Analytical and Classical Mechanics of Integrable Mixed and Quadratic Liénard Type Oscillator Equations

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Abstract

The Lagrangian description of a dynamical system from the equation of motion consists of an inverse problem in mechanics. This problem is solved for a class of exactly integrable mixed and quadratic Liénard type oscillator equations from a given first integral of motion. The dynamics of this class of equations, which contains the generalized modified Emden equation, also known as the second-order Riccati equation, and the inverted versions of the Mathews-Lakshmanan equations, is then investigated from Hamiltonian and Lagrangian points of view.

1. Consider the general class of integrable mixed Liénard-type oscillator equation

$$\ddot{x}+\frac{g'(x)}{g(x)}\dot{x}^2+\frac{f'(x)}{g(x)}x\dot{x}+a\frac{f(x)}{(g(x))^2}-\frac{(f(x))^2}{(g(x))^2}x=0 \tag{1}$$

generated from the first integral of motion

$$a(x,\dot{x})=\dot{x}g(x)+x\,f(x) \tag{2}$$

where dot denotes differentiation with respect to time and prime means differentiation with respect to x , $g(x)\neq 0$ and $f(x)$ are arbitrary functions of x. In this context, the Lagrangian for the equation (1) may then be computed as [1]

$$L(t,x,\dot{x})=\dot{x}g(x)\ln(\dot{x})-x\,f(x)+K\dot{x} \tag{3}$$

where $\ln$ holds for the natural logarithm, and K is an arbitrary constant. That being so, it is required to check the equivalence between the equation (1) and the Euler-Lagrange equation from (3). In this perspective the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0 \tag{4}$$

gives, knowing

$$\frac{\partial L}{\partial \dot{x}}=g(x)[1+\ln(\dot{x})]+K \tag{5}$$

and

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$$\frac{\partial L}{\partial x} = \dot{x}g'(x)\ln(\dot{x}) - f(x) - x f'(x) \tag{6}$$

after a few mathematical treatment, the expected equation (1). The preceding equation (5) gives the conjugate momentum p as

$$p = g(x)\ln(\dot{x}) + g(x) + K \tag{7}$$

such that the Hamiltonian

$$H(p,x) = p\dot{x} - L(x,\dot{x}) \tag{8}$$

becomes

$$H(p,x) = \dot{x}g(x) + x f(x) \tag{9}$$

which is, as expected, equal to (2). Eliminating $\dot{x}$ from (9) by using (7), then the Hamiltonian (9) takes the form

$$H(p,x) = \frac{g(x)}{e} e^{(\frac{p-K}{g(x)})} + x f(x) \tag{10}$$

In this perspective the Hamiltonian equations

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial x} \end{cases} \tag{11}$$

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$$\begin{cases} \dot{x} = \dfrac{1}{e} e^{(\frac{p-K}{g(x)})} \\ \dot{p} = \dfrac{g'(x)}{e} e^{(\frac{p-K}{g(x)})} \left[\dfrac{p-K}{g(x)} - 1 \right] - \left[f(x) + x f'(x) \right] \end{cases} \tag{12}$$

So with that, some examples may be given to illustrate the application of the current theory.

2. Application

2.1 Let $g(x) = a_1 x^m$, and $f(x) = a_1^2 x^{2m+1}$, where the exponent m is a real number. So, the equation (1) reduces to

$$\ddot{x} + m\frac{\dot{x}^2}{x} + (2m+1)a_1 x^{m+1}\dot{x} + ax - a_1^2 x^{2m+3} = 0 \tag{13}$$

The equation (13) consists of a generalized mixed Liénard-type equation. Now, substitution of $m=0$, into the equation (13), leads immediately to the generalized modified Emden type equation with a linear forcing term, also known as a second-order Riccati equation, that is.

$$\ddot{x}+a_1x\dot{x}+ax-a_1^2x^3=0 \tag{14}$$

Also, $m=-\frac{1}{2}$, gives, taking into account the equation (13)

$$\ddot{x}-\frac{1}{2}\frac{\dot{x}^2}{x}+ax-a_1^2x^2=0 \tag{15}$$

This equation (15) is known as a quadratic Liénard-type differential equation. The analytical description of these equations is secured by the equations (3), (10) and (12).

2.2 Case 1: $f(x)=1$

The equation (1) becomes in this case the exactly integrable quadratic Liénard-type nonlinear dissipative oscillator equation

$$\ddot{x}+\frac{g'(x)}{g(x)}\dot{x}^2+\frac{(a-x)}{(g(x))^2}=0 \tag{16}$$

By choosing $g(x)=\sqrt{1\pm\mu x^2}$, where μ is an arbitrary parameter, a physically important quadratic Liénard-type differential equation may be obtained as

$$\ddot{x}\pm\frac{\mu x}{1\pm\mu x^2}\dot{x}^2+\frac{(a-x)}{1\pm\mu x^2}=0 \tag{17}$$

since for $a=0$, one may obtain the inverted versions of the Mathews-Lakshmanan oscillator equations.

The Hamiltonian and Lagrangian description of (17) is then assured by the general relationships (3), (10) and (12).

2.2 Case 2: $g(x)=1$

The equation (1) gives the general class of exactly solvable Liénard nonlinear dissipative oscillator equations

$$\ddot{x}+x\dot{x}f'(x)+af(x)-x(f(x))^2=0 \tag{18}$$

Substitution of $f(x)=x^l$, gives the generalized modified Emden-type equation with nonlinear forcing function, also called generalized second-order Riccati equation, viz

$$\ddot{x}+lx^l\dot{x}-x^{2l+1}+ax^l=0 \tag{19}$$

where l is an arbitrary parameter. It is worth to note that a generalization of (1) and (3) may be written in the form

$$\ddot{x}+\frac{g'(x)}{g(x)}\dot{x}^2+x^l\frac{f'(x)}{g(x)}\dot{x}+a\,l\,x^{l-1}\frac{f(x)}{\left(g(x)\right)^2}-l\,x^{2l-1}\frac{\left(f(x)\right)^2}{\left(g(x)\right)^2}=0 \tag{20}$$

and

$$L\left(x,\dot{x}\right)=\dot{x}g(x)\ln(\dot{x})-x^l f(x)+K\dot{x} \tag{21}$$

respectively, where l and K are arbitrary parameters, from the first integral

$$a\left(x,\dot{x}\right)=\dot{x}g(x)+x^l f(x) \tag{22}$$

Finally, a more generalization may be computed from the first integral of motion

$$a_1\left(x,\dot{x}\right)=\dot{x}g(x)+ax^l\int f(x)dx \tag{23}$$

References

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