t}}] \wedge [\boldsymbol{p} - \boldsymbol{c}_3].
\end{aligned}$$

Because $\boldsymbol{t} = r_1\hat{\boldsymbol{t}}$ and $\boldsymbol{c}_3 = (r_1 + r_3)\,\hat{\boldsymbol{t}}$, $\boldsymbol{t} \wedge \boldsymbol{c}_3 = 0$. We can expand $[\boldsymbol{p} - \boldsymbol{t}]^2$ as $p^2 - 2\boldsymbol{p} \cdot \boldsymbol{t} + {r_1}^2$. Using all of these ideas, (3.3) becomes (after simplification)

$$2\left([\boldsymbol{p} - \boldsymbol{t}] \cdot [\boldsymbol{p} - \boldsymbol{c}_3]\right)\boldsymbol{p} \wedge \boldsymbol{t} - \left(p^2 - 2\boldsymbol{p} \cdot \boldsymbol{t} + {r_1}^2\right)\boldsymbol{p} \wedge \boldsymbol{t} = 0.$$

For $\boldsymbol{p} \wedge \hat{\boldsymbol{t}} \neq 0$, that equation becomes, after further simplification,

$$p^2 - {r_1}^2 - 2\boldsymbol{p} \cdot \boldsymbol{c}_3 + 2\boldsymbol{t} \cdot \boldsymbol{c}_3 = 0.$$

Recalling that $\boldsymbol{c}_3 = (r_1 + r[3])\,\hat{\boldsymbol{t}}$, and substituting the expression that we derived for $r_1 + r_3$ in (3.1), then expanding and simplifying,

$$\left[\boldsymbol{c}_2^2 - (r_2 - r_1)^2\right]\boldsymbol{p} \cdot \hat{\boldsymbol{t}} - \left(p^2 - {r_1}^2\right)\boldsymbol{c}_2 \cdot \hat{\boldsymbol{t}} = (r_2 - r_1)\left(p^2 - r_2 r_1\right) + r_1 c_2^2.$$

Finally, we rearrange that result and multiply both sides by r_1, giving

$$\left\{\left[\boldsymbol{c}_2^2 - (r_2 - r_1)^2\right]\boldsymbol{p} - \left(p^2 - {r_1}^2\right)\boldsymbol{c}_2\right\} \cdot \boldsymbol{t} = r_1\left[(r_2 - r_1)\left(p^2 - r_2 r_1\right) + r_1 c_2^2\right]. \tag{3.4}$$

Defining $\boldsymbol{u} = \left[\boldsymbol{c}_2^2 - (r_2 - r_1)^2\right] \boldsymbol{p} - \left(p^2 - {r_1}^2\right) \boldsymbol{c}_2$, we can transform that result into

$$\boldsymbol{P}_{\hat{\boldsymbol{u}}}\left(\boldsymbol{t}\right) = \frac{r_1 \left[(r_2 - r_1)\left(p^2 - r_2 r_1\right) + r_1 c_2^2\right]}{\|\boldsymbol{u}\|}, \tag{3.5}$$

where $\boldsymbol{P}_{\hat{\boldsymbol{u}}}\left(\boldsymbol{t}\right)$ is the projection of $\boldsymbol{t}$ upon $\hat{\boldsymbol{u}}$. As described in detail in [2], there are two vectors that fulfill that condition. Labeled $\hat{\boldsymbol{t}}$ and $\hat{\boldsymbol{t}}'$ in Fig. 3.3, they are the vectors from the center of $\mathcal{C}_1$ to the points of tangency with the two solution circles shown. Readers are encouraged to derive this same solution using the

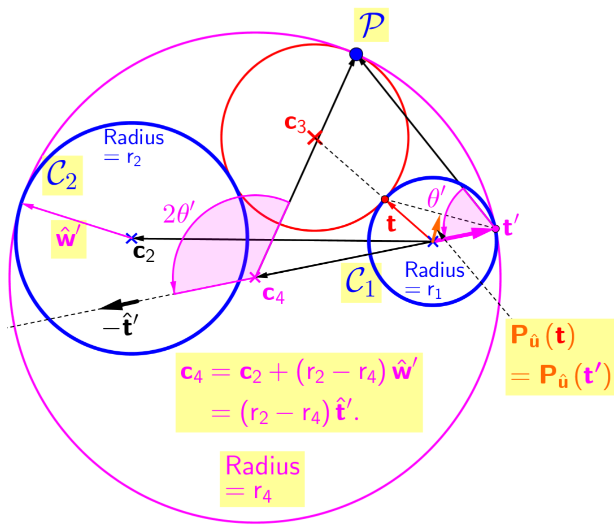

Figure 3.3: The solution circles that enclose both of the givens, and that enclose neither. See text for definitions of $\boldsymbol{u}$ and $\boldsymbol{P}_{\hat{\boldsymbol{u}}}\left(\boldsymbol{t}\right)$.

magenta circle as the starting point.

4 Literature Cited

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