Recurrent Formulas of the Generalized Fibonacci Sequences of Third & Fourth Order

A. D.Godase

Department of Mathematics,

V.P.College, Vaijapur, Aurangabad - 423701. (Maharashtra, India)

ashokgodse2012@gmail.com

Abstract

Coupled Fibonacci sequences involve two sequences of integers in which the elements of one sequence are part of the generalization of the other and vice versa. K. T. Atanassov was first introduced coupled Fibonacci sequences of second order in additive form. There are 8 different schemes of generalization for the Tribonacci sequences in the case of two sequences & there are 16 different schemes of generalization for the Tetranacci sequences in the case of two sequences [1]. I introduce their recurrent formulas below.

Mathematics Subject Classification: 11B39, 11B37

Keywords: Fibonacci sequence, multiplicative Tribonacci sequence, multiplicative Tetranacci sequence.

1. INTRODUCTION:

In the recent years much work has been done in this field but its multiplicative form is less known. The coupled Fibonacci sequence was first introduced by K. T. Atanassov and also discussed many curious properties and new direction of generalization of Fibonacci sequence in [1]. He defined and studied about four different ways to generate coupled sequences and called them coupled Fibonacci sequences (or 2-F sequences). K. T. Atanassov [1] notifies four different schemes in multiplicative form for coupled Fibonacci sequences.

2. RECURRENT FORMULAS OF THE GENERALIZED MULTIPLICATIVE FIBONACCI SEQUENCE OF THIRD ORDER:

We can construct 8 different schemes of generalized multiplicative Fibonacci sequence in the case of two sequences. We introduce their recurrent formulas below. Everywhere let, $X_0 = C_0, Y_0 = C_1, X_1 = C_2, Y_1 = C_3, X_2 = C_4, Y_2 = C_5$ and assume that $n \geq 0$ is a natural number, where $C_0, C_1, C_2, C_3, C_4, C_5$ are given constants and Z *is* one of the symbols X or Y.

The different schemes are as follows:

$$T_1 : \begin{cases} X_{n+3} = X_{n+2}X_{n+1}X_n \\ Y_{n+3} = Y_{n+2}Y_{n+1}Y_n \end{cases}, \qquad T_2 : \begin{cases} X_{n+3} = X_{n+2}X_{n+1}Y_n \\ Y_{n+3} = Y_{n+2}Y_{n+1}X_n \end{cases},$$

$$T_3 : \begin{cases} X_{n+3} = X_{n+2}Y_{n+1}X_n \\ Y_{n+3} = Y_{n+2}X_{n+1}Y_n \end{cases}, \qquad T_4 : \begin{cases} X_{n+3} = X_{n+2}Y_{n+1}Y_n \\ Y_{n+3} = Y_{n+2}X_{n+1}X_n \end{cases},$$

$$T_5 : \begin{cases} X_{n+3} = Y_{n+2}X_{n+1}X_n \\ Y_{n+3} = X_{n+2}Y_{n+1}Y_n \end{cases}, \qquad T_6 : \begin{cases} X_{n+3} = Y_{n+2}X_{n+1}Y_n \\ Y_{n+3} = X_{n+2}Y_{n+1}X_n \end{cases},$$

$$T_7 : \begin{cases} X_{n+3} = Y_{n+2}Y_{n+1}X_n \\ Y_{n+3} = X_{n+2}X_{n+1}Y_n \end{cases}, \qquad T_8 : \begin{cases} X_{n+3} = Y_{n+2}Y_{n+1}Y_n \\ Y_{n+3} = X_{n+2}X_{n+1}X_n \end{cases}$$

The first scheme is trivial. All of the others are nontrivial; they have the following recurrent formulas for n $\geq$ 0:

----***For*** $T_1 : Z_{n+6} = Z_{n+5}Z_{n+4}Z_{n+3}$,

----***For*** $T_2 : Z_{n+6} = \dfrac{Z_{n+5}^2 Z_{n+4} Z_n}{Z_{n+3}^2 Z_{n+2}}$,

----***For*** $T_3 : Z_{n+6} = \dfrac{Z_{n+5}^2 Z_{n+3}^2}{Z_{n+4} Z_{n+2} Z_n}$,

----***For*** $T_4 : Z_{n+6} = \dfrac{Z_{n+5}^2 Z_{n+2} Z_{n+1}^2 Z_n}{Z_{n+4}}$,

----**For** $T_5 : Z_{n+6} = \dfrac{Z^3_{n+4} Z^2_{n+3}}{Z_{n+2} Z^2_{n+1} Z_n}$,

----**For** $T_6 : Z_{n+6} = Z^3_{n+4} Z_{n+2} Z_n$,

----**For** $T_7 : Z_{n+6} = \dfrac{Z_{n+4} Z^4_{n+3} Z_{n+2}}{Z_n}$,

----**For** $T_8 : Z_{n+6} = Z_{n+4} Z^2_{n+3} Z^3_{n+2} Z^2_{n+1} Z_n$

3. RECURRENT FORMULAS OF THE GENERALIZED FIBONACCI SEQUENCE OF THIRD ORDER:

We can construct 8 different schemes of generalized Fibonacci sequence of third order in the case of two sequences. We introduce their recurrent formulas below. Everywhere let $X_0 = C_0, Y_0 = C_1, X_1 = C_2, Y_1 = C_3, X_2 = C_4, Y_2 = C_5$ and assume that $n \geq 0$ is a natural number, where $C_0, C_1, C_2, C_3, C_4, C_5$ are given constants and Z is one of the symbols X or Y.

The different schemes are as follows:

$T_1 : \begin{cases} X_{n+3} = X_{n+2} + X_{n+1} + X_n \\ Y_{n+3} = Y_{n+2} + Y_{n+1} + Y_n \end{cases}$, $T_2 : \begin{cases} X_{n+3} = X_{n+2} + X_{n+1} + Y_n \\ Y_{n+3} = Y_{n+2} + Y_{n+1} + X_n \end{cases}$,

$T_3 : \begin{cases} X_{n+3} = X_{n+2} + Y_{n+1} + X_n \\ Y_{n+3} = Y_{n+2} + X_{n+1} + Y_n \end{cases}$, $T_4 : \begin{cases} X_{n+3} = X_{n+2} + Y_{n+1} + Y_n \\ Y_{n+3} = Y_{n+2} + X_{n+1} + X_n \end{cases}$,

$T_5 : \begin{cases} X_{n+3} = Y_{n+2} + X_{n+1} + X_n \\ Y_{n+3} = X_{n+2} + Y_{n+1} + Y_n \end{cases}$, $T_6 : \begin{cases} X_{n+3} = Y_{n+2} + X_{n+1} + Y_n \\ Y_{n+3} = X_{n+2} + Y_{n+1} + X_n \end{cases}$,

$T_7 : \begin{cases} X_{n+3} = Y_{n+2} + Y_{n+1} + X_n \\ Y_{n+3} = X_{n+2} + X_{n+1} + Y_n \end{cases}$, $T_8 : \begin{cases} X_{n+3} = Y_{n+2} + Y_{n+1} + Y_n \\ Y_{n+3} = X_{n+2} + X_{n+1} + X_n \end{cases}$

The first scheme is trivial. All of the others are nontrivial; they have the following recurrent formulas for n ≥ 0:

----**For** $T_1 : Z_{n+6} = Z_{n+5} + Z_{n+4} + Z_{n+3}$,

----**For** $T_2 : Z_{n+6} = 2Z_{n+5} + Z_{n+4} - 2Z_{n+3} - Z_{n+2} + Z_n$,

----**For** $T_3 : Z_{n+6} = 2Z_{n+5} - Z_{n+4} + 2Z_{n+3} - Z_{n+2} - Z_n$,

----*For* $T_4 : Z_{n+6} = 2Z_{n+5} - Z_{n+4} + Z_{n+2} + 2Z_{n+1} + Z_n$,

----*For* $T_5 : Z_{n+6} = 3Z_{n+4} + 2Z_{n+3} - Z_{n+2} - 2Z_{n+1} - Z_n$,

----*For* $T_6 : Z_{n+6} = 3Z_{n+4} + Z_{n+2} + Z_n$,

----*For* $T_7 : Z_{n+6} = Z_{n+4} + 4Z_{n+3} + Z_{n+2} - Z_n$,

----*For* $T_8 : Z_{n+6} = Z_{n+4} + 2Z_{n+3} + 3Z_{n+2} + 2Z_{n+1} + Z_n$,

4. RECURRENT FORMULAS OF THE GENERALIZED MULTIPLICATIVE FIBONACCI SEQUENCE OF FOURTH ORDER:

We can construct 16 different schemes of generalized multiplicative Fibonacci sequence in the case of two sequences. We introduce their recurrent formulas below. Everywhere let, $X_0 = C_0, Y_0 = C_1, X_1 = C_2, Y_1 = C_3, X_2 = C_4, Y_2 = C_5, X_3 = C_6, Y_3 = C_7$, and assume that $n \geq 0$ is a natural number, where $C_0, C_1, C_2, C_3, C_4, C_5, C_6, C_7$ are given constants and Z *is* one of the symbols X or Y.

The different schemes are as follows:

$$T_1 : \begin{cases} X_{n+4} = X_{n+3}X_{n+2}X_{n+1}X_n \\ Y_{n+4} = Y_{n+3}Y_{n+2}Y_{n+1}Y_n \end{cases}, \quad T_2 : \begin{cases} X_{n+4} = X_{n+3}X_{n+2}X_{n+1}Y_n \\ Y_{n+4} = Y_{n+3}Y_{n+2}Y_{n+1}X_n \end{cases},$$

$$T_3 : \begin{cases} X_{n+4} = X_{n+3}X_{n+2}Y_{n+1}X_n \\ Y_{n+4} = Y_{n+3}Y_{n+2}X_{n+1}Y_n \end{cases}, \quad T_4 : \begin{cases} X_{n+4} = X_{n+3}X_{n+2}Y_{n+1}Y_n \\ Y_{n+4} = Y_{n+3}Y_{n+2}X_{n+1}X_n \end{cases},$$

$$T_5 : \begin{cases} X_{n+4} = X_{n+3}Y_{n+2}X_{n+1}X_n \\ Y_{n+4} = Y_{n+3}X_{n+2}Y_{n+1}Y_n \end{cases}, \quad T_6 : \begin{cases} X_{n+4} = X_{n+3}Y_{n+2}X_{n+1}Y_n \\ Y_{n+4} = Y_{n+3}X_{n+2}Y_{n+1}X_n \end{cases},$$

$$T_7 : \begin{cases} X_{n+4} = Y_{n+3}X_{n+2}X_{n+1}X_n \\ Y_{n+4} = X_{n+3}Y_{n+2}Y_{n+1}Y_n \end{cases}, \quad T_8 : \begin{cases} X_{n+4} = Y_{n+3}X_{n+2}X_{n+1}Y_n \\ Y_{n+4} = X_{n+3}Y_{n+2}Y_{n+1}X_n \end{cases},$$

$$T_9 : \begin{cases} X_{n+4} = X_{n+3}Y_{n+2}Y_{n+1}X_n \\ Y_{n+4} = Y_{n+3}X_{n+2}X_{n+1}Y_n \end{cases}, \quad T_{10} : \begin{cases} X_{n+4} = X_{n+3}Y_{n+2}Y_{n+1}Y_n \\ Y_{n+4} = Y_{n+3}X_{n+2}X_{n+1}X_n \end{cases},$$

$$T_{11} : \begin{cases} X_{n+4} = Y_{n+3}Y_{n+2}X_{n+1}X_n \\ Y_{n+4} = X_{n+3}X_{n+2}Y_{n+1}Y_n \end{cases}, \quad T_{12} : \begin{cases} X_{n+4} = Y_{n+3}Y_{n+2}X_{n+1}Y_n \\ Y_{n+4} = X_{n+3}X_{n+2}Y_{n+1}X_n \end{cases},$$

$$T_{13}: \begin{cases} X_{n+4}=Y_{n+3}X_{n+2}Y_{n+1}X_n \\ Y_{n+4}=X_{n+3}Y_{n+2}X_{n+1}Y_n \end{cases}, T_{14}: \begin{cases} X_{n+4}=Y_{n+3}X_{n+2}Y_{n+1}Y_n \\ Y_{n+4}=X_{n+3}Y_{n+2}X_{n+1}X_n \end{cases},$$

$$T_{15}: \begin{cases} X_{n+4}=Y_{n+3}Y_{n+2}Y_{n+1}X_n \\ Y_{n+4}=X_{n+3}X_{n+2}X_{n+1}Y_n \end{cases}, T_{16}: \begin{cases} X_{n+4}=Y_{n+3}Y_{n+2}Y_{n+1}Y_n \\ Y_{n+4}=X_{n+3}X_{n+2}X_{n+1}X_n \end{cases}$$

The first scheme is trivial. All of the others are nontrivial; they have the following recurrent formulas for n ≥ 0:

----For $T_1 : Z_{n+8} = Z_{n+7}Z_{n+6}Z_{n+5}Z_{n+4}$,

----For $T_2 : Z_{n+8} = \frac{Z_{n+7}^2 Z_{n+6} Z_n}{Z_{n+4}^3 Z_{n+3}^2 Z_{n+2}}$,

----For $T_3 : Z_{n+8} = \frac{Z_{n+7}^2 Z_{n+6} Z_{n+4}}{Z_{n+5}^2 Z_{n+3}^2 Z_{n+2} Z_n}$,

----For $T_4 : Z_{n+8} = \frac{Z_{n+7}^2 Z_{n+6} Z_{n+2} Z_{n+1}^2 Z_n}{Z_{n+5}^2 Z_{n+4}}$,

----For $T_5 : Z_{n+8} = \frac{Z_{n+7}^2 Z_{n+5}^2 Z_{n+4}}{Z_{n+6} Z_{n+3}^2 Z_{n+2} Z_{n+1}^2 Z_n}$,

----For $T_6 : Z_{n+8} = \frac{Z_{n+7}^2 Z_{n+5}^2 Z_{n+1} Z_n}{Z_{n+6} Z_{n+4}}$,

----For $T_7 : Z_{n+8} = \frac{Z_{n+6}^2 Z_{n+5}}{Z_{n+3}^2 Z_{n+2}^3 Z_{n+1}^2 Z_n}$,

----For $T_8 : Z_{n+8} = \frac{Z_{n+6}^3 Z_{n+5}^2 Z_n}{Z_{n+4} Z_{n+2}}$,

----For $T_9 : Z_{n+8} = \frac{Z_{n+7}^2 Z_{n+4}^3 Z_{n+2}}{Z_{n+6} Z_n}$,

----For $T_{10} : Z_{n+8} = \frac{Z_{n+7}^2 Z_{n+4} Z_{n+3}^2 Z_{n+2}^3 Z_{n+1}^2 Z_n}{Z_{n+6}}$,

----For $T_{11} : Z_{n+8} = \frac{Z_{n+6} Z_{n+5}^4 Z_{n+4}^3}{Z_{n+2} Z_{n+1}^2 Z_n}$,

----For $T_{12} : Z_{n+8} = Z_{n+6}^3 Z_{n+4} Z_{n+3}^2 Z_{n+2} Z_{n+1}^2 Z_n$,

----For $T_{13} : Z_{n+8} = \frac{Z_{n+7}^2 Z_{n+5}^2 Z_{n+2} Z_n}{Z_{n+6} Z_{n+4}}$,

----For $T_{14}: Z_{n+8} = Z_{n+6} Z^4_{n+5} Z_{n+4} Z^2_{n+3} Z_{n+2} Z_n$,

----For $T_{15}: Z_{n+8} = \dfrac{Z_{n+6} Z^2_{n+5} Z^5_{n+4} Z^2_{n+3} Z_{n+2}}{Z_n}$,

----For $T_{16}: Z_{n+8} = Z_{n+6} Z^2_{n+5} Z^2_{n+4} Z^3_{n+3} Z^3_{n+2} Z^2_{n+1} Z_n$

5. RECURRENT FORMULAS OF THE GENERALIZED FIBONACCI SEQUENCE OF FOURTH ORDER:

We can construct 16 different schemes of generalized Fibonacci sequence of fourth order in the case of two sequences. We introduce their recurrent formulas below. Everywhere let, $X_0 = C_0, Y_0 = C_1, X_1 = C_2, Y_1 = C_3, X_2 = C_4, Y_2 = C_5, X_3 = C_6, Y_3 = C_7$, and assume that $n \geq 0$ is a natural number, where $C_0, C_1, C_2, C_3, C_4, C_5, C_6, C_7$ are given constants and Z is one of the symbols X or Y.

The different schemes are as follows:

$T_1 : \begin{cases} X_{n+4} = X_{n+3} + X_{n+2} + X_{n+1} + X_n \\ Y_{n+4} = Y_{n+3} + Y_{n+2} + Y_{n+1} + Y_n \end{cases}$, $T_2 : \begin{cases} X_{n+4} = X_{n+3} + X_{n+2} + X_{n+1} + Y_n \\ Y_{n+4} = Y_{n+3} + Y_{n+2} + Y_{n+1} + X_n \end{cases}$,

$T_3 : \begin{cases} X_{n+4} = X_{n+3} + X_{n+2} + Y_{n+1} + X_n \\ Y_{n+4} = Y_{n+3} + Y_{n+2} + X_{n+1} + Y_n \end{cases}$, $T_4 : \begin{cases} X_{n+4} = X_{n+3} + X_{n+2} + Y_{n+1} + Y_n \\ Y_{n+4} = Y_{n+3} + Y_{n+2} + X_{n+1} + X_n \end{cases}$,

$T_5 : \begin{cases} X_{n+4} = X_{n+3} + Y_{n+2} + X_{n+1} + X_n \\ Y_{n+4} = Y_{n+3} + X_{n+2} + Y_{n+1} + Y_n \end{cases}$, $T_6 : \begin{cases} X_{n+4} = X_{n+3} + Y_{n+2} + X_{n+1} + Y_n \\ Y_{n+4} = Y_{n+3} + X_{n+2} + Y_{n+1} + X_n \end{cases}$,

$T_7 : \begin{cases} X_{n+4} = Y_{n+3} + X_{n+2} + X_{n+1} + X_n \\ Y_{n+4} = X_{n+3} + Y_{n+2} + Y_{n+1} + Y_n \end{cases}$, $T_8 : \begin{cases} X_{n+4} = Y_{n+3} + X_{n+2} + X_{n+1} + Y_n \\ Y_{n+4} = X_{n+3} + Y_{n+2} + Y_{n+1} + X_n \end{cases}$,

$T_9 : \begin{cases} X_{n+4} = X_{n+3} + Y_{n+2} + Y_{n+1} + X_n \\ Y_{n+4} = Y_{n+3} + X_{n+2} + X_{n+1} + Y_n \end{cases}$, $T_{10} : \begin{cases} X_{n+4} = X_{n+3} + Y_{n+2} + Y_{n+1} + Y_n \\ Y_{n+4} = Y_{n+3} + X_{n+2} + X_{n+1} + X_n \end{cases}$,

$T_{11} : \begin{cases} X_{n+4} = Y_{n+3} + Y_{n+2} + X_{n+1} + X_n \\ Y_{n+4} = X_{n+3} + X_{n+2} + Y_{n+1} + Y_n \end{cases}$, $T_{12} : \begin{cases} X_{n+4} = Y_{n+3} + Y_{n+2} + X_{n+1} + Y_n \\ Y_{n+4} = X_{n+3} + X_{n+2} + Y_{n+1} + X_n \end{cases}$,

$T_{13} : \begin{cases} X_{n+4} = Y_{n+3} + X_{n+2} + Y_{n+1} + X_n \\ Y_{n+4} = X_{n+3} + Y_{n+2} + X_{n+1} + Y_n \end{cases}$, $T_{14} : \begin{cases} X_{n+4} = Y_{n+3} + X_{n+2} + Y_{n+1} + Y_n \\ Y_{n+4} = X_{n+3} + Y_{n+2} + X_{n+1} + X_n \end{cases}$,

$T_{15} : \begin{cases} X_{n+4} = Y_{n+3} + Y_{n+2} + Y_{n+1} + X_n \\ Y_{n+4} = X_{n+3} + X_{n+2} + X_{n+1} + Y_n \end{cases}$, $T_{16} : \begin{cases} X_{n+4} = Y_{n+3} + Y_{n+2} + Y_{n+1} + Y_n \\ Y_{n+4} = X_{n+3} + X_{n+2} + X_{n+1} + X_n \end{cases}$

The first scheme is trivial. All of the others are nontrivial; they have the following recurrent formulas for n ≥ 0:

----For $T_1 : Z_{n+8} = Z_{n+7} + Z_{n+6} + Z_{n+5} + Z_{n+4}$,

----For $T_2 : Z_{n+8} = 2Z_{n+7} + Z_{n+6} - 3Z_{n+4} - 2Z_{n+3} - Z_{n+2}$,

----For $T_3 : Z_{n+8} = 2Z_{n+7} + Z_{n+6} - 2Z_{n+5} + Z_{n+4} - 2Z_{n+3} - 2Z_{n+2} - Z_n$,

----For $T_4 : Z_{n+8} = 2Z_{n+7} + Z_{n+6} - Z_{n+5} - 2Z_{n+4} - Z_{n+3} + Z_{n+1} + Z_n$,

----For $T_5 : Z_{n+8} = 2Z_{n+7} - Z_{n+6} + 2Z_{n+5} + Z_{n+4} - 2Z_{n+3} - Z_{n+2} - 2Z_{n+1} - Z_n$,

----For $T_6 : Z_{n+8} = 2Z_{n+7} - Z_{n+6} + 2Z_{n+5} - Z_{n+4} + Z_{n+2} + Z_n$,

----For $T_7 : Z_{n+8} = Z_{n+7} + 2Z_{n+6} + 2Z_{n+5} - 3Z_{n+3} - 3Z_{n+2} - 2Z_{n+1} - Z_n$,

----For $T_8 : Z_{n+8} = 3Z_{n+6} + 2Z_{n+5} - Z_{n+4} - Z_{n+2} + Z_n$,

----For $T_9 : Z_{n+8} = 2Z_{n+7} - Z_{n+6} - 2Z_{n+4} + Z_{n+3} + Z_{n+2} - Z_n$,

----For $T_{10} : Z_{n+8} = 2Z_{n+7} - Z_{n+6} + Z_{n+4} + 2Z_{n+3} + 3Z_{n+2} + 2Z_{n+1} + Z_n$,

----For $T_{11} : Z_{n+8} = Z_{n+6} + 4Z_{n+5} + 3Z_{n+4} - Z_{n+2} - Z_{n+1} - Z_n$,

----For $T_{12} : Z_{n+8} = 3Z_{n+6} + Z_{n+4} + 2Z_{n+3} + Z_{n+2} + 2Z_{n+1} + Z_n$,

----For $T_{13} : Z_{n+8} = 3Z_{n+6} + 3Z_{n+4} - Z_{n+2} - Z_n$,

----For $T_{14} : Z_{n+8} = Z_{n+6} + 4Z_{n+5} + Z_{n+4} + 2Z_{n+3} + Z_{n+2} + Z_n$,

----For $T_{15} : Z_{n+8} = Z_{n+6} + 2Z_{n+5} + 5Z_{n+4} + 2Z_{n+3} + Z_{n+2} - Z_n$,

----For $T_{16} : Z_{n+8} = Z_{n+6} + 2Z_{n+5} + 3Z_{n+4} + 4Z_{n+3} + 3Z_{n+2} + 2Z_{n+1} + Z_n$

Conclusion:

In this paper I introduced recurrent formulas for coupled Fibonacci sequences of third & fourth order under different schemes. The proofs for these facts can be shown by induction. An open problem is the construction of an explicit formula for each of the schemes given above.

Acknowledgement:

I am thankful to anonymous referees for valuable suggestions.

References:

[1]. K. Atanassov, L. Atanassov & D. Sasselov. "A New Perspective to the Generalization of
The Fibonacci Sequence." *Fibonacci Quarterly 23.1* (1985):21-28.

[2]. K. Atanassov. "On a Generalization of the Fibonacci Sequence in the Case of
Three Sequences." *Fibonacci Quarterly 27.1* (1989):7-10.

[3]. J.-Z. Lee & J. -S. Lee. "Some Properties of the Generalization of the Fibonacci
Sequence. "Fibonacci *Quarterly 25. 2* (1987):111-17.