# Elementary formulas for Catalan’s constant

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**abstract**

In this note we give some formulas for Catalan’s constant:

$$G=\int_0^1 \frac{\tan^{-1} x}{x}\,dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=0.915965594177\ldots$$

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## 1. Formulas for Catalan’s constant

$$G=\sum_{n=1}^{\infty}\frac{1}{n^2}\operatorname{Re}\left(i^{n-1}F(n\,,\,n\,;n+1\,;-i)\right) \tag{1}$$

$$G=\sum_{n=1}^{\infty}\frac{1}{n^2}\operatorname{Re}\left(-i\left(\frac{1+i}{2}\right)^n F(1\,,\,n\,;n+1\,;(1+i)/2)\right) \tag{2}$$

$$G=\sum_{n=1}^{\infty}\frac{1}{n^2}\operatorname{Re}\left(\left(\frac{1+i}{2}\right)^{n-1} F(1\,,\,1\,;n+1\,;-i)\right) \tag{3}$$

$$G=\sum_{n=0}^{\infty}\frac{(-1)^n\,2^{-2n}}{(2n+1)^2}\operatorname{Re}(F(2n+1\,,\,2n+1\,;2n+2\,;-i/2)) \tag{4}$$

$$G=2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\operatorname{Re}\left(\left(\frac{2-i}{5}\right)^{2n+1} F(1\,,\,2n+1\,;2n+2\,;(1+2i)/5)\right) \tag{5}$$

$$G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\operatorname{Re}\left(\left(\frac{2-i}{5}\right)^{2n} F(1\,,\,1\,;2n+2\,;-i/2)\right) \tag{6}$$

$$G=\frac{1}{2}\int_0^1\left(\sum_{n=0}^{\infty}\frac{x}{\cosh(x+n)}\right)dx+A \tag{7}$$

donde

$$A=\sum_{n=1}^{\infty} n\tan^{-1}\left(\frac{\operatorname{senh}(1/2)}{\cosh(n+(1/2))}\right)=\sum_{n=1}^{\infty}\tan^{-1}(e^{-n}) \tag{8}$$

$$G=2\tan^{-1}\left(\frac{1}{2}\right)+\sum_{n=1}^{\infty}\frac{c_n}{2n+1} \tag{9}$$

donde

$$c_n = 2\tan^{-1}\left(\frac{1}{2}\right) - 2\sum_{k=0}^{2n-1} \frac{(-1)^k}{2n-k} \binom{2n}{k} \operatorname{Im}\left(\left(1+\frac{i}{2}\right)^{k-2n}\right) \tag{10}$$

$$G = \sum_{n=1}^{\infty} \frac{2^{-n}}{n} \left(\sum_{k=1}^{n} \frac{1}{k}\right) \operatorname{Im}((1+i)^n) \tag{11}$$

$$G = \sum_{n=1}^{\infty} \frac{2^{-n}}{n} \left(\sum_{k=1}^{n} \frac{1}{k}\right) \sum_{m=0}^{[(n-1)/2]} (-1)^m \binom{n}{2m+1} \tag{12}$$

$$G = \sum_{n=1}^{\infty} \frac{2^{-n}}{n} \left(\sum_{k=1}^{n} \frac{1}{k}\right) c_n \tag{13}$$

donde

$$c_{n+2} = 2(c_{n+1} - c_n)\ ,\ \ c_1 = 1\ ,\ \ c_2 = 2 \tag{14}$$

$$G = 2\sum_{n=0}^{\infty} \frac{1}{2n+1} \int_0^1 \frac{x^{2n} \operatorname{sen}((2n+1)x)}{\left(\operatorname{sen} x + \sqrt{(\operatorname{sen} x)^2 + x^2}\right)^{2n+1}}\, dx \tag{15}$$

$$G = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^1 \frac{x^{2n} \cos((2n+1)x)}{\left(\cos x + \sqrt{(\cos n\, x)^2 + x^2}\right)^{2n+1}}\, dx \tag{16}$$

$$G = \sum_{n=2}^{\infty} (\ln n) \tan^{-1}\left(\frac{1}{n^2+n+1}\right) + \int_0^1 \left(\sum_{n=1}^{\infty} \frac{\ln(1+(x/n))}{1+n^2+2xn+x^2}\right) dx \tag{17}$$

$$G = \frac{1}{2}\int_0^1 \frac{x}{\cosh x}\, dx + 2\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)\, e^{-(2n+1)}}{(2n+1)^2} \tag{18}$$

$$G = \frac{1}{1+\cosh 1} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \left(-\frac{2}{1+\cosh 1}\right)^k \binom{n}{k} c_k + 2\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)\, e^{-(2n+1)}}{(2n+1)^2} \tag{19}$$

donde

$$c_k = \int_0^1 x (\cosh x)^k\, dx \tag{20}$$

$$G = \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{1+e^{-2}}{3+e^{-2}}\right)^{n-k} \left(\frac{2}{3+e^{-2}}\right)^{k+1} \left(\frac{1-2(k+1)\,e^{-(2k+1)}}{(2k+1)^2}\right) + 2\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)\, e^{-(2n+1)}}{(2n+1)^2} \tag{21}$$

$$G = \sum_{n=0}^{\infty} e^{-n} \int_0^1 \frac{x\, e^{-x}}{1+e^{-2x-2n}}\, dx + \sum_{n=0}^{\infty} e^{-n-1} \sum_{k=0}^{n} \frac{(-1)^k (n-k+1)\, e^{-k(2n-2k+1)} \left(1-e^{-2k-1}\right)}{2k+1} \tag{22}$$

$$G = \sum_{n=0}^{\infty} 2^{-n-1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k f_k}{2k+1} - \sum_{n=0}^{\infty} \frac{g_n}{(n+1)!} \tag{23}$$

donde

$$f_n = -e(2n-1) + 2n(2n-1) f_{n-1}\ ,\ \ f_0 = e - 1 \tag{24}$$

$$g_n = \frac{\pi}{2n+2} - \frac{1}{n(n+1)} - \frac{n-1}{n+1} g_{n-2}\ ,\ \ g_0 = \frac{\pi}{4} - \frac{\ln 2}{2}\ ,\ \ g_1 = \frac{\pi}{4} - \frac{1}{2} \tag{25}$$

$$G = \sum_{n=0}^{\infty} \frac{1}{2n+1} \sum_{k=n}^{\infty} \frac{(-1)^k\, 2^{-2k}}{2k+1} \binom{2k}{2n} \tag{26}$$

$$G = \sum_{n=0}^{\infty} \sum_{k=[n/2]}^{n} \frac{(-1)^k\, 2^{-2k}}{(2k+1)(2n-2k+1)} \binom{2k}{2n-2k} \tag{27}$$

$$G = \sum_{n=0}^{\infty} (-1/4)^n \sum_{k=0}^{[n/2]} \frac{(-1)^k\, 2^{2k}}{(2k+1)(2n-2k+1)} \binom{2n-2k}{2k} \tag{28}$$

$$G = \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\pi}{4} - a_n \right) \tag{29}$$

donde

$$a_{n+2} = \frac{(4n+2)\, a_{n+1} - (3n-1)\, a_n + (n-1)\, a_{n-1}}{2n+2}, \;\; a_1 = 0, \;\; a_2 = 1/2, \;\; a_3 = 3/4 \tag{30}$$

$$G = \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\pi}{4} - \sum_{k=1}^{n-1} \frac{2^{-k}}{k} \sum_{m=0}^{k-1} (-1)^m \binom{k}{2m+1} \right) \tag{31}$$

$$G = \sum_{n=1}^{\infty} \frac{2^{-n-1}}{n} \left( i(1-i)^n\, \Phi\!\left( \frac{1-i}{2}, 1, n \right) - i(1+i)^n\, \Phi\!\left( \frac{1+i}{2}, 1, n \right) \right) \tag{32}$$

donde

$$\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \;\;, \text{ Lerch transcendent} \tag{33}$$

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n\, a^{2n+1}}{(2n+1)^2} - (\ln a) \tan^{-1} a + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)} \left( \frac{1-a}{1+a} \right)^{2n+2} f(a, n) \tag{34}$$

donde

$$f(a, n) = \frac{1}{a} F(1, 2n+2 \,;\, 2n+3 \,;\, -p) + a\, F(1, 2n+2 \,;\, 2n+3 \,;\, q) \tag{35}$$

$$p = \frac{1-a}{a+a^2}, \;\; q = \frac{a-a^2}{1+a}, \;\; \sqrt{2} - 1 < a \le 1 \tag{36}$$

$$f(a, n) = \frac{1+a}{1+a^2} F(1, 1 \,;\, 2n+3 \,;\, r) + a\, F(1, 2n+2 \,;\, 2n+3 \,;\, q) \tag{37}$$

$$r = \frac{1-a}{1+a^2}, \;\; q = \frac{a-a^2}{1+a}, \;\; 0 < a \le 1 \tag{38}$$

$$G = \frac{1}{2} \sum_{n=0}^{\infty} \int_0^{\pi/2} \left( 1 - \frac{\operatorname{sen} x}{x} \right)^n dx = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int_0^{\pi/2} \left( \frac{\operatorname{sen} x}{x} \right)^k dx \tag{39}$$

$$G = \sum_{n=0}^{\infty} (-1)^n \int_0^1 \left( \frac{x}{\tan^{-1} x} - 1 \right)^n dx = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int_0^1 \left( \frac{x}{\tan^{-1} x} \right)^k dx \tag{40}$$

$$G = \frac{e^{\pi/2} - 1}{2} - \frac{1}{2} \sum_{n=2}^{\infty} \frac{2^{n/2}}{n!} \operatorname{sen}\!\left( \frac{n\pi}{4} \right) \int_0^{\pi/2} \frac{x^n}{\operatorname{sen} x} dx \tag{41}$$

$$G = 2 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n+k}\, 17^{-k}\, (2/17)^{2n-2k+1}\, u(n, k)}{2n-2k+1} \binom{2n-k}{k} \sum_{m=0}^{k} \frac{(-1)^m\, 2^m}{2n-2k+m+1} \binom{k}{m} \tag{42}$$

donde

$$u(n, k) = \operatorname{Re}\!\left( (1+4i)^k\, (4-i)^{2n-2k+1} \right) \tag{43}$$

$$G = \int_1^y \frac{\ln x}{1+x^2}\, dx - \int_0^{1/y} \frac{\ln x}{1+x^2}\, dx \ , \ 1 \le y \le \infty \tag{44}$$

$$G = \frac{z^2}{2\cosh z} + \frac{1}{2}\int_z^{\infty} \frac{x}{\cosh x}\, dx - \frac{1}{2}\int_0^{z/\cosh z} x \cosh(x \cosh(x \cosh(x \ldots)))\, dx \tag{45}$$

donde

$$0 \le z \le z^* = 1.199678640257733 \ldots,\ \ z^* \tanh z^* = 1 \tag{46}$$

$$G = \int_0^{\infty} e^{-x - x\, e^{-2x - 2x\, e^{-2x - 2x\, e^{-2x - \ldots}}}}\, dx \tag{47}$$

Observación 1 : En las fórmulas (1), (2), (3), (4), (5), (6), (35), (37), aparece la función hipergeométrica de Gauss :

$$F(a,\ b\ ;c\ ;x) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \frac{x^n}{n!}\ ,\ \ -1 < x < 1 \tag{48}$$

Observación 2 : $w = x + i\,y$, $i = \sqrt{-1}$, $x,\ y \in \mathbb{R}$, $x = \mathrm{Re}(w)$, $y = \mathrm{Im}(w)$.

# References


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