Solving Boolean Equation

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Abstract

If $\forall P\text{: proposition}$, $B(P)$ is the truth value(0 or 1) of P then we can solve a boolean equation by using these below.

$$B(p_1 \vee p_2 \vee \dots \vee p_n) \equiv 1 + \prod_{k=1}^{n}(1 + p_k)\ (mod\ 2)$$

$$\left\{(x_1, x_2, \dots, x_n) \mid \prod_{i=1}^{n} B(x_i) \equiv 0(mod\ 2)\right\} = \left(\bigcap_{i=1}^{n}\{\ (x_i) \mid B(x_i) \equiv 1(mod\ 2)\}\right)^c$$

$$= \{(x_1, x_2, \dots, x_n)\ |(1,1,1, \dots ,1)\}^c$$

Introduction

We have started to study because we don’t know how to apply many Boolean laws to Boolean algebra. So, We study to redefine Boolean algebra using by mod 2, to reprove Boolean laws, and we study the method of solving Boolean equation.

1. Redefine Boolean algebra and basical theorem

Definition 1.1) $\forall P$: proposition

- $B(P) = \begin{cases} 0, & \text{truth value of } P \text{ is } False \\ 1, & \text{truth value of } P \text{ is } True \end{cases}$
- If $B(P) = z(0 \text{ or } 1)$ then $B(P) \equiv z(mod\ 2)$

(But, we can omit $B(\quad)$ like as $B(P) \equiv P(mod\ 2)$ for usability.)

$Ex)\ P \equiv p \vee q(p, q$: proposition$)$, If the truth value of P is $True$

then $B(P) \equiv B(p \vee q) \equiv P \equiv 1(mod\ 2)$

Theorem 1.1) $\forall n \in \mathbb{N}, \forall P$: proposition

$$\begin{cases} 2nB(P) \equiv 2nP \equiv 0\ (\ mod\ 2\ ) \\ (2n+1)B(P) \equiv B(P) \equiv (2n+1)P \equiv P\ (\ mod\ 2\ ) \end{cases}$$

$$\big(B(P)\big)^n \equiv P^n \equiv B(P) \equiv P(\ mod\ 2\ )$$

$$-B(P) \equiv -P \equiv B(P) \equiv P\ (\ mod\ 2\ )$$

Proof)

$$\begin{cases} 2nB(P) \equiv 2 \times nB(P) \equiv 0 \times nB(P) \equiv 0(\ mod\ 2\ ) \\ (2n+1)B(P) \equiv 2nB(P) + B(P) \equiv 0 + B(P) \equiv B(P)(\ mod\ 2\ ) \end{cases}$$

$$\begin{cases} B(P) \equiv 1(\ mod\ 2\ ) \Rightarrow \big(B(P)\big)^n \equiv (1)^n \equiv 1 \equiv B(P)\ (\ mod\ 2\ ) \\ B(P) \equiv 0(\ mod\ 2\ ) \Rightarrow \big(B(P)\big)^n \equiv (0)^n \equiv 0 \equiv B(P)\ (\ mod\ 2\ ) \end{cases}$$

$$\begin{cases} B(P) \equiv 1(\ mod\ 2\ ) \Rightarrow -B(P) \equiv -1 \times 1 \equiv -1 \equiv 0 - 1 \equiv 2 - 1 \equiv 1 \equiv B(P)\ (\ mod\ 2\ ) \\ B(P) \equiv 0(\ mod\ 2\ ) \Rightarrow -B(P) \equiv -1 \times 0 \equiv 0 \equiv B(P)\ (\ mod\ 2\ ) \end{cases}$$ ■

Theorem 1.2) $\forall p, q$: proposition

$B(c) \equiv 0(mod\ 2), B(t) \equiv 1(mod\ 2)(where, c$: contradiction, t : tautology$)$

$B({\sim}p) \equiv 1 + B(p) \equiv 1 + p(\ mod\ 2\ )$

$B(p \wedge q) \equiv B(p)B(q) \equiv pq(\ mod\ 2\ )$

$B(p \vee q) \equiv B(p) + B(q) + B(p)B(q) \equiv p + q + pq(\ mod\ 2\ )$

$B(p \to q) \equiv 1 + B(p) + B(p)B(q) \equiv 1 + p + pq(\ mod\ 2\ )$

$B(p \leftrightarrow q) \equiv 1 + B(p) + B(q) \equiv 1 + p + q(mod\ 2)$

Proof)

p	q	${\sim}p$	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$
		$1+p$	pq	$p+q+pq$	$1+p+pq$	$1+p+q$
0	0	1	0	0	1	1
0	1	1	0	1	1	0
1	0	0	0	1	0	0
1	1	0	1	1	1	1

■

Theorem 1.3) $\forall p, q$: proposition

$B(p \text{ XOR } q) \equiv B(p) + B(q) \equiv p + q (mod\ 2)$

$B(p \text{ NAND } q) \equiv 1 + B(p)B(q) \equiv 1 + pq (mod\ 2)$

$B(p \text{ NOR } q) \equiv 1 + B(p) + B(q) + B(p)B(q) \equiv 1 + p + q + pq (mod\ 2)$

$B(p \text{ XNOR } q) \equiv 1 + B(p) + B(q) \equiv 1 + p + q (mod\ 2)$

Proof)

p	q	p XOR q	p NAND q	p NOR q	p XNOR q
		$p + q$	$1 + pq$	$1 + p + q + pq$	$1 + p + q$
0	0	0	1	1	1
0	1	1	1	0	0
1	0	1	1	0	0
1	1	0	0	0	1

■

Corollary 1.1) $\forall p, q$: proposition

$B(p \Rightarrow q) \Leftrightarrow p + pq \equiv 0(mod\ 2)$

$B(p \Leftrightarrow q) \Leftrightarrow p + q \equiv 0(mod\ 2)$

Proof)

$B(p \Rightarrow q) \Leftrightarrow B(p \rightarrow q) \equiv 1(mod\ 2) \Leftrightarrow 1 + p + pq \equiv 1(mod\ 2) \Leftrightarrow p + pq \equiv 0(mod\ 2)$

$B(p \Leftrightarrow q) \Leftrightarrow B(p \leftrightarrow q) \equiv 1(mod\ 2) \Leftrightarrow 1 + p + q \equiv 1(mod\ 2) \Leftrightarrow p + q \equiv 0(mod\ 2)$ ■

Corollary 1.2) $\forall p, q$: proposition,

$B(p \vee q) \equiv p + q + pq \equiv 1 + (1 + p)(1 + q)(mod\ 2)$

$B(p \rightarrow q) \equiv 1 + p + pq \equiv 1 + p(1 + q)(mod\ 2)$

$B(p \leftrightarrow q) \equiv 1 + p + q \equiv 1 + (1 + pq)(p + q + pq) \equiv (1 + p + pq)(1 + q + qp)(mod\ 2)$

$B(p \text{ XOR } q) \equiv p + q \equiv (1 + pq)(p + q + pq) \equiv 1 + (1 + p + pq)(1 + q + qp)(mod\ 2)$

Proof)

$B(p \vee q) \equiv p + q + pq \equiv 1 + 1 + p + q + pq \equiv 1 + p(1 + q) + 1 + q$

$\equiv 1 + (1 + p)(1 + q)(mod\ 2)$

$B(p \rightarrow q) \equiv 1 + p + pq \equiv 1 + p(1 + q)(mod\ 2)$

$B(p \leftrightarrow q) \equiv 1 + p + q \equiv 1 + p + q + pq + pq \equiv (1 + pq) + (p + q + pq)$

$\equiv (1 + pq) + (p + q + pq) + ppq + qpq + pqpq + pq$

$\equiv (1 + pq) + (p + q + pq) + (p + q + pq)pq + pq$

$\equiv (1 + pq) + (p + q + pq)(1 + pq) + pq$

$\equiv (1 + pq + pq) + (p + q + pq)(1 + pq) \equiv 1 + (1 + pq)(p + q + pq)$

$\equiv (1 + p + pq) + q(1 + p + pq) + pq(1 + p + pq)$

$\equiv (1 + p + pq)(1 + q + pq)(mod\ 2)$

$B(p \text{ XOR } q) \equiv p + q \equiv 1 + 1 + p + q \equiv 1 + B(p \leftrightarrow q) \equiv (1 + pq)(p + q + pq)$

$\equiv 1 + (1 + p + pq)(1 + q + pq)(mod\ 2)$ ■

Corollary 1.3) $\forall \boldsymbol{p_k}$: proposition,

1) $B(p_1 \wedge p_2 \wedge \dots \wedge p_n) \equiv \prod_{k=1}^{n} p_k \ (mod\ 2)$

2) $B(p_1 \vee p_2 \vee \dots \vee p_n) \equiv 1 + \prod_{k=1}^{n} (1 + p_k) \ (mod\ 2)$

Proof)

1) $B(p_1 \wedge p_2) \equiv \prod_{k=1}^{2} p_k \ (mod\ 2)\ by\ Theorem 1.2$

$$Assume\ B(p_1 \wedge p_2 \wedge \dots \wedge p_n) \equiv \prod_{k=1}^{n} p_k \ (mod\ 2) \Rightarrow$$

$$B(p_1 \wedge p_2 \wedge \dots \wedge p_n \wedge p_{n+1}) \equiv B\big((p_1 \wedge p_2 \wedge \dots \wedge p_n) \wedge p_{n+1}\big) \equiv \left(\prod_{k=1}^{n} p_k\right) p_{n+1}$$

$$\equiv \prod_{k=1}^{n+1} p_k \ (mod\ 2)$$

2) $B(p_1 \vee p_2) \equiv 1 + \prod_{k=1}^{2} (1 + p_k) \ (mod\ 2)\ by\ Corollary 1.2$

$$Assume\ B(p_1 \vee p_2 \vee \dots \vee p_n) \equiv 1 + \prod_{k=1}^{n} (1 + p_k) \ (mod\ 2) \Rightarrow$$

$$B(p_1 \vee p_2 \vee \dots \vee p_n \vee p_{n+1}) \equiv B\big((p_1 \vee p_2 \vee \dots \vee p_n) \vee p_{n+1}\big)$$

$$\equiv 1 + \left(1 + \left(1 + \prod_{k=1}^{n} (1 + p_k)\right)\right)(1 + p_{n+1})$$

$$\equiv 1 + \left(1 + 1 + \prod_{k=1}^{n} (1 + p_k)\right)(1 + p_{n+1}) \equiv 1 + \left(\prod_{k=1}^{n} (1 + p_k)\right)(1 + p_{n+1})$$

$$\equiv 1 + \left(\prod_{k=1}^{n+1} (1 + p_k)\right) (mod\ 2) \ \blacksquare$$

2. Redefine Boolean laws

2.1) $\boldsymbol{p \to q \equiv {\sim}(p \wedge {\sim}q)}$

Proof)

$B(p \to q) \equiv 1 + p + pq (mod\ 2)$

$B\big({\sim}(p \wedge {\sim}q)\big) \equiv 1 + B(p \wedge {\sim}q) \equiv 1 + p(1+q) \equiv 1 + p + pq (mod\ 2)$ ■

2.2) $\boldsymbol{p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)}$

Proof)

$B(p \leftrightarrow q) \equiv 1 + p + q (mod\ 2)$

$B\big((p \to q) \wedge (q \to p)\big) \equiv B(p \to q)B(q \to p) \equiv (1 + p + pq)(1 + q + qp)$

$\equiv 1 + q + qp + p + pq + p^2q + pq + pq^2 + p^2q^2$

$\equiv 1 + q + qp + p + pq + pq + pq + pq + pq \equiv 1 + q + p + 6pq$

$\equiv 1 + p + q (mod\ 2)$ ■

2.3) $\boldsymbol{{\sim}({\sim}p) \equiv p}$ **(Law of Double Negation)**

Proof)

$B\big({\sim}({\sim}p)\big) \equiv 1 + B({\sim}p) \equiv 1 + (1 + p) \equiv 2 + p \equiv p (mod\ 2)$ ■

2.4) $\boldsymbol{p \wedge p \equiv p,\ \ p \vee p \equiv p}$ **(Laws of Idempotency)**

Proof)

1) $B(p \wedge p) \equiv pp \equiv p^2 \equiv p (mod\ 2)$ ■

2) $B(p \vee p) \equiv p + p + pp \equiv p + p + p^2 \equiv p + p + p \equiv 3p \equiv p (mod\ 2)$ ■

2.5) $\boldsymbol{p \to q \equiv {\sim}q \to {\sim}p}$ **(Contrapositive Law)**

Proof)

$B(p \to q) \equiv 1 + p + pq (mod\ 2)$

$B({\sim}q \to {\sim}p) \equiv 1 + B({\sim}q) + B({\sim}q)B({\sim}p) \equiv 1 + (1 + q) + (1 + q)(1 + p)$

$\equiv 1 + (1 + q) + (1 + p + q + qp) \equiv 3 + 2q + p + pq$

$\equiv 1 + p + pq(\mathrm{mod}\ 2)$ ■

2.6) $\boldsymbol{{\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q, {\sim}(\ p \vee q) \equiv {\sim}p \wedge {\sim}q}$ **(De Morgan's Laws)**

Proof)

1) $B\big({\sim}(p \wedge q)\big) \equiv 1 + B(p \wedge q) \equiv 1 + pq (mod\ 2)$

$B({\sim}p \vee {\sim}q) \equiv B({\sim}p) + B({\sim}q) + B({\sim}p)B({\sim}q) \equiv (1 + p) + (1 + q) + (1 + p)(1 + q)$

$\equiv (1 + p) + (1 + q) + (1 + p + q + qp) \equiv 3 + 2p + 2q + qp$

$\equiv 1 + pq (mod\ 2)$ ■

2) $B\big({\sim}(\ p \vee q)\big) \equiv 1 + B(\ p \vee q) \equiv 1 + (p + q + pq)(mod\ 2)$

$B({\sim}p \wedge {\sim}q) \equiv B(\ {\sim}p)B(\ {\sim}q) \equiv (1 + p)(1 + q) \equiv 1 + q + p + pq (mod\ 2)$■

2.7) $\boldsymbol{p \wedge q \equiv q \wedge p}$, $\boldsymbol{p \vee q \equiv q \vee p}$ (**Commutative Laws**)

Proof)

1) $B(p \wedge q) \equiv pq (mod\ 2), B(q \wedge p) \equiv qp (mod\ 2)$ ■

2) $B(p \vee q) \equiv p + q + pq (mod\ 2), B(q \vee p) \equiv q + p + qp (mod\ 2)$ ■

2.8) $\boldsymbol{(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)}$, $\boldsymbol{(p \vee q) \vee r \equiv p \vee (q \vee r)}$(**Associative Laws**)

Proof)

1) $B\big((p \wedge q) \wedge r\big) \equiv (pq)r \equiv pqr (mod\ 2), B\big(p \wedge (q \wedge r)\big) \equiv p(qr) \equiv pqr (mod\ 2)$ ■

2) $B\big((p \vee q) \vee r\big) \equiv B(p \vee q) + r + B(p \vee q)r \equiv B(p \vee q)(1 + r) + r$

$$\equiv (p + q + pq)(1 + r) + r \equiv (p + q + pq)(1 + r) + r + 1 + 1$$

$$\equiv (p + q + pq + 1)(1 + r) + 1 \equiv \big((p + 1) + q(1 + p)\big)(1 + r) + 1$$

$$\equiv (p + 1)(1 + q)(1 + r) + 1 (mod\ 2)$$

$B\big(p \vee (q \vee r)\big) \equiv p + B(q \vee r) + pB(q \vee r) \equiv p + (1 + p)B(q \vee r)$

$$\equiv p + (1 + p)(q + r + qr) \equiv 1 + 1 + p + (1 + p)(q + r + qr)$$

$$\equiv 1 + (1 + p)(1 + q + r + qr) \equiv 1 + (1 + p)\big(1 + r + q(1 + r)\big)$$

$$\equiv 1 + (1 + p)(1 + q)(1 + r) (mod\ 2)$$ ■

2.9) $\boldsymbol{p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)}$,

$\boldsymbol{p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)}$(**Distributive Laws**)

Proof)

1) $B\big(p \wedge (q \vee r)\big) \equiv pB(q \vee r) \equiv p(q + r + qr) \equiv pq + pr + pqr (mod\ 2)$

$B\big((p \wedge q) \vee (p \wedge r)\big) \equiv B(p \wedge q) + B(p \wedge r) + B(p \wedge q)B(p \wedge r)$

$$\equiv (pq) + (pr) + (pq)(pr) \equiv (pq) + (pr) + (p^2qr)$$

$$\equiv (pq) + (pr) + (pqr) \equiv pq + pr + pqr (mod\ 2)$$ ■

2) $B\big(p \vee (q \wedge r)\big) \equiv p + B(q \wedge r) + pB(q \wedge r) \equiv p + qr + pqr (mod\ 2)$

$B((p \vee q) \wedge (p \vee r)) \equiv B(p \vee q)B(p \vee r) \equiv (p + q + pq)(p + r + pr)$

$$\equiv p^2 + qp + p^2q + pr + qr + pqr + p^2r + qpr + p^2qr$$

$$\equiv p + qp + pq + pr + qr + pqr + pr + qpr + pqr$$

$$\equiv p + 2pq + 2pr + qr + 3pqr \equiv p + qr + pqr (mod\ 2)$$ ■

2.10) $\boldsymbol{p \wedge t \Leftrightarrow p, p \vee t \Leftrightarrow t, p \vee c \Leftrightarrow p, p \wedge c \Leftrightarrow c, c \Rightarrow p, p \Rightarrow t}$

Proof)

1) $B(p \wedge t \leftrightarrow p) \equiv 1 + B(p \wedge t) + B(p) \equiv 1 + (p \times 1) + p \equiv 1 + p + p \equiv 1 + 2p$
$\equiv 1 (mod\ 2)$ ■

2) $B(p \vee t \leftrightarrow t) \equiv 1 + B(p \vee t) + B(t) \equiv 1 + (p + 1 + p \times 1) + 1 \equiv 1 + (p + 1 + p) + 1$
$\equiv 3 + 2p \equiv 1 (mod\ 2)$ ■

3) $B(p \vee c \leftrightarrow p) \equiv 1 + B(p \vee c) + B(p) \equiv 1 + (p + 0 + p \times 0) + p \equiv 1 + (p) + p$
$\equiv 1 + 2p \equiv 1 (mod\ 2)$ ■

4) $B(p \wedge c \leftrightarrow c) \equiv 1 + B(p \wedge c) + B(c) \equiv 1 + (p \times 0) + 0 \equiv 1 + (0) + 0 \equiv 1 (mod\ 2)$ ■

5) $B(c \rightarrow p) \equiv 1 + B(c) + B(c)p \equiv 1 + (0) + (0 \times p) \equiv 1 + (0) + (0) \equiv 1 (mod\ 2)$ ■

6) $B(p \rightarrow t) \equiv 1 + p + pB(t) \equiv 1 + p + (p \times 1) \equiv 1 + p + (p) \equiv 1 + 2p \equiv 1 (mod\ 2)$ ■

2.11) $\boldsymbol{p \wedge \sim p \equiv c, p \vee \sim p \equiv t}$

Proof)

1) $B(p \wedge \sim p) \equiv p(1 + p) \equiv p + p^2 \equiv p + p \equiv 2p \equiv 0 \equiv B(c) (mod\ 2)$ ■

2) $B(p \vee \sim p) \equiv p + (1 + p) + p(1 + p) \equiv p + 1 + p + p + p^2 \equiv 1 + 3p + p \equiv 1 + 4p \equiv 1$
$\equiv B(t) (mod\ 2)$ ■

2.12) $\boldsymbol{p \Rightarrow p \vee q}$ (**Law of addition**)

Proof)

$B(p \rightarrow p \vee q) \equiv 1 + p + pB(p \vee q) \equiv 1 + p + p(p + q + pq) \equiv 1 + p + (p^2 + pq + p^2 q)$
$\equiv 1 + p + (p + pq + pq) \equiv 1 + 2p + 2pq \equiv 1 (mod\ 2)$ ■

2.13) $\boldsymbol{p \wedge q \Rightarrow p, p \wedge q \Rightarrow q}$ (**Laws of Simplification**)

Proof)

1) $B(p \wedge q \rightarrow p) \equiv 1 + B(p \wedge q) + B(p \wedge q)p \equiv 1 + pq + pq(p) \equiv 1 + pq + p^2 q$
$\equiv 1 + pq + pq \equiv 1 + 2pq \equiv 1 (mod\ 2)$ ■

2) $B(p \wedge q \rightarrow q) \equiv 1 + B(p \wedge q) + B(p \wedge q)q \equiv 1 + pq + pq(q) \equiv 1 + pq + pq^2$
$\equiv 1 + pq + pq \equiv 1 + 2pq \equiv 1 (mod\ 2)$ ■

2.14) $\boldsymbol{p \wedge q \rightarrow r \equiv p \rightarrow (q \rightarrow r)}$ (**Exportation Law**)

Proof)

$B(p \wedge q \rightarrow r) \equiv 1 + B(p \wedge q) + B(p \wedge q)r \equiv 1 + pq + pqr (mod\ 2)$

$B\big(p \rightarrow (q \rightarrow r)\big) \equiv 1 + p + pB(q \rightarrow r) \equiv 1 + p + p(1 + q + qr) \equiv 1 + p + p + pq + pqr$
$\equiv 1 + 2p + pq + pqr \equiv 1 + pq + pqr (mod\ 2)$ ■

2.15) $(\boldsymbol{p} \rightarrow \boldsymbol{q}) \wedge (\boldsymbol{q} \rightarrow \boldsymbol{r}) \Rightarrow (\boldsymbol{p} \rightarrow \boldsymbol{r})$(**Transitive Law**)

Proof)

$$\begin{aligned}
B\big((p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)\big) &\equiv 1 + B\big((p \rightarrow q) \wedge (q \rightarrow r)\big)\big(1 + B(p \rightarrow r)\big) \\
&\equiv 1 + (1 + p + pq)(1 + q + qr)\big(1 + (1 + p + pr)\big) \\
&\equiv 1 + (1 + p + pq)(1 + q + qr)(2 + p + pr) \\
&\equiv 1 + (1 + p + pq)(1 + q + qr)(p + pr) \\
&\equiv 1 + (1 + p + pq)(1 + q + qr)(1 + r)p \\
&\equiv 1 + (1 + p + pq + q + pq + pq^2 + qr + pqr + pq^2r)(1 + r)p \\
&\equiv 1 + (1 + p + pq + q + pq + pq + qr + pqr + pqr)(1 + r)p \\
&\equiv 1 + (1 + p + 3pq + q + qr + 2pqr)(1 + r)p \\
&\equiv 1 + (1 + p + pq + q + qr)p(1 + r) \\
&\equiv 1 + (p + p^2 + p^2q + pq + pqr)(1 + r) \\
&\equiv 1 + (p + p + pq + pq + pqr)(1 + r) \equiv 1 + (2p + 2pq + pqr)(1 + r) \\
&\equiv 1 + (pqr)(1 + r) \equiv 1 + (pqr + pqr^2) \equiv 1 + (pqr + pqr) \equiv 1 + (2pqr) \\
&\equiv 1 (mod\ 2) \blacksquare
\end{aligned}$$

2.16) $(\boldsymbol{p} \rightarrow \boldsymbol{q}) \Leftrightarrow (\boldsymbol{p} \wedge {\sim}\boldsymbol{q} \rightarrow \boldsymbol{q} \wedge {\sim}\boldsymbol{q})$(**Reductio ad Absurdum**)

Proof)

$$\begin{aligned}
B\big((p \rightarrow q) \leftrightarrow (p \wedge {\sim}q \rightarrow q \wedge {\sim}q)\big) &\equiv 1 + B(p \rightarrow q) + B(p \wedge {\sim}q \rightarrow q \wedge {\sim}q) \\
&\equiv 1 + (1 + p + pq) + \Big(1 + B(p \wedge {\sim}q)\big(1 + B(q \wedge {\sim}q)\big)\Big) \\
&\equiv 1 + (1 + p + pq) + \Big(1 + \big(p(1 + q)\big)\big(1 + q(1 + q)\big)\Big) \\
&\equiv 1 + (1 + p + pq) + \big(1 + (p + pq)(1 + q + q^2)\big) \\
&\equiv 1 + (1 + p + pq) + \big(1 + (p + pq)(1 + q + q)\big) \\
&\equiv 1 + (1 + p + pq) + \big(1 + (p + pq)(1 + 2q)\big) \\
&\equiv 1 + (1 + p + pq) + \big(1 + (p + pq)\big) \equiv 1 + 2(1 + p + pq) \equiv 1 (mod\ 2) \blacksquare
\end{aligned}$$

2.17) $(\boldsymbol{p} \vee \boldsymbol{q}) \wedge \sim\boldsymbol{p} \Rightarrow \boldsymbol{q}$ (**Disjunctive Syllogism**)

Proof)

$$B\big((p \vee q) \wedge \sim p \rightarrow q\big) \equiv 1 + B\big((p \vee q) \wedge \sim p\big)(1+q) \equiv 1 + B(p \vee q)(1+p)(1+q)$$
$$\equiv 1 + (p + q + pq)(1+p)(1+q)$$
$$\equiv 1 + (p + p^2 + q + qp + pq + p^2q)(1+q)$$
$$\equiv 1 + (p + p + q + qp + pq + pq)(1+q) \equiv 1 + (2p + q + 3pq)(1+q)$$
$$\equiv 1 + (q + pq)(1+q) \equiv 1 + (q + q^2 + pq + pq^2)$$
$$\equiv 1 + (q + q + pq + pq) \equiv 1 + (2q + 2pq) \equiv 1 (mod\ 2)\ \blacksquare$$

2.18) $(\boldsymbol{p} \rightarrow \boldsymbol{q}) \wedge \boldsymbol{p} \Rightarrow \boldsymbol{q}$ (**Modus Ponens**)

Proof)

$$B\big((p \rightarrow q) \wedge p \rightarrow q\big) \equiv 1 + B\big((p \rightarrow q) \wedge p\big)(1+q) \equiv 1 + \big((1 + p + pq)p\big)(1+q)$$
$$\equiv 1 + (p + p^2 + p^2q)(1+q) \equiv 1 + (p + p + pq)(1+q)$$
$$\equiv 1 + (2p + pq)(1+q) \equiv 1 + pq(1+q) \equiv 1 + pq + pq^2 \equiv 1 + pq + pq$$
$$\equiv 1 + 2pq \equiv 1 (mod\ 2)\ \blacksquare$$

2.19) $(\boldsymbol{p} \rightarrow \boldsymbol{q}) \wedge \sim\boldsymbol{q} \Rightarrow \sim\boldsymbol{p}$(**Modus Tollens**)

Proof)

$$B\big((p \rightarrow q) \wedge \sim q \rightarrow \sim p\big) \equiv 1 + B\big((p \rightarrow q) \wedge \sim q\big)\big(1 + B(\sim p)\big)$$
$$\equiv 1 + (1 + p + pq)(1+q)\big(1 + (1+p)\big)$$
$$\equiv 1 + (1 + p + pq)(1+q)(2+p) \equiv 1 + \big(1 + p(1+q)\big)(1+q)p$$
$$\equiv 1 + \big((1+q) + p(1+q)^2\big)p \equiv 1 + \big((1+q) + p(1+q)\big)p$$
$$\equiv 1 + (1+q)(1+p)p \equiv 1 + (1+q)(p+p^2) \equiv 1 + (1+q)(p+p)$$
$$\equiv 1 + (1+q)(2p) \equiv 1 + (1+q)(0) \equiv 1 (mod\ 2)\ \blacksquare$$

2.20) $(\boldsymbol{p} \to \boldsymbol{q}) \wedge (\boldsymbol{r} \to \boldsymbol{s}) \Rightarrow (\sim\boldsymbol{q} \vee \sim\boldsymbol{s} \to \sim\boldsymbol{p} \vee \sim\boldsymbol{r})$,

$(\boldsymbol{p} \to \boldsymbol{q}) \wedge (\boldsymbol{r} \to \boldsymbol{s}) \Rightarrow (\sim\boldsymbol{q} \wedge \sim\boldsymbol{s} \to \sim\boldsymbol{p} \wedge \sim\boldsymbol{r})$(**Destructive Dilemmas**)

Proof)

1) $B\big((p \to q) \wedge (r \to s) \to (\sim q \vee \sim s \to \sim p \vee \sim r)\big)$

$\equiv 1 + B\big((p \to q) \wedge (r \to s)\big)\big(1 + B(\sim q \vee \sim s \to \sim p \vee \sim r)\big)$

$\equiv 1 + (1 + p + q)(1 + r + s)\Big(1 + \Big(1 + B(\sim q \vee \sim s)\big(1 + B(\sim p \vee \sim r)\big)\Big)\Big)$

$\equiv 1 + (1 + p + q)(1 + r + s)\Big(1 + \Big(1 + (1 + qs)\big(1 + (1 + pr)\big)\Big)\Big)$

($\because \forall x, y$: proposition, $B(\sim x \vee \sim y) \equiv B(\sim x) + B(\sim y) + B(\sim x)B(\sim y)$

$\equiv (1 + x) + (1 + y) + (1 + x)(1 + y) \equiv 2 + x + y + (1 + x + y + xy)$

$\equiv 1 + 2x + 2y + xy \equiv 1 + xy (mod\ 2)$)

$\equiv 1 + (1 + p + q)(1 + r + s)(1 + qs)pr$

$\equiv 1 + \big((1 + p + q + r + pr + qr + s + ps + qs)$

$+ (qs + pqs + q^2s + rqs + prqs + q^2rs + qs^2 + ps^2q + q^2s^2)\big)pr$

$\equiv 1 + \big((1 + p + q + r + pr + qr + s + ps + qs)$

$+ (qs + pqs + qs + rqs + prqs + qrs + qs + psq + qs)\big)pr$

$\equiv 1 + (1 + p + q + r + pr + qr + s + ps + 5qs + 2pqs + 2rqs + prqs)pr$

$\equiv 1 + (1 + p + q + r + pr + qr + s + ps + qs + prqs)pr$

$\equiv 1 + (pr + p^2r + qpr + r^2p + p^2r^2 + qpr^2 + spr + p^2sr + qspr + p^2r^2qs)$

$\equiv 1 + (pr + pr + qpr + rp + pr + qpr + spr + psr + qspr + prqs)$

$\equiv 1 + (2pr + 2qpr + 2pr + 2spr + 2qspr) \equiv 1 (mod\ 2)$ ■

2) $B\big((p \to q) \wedge (r \to s) \to (\sim q \wedge \sim s \to \sim p \wedge \sim r)\big)$

$\equiv 1 + B\big((p \to q) \wedge (r \to s)\big)\big(1 + B(\sim q \wedge \sim s \to \sim p \wedge \sim r)\big)$

$\equiv 1 + (1 + p + q)(1 + r + s)\Big(1 + \Big(1 + (1 + q)(1 + s)\big(1 + (1 + p)(1 + r)\big)\Big)\Big)$

$\equiv 1 + (1 + p + q)(1 + r + s)\Big(2 + (1 + q)(1 + s)\big(1 + (1 + p)(1 + r)\big)\Big)$

$\equiv 1 + (1 + p + q)(1 + r + s)(1 + q)(1 + s)\big(1 + (1 + p)(1 + r)\big)$

$\equiv 1 + \big(p + (1 + q)\big)(1 + q)\big(r + (1 + s)\big)(1 + s)\big(1 + (1 + p)(1 + r)\big)$

$\equiv 1 + (p(1 + q) + (1 + q)^2)(r(1 + s) + (1 + s)^2)\big(1 + (1 + p)(1 + r)\big)$

$\equiv 1 + \big(p(1 + q) + (1 + q)\big)\big(r(1 + s) + (1 + s)\big)\big(1 + (1 + p)(1 + r)\big)$

$\equiv 1 + (1 + p)(1 + q)(1 + r)(1 + s)\big(1 + (1 + p)(1 + r)\big)$

$\equiv 1 + (1 + p)(1 + q)(1 + r)(1 + s) + (1 + p)^2(1 + q)(1 + r)^2(1 + s)$

$\equiv 1 + (1 + p)(1 + q)(1 + r)(1 + s) + (1 + p)(1 + q)(1 + r)(1 + s)$

$\equiv 1 + 2(1 + p)(1 + q)(1 + r)(1 + s) \equiv 1 (mod\ 2)$ ■

2.21) $(\boldsymbol{p} \to \boldsymbol{q}) \wedge (\boldsymbol{r} \to \boldsymbol{s}) \Rightarrow (\boldsymbol{p} \vee \boldsymbol{r} \to \boldsymbol{q} \vee \boldsymbol{s})$,

$(\boldsymbol{p} \to \boldsymbol{q}) \wedge (\boldsymbol{r} \to \boldsymbol{s}) \Rightarrow (\boldsymbol{p} \wedge \boldsymbol{r} \to \boldsymbol{q} \wedge \boldsymbol{s})$(**Constructive Dilemmas**)

Proof)

1) $B\big((p \to q) \wedge (r \to s) \to (p \vee r \to q \vee s)\big)$

$\equiv 1 + B\big((p \to q) \wedge (r \to s)\big)\big(1 + B(p \vee r \to q \vee s)\big)$

$\equiv 1 + B(p \to q)B(r \to s)\Big(1 + \Big(1 + B(p \vee r)\big(1 + B(q \vee s)\big)\Big)\Big)$

$\equiv 1 + B(p \to q)B(r \to s)\Big(2 + B(p \vee r)\big(1 + B(q \vee s)\big)\Big)$

$\equiv 1 + B(p \to q)B(r \to s)B(p \vee r)\big(1 + B(q \vee s)\big)$

$\equiv 1 + (1 + p + pq)(1 + r + rs)(p + r + pr)\big(1 + (q + s + qs)\big)$

$\equiv 1 + (1 + p + pq)(1 + r + rs)(p + r + pr)(1 + q)(1 + s)$

$\equiv 1 + \big(1 + p(1 + q)\big)(1 + q)\big(1 + r(1 + s)\big)(1 + s)(p + r + pr)$

$\equiv 1 + \big((1 + q) + p(1 + q)^2\big)\big((1 + s) + r(1 + s)^2\big)(p + r + pr)$

$\equiv 1 + \big((1 + q) + p(1 + q)\big)\big((1 + s) + r(1 + s)\big)(p + r + pr)$

$\equiv 1 + (1 + q)(1 + p)(1 + s)(1 + r)(1 + 1 + p + r + pr)$

$\equiv 1 + (1 + q)(1 + p)(1 + s)(1 + r)\big(1 + (1 + p)(1 + r)\big)$

$\equiv 1 + (1 + q)(1 + p)(1 + s)(1 + r) + (1 + q)(1 + p)^2(1 + s)(1 + r)^2$

$\equiv 1 + (1 + q)(1 + p)(1 + s)(1 + r) + (1 + q)(1 + p)(1 + s)(1 + r)$

$\equiv 1 + 2(1 + q)(1 + p)(1 + s)(1 + r) \equiv 1(mod\ 2)$ ■

2) $B\big((p \to q) \wedge (r \to s) \to (p \wedge r \to q \wedge s)\big)$

$\equiv 1 + B\big((p \to q) \wedge (r \to s)\big)\big(1 + B(p \wedge r \to q \wedge s)\big)$

$\equiv 1 + B(p \to q)B(r \to s)\Big(1 + \Big(1 + B(p \wedge r)\big(1 + B(q \wedge s)\big)\Big)\Big)$

$\equiv 1 + B(p \to q)B(r \to s)\Big(2 + B(p \wedge r)\big(1 + B(q \wedge s)\big)\Big)$

$\equiv 1 + B(p \to q)B(r \to s)B(p \wedge r)\big(1 + B(q \wedge s)\big)$

$\equiv 1 + (1 + p + pq)(1 + r + rs)(pr)(1 + qs)$

$\equiv 1 + (1 + r + rs + p + pr + prs + pq + pqr + pqrs)(pr)(1 + qs)$

$\equiv 1 + (pr + pr^2 + r^2sp + p^2r + p^2r^2 + p^2r^2s + p^2qr + p^2qr^2 + p^2qr^2s)(1 + qs)$

$\equiv 1 + (pr + pr + rsp + pr + pr + prs + pqr + pqr + pqrs)(1 + qs)$

$\equiv 1 + (4pr + 2rsp + 2pqr + pqrs)(1 + qs)$

$\equiv 1 + (pqrs)(1 + qs) \equiv 1 + pqrs + pq^2rs^2 \equiv 1 + pqrs + pqrs \equiv 1 + 2pqrs$

$\equiv 1(mod\ 2)$ ■

3. Solving Boolean equation

Lemma 3.1) $\forall n, i \in \mathbb{N}, \forall x_i$: proposition

$$\left\{ (x_1, x_2, \dots, x_n) \mid \prod_{i=1}^{n} B(x_i) \equiv 1(mod\ 2) \right\} = \left(\bigcap_{i=1}^{n} \{ (x_i) \mid B(x_i) \equiv 1(mod\ 2) \} \right)$$

$$= \{(x_1, x_2, \dots, x_n) \mid (1,1,1, \dots, 1)\}$$

$$\left\{ (x_1, x_2, \dots, x_n) \mid \prod_{i=1}^{n} B(x_i) \equiv 0(mod\ 2) \right\} = \left(\bigcap_{i=1}^{n} \{ (x_i) \mid B(x_i) \equiv 1(mod\ 2) \} \right)^c$$

$$= \{(x_1, x_2, \dots, x_n) \mid (1,1,1, \dots, 1)\}^c$$

Proof)

$Let\ \forall n, i \in \mathbb{N}, \forall x_i$: proposition

$For\ 1 \le i \le n, \{ (x_i) \mid (0), (1)\} = \{ x_i \mid 0,1 \}$

$$Assume\ \exists B(x_i) \equiv 0(mod\ 2) \Rightarrow \prod_{i=1}^{n} B(x_i) \equiv B(x_1)B(x_2) \dots B(x_n) \equiv 0(mod\ 2)$$

$$Therefore\ \prod_{i=1}^{n} B(x_i) \equiv 1(mod\ 2) \Rightarrow \forall B(x_i) \equiv 1(mod\ 2)$$

$$\Rightarrow \left\{ (x_1, x_2, \dots, x_n) \mid \prod_{i=1}^{n} B(x_i) \equiv 1(mod\ 2) \right\} = \left(\bigcap_{i=1}^{n} \{ (x_i) \mid B(x_i) \equiv 1(mod\ 2) \} \right)$$

$$= \{(x_1, x_2, \dots, x_n) \mid (1,1,1, \dots, 1)\}$$

$$And \prod_{i=1}^{n} B(x_i) \equiv 0(mod\ 2) \Rightarrow \sim\left(\forall B(x_i) \equiv 1(mod\ 2)\right)$$

$$\Rightarrow \left\{ (x_1, x_2, \dots, x_n) \mid \prod_{i=1}^{n} B(x_i) \equiv 0(mod\ 2) \right\} = \left(\bigcap_{i=1}^{n} \{ (x_i) \mid B(x_i) \equiv 1(mod\ 2) \} \right)^c$$

$$= \{(x_1, x_2, \dots, x_n) \mid (1,1,1, \dots, 1)\}^c \ \blacksquare$$

Method) Steps of solving Boolean equation

$Step$ 1) We change Boolean equation to product of mod 2 using by 1.∗.

$Step$ 2) We get sets of solution using by Lemma3.1.

Example) $\forall n, i \in \mathbb{N}, \forall x_i$: proposition

- Ex 1) $B(x_1 \wedge \sim x_2 \wedge x_3) \equiv 1(mod\ 2)$

  $Step$ 1)

  $B(x_1 \wedge \sim x_2 \wedge x_3) \equiv 1(mod\ 2) \Rightarrow x_1(1 + x_2)x_3 \equiv 1(mod\ 2)$

  $Step$ 2)

$$\Rightarrow \begin{pmatrix} \{(x_1) \mid x_1 \equiv 1(mod\ 2)\} \\ \cap\ \{(x_2) \mid 1 + x_2 \equiv 1(mod\ 2)\} \\ \cap\ \{(x_3) \mid x_3 \equiv 1(mod\ 2)\} \end{pmatrix} = \begin{pmatrix} \{(x_1) \mid x_1 \equiv 1(mod\ 2)\} \\ \cap\ \{(x_2) \mid x_2 \equiv 0(mod\ 2)\} \\ \cap\ \{(x_3) \mid x_3 \equiv 1(mod\ 2)\} \end{pmatrix}$$

$$= \{(x_1, x_2, x_3) \mid (1,0,1)\}$$

- Ex 2) $B(x_1 \vee x_2 \vee x_3) \equiv 1(mod\ 2)$

  $Step$ 1)

  $B(x_1 \vee x_2 \vee x_3) \equiv 1(mod\ 2) \Rightarrow 1 + (1 + x_1)(1 + x_2)(1 + x_3) \equiv 1(mod\ 2)$

  $\Rightarrow (1 + x_1)(1 + x_2)(1 + x_3) \equiv 0(mod\ 2)$

  $Step$ 2)

$$\Rightarrow \begin{pmatrix} \{(x_1) \mid 1 + x_1 \equiv 1(mod\ 2)\} \\ \cap\ \{(x_2) \mid 1 + x_2 \equiv 1(mod\ 2)\} \\ \cap\ \{(x_3) \mid 1 + x_3 \equiv 1(mod\ 2)\} \end{pmatrix}^c = \begin{pmatrix} \{(x_1) \mid x_1 \equiv 0(mod\ 2)\} \\ \cap\ \{(x_2) \mid x_2 \equiv 0(mod\ 2)\} \\ \cap\ \{(x_3) \mid x_3 \equiv 0(mod\ 2)\} \end{pmatrix}^c$$

$$= \{(x_1, x_2, x_3) \mid (0,0,0)\}^c$$

$$= \{(x_1, x_2, x_3) \mid (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$$

- $Ex\ 3)\ B\big(x_1 \vee (\sim x_2 \wedge x_3)\big) \equiv 1(mod\ 2)$

  $Step\ 1)$

  $$B\big(x_1 \vee (\sim x_2 \wedge x_3)\big) \equiv 1(mod\ 2) \Rightarrow 1 + (1 + x_1)\big(1 + B(\sim x_2 \wedge x_3)\big) \equiv 1(mod\ 2)$$
  $$\Rightarrow 1 + (1 + x_1)(1 + (1 + x_2)x_3) \equiv 1(mod\ 2) \Rightarrow (1 + x_1)(1 + (1 + x_2)x_3) \equiv 0(mod\ 2)$$

  $Step\ 2)$

  $$\Rightarrow \begin{pmatrix} \{(x_1) \mid 1 + x_1 \equiv 1(mod\ 2)\} \\ \cap\ \{(x_2, x_3) \mid 1 + (1 + x_2)x_3 \equiv 1(mod\ 2)\} \end{pmatrix}^c$$
  $$= \begin{pmatrix} \{(x_1) \mid x_1 \equiv 0(mod\ 2)\} \\ \cap\ \{(x_2, x_3) \mid (1 + x_2)x_3 \equiv 0(mod\ 2)\} \end{pmatrix}^c$$
  $$= \begin{pmatrix} \{(x_1) \mid x_1 \equiv 0(mod\ 2)\} \\ \cap \begin{pmatrix} \{(x_2) \mid 1 + x_2 \equiv 1(mod\ 2)\} \\ \cap\ \{(x_3) \mid x_3 \equiv 1(mod\ 2)\} \end{pmatrix}^c \end{pmatrix}^c = \begin{pmatrix} \{(x_1) \mid x_1 \equiv 0(mod\ 2)\} \\ \cap \begin{pmatrix} \{(x_2) \mid x_2 \equiv 0(mod\ 2)\} \\ \cap\ \{(x_3) \mid x_3 \equiv 1(mod\ 2)\} \end{pmatrix}^c \end{pmatrix}^c$$
  $$= \begin{pmatrix} \{(x_1) \mid (0)\} \\ \cap\ \{(x_2, x_3) \mid (0,1)\}^c \end{pmatrix}^c = \begin{pmatrix} \{(x_1) \mid (0)\} \\ \cap\ \{(x_2, x_3) \mid (0,0), (1,0), (1,1)\} \end{pmatrix}^c$$
  $$= \{(x_1, x_2, x_3) \mid (0,0,0), (0,1,0), (0,1,1)\}^c$$
  $$= \{(x_1, x_2, x_3) \mid (0,0,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$$

- $Ex\ 4)\ B\big((x_1 \wedge x_2 \wedge \sim x_3) \vee (\sim x_1 \wedge x_2 \wedge \sim x_3)\big) \equiv 1(mod\ 2)$

  $Step\ 1)$

  $$B\big((x_1 \wedge x_2 \wedge \sim x_3) \vee (\sim x_1 \wedge x_2 \wedge \sim x_3)\big) \equiv 1(mod\ 2)$$
  $$\Rightarrow 1 + \big(1 + B(x_1 \wedge x_2 \wedge \sim x_3)\big)\big(1 + B(\sim x_1 \wedge x_2 \wedge \sim x_3)\big) \equiv 1(mod\ 2)$$
  $$\Rightarrow 1 + \big(1 + x_1x_2(1 + x_3)\big)\big(1 + (1 + x_1)x_2(1 + x_3)\big) \equiv 1(mod\ 2)$$
  $$\Rightarrow \big(1 + x_1x_2(1 + x_3)\big)\big(1 + (1 + x_1)x_2(1 + x_3)\big) \equiv 0(mod\ 2)$$

  $Step\ 2)$

  $$\Rightarrow \begin{pmatrix} \{(x_1, x_2, x_3) \mid 1 + x_1x_2(1 + x_3) \equiv 1(mod\ 2)\} \\ \cap\ \{(x_1, x_2, x_3) \mid 1 + (1 + x_1)x_2(1 + x_3) \equiv 1(mod\ 2)\} \end{pmatrix}^c$$
  $$= \begin{pmatrix} \{(x_1, x_2, x_3) \mid x_1x_2(1 + x_3) \equiv 0(mod\ 2)\} \\ \cap\ \{(x_1, x_2, x_3) \mid (1 + x_1)x_2(1 + x_3) \equiv 0(mod\ 2)\} \end{pmatrix}^c$$
  $$= \begin{pmatrix} \begin{pmatrix} \{(x_1) \mid x_1 \equiv 1(mod\ 2)\} \\ \cap\ \{(x_2) \mid x_2 \equiv 1(mod\ 2)\} \\ \cap\ \{(x_3) \mid (1 + x_3) \equiv 1(mod\ 2)\} \end{pmatrix}^c \\ \cap \begin{pmatrix} \{(x_1) \mid (1 + x_1) \equiv 1(mod\ 2)\} \\ \cap\ \{(x_2) \mid x_2 \equiv 1(mod\ 2)\} \\ \cap\ \{(x_3) \mid (1 + x_3) \equiv 1(mod\ 2)\} \end{pmatrix}^c \end{pmatrix}^c = \begin{pmatrix} \begin{pmatrix} \{(x_1) \mid x_1 \equiv 1(mod\ 2)\} \\ \cap\ \{(x_2) \mid x_2 \equiv 1(mod\ 2)\} \\ \cap\ \{(x_3) \mid x_3 \equiv 0(mod\ 2)\} \end{pmatrix}^c \\ \cap \begin{pmatrix} \{(x_1) \mid x_1 \equiv 0(mod\ 2)\} \\ \cap\ \{(x_2) \mid x_2 \equiv 1(mod\ 2)\} \\ \cap\ \{(x_3) \mid x_3 \equiv 0(mod\ 2)\} \end{pmatrix}^c \end{pmatrix}^c$$
  $$= \begin{pmatrix} \{(x_1, x_2, x_3) \mid (1,1,0)\}^c \\ \cap\ \{(x_1, x_2, x_3) \mid (0,1,0)\}^c \end{pmatrix}^c = \begin{pmatrix} \{(x_1, x_2, x_3) \mid (1,1,0)\} \\ \cup\ \{(x_1, x_2, x_3) \mid (0,1,0)\} \end{pmatrix}$$
  $$= \{(x_1, x_2, x_3) \mid (0,1,0), (1,1,0)\}$$

4. Application

4.1. Proof of validity

Example) H_1, H_2 : Hypothesis, C : Conclusion

H_1 : $x_1 \vee (x_2 \wedge x_3)$

H_2 : $x_1 \vee x_2 \rightarrow x_4$

C : $x_1 \vee x_4$

$H_1 \wedge H_2 \Rightarrow C \Leftrightarrow B(H_1 \wedge H_2 \rightarrow C) \equiv 1(mod\ 2)$

$\Rightarrow 1 + B\big(x_1 \vee (x_2 \wedge x_3)\big)B(x_1 \vee x_2 \rightarrow x_4)\big(1 + B(x_1 \vee x_4)\big) \equiv 1(mod\ 2)$

$B\big(x_1 \vee (x_2 \wedge x_3)\big)B(x_1 \vee x_2 \rightarrow x_4)\big(1 + B(x_1 \vee x_4)\big) \equiv 0(mod\ 2) \Rightarrow$

$$\big(1 + (1 + x_1)(1 + x_2x_3)\big)\Big(1 + \big(1 + (1 + x_1)(1 + x_2)\big)(1 + x_4)\Big)\Big(1 + \big(1 + (1 + x_1)(1 + x_4)\big)\Big)$$

$$\equiv \big(1 + (1 + x_1)(1 + x_2x_3)\big)\Big(1 + \big(1 + (1 + x_1)(1 + x_2)\big)(1 + x_4)\Big)\big((1 + x_1)(1 + x_4)\big)$$

$\equiv 0(mod\ 2)$

- Method 1) deployment

  $Let\ 1 + p \equiv p'\ for\ \forall p$: proposition

  $$\big(1 + (1 + x_1)(1 + x_2x_3)\big)\Big(1 + \big(1 + (1 + x_1)(1 + x_2)\big)(1 + x_4)\Big)\big((1 + x_1)(1 + x_4)\big)$$

  $\equiv (1 + x_1'(x_2x_3)')(1 + (1 + x_1'x_2')x_4')(x_1'x_4')$

  $\equiv (1 + x_1'(x_2x_3)')(1 + x_4' + x_1'x_2'x_4')(x_1'x_4')$

  $\equiv (1 + x_1'(x_2x_3)')(x_1'x_4' + x_1'x_4'x_4' + x_1'x_4'x_1'x_2'x_4')$

  $\equiv (1 + x_1'(x_2x_3)')(x_1'x_4' + x_1'x_4' + x_1'x_2'x_4') \equiv (1 + x_1'(x_2x_3)')(2x_1'x_4 + x_1'x_2'x_4')$

  $\equiv (1 + x_1'(x_2x_3)')(x_1'x_2'x_4') \equiv x_1'x_2'x_4' + x_1'x_2'x_4'x_1'(x_2x_3)'$

  $\equiv x_1'x_2'x_4' + x_1'x_2'x_4'(x_2x_3)' \equiv x_1'x_2'x_4' + x_1'x_2'x_4'(1 + x_2x_3)$

  $\equiv x_1'x_2'x_4' + x_1'x_2'x_4' + x_1'x_2'x_4'x_2x_3 \equiv 2x_1'x_2'x_4' + x_1'x_2'x_4'x_2x_3$

  $\equiv x_1'(1 + x_2)x_4'x_2x_3 \equiv x_1'(x_2 + x_2^2)x_4'x_3 \equiv x_1'(x_2 + x_2)x_4'x_3$

  $\equiv x_1'(2x_2)x_4'x_3 \equiv x_1'(0)x_4'x_3 \equiv 0(mod\ 2)$

  $\therefore\ B(H_1 \wedge H_2 \rightarrow C) \equiv 1(mod\ 2)$

- Method 2) using by solving method of 3.

$$\left(1+(1+x_1)(1+x_2x_3)\right)\left(1+\left(1+(1+x_1)(1+x_2)\right)(1+x_4)\right)\left((1+x_1)(1+x_4)\right) \equiv 0(mod\ 2)$$

$$\Rightarrow \left(\begin{array}{l} \{(x_1,x_2,x_3) \mid 1+(1+x_1)(1+x_2x_3) \equiv 1(mod\ 2)\} \\ \cap\ \{(x_1,x_2,x_4) \mid 1+\left(1+(1+x_1)(1+x_2)\right)(1+x_4) \equiv 1(mod\ 2)\} \\ \cap\ \{(x_1,x_4) \mid (1+x_1)(1+x_4) \equiv 1(mod\ 2)\} \end{array}\right)^c$$

$$= \left(\begin{array}{l} \{(x_1,x_2,x_3) \mid (1+x_1)(1+x_2x_3) \equiv 0(mod\ 2)\} \\ \cap\ \{(x_1,x_2,x_4) \mid \left(1+(1+x_1)(1+x_2)\right)(1+x_4) \equiv 0(mod\ 2)\} \\ \cap\ \{(x_1,x_4) \mid (1+x_1)(1+x_4) \equiv 1(mod\ 2)\} \end{array}\right)^c$$

$$= \left(\begin{array}{l} \left(\begin{array}{l} \{(x_1) \mid (1+x_1) \equiv 1(mod\ 2)\} \\ \cap\ \{(x_2,x_3) \mid (1+x_2x_3) \equiv 1(mod\ 2)\} \end{array}\right)^c \\ \cap \left(\begin{array}{l} \{(x_1,x_2) \mid 1+(1+x_1)(1+x_2) \equiv 1(mod\ 2)\} \\ \cap\ \{(x_4) \mid 1+x_4 \equiv 1(mod\ 2)\} \end{array}\right)^c \\ \cap \left(\begin{array}{l} \{(x_1) \mid (1+x_1) \equiv 1(mod\ 2)\} \\ \cap\ \{(x_4) \mid (1+x_4) \equiv 1(mod\ 2)\} \end{array}\right) \end{array}\right)^c$$

$$= \left(\begin{array}{l} \left(\begin{array}{l} \{(x_1) \mid x_1 \equiv 0(mod\ 2)\} \\ \cap\ \{(x_2,x_3) \mid x_2x_3 \equiv 0(mod\ 2)\} \end{array}\right)^c \\ \cap \left(\begin{array}{l} \{(x_1,x_2) \mid (1+x_1)(1+x_2) \equiv 0(mod\ 2)\} \\ \cap\ \{(x_4) \mid x_4 \equiv 0(mod\ 2)\} \end{array}\right)^c \\ \cap \left(\begin{array}{l} \{(x_1) \mid x_1 \equiv 0(mod\ 2)\} \\ \cap\ \{(x_4) \mid x_4 \equiv 0(mod\ 2)\} \end{array}\right) \end{array}\right)^c$$

$$= \left(\begin{array}{l} \left(\begin{array}{l} \{(x_1) \mid x_1 \equiv 0(mod\ 2)\} \\ \cap \left(\begin{array}{l} \{(x_2) \mid x_2 \equiv 1(mod\ 2)\} \\ \cap\ \{(x_3) \mid x_3 \equiv 1(mod\ 2)\} \end{array}\right)^c \end{array}\right)^c \\ \cap \left(\begin{array}{l} \left(\begin{array}{l} \{(x_1) \mid (1+x_1) \equiv 1(mod\ 2)\} \\ \cap\ \{(x_2) \mid (1+x_2) \equiv 1(mod\ 2)\} \end{array}\right)^c \\ \cap\ \{(x_4) \mid x_4 \equiv 0(mod\ 2)\} \end{array}\right)^c \\ \cap \left(\begin{array}{l} \{(x_1) \mid x_1 \equiv 0(mod\ 2)\} \\ \cap\ \{(x_4) \mid x_4 \equiv 0(mod\ 2)\} \end{array}\right) \end{array}\right)^c$$

$$= \left(\begin{array}{l} \left(\begin{array}{l} \{(x_1) \mid x_1 \equiv 0(mod\ 2)\} \\ \cap \left(\begin{array}{l} \{(x_2) \mid x_2 \equiv 1(mod\ 2)\} \\ \cap\ \{(x_3) \mid x_3 \equiv 1(mod\ 2)\} \end{array}\right)^c \end{array}\right)^c \\ \cap \left(\begin{array}{l} \left(\begin{array}{l} \{(x_1) \mid x_1 \equiv 0(mod\ 2)\} \\ \cap\ \{(x_2) \mid x_2 \equiv 0(mod\ 2)\} \end{array}\right)^c \\ \cap\ \{(x_4) \mid x_4 \equiv 0(mod\ 2)\} \end{array}\right)^c \\ \cap \left(\begin{array}{l} \{(x_1) \mid x_1 \equiv 0(mod\ 2)\} \\ \cap\ \{(x_4) \mid x_4 \equiv 0(mod\ 2)\} \end{array}\right) \end{array}\right)^c = \left(\begin{array}{l} \left(\begin{array}{l} \{(x_1) \mid (0)\} \\ \cap \left(\begin{array}{l} \{(x_2) \mid (1)\} \\ \cap\ \{(x_3) \mid (1)\} \end{array}\right)^c \end{array}\right)^c \\ \cap \left(\begin{array}{l} \left(\begin{array}{l} \{(x_1) \mid (0)\} \\ \cap\ \{(x_2) \mid (0)\} \end{array}\right)^c \\ \cap\ \{(x_4) \mid (0)\} \end{array}\right)^c \\ \cap \left(\begin{array}{l} \{(x_1) \mid (0)\} \\ \cap\ \{(x_4) \mid (0)\} \end{array}\right) \end{array}\right)^c$$

$$= \left(\begin{array}{l} \left(\begin{array}{l} \{(x_1) \mid (0)\} \\ \cap\ \{(x_2, x_3) \mid (1,1)\}^c \end{array} \right)^c \\ \cap \left(\begin{array}{l} \{(x_1, x_2) \mid (0,0)\}^c \\ \cap\ \{(x_4) \mid (0)\} \end{array} \right)^c \\ \cap\ (\{(x_1, x_4) \mid (0,0)\}) \end{array} \right)^c = \left(\begin{array}{l} \left(\begin{array}{l} \{(x_1) \mid (0)\} \\ \cap\ \{(x_2, x_3) \mid (0,0), (0,1), (1,0)\} \end{array} \right)^c \\ \cap \left(\begin{array}{l} \{(x_1, x_2) \mid (0,1), (1,0), (1,1)\} \\ \cap\ \{(x_4) \mid (0)\} \end{array} \right)^c \\ \cap\ (\{(x_1, x_4) \mid (0,0)\}) \end{array} \right)^c$$

$$= \left(\begin{array}{l} \{(x_1, x_2, x_3) \mid (0,0,0), (0,0,1), (0,1,0)\}^c \\ \cap\ \{(x_1, x_2, x_4) \mid (0,1,0), (1,0,0), (1,1,0)\}^c \\ \cap\ \{(x_1, x_4) \mid (0,0)\} \end{array} \right)^c$$

$$= \left(\begin{array}{l} \{(x_1, x_2, x_3) \mid (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\} \\ \cap\ \{(x_1, x_2, x_4) \mid (0,0,0), (0,0,1), (0,1,1), (1,0,1), (1,1,1)\} \\ \cap\ \{(x_1, x_4) \mid (0,0)\} \end{array} \right)^c$$

$$= \left(\begin{array}{l} \{(x_1, x_2, x_3, x_4) \mid (0,1,1,1), (1,0,0,1), (1,0,1,1), (1,1,0,1), (1,1,1,1)\} \\ \cap\ \{(x_1, x_4) \mid (0,0)\} \end{array} \right)^c$$

$$= \emptyset^c = U$$

$Let\ \{(x) | (*)\} = \{(x) | (0), (1)\} \Rightarrow U = \{(x_1, x_2, x_3, x_4) \mid (*,*,*,*)\}$

$\therefore \{(x_1, x_2, x_3, x_4) \mid B(H_1 \wedge H_2 \to C) \equiv 1 (mod\ 2)\} = \{(x_1, x_2, x_3, x_4) \mid (*,*,*,*)\}$

4.2. Simplification

Example)

x	y	z	$F(x,y,z)$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

$$B\Big(\big((\sim x)\wedge y\wedge(\sim z)\big)\vee\big(x\wedge(\sim y)\wedge(\sim z)\big)\vee\big(x\wedge y\wedge(\sim z)\big)\Big)\ \big(by\ SOP(Sum\ of\ Product)\big)$$

$$\equiv 1+\Big(1+B\big((\sim x)\wedge y\wedge(\sim z)\big)\Big)\Big(1+B\big(x\wedge(\sim y)\wedge(\sim z)\big)\Big)\Big(1+B\big(x\wedge y\wedge(\sim z)\big)\Big)$$

$$\equiv 1+\big(1+(1+x)y(1+z)\big)\big(1+x(1+y)(1+z)\big)\ \big(1+xy(1+z)\big)$$
$$\equiv 1+\big(1+(1+x)y(1+z)+x(1+y)(1+z)+(1+x)y(1+z)x(1+y)(1+z)\big)\big(1$$
$$+xy(1+z)\big)$$
$$\equiv 1+(1+(1+x)y(1+z)+x(1+y)(1+z)+(x+x^2)(y+y^2)(1+z)^2)\big(1$$
$$+xy(1+z)\big)$$
$$\equiv 1+\big(1+(1+x)y(1+z)+x(1+y)(1+z)+(x+x)(y+y)(1+z)\big)\big(1+xy(1+z)\big)$$
$$\equiv 1+\big(1+(1+x)y(1+z)+x(1+y)(1+z)+(2x)(2y)(1+z)\big)\big(1+xy(1+z)\big)$$
$$\equiv 1+\big(1+(1+x)y(1+z)+x(1+y)(1+z)\big)\big(1+xy(1+z)\big)$$

$$\equiv 1+\Big(1+\big((1+x)y+x(1+y)\big)(1+z)\Big)\ \big(1+xy(1+z)\big)$$

$$\equiv 1+\big(1+(y+xy+x+xy)(1+z)\big)\big(1+xy(1+z)\big)$$
$$\equiv 1+\big(1+(y+x+2xy)(1+z)\big)\big(1+xy(1+z)\big)$$
$$\equiv 1+\big(1+(y+x)(1+z)\big)\big(1+xy(1+z)\big)$$
$$\equiv 1+(1+(y+x)(1+z)+xy(1+z)+xy(y+x)(1+z)^2)$$
$$\equiv 1+\big(1+(y+x)(1+z)+xy(1+z)+xy(y+x)(1+z)\big)$$

$$\equiv 1+\Big(1+\big((y+x)+xy+xy(y+x)\big)(1+z)\Big)$$

$$\equiv 1+\big(1+(y+x+xy+xy^2+x^2y)(1+z)\big)$$
$$\equiv 1+\big(1+(y+x+xy+xy+xy)(1+z)\big)$$
$$\equiv 1+\big(1+(y+x+xy)(1+z)\big)\equiv 1+1+(y+x+xy)(1+z)$$
$$\equiv (y+x+xy)(1+z)(mod\ 2)$$
$$\therefore\ F(x,y,z)\equiv(y+x+xy)(1+z)\equiv B\big((x\vee y)\wedge(\sim z)\big)(mod\ 2)$$

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